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MACROSCOPIC ELECTRODYNAMICS

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The basic notions of classical electrodynamics of condensed media are studied: Maxwell's equations in electrodynamics, electrical and magnetic properties of substance, electromagnetic wave propagation, the radiation of charged particles in vacuum and media, the influence of strong bunches on condensed media.

The study aid is developed in the framework of Innovative Educational Programme of the TPU on the direction "Atomic power engineering, nuclear fuel cycle, safe handling with radioactive waste and spent nuclear fuel, accident prevention and counter-terrorism", and can be useful for the students majoring in other physical specialties.

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PREFACE

At present, various unique devices and installations function in different laboratories of the world. They generate, use and register different kinds of radiation: synchrotron, Vavilov-Cherenkov's, transition, radiation at channeling of electrons and positrons in crystals, and etc. Students and masters who choose the specialty associated with the operation of accelerators and other radiation sources, certainly should have knowledge of theoretical foundations and critical characteristics of these types of radiation. They should be familiar with the peculiarities of passing charged particle beam in vacuum, material media, and through the periodic structure. Possession of basic concepts and principles of electrodynamics will enable them to understand the requirements for the characteristics of particle beams in accelerators, but also creatively involved in the development of new programmes and carrying out experiments. This study aid can be considered as an introduction to the specified range. It consists of three chapters, tasks, tests and appendices.

The first chapter presents the electrodynamics in vacuum. Such an order of presentation when Maxwell's equations for vacuum are shown firstly allows to introduce necessary concepts and symbols in the future and to emphasize the experimental basis of Maxwell's equations. Description of all the chapters is in the light of these equations in the subsequent chapters.

In the second chapter it is shown with the help of simple models how to move from microscopic Maxwell's equations to the macroscopic ones of physical media: in electrostatics – for dielectrics, where the environment is composed of polar or nonpolar molecules, and magnetostatics– for magnets in explaining diamagnetism and paramagnetism. Then the general approach of averaging Maxwell's equations for the variable fields is set out, and material equations or equation of communication, without which the system of Maxwell's equations can not be considered complete, are considered. It is shown how you can set the type of permittivity medium with the simplest models. A large part of the second chapter is devoted to elucidate the physical meaning of permittivity. This is due to the importance of the values for the solutions of many problems of electrodynamics of media and the passage of radiation through the environment. Electrodynamic and thermodynamic methods have been used at considering properties of the electrical and magnetic media. Applied methods of quantum mechanics would require a substantial increase of volume of the study aid and should be done in special courses.

The third chapter is devoted to the theory of radiation of relativistic charged particles. There is derivation of Lienard–Wiehert's retarded potentials, on the basis of which the formula for the electromagnetic fields of fast moving point charge have been deduced. The expressions for the intensive radiation, as well as for the spectral-angular characteristics of radiation, have been received. Synchrotron and undulatory radiations are considered closely. Synchrotron radiation has unique properties: a continuous spectrum, lasting until the characteristic frequencies of gamma-radiation, high intensity and high degree of polarization. Due to this synchrotron radiation has become common use in various fields of science and technology. In the third chapter issues relating to other types of radiation and charged particles have been discussed: the transition radiation, coherent bremsstrahlung, radiation at channeling, etc. It should be emphasized that mathematical formalism and physical concepts developed in the theory of synchrotron radiation are also applicable in the analysis of characteristics of radiation of other types. It is suffice to point out such things as the formation of the radiation zone, or the length of coherence, polarization of radiation, coherence, arising from the emission of charged particles, formed in bunches, etc. It is expected that students are familiar with the special theory of relativity. Therefore the formulas of relativistic mechanics and relativistic electrodynamics, essential for understanding of the calculations, are available in the appendix.

Gaussian system of units is taken in the study aid, but the tasks and tests are formulated in such a way that students could use the international system of units (SI). The transition from one system of units to another can be done with the help of the tables in the appendices. Twelve tasks are developed for the each topics of the course, which consist of control questions, exercises and problems. Task execution contributes to better learning of the theoretical material. With the same purpose multiversion tests are given in a number of sections. The present study aid differs from the edition of 2007. The noticed misprints are removed, and tasks, tests and appendices are added.

Chapter 1
MICROSCOPIC ELECTRODYNAMICS IN VACUUM

1. Electrostatics in vacuum

1.1. Coulomb's Law. Electric field

One of the most important notions in the theory of electricity is the charge. It has the following fundamental characteristics: a) electric charges can be either positive or negative; b) an algebraically sum of charges doesn't change in any electrically isolated system, in other words there is the law of electric charge conservation; c) the charge of a body doesn't depend on the choice of the inertial frame of reference, in which it is measured.

The experiments on charge interaction show that the interaction force $\vec{F}$ between charged bodies depends on charges q, dimensions and shapes of the bodies, and their position in space. The main problem of electrostatics is to calculate the interaction force $\vec{F}$ between charged bodies. The problem simplifies owing to the fact that one can apply the superposition principle to electromagnetic interactions, which was proved by experiments:

$$\vec{F}_{A,B+C} = \vec{F}_{A,B} + \vec{F}_{A,C},$$

that is the interaction force between a charged body A and a system of charged bodies $B+C$ equals the sum of the forces acting between the charged bodies: *(A,B)* и *(A,C)*. The interaction force between any system of charges can be presented in the form of vector sum of forces acting between the separate discrete point charges

$$\vec{F} = \sum_{i,k} \vec{F}_{ik} \,. \tag{1.1}$$

Let $(q_1, \vec{r}_1)$ and $(q_2, \vec{r}_2)$ be two point charges, the positions of which in the chosen coordinate system are characterized by the radius-vectors $\vec{r}_1$ and $\vec{r}_2$, correspondingly.

The interaction force of two point charges in vacuum is directed along the straight line connecting these charges. This force is proportional to their values q_1 and q_2 and inversely proportional to the squared distance between them:

$$\vec{F}_{12} = q_1 q_2 \frac{1}{\left|\vec{r}_1 - \vec{r}_2\right|^2} \cdot \frac{\vec{r}_1 - \vec{r}_2}{\left|\vec{r}_1 - \vec{r}_2\right|} \,. \tag{1.2}$$

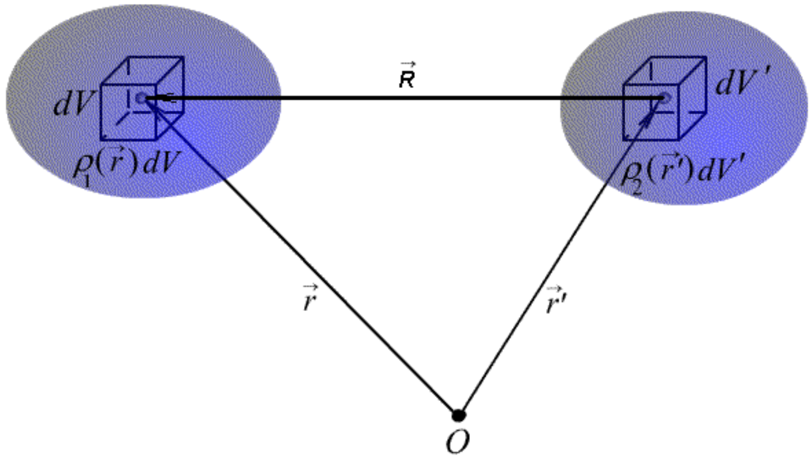


Fig. 1.1. The interaction of two point charges confined in volumes dV and dV.'

It is the attracting force for unlike charges and the repulsive force for like charges. Formula (1.2) expresses the basic qualitative law of electrostatics, known as Coulomb's law. The absolute or gaussian system of units was used to write this law (1.2). In the sequel it is more comfortable to refer to continuous distribution of charges. For calculation, a body is divided into infinitesimal volumes (or surfaces or lengths), and such a notion as the charge density is introduced – volumetric density ρ, surface density σ and line density λ, where

$$\rho = \frac{dq}{dV},\ \sigma = \frac{dq}{dS},\ \lambda = \frac{dq}{dl}, \tag{1.3}$$

and dq is the charge in volume element dV, on surface element dS on line element dl.

According to Fig. 1.1 and to the superposition principle, Coulomb's law should be written in the form:

$$\vec F = \int_{(1)}\int_{(2)} \rho_1(\vec r)dV\rho_2(\vec r')dV'\frac{\vec r - \vec r'}{|\vec r - \vec r'|^3} = \int_{(1)} dV \int_{(2)} dV'\rho_1(\vec r)\rho_2(\vec r')\frac{\vec R}{R^3}, \tag{1.4}$$

Where the integration in the expressions $\int_{(1)} dV$ and $\int_{(2)} dV'$ is over the volumes of the first and the second bodies and $\vec R = \vec r - \vec r'$.

Let's denote

$$\vec E(\vec r) = \int_{(2)} \rho_2(\vec r')\frac{\vec R}{R^3}dV', \tag{1.5}$$

Then the force $\vec F$ takes the form

$$\vec F(\vec r) = \int_{(1)} \rho_1(\vec r)\vec E(\vec r)dV, \tag{1.6}$$

And the task solution is found in two steps: at first $\vec{E}(\vec{r})$ is found, then $\vec{F}$, where $\rho_1(\vec{r})\vec{E}(\vec{r})dV$ – the force acting on the charge $\rho_1(\vec{r})dV$, and $\vec{E}$ – the force, working on a unit charge, electrostatic intensity.

Faraday came to the notion of electrostatic field in the 19th century. He asserted that the effect of a body on another one is made either through a contact or through the intermediate medium. So, he kept to the idea of a close-range interaction, which is opposite to the concept of long-range interaction. The latter was borrowed from the Newton's law of gravity and was developed in the works of Laplace, Ampere, Poisson, Gauss, Green, Francis Neumann, Charles Neumann, Veber, Kirchgoff and many other physicists and mathematicians.

Intuitive and qualitative Faraday's proofs got a strict mathematical form in Maxwell's theory.

Let's calculate the intensity $\vec{E}(\vec{r})$, made by the point charge q_0, which is in point $\vec{r}_0$. For this it is convenient to use the Dirac δ-delta function:

$$\rho(\vec{r}) = q_0 \cdot \delta(\vec{r} - \vec{r}_0)\,. \tag{1.7}$$

According to (1.5) and the property of δ-delta function

$$\vec{E}(\vec{r}) = \int \rho(\vec{r}\,')\frac{\vec{R}}{R^3}dV' = q_0 \int \delta(\vec{r}\,' - \vec{r}_0)\frac{\vec{r} - \vec{r}\,'}{\left|\vec{r} - \vec{r}\,'\right|^3}dV' =$$

$$= q_0 \frac{\vec{r} - \vec{r}\,'}{\left|\vec{r} - \vec{r}\,'\right|^3}\Bigg|_{\vec{r}\,' = \vec{r}_0} = q_0 \frac{\vec{r} - \vec{r}_0}{\left|\vec{r} - \vec{r}_0\right|^3}\,. \tag{1.8}$$

If there is a discrete collection of point charges, one can introduce for it a charge density distribution with the help of δ-delta function for it

$$\rho(\vec{r}) = \sum_{i=1}^{n} q_i \cdot \delta(\vec{r} - \vec{r}_i)\,, \tag{1.9}$$

which corresponds to n point charges, in points $\vec{r}_i$. Substituting (1.9) into (1.5) and integrating with the help of δ-function, we get

$$\vec{E}(\vec{r}) = \sum_{i=1}^{n} q_i \frac{\vec{r} - \vec{r}_i}{\left|\vec{r} - \vec{r}_i\right|^3}\,. \tag{1.10}$$

1.2. Integral and differential forms of Gauss's law

For electric field the calculation of $\vec{E}(\vec{r})$ the relation

$$\vec{E}(\vec{r}) = \int \rho(\vec{r}\,')\frac{\vec{R}}{R^3}dV' \tag{1.11}$$

is not very conveniently.

There is another integral relation, which gives much more possibilities. This is the Gauss's law of flux, one of the most important theorems in electrostatics. The base of the theorem is made by the notion of vector flux, which is one of the main notions of vector analysis. One needs certain symmetry of a task to apply Gauss' law of flux. Suppose a point charge q is at origin of coordinates (Fig. 1.2, Fig. 1.3.).

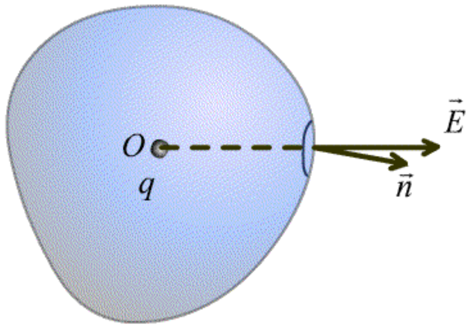


Fig. 1.2. Vector flux $\vec{E}$ through surface dS

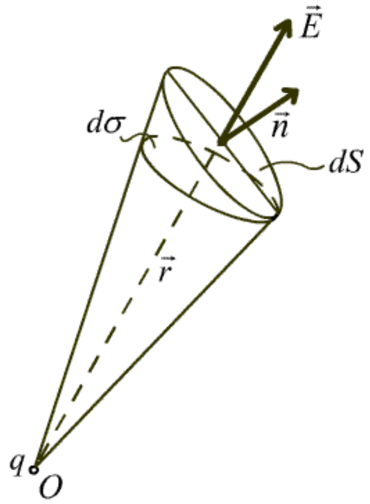


Fig. 1.3. Vector flux $\vec{E}$ through surface dS

Let us draw a closed surface S around the charge; we will choose the external normal to the surface as the positive normal. Let's calculate the vector flux of the intensity $\oint \vec{E}\,\vec{n}\,dS$ through the closed surface S of an arbitrary form. According to (1.8) the intensity made by the point charge at the observation point $\vec{r}$, on the surface S, is equal to

$$\vec{E} = q\frac{\vec{r}}{r^3}. \tag{1.12}$$

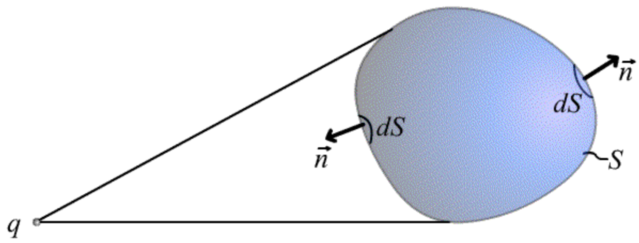


Fig. 1.4. To the derivation of the integral form of Gauss' law for the case when charge q is outside surface S

Consider an area of the surface dS (Fig. 1.4). From the figure it follows,

$$\left(\vec{E}\,\vec{n}\right)dS = \frac{q}{r^3}\left(\vec{r}\,\vec{n}\right)dS = \frac{q}{r^2}\cos\left(\vec{r},\vec{n}\right)dS = \frac{q}{r^2}d\sigma,$$

where $d\sigma = dS\cdot\cos\left(\vec{r},^\wedge\,\vec{n}\right)$.

On the other hand, $\frac{d\sigma}{r^2} = d\Omega$ - the value of the solid angle, at which the area $d\sigma$ is seen from the origin of coordinates. As a result we get

$$\oint \vec{E}\,\vec{n}\,dS = 4\pi q\,. \tag{1.13}$$

If a charge is out of the surface S, then from Fig. 1.4 it follows that the near and far surfaces are seen at the same solid angle, but $\cos(\vec{r},\vec{n})$ is negative for the near surface and positive for the far surface.

That is why

$$\oint (\vec{E}\,\vec{n})\,dS = 0\,. \tag{1.14}$$

In case of several charges

$$\vec{E} = \sum_i \vec{E}_i$$

and

$$\oint (\vec{E}\,\vec{n})\,dS = 4\pi q_{in} = 4\pi \int_V \rho(\vec{r})\,dV\,, \tag{1.15}$$

where V is the volume bounded by the surface S, q_{in} is the algebraic sum of all charges inside the closed surface. The charges that are outside this surface don't influence the quantity of the flux. The relation (1.15) is called the electrostatic Gauss's flux theorem.

The Gauss's theorem is a corollary of Coulomb's law, which by its form doesn't differ from the Newton's gravity law (in both cases the interaction force changes in inverse proportion to the squared distance). That's why the Gauss's theorem can be applied to gravitational fields. In this case a gravitational mass multiplied by the gravitational constant plays the role of the charge.

Let's apply Ostrogradsky and Gauss's theorem known from vector analysis:

$$\oint_S (\vec{E}\,\vec{n}) = \int_V div\,\vec{E}\,dV\,. \tag{1.16}$$

Comparing (1.15) and (1.16), we get the differential equation,

$$div\,\vec{E} = 4\pi\rho\,, \tag{1.17}$$

known as the differential Gauss's theorem.

1.3. Work of electric forces. Potential

Suppose a point charge q is in the origin of coordinates. Let's place a testing unit charge in its field $\vec{E}(\vec{r})$. In this case the field intensity made by the point charge q is

$$\vec{E}(\vec{r}) = q\frac{\vec{r}}{r^3}\,.$$

By the definition, the field intensity is equal to the force working on a unit charge

$$\vec{F} = \vec{E},$$

in this case, the following work is produced

$$d\vec{A} = \vec{E}d\vec{r} = \frac{q}{r^3}\vec{r}\,d\vec{r}\,. \tag{1.18}$$

Since

$$\vec{r}d\vec{r} = \frac{1}{2}d(\vec{r}\,\vec{r}) = \frac{1}{2}d(r^2) = r\,dr,$$

then

$$dA = \frac{q}{r^2}dr = -d\left(\frac{q}{r}\right) = -d\varphi(r). \tag{1.19}$$

The forces of the electrostatic central field are conservative, that is the work of the field done to move a unit charge along a closed path is equal to zero:

$$\oint dA = -\oint d\varphi(r) = 0. \tag{1.20}$$

The quantity introduced in (1.19)

$$\varphi(r) = \frac{q}{r} \tag{1.21}$$

is called the electrostatic potential of the field of a point charge q.

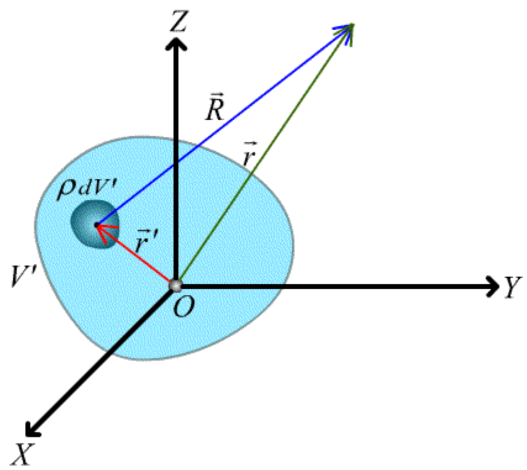


Fig. 1.5. The charge distribution in volume V. Vector $\vec{r}'$ is drawn to the point containing a point charge, vector $\vec{r}$ is drawn to the observation point

For a continuous charge distribution (Fig. 1.5)

$$\varphi(\vec{r}) = \int \frac{\rho(\vec{r}')dV'}{|\vec{r} - \vec{r}'|}. \tag{1.22}$$

It should be noted that the point charge potential $\varphi(\vec{r})$, as follows from (1.19), is defined accurate to an additive constant, which doesn't play any physical role. This constant is equated to the potential value at infinity which is equal to zero. From relations (1.18) and (1.19) it also follows that

$$\oint \vec{E}\, d\vec{r} = 0\,. \tag{1.23}$$

By the Stock's theorem from the vector analysis it is known that the vector circulation on a closed contour is equal to the vector flux through the surface taut on that contour. (Fig. 1.6)

$$\oint_L \vec{E}\, d\vec{r} = \int_S \left(\vec{n} \cdot rot\,\vec{E}\right) dS\,. \tag{1.24}$$

Taking into account (1.23) and (1.24) we come to the differential equation

$$rot\,\vec{E} = 0\,, \tag{1.25}$$

which is valid for all points of the space.

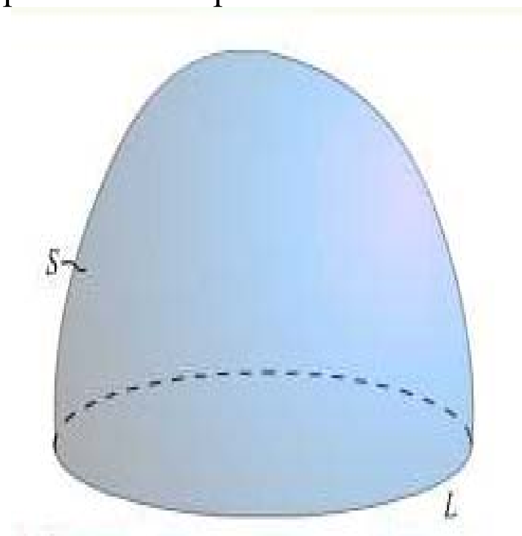


Fig. 1.6. Surface S strained on contour L

The task on computing the forces acting between charges reduces thus to computing the field $\vec{E}(\vec{r})$ from the system of differential equations

$$\begin{cases} div\,\vec{E} = 4\pi\rho, \\ rot\,\vec{E} = 0. \end{cases} \tag{1.26}$$

1.4. Maxwell's equation in electrostatics

According to Helmgoltz' theorem, any vector field $\vec{G}(\vec{r})$ can be found if $div\,\vec{G}$ and $rot\,\vec{G}$ are known.

If the density of the charge distribution ρ in (1.26) is given, these conditions are satisfied, and the field $\vec{E}(\vec{r})$ can be defined.

Equation (1.26) is Maxwell's equation in electrostatics. From the second equation it follows that

$$\vec{E} = -\nabla\varphi, \tag{1.27}$$

as $rot\ grad\,\varphi = 0$. From the first equation, in view of the relation

$$div\ grad = \nabla^2,$$

where ∇^2 is the Laplace operator denoted sometimes as $\Delta \equiv \nabla^2$, we get the Poisson's equation:

$$\nabla^2\varphi = -4\pi\rho. \tag{1.28}$$

In the space regions where there are no charges, it turns into the Laplace's equation

$$\nabla^2\varphi = 0. \tag{1.29}$$

In many cases it is more preferable to calculate first the potential $\varphi(\vec{r})$ from the equation (1.28), and then to find the field intensity $\vec{E}(\vec{r})$ by formula (1.27).

The solution of Poisson's equation has the form:

$$\varphi(r) = \int \frac{\rho(\vec{r}\,')\,dV'}{|\vec{r} - \vec{r}\,'|}. \tag{1.30}$$

If ρ is known, we find the solution for $\varphi(\vec{r})$ by formula (1.30), and then calculate $\vec{E}(\vec{r}) = -\nabla\varphi$. Thus, in electrostatics the main task of computing the forces interacting between the charged bodies is solved:

1) by Coulomb's law;
2) by Maxwell's equation (on the basis of Helmgoltz's theorem).

There are some ways to solve electrostatic tasks:

a) the method based on the application of field superposition principle;
b) the calculation of electric field with the help of the Gauss's law;
c) integration of Laplace and Poisson's equations;
d) the method of electrical pictures.

2. Magnetostatics in vacuum

2.1. Biot-Savart and Ampere's laws

While studying magnetic phenomena some experimental facts have been established. They form the basis of the contemporary electromagnetic theory. These facts are:

1) Force $\vec{F}$, working on a point charge q, which moves at a speed $\vec{\upsilon}$, consists of two components: electric and magnetic and is called Lorenz's force:

$$\vec{F} = q\left\{\vec{E} + \frac{1}{c}\left[\vec{\upsilon}\vec{B}\right]\right\}. \tag{2.1}$$

Here $\vec{E}$ and $\vec{B}$ are the intensity of electric and magnetic fields correspondingly. Formula (2.1) is valid not only for direct but also for alternating electric and magnetic fields. It is possible to determine the magnitudes and directions of vectors $\vec{E}$ and $\vec{B}$ by the Lorenz's force working on a charge. That is why, expression (2.1) can be regarded as the definition of electric and magnetic fields.

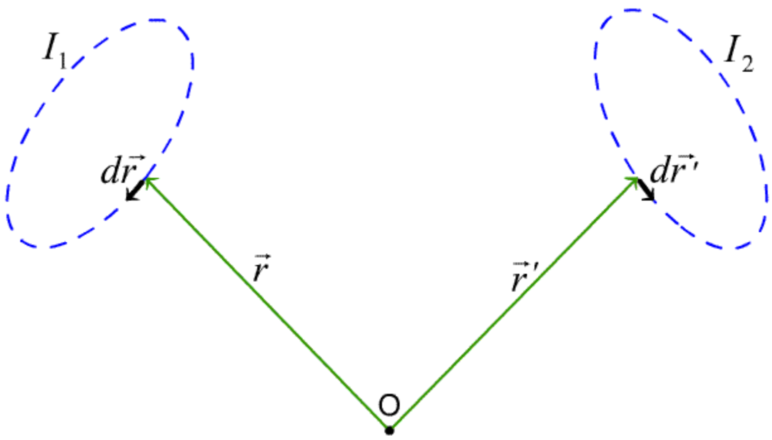


Fig. 2.1. The interactions two infinitely small elements of linear currents

2) Moving charges (currents) induce a magnetic field.

It is more convenient to experiment with magnetic action on moving charges using currents that present the motion of many moving charged particles but not separate charges.

Suppose that there are two currents I_1 and I_2, flowing along the corresponding contours (Fig. 2.1). As known, they interact with each other, and the force of interaction depends on these currents, their forms, directions etc. The main task is to calculate the currents interaction force $\vec{F}_{12}$. It was proved in experiments that for the magnetic interactions the superposition principal is valid:

$$\vec{F}_{12} = \oint_{(1)} d\vec{r} \oint_{(2)} d\vec{r}' .$$

There is a similar formula in electrostatics for interaction of charges:

$$\vec{F}_{12} = \int_{(1)} dV \oint_{2} dV' .$$

According to (1.6) and (1.5) the calculation $\vec{F}_{12}$ in electrostatics was done in two stages: first the intensity $\vec{E}(\vec{r})$ was calculated by formula (1.5), then the interaction force was calculated by formula (1.6):

$$\vec{E}(\vec{r}) = \int_{(2)} \rho_2(\vec{r}')\frac{\vec{R}}{R^3} dV' ,$$

$$\vec{F}_{12} = \int_{(1)} \rho_1(\vec{r})\vec{E}(\vec{r}) dV ,$$

where $\rho_2(\vec{r}')\dfrac{\vec{R}}{R^3}dV' = d\vec{E}$ is the field, made by an infinitely small charge $\rho_2(\vec{r}')dV'$, and $\rho_1(\vec{r})\vec{E}(\vec{r})dV = d\vec{F}$ is the force of the field $\vec{E}(r)$ acting on the infinitely small charge $\rho_1(\vec{r})dV$.

Thus, the following relations are valid for the interaction of charges

$$\vec{F}_{12} = \int_{(1)} d\vec{F}\,, \tag{2.2}$$

$$d\vec{F} = \rho_1(\vec{r})\vec{E}(\vec{r})dV \tag{2.3}$$

and

$$\vec{E}(\vec{r}) = \int_{(2)} d\vec{E}\,, \tag{2.4}$$

$$d\vec{E} = \rho_2(\vec{r}')\frac{\vec{R}}{R^3}dV'\,. \tag{2.5}$$

Similarly we have for current interaction:

$$\vec{F}_{12} = \oint_{(1)} d\vec{F}\,, \tag{2.6}$$

$$d\vec{F} = \frac{I_1}{c}\left[d\vec{r}, \vec{B}\right], \tag{2.7}$$

$$\vec{B} = \oint_{(2)} d\vec{B}\,, \tag{2.8}$$

$$d\vec{B} = \frac{I_2}{c}\left[d\vec{r}', \frac{\vec{R}}{R^3}\right], \tag{2.9}$$

where $\vec{R} = \vec{r} - \vec{r}'$.

Formula (2.7) gives the force acting on a linear element of current. This formula is called Ampere's law, discovered by Ampere. The force acting on a conductor of a finite length is found by integrating by formulas (2.6) and (2.7) over the finite length of the conductor. Formulas (2.8) and (2.9) express the Biot-Savart law to compute the magnetic field quantities $\vec{B}$ and $d\vec{B}$, made by a linear current I_2. Linear currents are the currents whose transverse section is small in comparison with the contour length. In case of currents of finite section, one can apply decomposition of a current on set of indefinitely thin strings of a current (Fig. 2.2):

Threadlike currents satisfy to a condition of linearity, therefore to them formulas (2.6–2.7) with formal replacement are applicable

$$\oint \frac{I[d\vec{r}\ldots]}{c} \leftrightarrow \int \frac{[\vec{j}\ldots]}{c} dV \qquad (2.10)$$

Here

$$I = \int dI \equiv \int \vec{j} \cdot d\vec{S},$$

and the quantity $\vec{j}$, density of a current, determined as

$$\lim_{\Delta S \to 0} = \frac{dI}{dS}$$

and

$$\vec{j} \perp d\vec{S}.$$

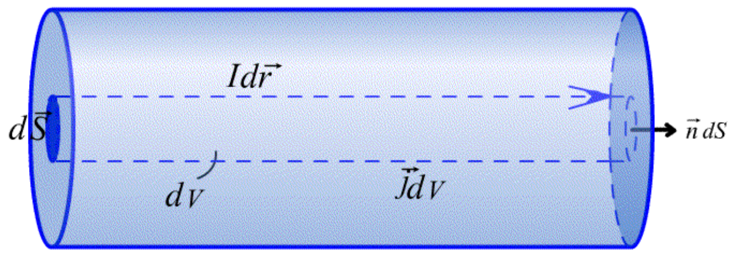


Fig. 2.2. Decomposition of a current of finite section on set of indefinitely thin strings of a current

The quantities $Id\vec{r}$, $\vec{j}dV$ are, accordingly, elements linear and volume currents:

$$I\,d\vec{r} = j\,dS\,d\vec{r} = \vec{j}\,dV.$$

Here we used the fact that the direction of a current density $\vec{j}$ coincides with that of contour element $d\vec{r}$. As a result of the transition from the linear currents to the current of finite section by formula (2.10) the integration $\oint d\vec{r}\ldots$ reduces to the integration $\int dV\ldots$, and we get the Biot-Savart law and Ampere's law for currents of finite section.

Ampere's law:

$$\vec{F}_{12} = \int \frac{1}{c}\left[\vec{j}_1\,\vec{B}\right] dV. \qquad (2.11)$$

The Biot-Savart law:

$$\vec{B} = \int \frac{1}{c}\left[\vec{j}_2\,\frac{\vec{R}}{R^3}\right] dV'. \qquad (2.12)$$

According to (2.12), magnetic field $\vec{B}(\vec{r})$ is a sum of fields made by the separate bulk elements of the current $\vec{j}_2 dV$.

Consider the expression

$$d\vec{F} = \frac{1}{c}\left[\vec{j}\,\vec{B}\right]dV. \tag{2.13}$$

It gives the force acting on the bulk element of current $\vec{j}_2 dV$. If ρ is a charge density, and $\vec{u}$ is an average speed of charge motion, then

$$\vec{j} = \rho \cdot \vec{u}. \tag{2.14}$$

On the other hand, density ρ can be expressed through the concentration n and the charge e of current carriers:

$$\rho = en. \tag{2.15}$$

Then we get

$$\vec{u} = \frac{\sum_i \vec{\upsilon}_i}{n\,dV}. \tag{2.16}$$

Having substituted the expression for $\vec{j}$ (2.14) in the formula for $d\vec{F}$ (2.13) and using (2.15) and (2.16), we find

$$d\vec{F} = \frac{1}{c}\left[en\frac{1}{n\,dV}\sum_i\left[\vec{\upsilon}_i \cdot \vec{B}\right]\right]dV = \sum_i \frac{e}{c}\left[\vec{\upsilon}_i \cdot \vec{B}\right]. \tag{2.17}$$

Force $d\vec{F}$ is expressed through the sum of Lorenz's forces, which act on the charges moving at speeds $\vec{\upsilon}_i$ from magnetic field $\vec{B}$:

$$\vec{f}_i = \frac{e}{c}\left[\vec{\upsilon}_i \cdot \vec{B}\right]. \tag{2.18}$$

Thus, actually, Lorenz's force action on the point charges appears through Ampere's law.

2.2. Vector potential of magnetic field

According to the Biot-Savart law, magnetic field $\vec{B}(\vec{r})$ is calculated by the formula

$$\vec{B}(\vec{r}) = \frac{1}{c}\int\left[\vec{j}(\vec{r}\,')\frac{\vec{R}}{R^3}\right]dV', \tag{2.19}$$

where $\vec{R} = \vec{r} - \vec{r}\,'$, vector $\vec{r}$ is drawn to the observation point, and $\vec{r}\,'$ – to the point of the volume element dV', in which the bulk current $\vec{j}(\vec{r}\,')$ is (Fig. 2.3).

As

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}, \tag{2.20}$$

The formula can be simplified noting that

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \tag{2.21}$$

and

$$\left[\vec{j}(\vec{r}\,')\frac{\vec{R}}{R^3}\right] = \left[\nabla\left(\frac{1}{R}\right), \vec{j}(\vec{r}\,')\right]. \tag{2.22}$$

Let us calculate $rot\left(\frac{1}{R}\vec{j}(\vec{r}\,')\right)$:

$$rot\left(\frac{1}{R}\vec{j}(\vec{r}\,')\right) = \left[\nabla\left(\frac{1}{R}\right), \vec{j}(\vec{r}\,')\right] + \frac{1}{R} rot\,\vec{j}(\vec{r}). \tag{2.23}$$

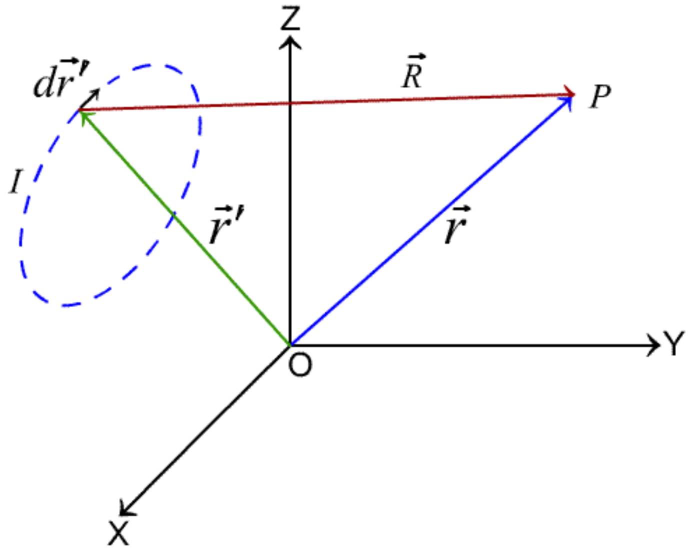


Fig. 2.3. To the calculation of magnetic field $\vec{B}$ created by current I in point P

The second term vanishes, as vector $\vec{j}$ depends on $\vec{r}\,'$, but not on vector $\vec{r}$, whose coordinates are used in the derivation. From formulas (2.22) and (2.23) it follows that

$$\left[\vec{j}(\vec{r}\,')\frac{\vec{R}}{R^3}\right] = rot\left(\frac{1}{R}\vec{j}(\vec{r}\,')\right) \tag{2.24}$$

and

$$\vec{B}(\vec{r}) = rot\,\frac{1}{c}\int\frac{\vec{j}(\vec{r}\,')}{R}dV'. \tag{2.25}$$

Denote

$$\vec{A}(\vec{r}) = \frac{1}{c}\int\frac{\vec{j}(\vec{r})}{R}dV'. \tag{2.26}$$

This quantity $\vec{A}(\vec{r})$ is called the vector potential of a magnetic field, and field $\vec{B}(\vec{r})$ is calculated by the formula:

$$\vec{B}(\vec{r}) = rot\left(\vec{A}(\vec{r})\right). \tag{2.27}$$

First, vector potential $\vec{A}(\vec{r})$ is calculated on the basis of (2.26) by a given current distribution $\vec{j}(\vec{r}\,')$, and then the field $\vec{B}(\vec{r}) = rot\,\vec{A}$ is computed.

For electrostatics the field calculations are also done in two steps:

$$\vec{E} = -\nabla\varphi,$$

where

$$\varphi = \int \frac{\rho(\vec{r}\,')dV'}{R},$$

with the scalar potential $\varphi(\vec{r})$ satisfying the Poisson's equation

$$\nabla^2\varphi = -4\pi\rho\,.$$

One can easily make sure that the vector potential satisfies the equation

$$\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{j}\,. \tag{2.28}$$

2.3. Maxwell's equation for magnetostatics

The main task of magnetostatics is to find the force $\vec{F}$, with which the currents interact. This task is solved consecutively. First, one finds the field $\vec{B}(\vec{r})$, and then the force $\vec{F}$. According to Helmgoltz's theorem, to find the field $\vec{B}(\vec{r})$ it is necessary to know $div\,\vec{B}$ and $rot\,\vec{B}$. From the Biot-Savart law it follows that $\vec{B} = rot\,\vec{A}$. Taking the divergence of vector $\vec{B}$, we obtain

$$div\,\vec{B} = 0, \tag{2.29}$$

as $div\,rot\,\vec{A} = 0$.

Let us calculate $rot\,\vec{B}$:

$$rot\,\vec{B} = rot\,rot\,\vec{A} = grad\,div\,\vec{A} - \nabla^2\vec{A} = grad\,div\,\vec{A} + \frac{4\pi}{c}\vec{j}\,. \tag{2.30}$$

While calculating $\vec{B}$, one can substitute $\vec{A}$ for $\vec{A}' = \vec{A} + \nabla x$, where x is an arbitrary function depending on $\vec{r}$, as $rot\;grad\;x = 0$.

Therefore $rot\,\vec{A}' = rot\,\vec{A}$.

Vector potentials $\vec{A}$ and $\vec{A}'$ give the same field $\vec{B}(\vec{r})$. One can suggest that in the ensemble $\{\vec{A}'\}$ there is such a potential that $div\,\vec{A} = 0$.

Indeed, if $div\,\vec{A}' \neq 0$, for example

$$div\,\vec{A}' = f(\vec{r}), \tag{2.31}$$

then one can choose

$$\vec{A}' = \vec{A} + \nabla x, \tag{2.32}$$

where function *x*, by virtue of arbitrariness of its choice, satisfies the equation

$$\nabla^2 x = -f. \tag{2.33}$$

In this case

$$div\,\vec{A}' = div\,\vec{A} + div\,grad\,x = f(\vec{r}) + \nabla^2 x = 0 \tag{2.34}$$

and

$$rot\,\vec{B} = \frac{4\pi}{c}\vec{j}. \tag{2.35}$$

As a result we obtained the Maxwell's equations for magnetostatics:

$$div\,\vec{B} = 0, \quad rot\,\vec{B} = \frac{4\pi}{c}\vec{j}. \tag{2.36}$$

3. Electrostatic and magnetic fields at a far distance from sources

As we established, electrostatic field $\vec{E}$ and magnetic field $\vec{B}$ are calculated in electrostatics and magnetostatics correspondingly in two steps:

a) for the electrostatic field we compute potential $\varphi(\vec{r})$ by formula (1.22) and then we calculate electric field intensity $\vec{E}(\vec{r})$ by formula (1.27);
b) for the magnetic field, we find vector potential $\vec{A}(\vec{r})$ by formula (2.26) and then we calculate magnetic field $\vec{B}(\vec{r})$ by (2.27).

1. If the distribution of charges, which make the field, is known, then the potential $\varphi(\vec{r})$ is found by integration in (1.22). Under the condition that the density $\rho(\vec{r})$ is other than zero in the limited area of space and decreases sufficiently fast in any direction as $\vec{r} \to \infty$. However, the integral in analytic form can be taken only for a comparatively simple function $\rho(\vec{r}')$. The form of function $\varphi(\vec{r})$ can be quite accurately determined to calculate the electrostatic field at far distances from the charge system. It occurs under the condition $r >> r'_{max}$ (Fig. 1.5).

Writing R in the form

$$R = \sqrt{(\vec{r} - \vec{r}')^2} = \sqrt{r^2 - 2\vec{r}\vec{r}' + r'^2} = r\sqrt{1 - 2\frac{\vec{r}\vec{r}'}{r^2} + \frac{r'^2}{r^2}}. \tag{3.1}$$

And making expansion of R into a series in powers of small parameter r'/r one gets

$$\frac{1}{R}=\frac{1}{r}+\frac{\vec{r}\vec{r}'}{r^3}+\frac{3}{2r}\left(\frac{\vec{r}\vec{r}'}{r^2}\right)^2-\frac{r'^2}{2r^3}+\dots . \quad (3.2)$$

Denoting coordinates $x=x_1$, $y=x_2$, $z=x_3$ and introducing tensor notations, we get

$$\varphi(\vec{r})=\frac{q}{r}+\frac{\vec{p}\ \vec{r}}{r^3}+\frac{Q_{ij}\ x_i\ x_j}{2\ r^5}. \quad (3.3)$$

Here

$$q=\int\rho(\vec{r}')dV' \quad (3.4)$$

is the total charge of the system;

$$\vec{p}=\int\vec{r}'\rho(\vec{r}')dV' \quad (3.5)$$

is a dipole moment of the system;

$$Q_{ij}=\int\rho(\vec{r}')\left(3x_i'x_j'-r'^2\delta_{ij}\right) \quad (3.6)$$

is a tensor of the quadrupole moment of the charge system.

The power series expansion (3.3) is called the electrostatic expansion into multipole moments, or multipoles. Every subsequent expansion term is minor in comparison with the preceding one. The principal term of the expansion is a point charge potential $\varphi_0=q/r$: at far distances the details of the charge distribution become unimportant, and the system makes a potential the same as a point charge q, which is in the origin of coordinates.

If $q=0$, the most important term in the expansion (3.3) is the second term containing a dipole moment, which is defined by formula (3.4) for a continuous charge distribution. In case of a discrete charge system the dipole moment is defined as

$$\vec{p}=\sum_i q_i\cdot\vec{r}_i . \quad (3.7)$$

In special case, when there is a system of two charges equal in magnitude but with opposite signs, we get from (3.7)

$$\vec{p}=(+q)\vec{r}_+ +(-q)\vec{r}_- =q\vec{l} , \quad (3.8)$$

where

$$\vec{l}=\vec{r}_+ -\vec{r}_- . \quad (3.9)$$

The system drawn in Fig. 3.1. is called an elementary dipole.

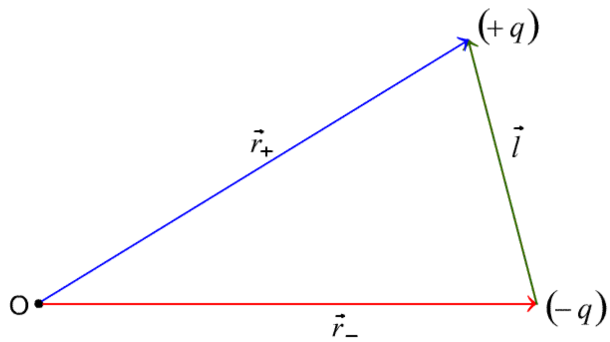


Fig. 3.1. The elementary dipole

The electric field intensity $\vec{E}$ is calculated by formula (1.27):

$$\vec{E} = -\nabla\varphi = \vec{E}_q + \vec{E}_p \, , \tag{3.10}$$

where

$$\vec{E}_q = -\nabla \frac{q}{r} \, , \tag{3.11}$$

$$\vec{E}_p = -\nabla \frac{\vec{p}\,\vec{r}}{r^3} \, . \tag{3.12}$$

As an exercise it is suggested to calculate:

a) $\vec{E}_q$ and $\vec{E}_p$ in Cartesian coordinate frame;

б) in spherical coordinate frame $(\vec{r},\theta,\varphi)$ with polar axe directed along vector $\vec{p}$, it is necessary to find the components of vector $\vec{E}$: E_r, E_θ and E_φ.

2. For magnetostatics, at the first stage, the calculation of field $\vec{B}(\vec{r})$ reduces to the potential calculation $\vec{A}(\vec{r})$ at a far distance from the current system (Fig. 2.3).

In this case the currents can be regarded as “thin” (linear) and according to formula (2.10), relating linear and bulk currents, the expression for $\vec{A}(\vec{r})$ (2.26) takes the form

$$\vec{A}(\vec{r}) = \frac{J}{c}\oint\frac{d\vec{r}\,'}{R} \approx \vec{A}_0 + \vec{A}_m \, . \tag{3.13}$$

Let’s calculate the last two terms $\vec{A}_0$ and $\vec{A}_m$ in approximation; when for value $\frac{1}{R}$ the first two terms of the expansion are taken (3.2). As a result we get

$$\vec{A}_0 = 0 \, , \tag{3.14}$$

because $\oint d\vec{r}\,' = 0$, and

$$\vec{A}_m = \frac{J}{c}\oint d\vec{r}\,'\frac{(\vec{r}\cdot\vec{r}\,')}{r^3}. \qquad (3.15)$$

Consider $\oint d\vec{r}\,'(\vec{r}\cdot\vec{r}\,')$.

The integrand can be transformed as:

$$d\vec{r}\,'(\vec{r}\,\vec{r}\,') = \frac{1}{2}d\vec{r}\,'(\vec{r}\,\vec{r}\,') + \frac{1}{2}d\vec{r}\,'(\vec{r}\,\vec{r}\,') + \frac{1}{2}\vec{r}\,'(\vec{r}\,d\vec{r}\,') - \frac{1}{2}\vec{r}\,'(\vec{r}\,d\vec{r}\,'). \qquad (3.16)$$

The first and the fourth terms in (3.16) are a double vector product

$$\frac{1}{2}[\vec{r}\,[d\vec{r}\,'\times\vec{r}\,']],$$

and the rest of the terms yield the differential with respect to coordinates of vector $\vec{r}\,'$.

As the integral (3.15) is over a closed contour, so

$$\oint\frac{1}{2}d'\,\vec{r}\,'(\vec{r}\,\vec{r}\,') = 0 \text{ and}$$

$$\vec{A}_m = \frac{J}{2cr^3}\oint[\vec{r}\,[d\vec{r}\,'\times\vec{r}\,']] = \left[\frac{\vec{r}}{r^3}\,, -\frac{J}{2c}\oint[\vec{r}\,'\,d\vec{r}\,']\right]. \qquad (3.17)$$

Let's denote

$$\vec{m} = \frac{J}{2c}\oint[\vec{r}\,'\,d\vec{r}\,'] = \frac{1}{2c}\int[\vec{r}\,'\,\vec{j}(\vec{r}\,')]dV'. \qquad (3.18)$$

We returned to the bulk current in formula (3.18) in the latter expression. The value $\vec{m}$ in (3.18) is called a magnetic dipole current moment; vector potential $\vec{A}_m$ (3.17) is expressed through this moment in the dipole approximation

$$\vec{A}_m = \frac{[\vec{m}\,\vec{r}]}{r^3}. \qquad (3.19)$$

Let's compare it with the electrostatic potential in dipole approximation (in formula 3.3)):

$$\varphi_p = \frac{(\vec{p}\,\vec{r})}{r^3}.$$

Let us calculate the magnetic field $\vec{B}(\vec{r})$ in the above mentioned approximation:

$$\vec{B}(\vec{r}) = rot\frac{1}{r^3}[\vec{m}\,\vec{r}]. \qquad (3.20)$$

Applying the rotor calculating rule for the composite functions (see the appendix "Field theory elements") we find

$$\vec{B}(\vec{r}) = \frac{3\vec{r}(\vec{m}\,\vec{r})}{r^5} - \frac{\vec{m}}{r^3}. \qquad (3.21)$$

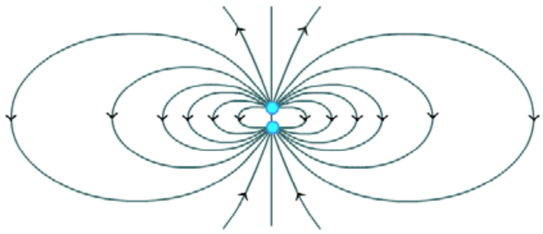


Fig. 3.2. The electric field of two equal charges opposite in sign. At far distance from charges it coincides with the field of electric dipole

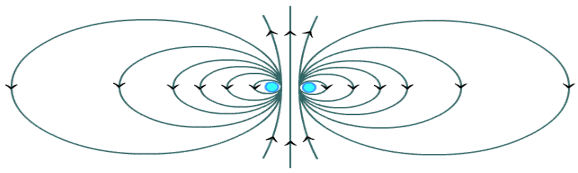


Fig. 3.3. The magnetic field of a loop with current. At a far distance from the loop the field coincides with the field of magnetic dipole

Formula (3.21) coincides (up to notations) with the field intensity of electric dipole (3.12), in which the corresponding calculations are done. The same are the shapes of the field lines of electric and magnetic dipoles (Fig. 3.2, Fig. 3.3).

3.1. The systems of charges and currents in external electric and magnetic fields

Let a charge system be in external electric field $\vec{E}(\vec{r})$, weakly changed within the volume V of the system (Fig. 1.5).

Let's compute the force of field $\vec{E}(\vec{r})$ acting on the charges by formula

$$\vec{F} = \int\limits_V \rho(\vec{r})\vec{E}(\vec{r})dV. \qquad (1.6')$$

Let's make the expansion $\vec{E}(\vec{r})$ in the neighborhood of the coordinate origin in Taylor's series:

$$\vec{E}(\vec{r}) \approx \vec{E}(0) + \left(\vec{r}\,\vec{\nabla}\right)\vec{E}(\vec{r})\Big|_{\vec{r}=0}. \qquad (3.22)$$

And substituting (3.22) into (1.6′), we get

$$\vec{F} = q\vec{E}(0) + \left(\vec{p}\,\vec{\nabla}\right)\vec{E}(0).$$

If the center of the charge cloud is not in the coordinate origin but in point $\vec{r}$, formula (3.22) will take the form

$$\vec{F} = q\vec{E}(\vec{r}) + (\vec{p}\vec{\nabla})\vec{E}(\vec{r}). \tag{3.22'}$$

If we measure the particles deviation caused by the force (3.22') acting on the charge system, one can find the charge q and the dipole electrical moment of the charge system.

The total force working on the charge system consists of separate forces $d\vec{F}$, which work on smaller parts of the system characterized by vector $\vec{r}$. That's why there appeared moments of forces

$$d\vec{N} = [\vec{r}\, d\vec{F}] \tag{3.23}$$

and the total moment

$$\vec{N} = \int_V [\vec{r}\, \vec{E}]\rho\, dV . \tag{3.24}$$

Taking into account only the first term of the expansion (3.22) in formula (3.24) for $\vec{E}(\vec{r})$, we find

$$\vec{N} \approx \int_V [\vec{r}\, \vec{E}(0)]\rho\, dV = [\vec{p}\, \vec{E}(0)] \approx [\vec{p}\, \vec{E}(\vec{r})]. \tag{3.25}$$

A system of charges is characterized by a dipole moment $\vec{p}$, which has a definite orientation with respect to $\vec{E}$. That's why the dipole is influenced by the force, which causes its movement in the space, as well as by force moment $\vec{N}$, which orientates it along the field.

The same conclusions are true for the system of the currents that are in external field $\vec{B}(\vec{r})$. In this case

$$\vec{F} = (\vec{m}\nabla)\vec{B} \tag{3.26}$$

and

$$\vec{N} = [\vec{m}\, \vec{B}], \tag{3.27}$$

where $\vec{m}$ is a magnetic dipole moment of the current system.

3.2. The energy of interaction of charges and currents with an external field

If a charge system $\{q_i\}$ is in an external field $\vec{E}(\vec{r})$, then the work done by the field for the infinitely slight displacement of charges is defined by the formulas:

$$dA = \sum_i \vec{F}_i \cdot d\vec{r}_i = \sum_i q_i \cdot \vec{E}(\vec{r}_i) \cdot d\vec{r}_i = \sum_i q_i \cdot (-\nabla \varphi(\vec{r}_i))d\vec{r}_i = -\sum_i q_i \cdot d\varphi(\vec{r}_i) =$$
$$= -d\sum q_i \cdot \varphi(\vec{r}_i) = -dW . \tag{3.28}$$

The value

$$W = \sum_i q_i \cdot \varphi(\vec{r}_i) \tag{3.29}$$

is the interaction energy of the system of point charges $\{q_i\}$ with the external field.

For a continuous charge distribution

$$W = \int_V \rho(\vec{r})\varphi(\vec{r})\,dV. \tag{3.30}$$

If the volume V, where the charges are, is small, one can think that the potential $\varphi(\vec{r})$ changes slightly within the system size. Doing the expansion $\varphi(\vec{r})$ in Taylor's series

$$\varphi(\vec{r}) \approx \varphi(0) + (\vec{r}\,\vec{\nabla})\varphi(0) = \varphi(0) - (\vec{r}, \vec{E}(0)) \tag{3.31}$$

And substituting (3.31) in formula (3.30), we get

$$W = q\,\varphi(0) - \vec{p}\cdot\vec{E}(0). \tag{3.32}$$

If the center of the charge cloud is not in the coordinate origin but in point $\vec{r}$, then

$$W = q\,\varphi(\vec{r}) - \vec{p}\cdot\vec{E}(\vec{r}). \tag{3.33}$$

So, according to (3.33), the interaction energy W is defined by the interaction energy of the point charge q of the cloud and by the interaction energy of the dipole moment of the system with field $\vec{E}(\vec{r})$.

For the current in the magnetic field

$$W = -\vec{m}\,\vec{B}. \tag{3.34}$$

As compared with (3.33) there is no summand similar to the first term in (3.33), because the magnetic "charges", monopoles, haven't been found experimentally.

4. Alternating electromagnetic field

4.1. The law of electric charge conservation

Suppose, that charges move in arbitrary volume V with the surface S, limiting this volume (Fig. 4.1), and the charge density can change in time, and by the time t the total charge in volume V is

$$q(t) = \int_V \rho(\vec{r}, t)\,dV. \tag{4.1}$$

Then by the time $t + \Delta t$ the charge is defined by the expression

$$q(t + \Delta t) = \int_V \rho(\vec{r}, t + \Delta t)\,dV. \tag{4.2}$$

Let's evaluate the balance of charges incoming and outgoing through surface element ΔS for time Δt at speed $\vec{\upsilon}$. Let's denote the normal to the surface element by $\vec{n}$. As is obvious from Fig. 4.2, the change of the charge quantity for time interval Δt is determined by vector flux $\vec{\upsilon}$ through the surface element ΔS for time interval Δt:

$$\rho(t)\cdot \Delta S\,\upsilon\,\Delta t\cdot\cos\alpha = \Delta S\,\Delta t\left(\vec{\upsilon}\,\vec{n}\right), \tag{4.3}$$

and through the whole surface S:

$$\Delta q = q(t+\Delta t) - q(t) = -\Delta t \oint_S \vec{\upsilon}\,\vec{n}\,\rho(\vec{r},t)\,dS\,. \tag{4.4}$$

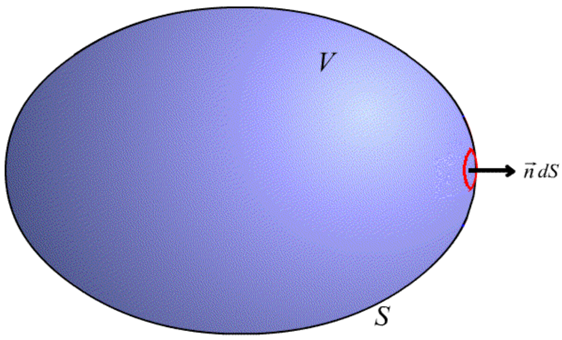


Fig. 4.1. To calculation of balance of charges,
moving with velocity $\vec{v}$ in the areas V limited to surface.
Density of changes eventually

Passing to the limit under the sign of integrals (4.1) and (4.2)

$$\lim_{\Delta t\to 0}\frac{\rho(\vec{r},t+\Delta t) - \rho(\vec{r},t)}{\Delta t} = \frac{\partial\rho}{\partial t}, \tag{4.5}$$

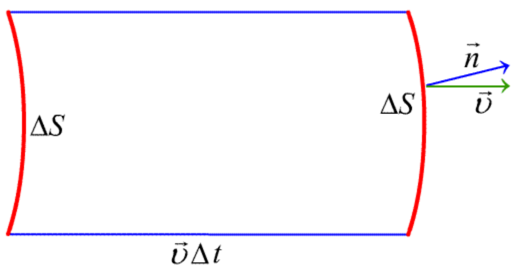
we get from (4.4)

$$\int_V \frac{\partial\rho}{\partial t}dV = -\oint_S \vec{j}\,\vec{n}\,dS\,, \tag{4.6}$$

where

$$\vec{j} = \rho\vec{\upsilon} \tag{4.7}$$

is the electric current density.



4.2. Vector flux $\vec{v}$ through surface element ΔS

Applying the Ostrogradsky and Gauss's theorem to the right-hand side of (4.6), we come to the differential form of the charge conservation law:

$$\frac{\partial \rho}{\partial t} + div\,\vec{j} = 0\,, \qquad (4.8)$$

which is valid at an arbitrary moment of time. Equation (4.6) is the integral form of the charge conservation law.

4.2. The electromagnetic induction law

Suppose there is a closed conductor that moves at speed $\vec{V}$ in a magnetic field. There are electrons inside the conductor, moving at a speed $\vec{\upsilon}'$ relative to the conductor. That's why, according to the speed composition law

$$\vec{\upsilon} = \vec{\upsilon}' + \vec{V}\,.$$

The electron with charge $-e$ is subjected to the Lorenz force of magnetic field $\vec{B}$:

$$\vec{f} = -\frac{e}{c}\left[\vec{\upsilon}\,\vec{B}\right] = -\frac{e}{c}\left[\vec{\upsilon}'\,\vec{B}\right] - \frac{e}{c}\left[\vec{V}\,\vec{B}\right]. \qquad (4.9)$$

The first term in (4.9) presents the force perpendicular to speed $\vec{\upsilon}'$, that is why the value $\left|\vec{\upsilon}'\right|$ doesn't change. The second term accelerates the electron at the expense of

$$\vec{E} = \frac{1}{c}\left[\vec{V}\,\vec{B}\right]. \qquad (4.10)$$

Let's calculate the electro-moving force (emf), defined as the circulation of vector $\vec{E}$ along the contour, which is the conductor

$$\varepsilon = \oint\limits_L \vec{E}\,d\vec{r} = \frac{1}{c}\oint\limits_L \left[\frac{d\vec{l}}{dt}\vec{B}\right]\cdot d\vec{r}\,. \qquad (4.11)$$

Here we use the expression for $\vec{E}$ (4.10) and the definition of $\vec{V}$ as a speed of the position change of the conductor in the magnetic field:

$$\vec{V} = \frac{d\vec{l}}{dt}\,. \qquad (4.12)$$

Consider the integrand in (4.11):

$$\left[d\vec{l}\,\vec{B}\right]\cdot d\vec{r} = \left[d\vec{r}\,d\vec{l}\right]\cdot\vec{B}. \qquad (4.13)$$

In formula (4.13) the property of cyclic permutation of vectors in a parallelepipedal product is used. Let the surface Σ rest on the contour, which is a conductor.

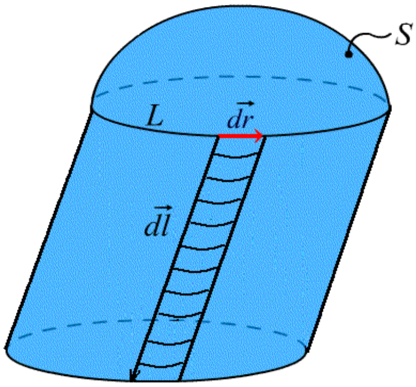


Fig. 4.3. The figure shows the forward motions of contour L, which shifts on $d\vec{l}$ for time dt, surface S bears on contour L. Value $d\vec{r}$ presents element of contour L

From Fig. 4.3, it follows that

$$\left[d\vec{r}\,d\vec{l}\right] = -\vec{n}\,d\sigma, \tag{4.14}$$

where $d\sigma$ is the magnitude of the square, which is "shaded" by contour element $d\vec{r}$ in the space when being shifted by vector $d\vec{l}$, and $\vec{n}$ is a normal vector to the surface.

As a result we get

$$\frac{1}{c}\oint_L \left[\frac{d\vec{l}}{dt}\vec{B}\right] d\vec{r} = -\frac{1}{c}\frac{1}{dt}\left\{\int_\sigma \left(\vec{n}\,\vec{B}\right) d\sigma + \int_S \left(\vec{n}\,\vec{B}\right) dS - \int_S \left(\vec{n}\,\vec{B}\right) dS\right\} =$$

$$= -\frac{1}{c}\frac{1}{dt}\left\{\int_{S+\sigma} \left(\vec{n}\,\vec{B}\right) dS - \int_S \left(\vec{n}\,\vec{B}\right) dS\right\}. \tag{4.15}$$

We added and subtracted the integral over an arbitrary surface S, resting on contour L in formula (4.15).

Vector flux $\vec{B}$ through the surface S is the magnetic flux Φ through contour L:

$$\Phi = \int_S \left(\vec{n}\,\vec{B}\right) dS, \tag{4.16}$$

And the expression in the curly brackets in (4.15) is inaccurate as the change of magnetic flux $d\Phi$. From formulas (4.11), (4.16) and (4.15) it follows that

$$\varepsilon = -\frac{1}{c}\frac{\partial \Phi}{\partial t}. \tag{4.17}$$

That is, the emf in the contour appears due to the change of magnetic flux through the contour. Formula (4.17) presents the electromagnetic induction law in the integral form.

Let's formulate this law in the differential form.

The conductor may remain motionless, but in the course of time the magnetic field $\vec{B}(\vec{r},t)$ changes and so does the flux consequently:

$$\varepsilon = -\frac{1}{c}\frac{\partial \Phi}{\partial t}. \qquad (4.18)$$

Writing $\phi(t) = \int \vec{n}\,\vec{B}(\vec{r},t)dS$ and $\phi(t+\Delta t) = \int \vec{n}\,\vec{B}(\vec{r},t+\Delta t)dS$, and substituting these expressions into (4.18) we have:

$$\frac{d\Phi}{dt} = \int \vec{n}\frac{\partial \vec{B}}{\partial t}dS. \qquad (4.19)$$

On the other hand

$$\varepsilon = \oint_L \vec{E}d\vec{r} = \int_S \left(\vec{n}\,rot\,\vec{E}\right)dS \qquad (4.20)$$

according to the Stock's theorem.

From formulas (4.17), (4.19) and (4.20), the electromagnetic induction law follows in the differential form

$$rot\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}. \qquad (4.21)$$

4.3. Maxwell's equation for alternating electromagnetic field

Suppose, there is a system of moving charges in some region of space (Fig. 4.4). A moving electric charge is an electric current. As the charge never appears and never disappears, charge density $\rho(\vec{r},t)$ and current density $\vec{j}(\vec{r},t)$ obey the charge conservation law in the differential form, in other words the continuity equation:

$$\frac{\partial \rho}{\partial t} + div\,\vec{j} = 0. \qquad (4.8)$$

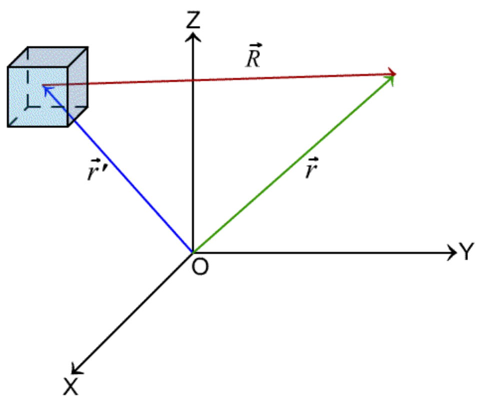


Fig. 4.4. The motion of a charge system in some area of space

Besides, the following equation is valid for moving charges as well as for motionless ones

$$div\,\vec{E} = 4\pi\rho\,. \qquad (1.17)$$

The absence of magnetic charges leads to the equation

$$div\,\vec{B} = 0\,, \qquad (2.36a)$$

And according to the electromagnetic induction law, the alternating magnetic field is connected with the electric field. This connection is described by the equation

$$rot\,\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\,. \qquad (4.21)$$

According to the Helmgoltz's theorem, to find the fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ it is necessary to know their divergences and rotors.

It is known from magnetostatics that

$$rot\,\vec{B} = \frac{4\pi}{c}\vec{j}\,, \qquad (2.36b)$$

where $\vec{j}$ is the density of a conduction flux.

It is easy to make sure that the equation in the form (2.36.b) for changing in time charges and currents don't satisfy the continuity equation. Indeed, if we take div from both parts (2.36.b), we will get

$$div\,rot\,\vec{B} = \frac{4\pi}{c}div\,j\,,$$

that is

$$div\,\vec{j} = 0\,.$$

The right equation for $rot\,\vec{B}$ was suggested by Maxwell. He proceeded from the symmetry of electric $\vec{E}$ and magnetic $\vec{B}$ fields in the equations. The equation containing $rot\,\vec{B}$, he wrote as

$$rot\,\vec{B} = \frac{4\pi}{c}\left(\vec{j} + \vec{j}_{см}\right), \qquad (4.22)$$

where the density of displacement current was introduced:

$$\vec{j}_{см} = \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}\,. \qquad (4.23)$$

In this case the charge conservation law is valid (4.8) in the differential form, and the equations transform into the equations of magnetostatics and electrostatics if ρ and $\vec{j}$ don't depend on time.

Let us write Maxwell's equation for alternating fields:

$$rot\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \tag{4.24a}$$

$$div\,\vec{E} = 4\pi\,\rho, \tag{4.24b}$$

$$rot\,\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j}, \tag{4.24c}$$

$$div\,\vec{B} = 0. \tag{4.24d}$$

Maxwell's hypothesis about displacement currents reduces into the assertion that alternating electric fields are the sources of magnetic fields. This discovery belongs purely to Maxwell and it is similar to the discovery of electromagnetic induction. The latter means that alternating magnetic fields induce electric fields.

Let us formulate a system of Maxwell's equations in the integrated form:

$$\oint_L \vec{E}d\vec{r} = -\frac{1}{c}\int_S \frac{\partial \vec{B}}{\partial t}d\vec{S}, \tag{4.25a}$$

$$\oint_S \left(\vec{E}\,d\vec{S}\right) = 4\pi\int \rho\; dV, \tag{4.25b}$$

$$\oint_L \vec{B}\,d\vec{r} = \frac{4\pi}{c}\int_S \left(\vec{j} + \frac{1}{4\pi}\frac{\partial \vec{E}}{\partial t}\right)d\vec{S}, \tag{4.25c}$$

$$\int_S \left(\vec{B}\,d\vec{S}\right) = 0. \tag{4.25d}$$

4.4. The solution of Maxwell's equations

The solution of Maxwell's equations in electrostatics (1.26) is

$$\vec{E} = -\nabla\varphi, \tag{1.27}$$

where electric potential $\varphi(\vec{r})$ obeys the Poisson's equation

$$\nabla^2\varphi = -4\pi\rho, \tag{1.28}$$

which has the solution

$$\varphi(\vec{r}) = \int \frac{\rho(\vec{r}')\,dV'}{R}, \tag{1.30}$$

$$R = \left|\vec{r} - \vec{r}'\right|.$$

The solution of Maxwell's equations in magnetostatics (2.36) is

$$\vec{B}(\vec{r}) = rot\,\vec{A}(\vec{r}), \tag{2.27}$$

and vector potential $\vec{A}(\vec{r})$ is found as a solution of the equation

$$\nabla^2 \vec{A} = -\frac{4\pi}{c}\vec{j} \tag{2.28}$$

and has the form

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{\vec{j}(\vec{r}\,')}{R} dV'. \tag{2.26}$$

The set of Maxwell's equations (4.24) allows us to find the fields $\vec{E}$ and $\vec{B}$, which are expressed through scalar and vector potentials.

It immediately follows from the equation (4.24d) that

$$\vec{B}(\vec{r},t) = rot\,\vec{A}(\vec{r},t). \tag{4.26}$$

Substituting (4.26) into the equation (4.24a), we have

$$rot\left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0.$$

The equation is fulfilled if

$$\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\nabla\varphi.$$

So, the fields are expressed through the yet unknown potentials $\vec{A}(\vec{r},t)$ and $\varphi(\vec{r},t)$:

$$\vec{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \tag{4.27}$$

$$\vec{B} = rot\,\vec{A}. \tag{4.28}$$

Let's calculate $\vec{A}(\vec{r},t)$ and $\varphi(\vec{r},t)$. Let's substitute (4.27), (4.28) into the equation (4.24c):

$$rot\,rot\,\vec{A} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right). \tag{4.29}$$

As

$$rot\,rot\,\vec{A} = \vec{\nabla}\left(div\,\vec{A}\right) - \nabla^2\vec{A}, \tag{4.30}$$

the equation (4.29) takes the form

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} + \nabla\left(div\,\vec{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t}\right) = \frac{4\pi}{c}\vec{j}. \tag{4.31}$$

The substitution of fields $\vec{E}$ and $\vec{B}$ in equation (4.24b) gives

$$div\left(-\nabla\varphi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)=4\pi\rho.$$

Let us add to the left-hand side of the equation and subtract the term $\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}$, then we get

$$-\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi-\frac{1}{c}\frac{\partial}{\partial t}\left(div\,\vec{A}+\frac{1}{c}\frac{\partial\varphi}{\partial t}\right)=4\pi\rho. \qquad (4.32)$$

Operator $\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is called d'Alembert operator:

$$\Box=\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

Equations (4.31) and (4.32) in these notations take the form:

$$-\Box\vec{A}+\nabla\left(div\,\vec{A}+\frac{1}{c}\frac{\partial\varphi}{\partial t}\right)=\frac{4\pi}{c}\vec{j}, \qquad (4.31a)$$

$$-\Box\varphi-\frac{1}{c}\frac{\partial}{\partial t}\left(div\,\vec{A}+\frac{1}{c}\frac{\partial\varphi}{\partial t}\right)=4\pi\rho. \qquad (4.32a)$$

The equations will simplify if one can prove that

$$div\,\vec{A}+\frac{1}{c}\frac{\partial\varphi}{\partial t}=0. \qquad (4.33)$$

Really, having chosen the potentials

$$\varphi'=\varphi-\frac{1}{c}\frac{\partial}{\partial t}x(\vec{r},t), \qquad (4.34)$$

$$\vec{A}'=\vec{A}+\nabla x(\vec{r},t), \qquad (4.35)$$

where $x(\vec{r},t)$ is an arbitrary function, it is easy to make sure that physical fields $\vec{E}'$ and $\vec{B}'$,written through the new potentials, coincide with the fields $\vec{E}$ and $\vec{B}$, written through the former potentials, that is $\vec{E}'=\vec{E}$, $\vec{B}'=\vec{B}$.

Generally, there is an infinite set of potentials bringing to the same fields. One can assume that there are the potentials among them such that

$$div\,\vec{A}+\frac{1}{c}\frac{\partial\varphi}{\partial t}=0.$$

If

$$div\,\vec{A}+\frac{1}{c}\frac{\partial\varphi}{\partial t}=f(\vec{r},t)\neq 0,$$

the calculations with the potentials (4.34) and (4.35) give

$$div\,\vec{A}'+\frac{1}{c}\frac{\partial\varphi'}{\partial t}=div\left(\vec{A}+\nabla x\right)+\frac{1}{c}\frac{\partial}{\partial t}\left(\varphi-\frac{1}{c}\frac{\partial x}{\partial t}\right)=f\left(\vec{r},t\right).$$

Having chosen a function $x(\vec{r},t)$ which satisfies the equation

$$\Box x=-f,$$

we come to the condition $div\,\vec{A}'+\frac{1}{c}\frac{\partial\varphi'}{\partial t}=0$.

Thus, one can always think that the d'Alembert equations hold true

$$\varphi=-4\pi\rho, \tag{4.36}$$

$$\vec{A}=-\frac{4\pi}{c}\cdot\vec{j} \tag{4.37}$$

with the additional condition of Lorenz

$$div\,\vec{A}+\frac{1}{c}\frac{\partial\varphi}{\partial t}=0. \tag{4.38}$$

The equations (4.36), (4.37) and the condition (4.33) present a set of equations that are equivalent to Maxwell's equations.

4.5. Retarded potentials

Suppose that in the wave equations

$$\nabla^2\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-4\pi\,\rho\left(\vec{r},t\right), \tag{4.36}$$

$$\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}=-\frac{4\pi}{c}\vec{j}\left(\vec{r},t\right). \tag{4.37}$$

The sources ρ and $\vec{j}$ are arbitrary coordinate and time functions placed in a finite region of space. In this case, it is possible to find the solutions to the inhomogeneous equations (4.36), that is the potentials $\varphi(\vec{r},t)$, $\vec{A}(\vec{r},t)$, made by these sources in the whole space. Generally speaking, the fields made by charged particles can influence the motion of these particles, but this influence can be often ignored. The solution of a given task is found with the help of Green's function $G(\vec{r},t;\vec{r}',t')$, determined as the solution of the equation

$$\nabla^2G-\frac{1}{c^2}\frac{\partial^2G}{\partial t^2}=-4\pi\,\delta(\vec{r}-\vec{r}')\delta(t-t') \tag{4.38}$$

for the unbounded space.

Then the solution of the wave equations is written by means of Green's function in the form:

$$\vec{A}(\vec{r},t)=\frac{1}{c}\int G(\vec{r},t;\vec{r}',t')\vec{j}(\vec{r}',t')dV'dt'. \tag{4.39}$$

The same expression is also written for $\varphi(\vec{r},t)$. Operating with d'Alembertian on the potentials, it is easy to verify that the equations (4.36) are satisfied. A careful mathematical analysis shows that, actually, a large class of Green's functions satisfies the equation (4.38). Using these functions we get different solutions for $\vec{A}(\vec{r},t)$. To find the only solution for a given task it turns out to be enough to use the causality principle. According to this principle, the cause (the motion of a charge in the source), which results in the radiation, is prior to the consequence (the field excitation at the point of observation). The potentials that satisfy this condition are called retarded potentials. The mathematical proof of this unique choice is described in Jackson's monograph or in the course "Classical electrodynamics" by M.M. Bredov, V.V. Rumjanzev and I.N. Toptygin. We will only notice that Green's retarded function has the form:

$$G^R(\vec{r}-\vec{r}\,',t-t')=\frac{1}{R}\delta\left(t-t'-\frac{R}{c}\right), \tag{4.40}$$

where $\vec{R}=|\vec{r}-\vec{r}\,'|$ and the retarded potentials can be written as:

$$\varphi^R(\vec{r},t)=\int\frac{\rho\left(\vec{r}\,',t-\frac{R}{c}\right)}{R}dV', \tag{4.41a}$$

$$\vec{A}^R(\vec{r},t)=\frac{1}{c}\int\frac{\vec{j}\left(\vec{r}\,',t-\frac{R}{c}\right)}{R}dV'. \tag{4.41б}$$

From equations (4.41) it follows that the field at point $\vec{r}$ at moment t is defined by the state of the sources in the previous moment of time $t'=t-\frac{R}{c}$, which differs by the time of electromagnetic disturbance propagation from the source to the point of observation.

4.6. Electromagnetic field energy. Umov- Pointing's vector

Suppose, the charges distributed in the space with density $\rho(\vec{r},t)$ move in electromagnetic field $\vec{B}(\vec{r},t)$ and $\vec{E}(\vec{r},t)$. The charge in the volume dV is acted by Lorenz's force:

$$\rho\, dV\left\{\vec{E}+\frac{1}{c}\left[\vec{\upsilon}\,\vec{B}\right]\right\}. \tag{4.42}$$

And while the charges shift by the way $d\vec{r}$, the following work is done

$$\rho\, dV\left\{\vec{E}d\vec{r}+\frac{1}{c}\left[\vec{\upsilon}\,\vec{B}\right]d\vec{r}\right\}. \tag{4.43}$$

Dividing (4.43) by time dt, we get the work performed per unit of time

$$\rho\, dV\left\{\vec{E}\vec{\upsilon}+\frac{1}{c}\left[\vec{\upsilon}\,\vec{B}\right]\cdot\vec{\upsilon}\right\}=\vec{j}\vec{E}\,dV\ . \qquad (4.44)$$

Here we take into account that the second term equals zero and $\rho\vec{\upsilon}=j$.

The work done by the field on all the charges per time unit is equal to

$$\frac{dA}{dt}=\int\left(\vec{j}\,\vec{E}\right)dV=\frac{dT}{dt}, \qquad (4.45)$$

where T is the kinetic energy of all the charges of the observed system.

Let us transform Maxwell's equations (4.24.a) and (4.24.c). Let's make a scalar multiplication of $\vec{B}$ by $rot\,\vec{E}$ and subtract $\vec{E}\cdot rot\,\vec{B}$:

$$\vec{B}\cdot rot\,\vec{E}-\vec{E}\cdot rot\,\vec{B}=div\left[\vec{E}\,\vec{B}\right]. \qquad (4.46)$$

On the other hand we obtain

$$-\frac{1}{c}\vec{B}\frac{\partial\vec{B}}{\partial t}-\frac{1}{c}\vec{E}\frac{\partial\vec{E}}{\partial t}-\frac{4\pi}{c}\vec{j}\vec{E},$$

that is, we come to the relation

$$div\left[\vec{E}\,\vec{B}\right]=-\frac{4\pi}{c}\left(\vec{j}\vec{E}\right)-\frac{1}{2c}\frac{\partial}{\partial t}\left(B^2+E^2\right),$$

which is convenient to write down in the form

$$\frac{\partial}{\partial t}\frac{E^2+B^2}{8\pi}=-\left(\vec{j}\vec{E}\right)-div\frac{c}{4\pi}\left[\vec{E}\,\vec{B}\right]. \qquad (4.47)$$

Let's introduce the notations

$$\omega=\frac{E^2+B^2}{8\pi}, \qquad (4.48)$$

$$\vec{S}=\frac{c}{4\pi}\left[\vec{E}\,\vec{B}\right]. \qquad (4.49)$$

The physical sense of these values can be defined on taking integral over the space arguments in both parts of the equation (4.47). First let's integrate over the whole space, that is x, y, z change between $-\infty$ and $+\infty$:

$$\int\frac{\partial\omega}{\partial t}dV=\frac{d}{dt}\int\omega\, dV\equiv\frac{d}{dt}W, \qquad (4.50)$$

where

$$\int \omega \, dV = W , \tag{4.51}$$

$$\int_{\infty} (\vec{j}\,\vec{E}) dV = \frac{dT}{dt} , \tag{4.52}$$

$$\int_{\infty} div \, \vec{S} dV = \oint_{\infty} \vec{S}\,\vec{n}\, dS = 0 . \tag{4.53}$$

In the expression (4.53) the Ostrogradsky-Gauss' law of flux was used. This law reduces the integral of vector divergence $\vec{S} = \frac{c}{4\pi}[\vec{E}\,\vec{B}]$ to that of vector flux $\vec{S}$ through the unlimited surface. But fields $\vec{E}$ and $\vec{B}$, included in the expression for $\vec{S}$ at infinity are equal to zero by physical implication.

As a result we come to the equation

$$\frac{dW}{dt} = -\frac{dT}{dt} ,$$

or

$$\frac{d}{dt}(W + T) = 0 . \tag{4.54}$$

The constant value $W + T$ gives the total energy of moving charges T and the electromagnetic field energy W.

Value ω is the energy density of electromagnetic field. To find out the physical sense of vector $\vec{S}$, it is necessary to integrate the equation (4.47) over the infinite space:

$$\frac{d}{dt} W_V = -\frac{dT_V}{dt} - \oint \vec{S}\,\vec{n}\, dS . \tag{4.55}$$

The energy of electromagnetic field is spent on the kinetic energy of moving charges and on radiating energy of the field through the surface, which bounds the observed volume V. That is why the value $(\vec{S}\,\vec{n})dS$ is the electromagnetic field energy that is emitted through surface dS in a time unit. Vector $\vec{S}$ shows the direction of electromagnetic energy propagation. $S = |\vec{S}|$ is the quantity of energy which goes through 1 cm^2 per 1 sec. Value $\vec{S}$ is called the energy flux or Umov -Pointing's vector. Vector $\vec{S}$ depends on the position in the space at a given moment of time:

$$\vec{S} \equiv \vec{S}(\vec{r}, t) .$$

The above-mentioned formulae express the energy conservation law for the charges moving in the electromagnetic field made by these charges.

5. The Propagation of electromagnetic waves in vacuum and in media

5.1. Wave equations

It follows from Maxwell's electrodynamics that there is a principally new physical phenomenon, which was discovered by Maxwell. This phenomenon is electromagnetic waves or disturbances which propagate in space at a certain speed.

Let's consider Maxwell's equations (4.24), written for vacuum in the presence of electric charges with density ρ and electric currents with density $\vec{j}$. The equations are asymmetric with respect to the fields $\vec{E}$ and $\vec{B}$, and this asymmetry is conditioned by currents and charges and by the absence of magnetic charges. Let $\rho = 0$, $\vec{j} = 0$ in these equations. The equations take a symmetric form:

$$rot\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \tag{5.1}$$

$$div\,\vec{E} = 0, \tag{5.2}$$

$$rot\,\vec{B} = -\frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \tag{5.3}$$

$$div\,\vec{B} = 0. \tag{5.4}$$

The term with displacement current $\frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ plays the most important role here. Its presence in equation (5.3), as well as the similar term for magnetic field in equation (5.1), means the possibility of electromagnetic waves to appear. The alternating in time magnetic field makes an alternating electric field, which makes an alternating magnetic field, and so and so forth (see Fig. 5.1).

One of the greatest Maxwell's discoveries is his statement that electromagnetic waves can propagate at a long distance from the source where they can be registered by an appropriate device.

Developing the mathematical theory of the electromagnetic wave, Maxwell found out that the electromagnetic waves propagate at a speed equal to velocity of light which had been known at that time thanks to Romer's measuring. He compared these facts and also understood that both the electromagnetic radiation and the light are of the same wave nature, which allowed him to refer the light to the electromagnetic phenomena. Different light theories had existed before Maxwell's researches. But only Maxwell's theory managed to explain all the light phenomena known at that time. Specifically he managed to predict the behavior of the light passing through different mediums. As is known, the sunlight contains all colors from red to violet. In combination they make a sense of white color. In other words, it means that the white color (sun light) is a combination of a wide spectrum of frequencies (rations) in the interval $(4 \div 10)^{14}$ hertz. The further discoveries and studies of ultra-violet, infra-red and x-rays radiation (the latter was discovered by Roentgen in 1895) as well as γ-radiation led to the idea that, all the types of radiation are of electromagnetic nature. The spectrum of the electromagnetic radiation is in the frequency interval $10^3 \div 10^{23}$ hertz.

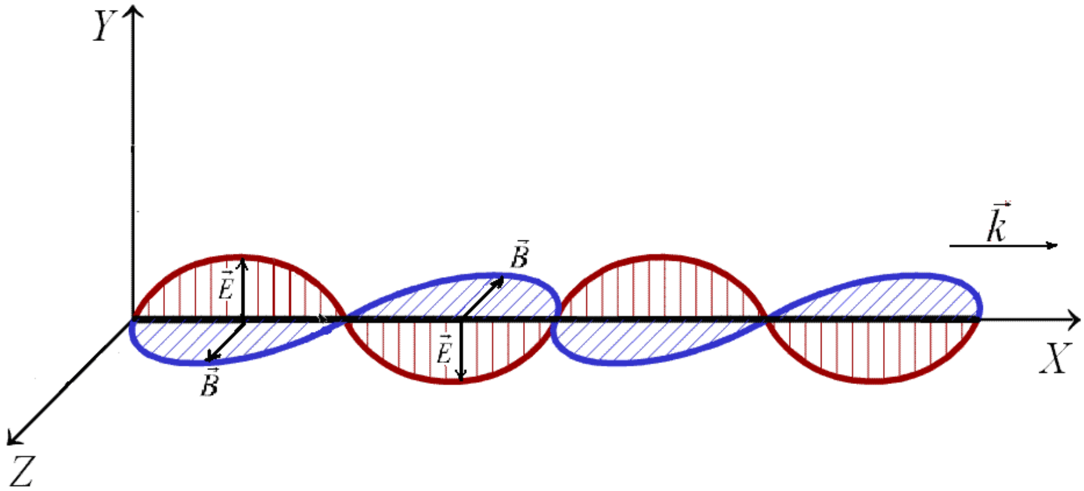


Fig. 5.1. The electromagnetic wave propagation. Vectors $\vec{E}$, $\vec{B}$ and $\vec{k}$ are mutually perpendiculars vectors. Vectors $\vec{E}$ and $\vec{B}$ make synchronic harmonic oscillations in plates ZY and XY correspondingly

The experimental proof of electromagnetic waves existence was done by Enrich Hertz. He made the electromagnetic wave generator (the Hertz dipole) and received them by means of a resonator (the Hertz resonator). The transmitter and the receiver were at a distance from each other. The further discoveries changed the human's way of life and the technologies of human society:

- William Crooks invented the wireless telegraph in 1892;
- Oliver Joseph Lodge managed to transmit the electromagnetic waves at short distances in 1894;
- the Russian scientist Popov A.S. invented the radio but didn't obtain a patent for his discovery;
- Guljemo Markoni registered his idea about the transmission of electromagnetic waves at far distances in 1897 and soon fulfilled the first transmission of a human speech;
- in 1907 Lee de Forest invented the first electronic tube which makes it possible to transmit music and speech by the radio.

The list of discoveries and applications of Maxwell's electromagnetic theory could go on and on.

5.2. Plane waves

From equation (5.1 – 5.4) we get the equation of the second order for vectors $\vec{E}$ and $\vec{H}$ taken separately. Applying the operation *rot* to equation (5.1) and using the equation (5.3) we get:

$$rot\ rot\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t} rot\vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}. \tag{5.5}$$

We will write the left-hand side of the equation

$$rot\ rot\vec{E} = \left[\nabla\left[\nabla\vec{E}\right]\right] = \nabla\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E} = -\nabla^2\vec{E}\ , \tag{5.6}$$

as

$$div\vec{E} = \left(\nabla\cdot\vec{E}\right) = 0.$$

As a result we come to the d'Alembert equation:

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0. \tag{5.7}$$

A similar equation can be obtained for the field $\vec{B}$:

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0. \tag{5.8}$$

In the absence of charges and currents similar equations take place for the scalar and vector potentials φ and $\vec{A}$ from equations (4.34), obtained under the additional Lorenz's condition (4.33):

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \tag{5.9}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0. \tag{5.10}$$

Moreover, as we'll see in the next chapter, similar equations are valid also for electrodynamics in non-conducting mediums. For these mediums Maxwell's equations include dielectric ε and μ magnetic permeability.

Any Cartesian component of vectors $\vec{E}$, $\vec{B}$, $\vec{A}$ and the scalar potential φ satisfy the equation

$$\nabla^2 U(\vec{r}.t) - \frac{1}{\upsilon^2}\frac{\partial^2 U(\vec{r}.t)}{\partial t^2} = 0, \tag{5.11}$$

where the constant

$$\upsilon = \frac{c}{\sqrt{\mu\varepsilon}}, \tag{5.12}$$

which has a speed dimension in the wave equation(5.11), is a characteristic of the medium. For vacuum $\upsilon = c$, where c is the velocity of light.

The wave equation (5.11) has a solution in the form of a plane wave:

$$u(\vec{r},t) = e^{i\vec{k}\vec{r} - i\omega t}, \tag{5.13}$$

where the frequency ω and the scalar of the wave vector $\vec{k}$ are related by the formula:

$$k = \frac{\omega}{\upsilon} = \sqrt{\mu\varepsilon}\,\frac{\omega}{c}. \tag{5.14}$$

For vacuum

$$k = \frac{\omega}{c}. \tag{5.15}$$

If we study the waves propagating in one direction, for example along the axis x, the general solution of the equation (5.11) will be:

$$u_k(x,t) = Ae^{ik(x-\upsilon t)} + Be^{-ik(x+\upsilon t)}. \tag{5.16}$$

For the speed υ, which is independent on k (it takes place in the case of a non-dispersive medium for which the value $\mu\varepsilon$ doesn't depend on the frequency ω), one can use the expansion of the function $u(x,t)$ in Fourier integral and formulate the general solution of the wave equation with the help of the linear superposition $u_k(x,t)$:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{\upsilon^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{5.17}$$

In the form

$$u(x,t) = f(x - \upsilon t) + g(x + \upsilon t), \tag{5.18}$$

where f and g are arbitrary functions of the arguments $x - \upsilon t$ and $x + \upsilon t$ correspondingly.

One can easily check that functions f and g satisfy the equation of the form (5.7). Let's verify it for function $f(x - \upsilon t)$:

$$\frac{\partial f}{\partial t} = -\upsilon f', \quad \frac{\partial^2 f}{\partial t^2} = +\upsilon^2 f'', \quad \frac{\partial f}{\partial x} = f', \quad \frac{\partial^2 f}{\partial x^2} = f'', \tag{5.19}$$

where f' is an arbitrary function by the argument of the function.

From (5.19) we get

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{\upsilon^2}\frac{\partial^2 f}{\partial t^2} = 0.$$

The functions $f(x - \upsilon t)$ and $g(x + \upsilon t)$ describe the waves moving along the axis x in positive and negative directions correspondingly.

If function $f(x-\upsilon t)$ presented a certain curve (a) at a moment t, at the moment $t+\Delta t$ it would be the same curve shifted in the positive direction of axis x by $\upsilon\Delta t$. That is, it is a wave moving at speed υ in the positive direction along the axis x (see Fig. 5.1). Function $g(x+\upsilon t)$ presents the wave moving at a speed υ in the negative direction of the axis x. It explains the sense of the introduced notation:

$$\upsilon=\frac{c}{\sqrt{\mu\varepsilon}}.$$

The speed υ is called the phase velocity of wave.

Plane waves which satisfy the scalar equation (5.11), are defined by the formulas (5.13) and (5.14). Electric and magnetic fields $\vec{E}$ and $\vec{B}$ must correspond to the real parts of complex quantities:

$$\begin{aligned}\vec{E}(\vec{r},t)&=\left(\vec{e}_1E_0e^{i\vec{k}\vec{r}-i\omega t}\right)=\left(\vec{E}_0e^{i\vec{k}\vec{r}-i\omega t}\right),\\ \vec{B}(\vec{r},t)&=\left(\vec{e}_2B_0e^{i\vec{k}\vec{r}-i\omega t}\right)=\left(\vec{B}_0e^{i\vec{k}\vec{r}-i\omega t}\right).\end{aligned}\tag{5.20}$$

In another representation the vector $\operatorname{Re}\vec{E}(\vec{r},t)$ has the form

$$\operatorname{Re}\vec{E}(\vec{r},t)=\vec{\varepsilon}_0\cos\left(\vec{k}\vec{r}-\omega t+\varphi_0\right),\tag{5.21}$$

where

$$\vec{E}_0=\vec{\varepsilon}_0e^{i\varphi_0}=\vec{e}_1E_0\tag{5.22}$$

and $\vec{\varepsilon}_0$ is the real constant vector.

Similar relations are also true for vector $\vec{B}_0$. Here vectors $\vec{e}_1$ and $\vec{e}_2$ are unit constant vectors, E_0 and B_0 are complex amplitudes, which are constant in space and time.

From the equations

$$div\,\vec{E}=0\tag{5.2}$$

and

$$div\,\vec{E}=0\tag{5.4}$$

the following relations follow

$$\vec{e}_1\cdot\vec{k}=0,\tag{5.23}$$

$$\vec{e}_2\cdot\vec{k}=0,\tag{5.24}$$

which mean that vectors $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of the propagation of wave $\vec{k}$.

Such waves are called transverse waves.

From Maxwell's equation for the non-conducting medium without the sources in the unlimited space

$$rot\,\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0\,.$$

Taking into account (5.20), we get

$$i\left\{\left[\vec{k}\vec{e}_1\right]E_0-\frac{\omega}{c}\vec{e}_2B_0\right\}e^{i\vec{k}\vec{r}-i\omega t}=0\,.$$

This implies:

$$\vec{e}_2=\left[\hat{k}\,\vec{e}_1\right], \tag{5.25}$$

$$B_0=\sqrt{\mu\varepsilon}\,E_0\,, \tag{5.26}$$

where

$$\hat{k}=\vec{k}/k\,. \tag{5.27}$$

Thus, in virtue of the formulas (5.23)–(5.25) the vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{k}$ make a right system of orthogonal vectors. Formulas (5.20) show that vectors $\vec{E}$ and $\vec{B}$ oscillate in a phase, and the ratio $\vec{E}$ to $\vec{B}$ is constant. In vacuum

$$B_0=E_0\,. \tag{5.28}$$

It is easy to use these formulas while calculating the mean-square root values by the field period

$$T=\frac{2\pi}{\omega},$$

if

$$\vec{A}=\vec{A}_0e^{-i\omega t}\ \text{и}\ \vec{B}=\vec{B}_0e^{-i\omega t}\,, \tag{5.29}$$

then

$$\overline{A^2}=\left[\mathrm{Re}\left(\vec{A}_0e^{-i\omega t}\right)\right]=|A_0|^2/2=\left|\vec{A}\right|^2/2\,, \tag{5.30}$$

$$\overline{\vec{A}\vec{B}}=\left[\mathrm{Re}\left(\vec{A}_0e^{-i\omega t}\right)\right]\cdot\left[\mathrm{Re}\left(\vec{B}_0e^{-i\omega t}\right)\right]=\frac{1}{4}\left(\vec{A}_0\vec{B}_0^*+\vec{A}_0^*\vec{B}_0\right)=\frac{1}{2}\mathrm{Re}\left(\vec{A}_0\vec{B}_0^*\right). \tag{5.31}$$

For example, a time- average energy flux is defined as

$$\overline{\vec{S}}=\overline{\frac{c}{4\pi}\left[\vec{E}\,\vec{H}\right]}=\frac{c}{4\pi}\mathrm{Re}\left[\vec{E}\,\vec{H}^*\right]=\frac{1}{2}\frac{c}{4\pi}\left[\vec{E}\,\vec{H}^*\right] \tag{5.32}$$

and presents a flux (that is energy flowing through a square unit per a time unit), which equals

$$\overline{\vec{S}} = \frac{c}{8\pi}\sqrt{\frac{\varepsilon}{\mu}}\left|E_0\right|^2 \vec{e}_3\,, \tag{5.33}$$

where $\vec{e}_3$ is a unit vector in the direction $\vec{k}$:

$$\vec{e}_3 = \hat{k} = \frac{\vec{k}}{k}\,. \tag{5.34}$$

The period-average density of the energy,

$$\overline{\omega} = \frac{1}{2}\frac{1}{8\pi}\left(\varepsilon\,\vec{E}\vec{E}^* + \frac{1}{\mu}\vec{B}\vec{B}^*\right) \tag{5.35}$$

taking into account (5.20), (5.26), equals

$$\overline{\omega} = \frac{\varepsilon}{4\pi}\left|E_0\right|^2. \tag{5.36}$$

The speed of the energy flow is defined as the ratio of the absolute values (5.32) and (5.36) and turns out to be equal to

$$\upsilon = \frac{c}{\sqrt{\mu\,\varepsilon}}\,. \tag{5.37}$$

For vacuum $\mu = \varepsilon = 1$ and $\upsilon = c$.

5.3. Polarization of the plane wave

In this part we will study the properties of the amplitude $\vec{E}_0$ of a monochromatic plane wave in details. If the wave (5.20) is described with the help of the complex vector $\vec{E}_0$, then one can decompose $\vec{E}_0$ in two actual vectors:

$$\vec{E}_0 = \vec{E}_{01} + i\vec{E}_{02}\,. \tag{5.38}$$

Substituting (5.20) into the set of equations (5.1 – 5.4), we get a relation between the complex amplitude of the fields

$$\vec{B}_0 = \left[\hat{k}\,\vec{E}_0\right],\quad \vec{E}_0 = \left[\vec{B}_0\,\hat{k}\right],\quad \hat{k}\cdot\vec{E}_0 = 0,\quad \hat{k}\cdot\vec{B}_0 = 0\,, \tag{5.39}$$

where

$$\hat{k} = \frac{\vec{k}}{k} = c\frac{\vec{k}}{\omega} \tag{5.27}$$

is a unit vector in the direction of the wave propagation.

It follows from the conditions (5.39) that vectors $\vec{E}_{01}$ and $\vec{E}_{02}$ are perpendicular to vector $\vec{k}$, in all the other respects they can be thought as arbitrary ones. So let's factor out phase factor $e^{i\varphi_0}$ from (5.38), so that the rest two actual vectors may be mutually perpendicular:

$$\vec{E}_0 = \left(\vec{\varepsilon}_{01} + i\vec{\varepsilon}_{02}\right)e^{i\varphi_0}\,, \tag{5.40}$$

$$\vec{\varepsilon}_{01} \cdot \vec{\varepsilon}_{02} = 0 . \tag{5.41}$$

From (5.40) and (5.41) we obtain

$$\begin{aligned} \vec{\varepsilon}_{01} &= \vec{E}_{01} \cos\varphi_0 + \vec{E}_{02} \sin\varphi_0, \\ \vec{\varepsilon}_{02} &= -\vec{E}_{01} \sin\varphi_0 + \vec{E}_{02} \cos\varphi_0. \end{aligned} \tag{5.42}$$

The value of phase φ_0 is found from the conditions of perpendicularity (5.41):

$$tg\, 2\varphi_0 = \frac{2\vec{E}_{01} \cdot \vec{E}_0}{E_{01}^2 - E_{02}^2} . \tag{5.43}$$

Then the real part $\vec{E}$, which is the observed field, will be written in the form:

$$\vec{E} = \vec{\varepsilon}_1 \cos\left(\vec{k}r - \omega t + \varphi_0\right) - \vec{\varepsilon}_2 \sin\left(\vec{k}r - \omega t + \varphi_0\right). \tag{5.44}$$

Suppose vectors $\vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$ are directed along the axis x and y correspondingly. Then the equation of the curve which is described by the extreme point of vector $\vec{E}$, has the form

$$\frac{E_x^2}{\varepsilon_1^2} + \frac{E_y^2}{\varepsilon_2^2} = 1 . \tag{5.45}$$

That is, it is an ellipse with semi-axis ε_1 and ε_2. In this case they say that the wave has an elliptical polarization. If vectors $\vec{\varepsilon}_1$, $\vec{\varepsilon}_2$, $\hat{k}$ make either the right or the left triplet, the direction of the rotation is called the right or the left one correspondingly. The rotation directions of vectors $\vec{E}$ and $\vec{B}$ coincide, but the vector of the magnetic field describes an ellipse, its axes are turned relative to the axes of vector $\vec{E}$ by $\pi / 2$.

When $\vec{\varepsilon}_1 = \vec{\varepsilon}_2$ we get a circular polarization which also has two directions of rotation. When $\varepsilon_1 \neq 0$, $\varepsilon_2 = 0$ or $\varepsilon_1 = 0$, $\varepsilon_2 \neq 0$ the polarization becomes linear, and the electric and magnetic vectors oscillate along mutually perpendicular directions. So, with a given $\vec{k}$ there may be two independent polarizations of the plane monochromic wave: an elliptical (in the special case it is a circular polarization) with the right and left directions of rotation or two linear polarizations in mutually perpendicular directions.

6. Radiation of a non-relativistic system of charged particles

6.1. Electric dipole radiation

In section 4.5 we obtained the formulas (4.41) for the retarded potentials. The task is the following: it is necessary to find potentials by given distributions of sources ρ and $\vec{j}$ in a certain bounded region of space (Fig. 6.1), and then (with their help) to calculate fields $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$.

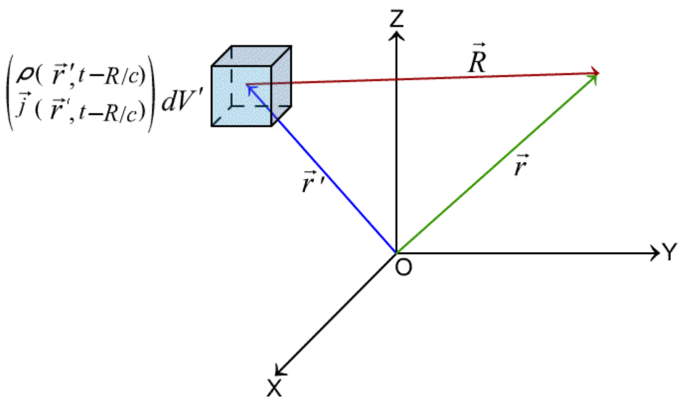


Fig. 6.1. The motion of a charge system in some area of space

The solution of this task considerably simplifies if the time of the propagation of electromagnetic disturbances in the range of a radiating charge system is small in comparison with the characteristic time of the motion of charged particles in the system:

$$l/c \ll T, \tag{6.1}$$

where l is a linear dimension of the system, T is the time during which the distribution of charges changes insignificantly. If the charges move periodically, then T is a period, and the inequation has the form

$$l \ll \lambda, \tag{6.2}$$

where $\lambda = cT$ is the length of radiation wave.

Dividing both parts of the inequation (6.2) by T and taking into account the characteristic speed of particles $\frac{l}{T} \approx \upsilon$, leads to the condition:

$$\upsilon/c \ll 1, \tag{6.3}$$

that means the requirement that the particles should have a non-relativistic speed. Besides, we assume that the system dimensions are small in comparison with the distance to the observation point:

$$l \ll r. \tag{6.4}$$

The region determined by the conditions $r \gg l$ and $r \gg \lambda$ is called a wave zone. One can simplify the retarded potentials in this region. First, let's consider the retarded potential

$$\varphi(\vec r, t) = \int \frac{1}{R} \rho\left(\vec r\,', t - \frac{R}{c}\right) dV',, \tag{6.5}$$

where $\vec R = \vec r - \vec r\,'$.

If r is large, then

$$R \approx r\left(1 - \frac{\vec r\,\vec r\,'}{r^2}\right) = r - \vec\Omega \vec r\,', \tag{6.6}$$

where

$$\vec{\Omega} = \vec{r}/r\,. \tag{6.7}$$

Present the charge density which is under the integral, in the form of expansion:

$$\rho\left(\vec{r}\,',t-\frac{r}{c}+\frac{\vec{\Omega}\vec{r}\,'}{c}\right) \equiv \rho\left(\vec{r}\,';\tau+\Delta\tau\right) \approx \rho\left(\vec{r}\,';\tau\right)+\frac{\partial\rho\left(\vec{r}\,';\tau\right)}{\partial\tau}\cdot\Delta\tau\,, \tag{6.8}$$

where

$$\tau = t-\frac{r}{c}\,, \tag{6.9}$$

$$\Delta\tau = \frac{\vec{\Omega}\vec{r}\,'}{c}\,. \tag{6.10}$$

It suffices to limit oneself to the zero approximation for R in the denominator under the integral (6.5)

$$R(r) \approx r \tag{6.11}$$

accurate to $l/r \ll 1$.

Substituting (6.8) into (6.5) and taking into account (6.9), (6.10), we get

$$\begin{aligned} \varphi(\vec{r},t) &= \frac{1}{r}\int\left\{\rho\left(\vec{r}\,';\tau\right)+\frac{\partial}{\partial\tau}\cdot\rho\left(\vec{r}\,';\tau\right)\cdot\frac{\vec{\Omega}\vec{r}\,'}{c}\right\}dV' = \\ &= \frac{q}{r}+\frac{\vec{\Omega}}{cr}\dot{\vec{p}}(\tau) = \frac{q}{r}+\frac{\vec{\Omega}\dot{\vec{p}}}{cr}. \end{aligned} \tag{6.12}$$

Here q is the charge of a system, $\vec{p}$ is the dipole moment of a system which depends on τ, and while calculating the second integral, with respect to the derivative time τ is taken outside the integral, with respect to the derivative τ is marked by the point:

$$\dot{\vec{p}} = \frac{\partial\vec{p}(\tau)}{\partial\tau}\,. \tag{6.13}$$

For the vector potential we find

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int\frac{\vec{j}\left(\vec{r};t-\frac{R}{c}\right)}{R}dV' \approx \frac{1}{cr}\int dV'\left\{\vec{j}\left(\vec{r}\,';\tau\right)+\frac{\partial}{\partial\tau}\cdot\vec{j}\left(\vec{r}\,';\tau\right)\cdot\frac{\vec{\Omega}\vec{r}\,'}{c}\right\}.$$

The second term is negligibly small, that’s why we get

$$\vec{A}(\vec{r},t) = \frac{1}{cr}\int\vec{j}\left(\vec{r}\,';t-\frac{r}{c}\right)dV'\,. \tag{6.14}$$

Then, we make use of the identity

$$\vec{a}\cdot\vec{j}=\vec{j}\cdot\nabla'(\vec{a}\cdot\vec{r}')=\nabla'\cdot\left[\vec{j}(\vec{a}\cdot\vec{r}')\right]-(\vec{a}\cdot\vec{r}')(\nabla'\cdot\vec{j}), \tag{6.15}$$

where $\vec{a}$ is a constant vector, to get

$$\vec{a}\cdot\int\vec{j}\left(\vec{r}',t-\frac{r}{c}\right)dV'=\int\nabla'\cdot\left\{\vec{j}(\vec{a}\cdot\vec{r}')\right\}dV'-\vec{a}\cdot\int\vec{r}'div'\vec{j}dV'. \tag{6.16}$$

The first integral in the right-hand side is equal to zero, therefore,

$$\vec{a}\cdot\int\vec{j}\left(\vec{r}',t-\frac{r}{c}\right)dV'=-\vec{a}\int\vec{r}'\cdot div'\,\vec{j}dV'. \tag{6.17}$$

Using the continuity equation

$$div'\,\vec{j}(\tau)+\frac{\partial\rho}{\partial\tau}=0,$$

we find from (6.17):

$$\vec{a}\cdot\int\vec{j}\left(\vec{r}',t-\frac{r}{c}\right)dV'=\dot{\vec{p}}(\tau) \tag{6.18}$$

and

$$\vec{A}(\vec{r},t)=\frac{\dot{\vec{p}}(\tau)}{cr}. \tag{6.19}$$

The formulas (6.12) and (6.19) give electromagnetic potentials at a far distance from the system of radiating charges, that is in a dipole approximation.

Let's find the electromagnetic fields in a dipole approximation.

For that it is necessary to calculate

$$\vec{B}=rot\,\vec{A} \tag{6.20}$$

and

$$\vec{E}=-\nabla\varphi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}. \tag{6.21}$$

For electrically neutral systems ($q = 0$) there is no first term in formula (6.12):

$$\varphi=\frac{\vec{\Omega}\dot{\vec{p}}}{cr}. \tag{6.22}$$

The vector potential is equal to

$$\vec{A}(\vec{r},t)=\frac{\dot{\vec{p}}(\tau)}{cr}. \tag{6.19}$$

While calculating the fields (6.20) and (6.21) one should take into account that $\dot{\vec{p}}\equiv\dot{\vec{p}}(\tau)$, where $\tau=t-\frac{r}{c}$ and $r=\sqrt{x^2+y^2+z^2}$, that is time τ depends on the coordinates of the observation point.

Let's calculate $\nabla\tau$:

$$\nabla\tau = -\frac{1}{c}\nabla r = -\frac{1}{c}\frac{\vec{r}}{r} = -\frac{\vec{\Omega}}{c}. \tag{6.23}$$

Applying the operator ∇ to function $f(\tau)$, we find

$$\nabla f = \frac{df}{d\tau}\nabla\tau = -\frac{\vec{\Omega}}{c}\frac{d}{d\tau}f(\tau), \tag{6.24}$$

that is

$$\nabla = -\frac{\vec{\Omega}}{c}\frac{d}{d\tau}. \tag{6.25}$$

If there is a vector function $\vec{F}(\tau)$, then

$$div\,\vec{F}(\tau) = \left(\vec{\nabla},\vec{F}\right) = -\frac{1}{c}\left(\vec{\Omega},\frac{d\vec{F}}{d\tau}\right) = -\frac{1}{c}\left(\vec{\Omega}\,\dot{\vec{F}}\right), \tag{6.26}$$

$$rot\,\vec{F}(\tau) = \left[\vec{\nabla},\vec{F}\right] = -\frac{1}{c}\left[\vec{\Omega}\,\dot{\vec{F}}\right]. \tag{6.27}$$

Let's find field $\vec{B}$:

$$\vec{B} = rot\,\vec{A} = rot\frac{1}{cr}\dot{\vec{p}} = \left[\nabla,\frac{\dot{\vec{p}}}{cr}\right] = \frac{1}{cr}\left[\nabla,\dot{\vec{p}}\right] + \left[\nabla\left(\frac{1}{cr}\right),\dot{\vec{p}}\right].$$

Dropping the second addend $\sim\frac{1}{r^2}$, we get

$$\vec{B} = \frac{1}{cr}\left[\nabla,\dot{\vec{p}}\right] = \frac{1}{c^2 r}\left[\vec{\Omega},\ddot{\vec{p}}\right] = \frac{1}{c^2 r}\left[\ddot{\vec{p}},\vec{\Omega}\right]. \tag{6.28}$$

Let's calculate the field $\vec{E}$.

We find $\frac{\partial\vec{A}}{\partial t}$:

$$\frac{\partial\vec{A}}{\partial t} = \frac{\partial}{\partial t}\frac{\dot{\vec{p}}}{cr} = \frac{1}{cr}\frac{\partial\dot{\vec{p}}}{\partial\tau}\frac{\partial\tau}{\partial t} = \frac{\ddot{\vec{p}}}{cr}. \tag{6.29}$$

Calculate $\nabla\varphi$:

$$\nabla\varphi = \nabla\left(\frac{1}{cr^2}\left(\vec{r}\,\dot{\vec{p}}\right)\right) = \frac{1}{cr^2}\nabla\left(\vec{r}\,\dot{\vec{p}}\right) + \left(\vec{r}\,\dot{\vec{p}}\right)\nabla\left(\frac{1}{cr^2}\right) \approx \frac{1}{cr^2}\nabla\left(\vec{r}\,\dot{\vec{p}}\right) =$$

$$= \frac{1}{cr^2}\left\{\left[\vec{r},rot\,\dot{\vec{p}}\right] + \left(\vec{r}\,\vec{\nabla}\right)\dot{\vec{p}} + \left[\dot{\vec{p}},rot\,\vec{r}\right] + \left(\dot{\vec{p}}\,\vec{\nabla}\right)\vec{r}\right\}. \tag{6.30}$$

The third term is equal to zero and

$$\left(\dot{\vec{p}}\,\vec{\nabla}\right)\vec{r} = \dot{\vec{p}}, \tag{6.31}$$

$$\left(\vec{r}\,\vec{\nabla}\right)\dot{\vec{p}} = -\frac{1}{c}\left(\vec{r}\,\vec{\Omega}\right)\ddot{\vec{p}}\,, \tag{6.32}$$

$$\left[\vec{r}, rot\,\dot{\vec{p}}\right] = \left[\vec{r}\left[\vec{\nabla}\,\dot{\vec{p}}\right]\right] = -\frac{1}{c}\left[\vec{r}\left[\vec{\Omega}\,\ddot{\vec{p}}\right]\right] = -\frac{1}{c}\vec{\Omega}\left(\vec{r}\,\ddot{\vec{p}}\right) + \frac{1}{c}\ddot{\vec{p}}\left(\vec{r}\,\vec{\Omega}\right). \tag{6.33}$$

Substituting (6.31)-(6.33) into (6.30), we find

$$\nabla\varphi = \frac{1}{cr^2}\left\{-\frac{1}{c}\vec{\Omega}\left(\vec{r}\,\ddot{\vec{p}}\right) + \dot{\vec{p}}\right\} = -\frac{1}{cr^2}\vec{\Omega}\left(\vec{\Omega}\,\ddot{\vec{p}}\right) + \frac{\dot{\vec{p}}}{cr^2}.$$

Neglecting the second term $\sim \frac{1}{r^2}$, we get

$$\nabla\varphi = -\frac{1}{cr^2}\vec{\Omega}\left(\vec{\Omega}\,\ddot{\vec{p}}\right)$$

and

$$\vec{E} = \frac{1}{cr^2}\vec{\Omega}\left(\vec{\Omega}\,\ddot{\vec{p}}\right) - \frac{1}{cr^2}\ddot{\vec{p}}\left(\vec{\Omega}\vec{\Omega}\right) = \frac{1}{cr^2}\left[\vec{\Omega}\left[\vec{\Omega}\,\ddot{\vec{p}}\right]\right] = -\left[\vec{\Omega}\,\vec{B}\right] = \left[\vec{B}\,\vec{\Omega}\right].$$

Thus, we get the expression for the fields in dipole approximation:

$$\vec{B} = \frac{1}{c^2 r}\left[\ddot{\vec{p}}, \vec{\Omega}\right],$$

$$\vec{E} = \left[\vec{B}\,\vec{\Omega}\right].$$

Vectors $\vec{E}$, $\vec{B}$ and $\vec{\Omega}$ are mutually perpendicular to each other, vectors $\vec{E}$ and $\vec{B}$ are lying in the plane perpendicular to the direction of the electromagnetic wave propagation (Fig. 5.1). The magnitudes of vectors $\vec{E}$ and $\vec{B}$ are qual: $E = B$.

Chapter 2
BASIC CONCEPTS
OF MACROSCOPIC ELECTRODYNAMICS

7. Electrostatics in dielectrics

A substance consists of atoms, atoms consist of charged particles. Atom is neutral, but in electric field charges can shift, which can cause important effects – the appearance of current in conductors, the polarization of dielectrics. Such a substance generates its own field, which adds to the external field.

7.1. Vector of Polarization

In the electric field the charges shift, it breaks the neutrality of the selected volume ΔV. A bound charge is $\sum_i q_i \neq 0$ and $\sum_i q_i \vec{r}_i \neq 0$. If the electric dipole moment is $\Delta\vec{p} \neq 0$, then the dielectric is polarized and

$$\vec{P} = lim_{\Delta V \to 0}\left(\frac{\Delta\vec{p}}{\Delta V}\right) \tag{7.1}$$

is the polarization vector.

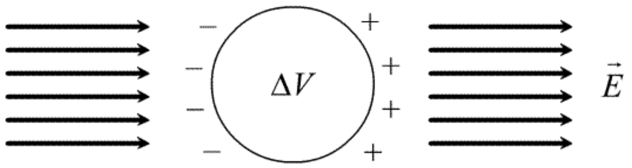


Fig. 7.1. The polarization of dielectrics in distinguished volume ΔV if there is field E

7.2. Polarization of non-polar molecules

Non-polar molecules are the molecules, for which the distribution centers of positive and negative charges coincide. The field shifts them to a certain value. In equilibrium the internal molecular field $\vec{E}_{in}$ is equal to the external field $\vec{E}_{out}$. For calculation $\vec{E}_{in}$ let's draw a sphere through the observation point and make use of the Gauss-Ostrogradsky integral theorem:

$$\oint_S \left(\vec{E}\vec{n}\right) dS = \int_V 4\pi \cdot \rho \cdot dV\,, \tag{7.2}$$

where $\vec{n} = \frac{\vec{r}}{r}$, $\vec{E}(\vec{r}) = E(r)\frac{\vec{r}}{r}$ – is a spherically symmetric field.

Calculating the integrals in both sides of the equation (7.2) yields

$$E(r)4\pi r^2 \approx 4\pi \cdot \rho(0)\frac{4}{3}\pi r^3 . \quad (7.3)$$

From the relation

$$E(r) = 4\pi \cdot \rho(0)\frac{r}{3} . \quad (7.4)$$

It follows that the shift is equal to

$$r_0 = \frac{3E}{4\pi\rho(0)} , \quad (7.5)$$

where $E = E_{out}$.

The electric moment of molecule is equal to

$$\vec{p} = \frac{z_e \cdot 3}{4\pi\rho(0)} \cdot \vec{E} \equiv \beta \cdot \vec{E} . \quad (7.6)$$

The proportionality coefficient β of $\vec{p}$ to $\vec{E}$ is called the molecular polarizability.

Multiplying the electric moment of molecule by the concentration n_0 yields the polarization vector of a unit volume of a substance:

$$\vec{P} = n_0\beta \cdot \vec{E} = \alpha\vec{E} , \quad (7.7)$$

where α is molecular polarizability, which characterizes the dielectric properties. Polarization vector $\vec{P}$ generates its own field. The experimental data show that there exists the proportionality $P \sim E$ which can be observed in weak fields. Thus, if a non-polar substance is placed in electric field, it is polarized and $\vec{P}$ generates its own field.

7.3. Orientation Polarization

In nature there are also polar molecules, which are dipoles at once, for example, NaCl. These molecules have electric dipole moment, but in the absence of field they are chaotically oriented and the sum of the electric moments is $\vec{P} = 0$. The field turns dipole molecules along itself, but thermal motion breaks the orientation, but $\vec{P} \neq 0$ because in equilibrium there is an excess of field oriented molecules.

To calculate the number of molecules oriented in a given direction, let's draw a sphere of a unit radius in the substance; let's mark it by parallels and meridians to get sections with the same square S, so that the solid angle at which the area is seen, should be equal. Consider ΔV .

Let $dN(\theta,\varphi)$ be a number of molecules within the solid angle $d\Omega = \frac{dS}{R^2}$.

If there is no external field, then all molecules $dN(\theta,\varphi)$ are equal, that is

$$dN = C_0 d\Omega . \tag{7.8}$$

By integrating of both sides of the equation (7.8), we find

$$C_0 = \frac{N}{4\pi},$$

that is C_0 does not depend on θ,φ.

Thus, $dN = \frac{N}{4\pi}\ d\Omega$. Thermal motion changes the orientation but any dN remains unchanged.

In field $\vec{E}$ dipole has the potential energy $W = -(\vec{p}\vec{E})$. Dipole tends to reach the condition with a minimal W, but thermal motion impedes this.

In the condition of thermodynamic (thermal) equilibrium the Boltsman's distribution is true:

$$dN = C_E e^{-W/kT} d\Omega = C_E e^{\vec{p}\cdot\vec{E}/kT} d\Omega ,$$

where $C_E = const$, which, if there is a field, is found from the condition

$$N = C_E \int e^{-W/kT} d\Omega .$$

If z-axis is $\vec{E}$-directed, then $\vec{p}\cdot\vec{E} = pE\cos\theta$, $\vec{n} = \vec{p}/p$, where

$$d\Omega = \sin\theta d\theta d\varphi,$$

and we find, that

$$dN = \frac{N}{2\pi}\frac{a}{e^a - e^{-a}} e^{a\cos\theta} . \tag{7.9}$$

Here

$$a = \frac{pE}{kT} . \tag{7.10}$$

Let's calculate:

$$\Delta\vec{P} = \sum_{i=1} \vec{P}_i = \sum_k N_k \vec{p}_k , \tag{7.11}$$

$$\Delta\vec{P} = \int \vec{p} dN = p\int \vec{n}\left(\frac{dN}{d\Omega}\right) d\Omega , \tag{7.12}$$

where $\vec{n} = \vec{p}/p$.

To find ΔP means to find the projection $\nabla\vec{P}$ onto $0z$-axis.

Since $N = n_0 \Delta V$, then for the polarization vector we find:

$$P = n_0 p \left(\frac{e^a + e^{-a}}{e^a - e^{-a}} - \frac{1}{a} \right). \qquad (7.13)$$

Formula (7.13) shows that $P \to n_0 p$ as $E \to \infty \, (a \to \infty)$.
For small $\vec{E}$ one obtains

$$P = \frac{n_0 p^2}{3kT} E = \alpha E . \qquad (7.14)$$

Where α molecular polarizability:

$$\alpha \equiv \alpha(T) = \frac{n_0 p^2}{3KT} . \qquad (7.15)$$

Polarized dielectric generates its own field, which adds to the external field. That is the field which should be found.

7.4. Field Potential in Dielectric

Dielectric is polarized in electric field. If the field is not strong, $\vec{P} = \alpha \vec{E}$, that is an additional field is generated (α can be thought to be known). If dipole is situated in the coordinate origin then the electric potential, generated by dipole at point $\vec{r}$, is equal to $\varphi_P(\vec{r}) = \frac{\vec{P}\vec{r}}{r^3}$; if it is situated in point $\vec{r}\,'$, then (see Fig. 7.2)

$$\varphi_P(\vec{r}) = \frac{\vec{P}\vec{R}}{R^3}, \; \vec{R} = \vec{r} - \vec{r}\,' .$$

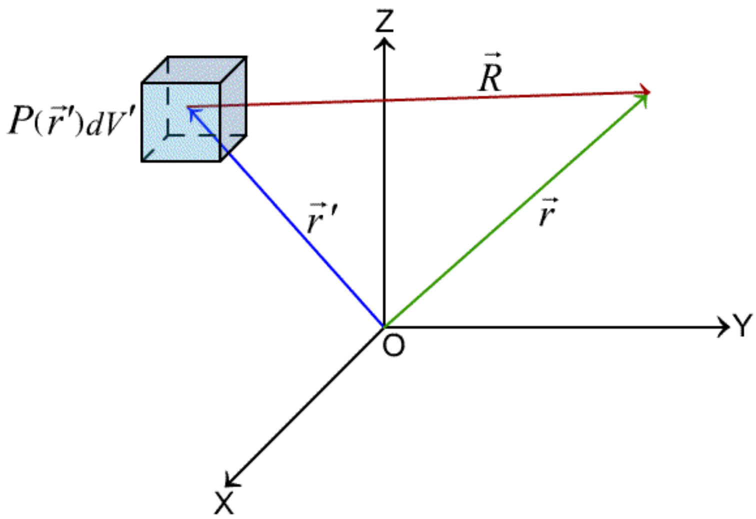


Fig. 7.2. To calculation of the potential created by bound charges in dielectric, taking place in a homogeneous field $\vec{E}$.
$\vec{P}(\vec{r}\,')$ is the polarization vector of the unit volume

Let's divide the polarized dielectric into regions dV'.

On the whole the electric moment of neutral volume element dV' is equal to:

$$\vec{P}dV' = \sum_{dV'} q_i \vec{r}_i' = \sum_{dV'} \vec{p}_i ,$$

where the summing is done over the charges (or over all the molecules) in volume dV'. By the definition $\vec{P}$ is the polarization vector of the unit volume, that is

$$\vec{P} = lim_{\Delta V \to 0} \frac{\Delta \vec{P}}{\Delta V},$$

therefore, it can be written as

$$\vec{P}dV' = \vec{P}(\vec{r}')dV' \text{ и } \varphi_P(\vec{r}) = \int \frac{\vec{P}\vec{R}}{R^3} dV' . \qquad (7.16)$$

One can easily verify that

$$\frac{\vec{R}}{R^3} = \nabla' \frac{1}{R}, \; div \frac{\vec{P}}{r} = \frac{1}{r} div' \vec{P} + \vec{P}\nabla(1/r) . \qquad (7.17)$$

Where primes over operators ∇' и div' signify that the differentiation is carried out with respect to the components of vector $\vec{r}' = (x', y', z')$, entering $R = |\vec{r} - \vec{r}'|$.

The integrations element can be written as

$$\frac{\vec{P}\vec{R}}{R^3} = \vec{P}\nabla'\left(\frac{1}{R}\right) = div' \frac{\vec{P}}{R} - \frac{1}{R} div' \vec{P}, \qquad (7.18)$$

$$\varphi_P(\vec{r}) = \int div'(\frac{\vec{P}}{R}) dV' - \int \frac{div' \vec{P}}{R} dV'.. \qquad (7.19)$$

By the Gauss-Ostrogradsky divergence theorem, integration over infinite contour implies

$$\int_{\infty} div'(\vec{P}/R) dV' = \oint \frac{\vec{n} \cdot \vec{P}}{R} dS = 0 . \qquad (7.20)$$

Here we took into account that $\vec{P}(\infty) = 0$. As a result we obtain

$$\varphi_p(\vec{r}) = -\int \frac{div' \vec{P}(\vec{r}) dV'}{R}.. \qquad (7.16')$$

Compare with $\varphi(\vec{r}) = \int \frac{\rho \cdot dV'}{R}$. Magnitude $-div' \vec{P}(\vec{r}')$ stands for the density of bound charges:

$$-div \vec{P}(\vec{r}') = \rho_{bound}(\vec{r}') . \qquad (7.21)$$

As a result we'll get the total field of free and bound charges:

$$\varphi(\vec{r}) = \int \frac{\rho - div'\vec{P}(\vec{r})dV'}{R} = \int \frac{(\rho - \rho_{bound})}{R} dV'. \qquad (7.22)$$

Thus, the appearance of polarization in dielectric results in the appearance of the total field.

7.5. Maxwell Equations in Dielectrics

To find the total field $\vec{E}$ by the Helmgoltz's theorem it's necessary to know $div\vec{E}$ and $rot\vec{E}$.

As known, the scalar potential

$$\varphi = \int \frac{\rho \cdot dV}{R}$$

is a solution of the Poisson's equation:

$$\nabla^2 \varphi = -4\pi \rho .$$

In a dielectric

$$\nabla^2 \varphi = -4\pi(\rho - div\vec{P}), \qquad (7.23)$$

where

$$\nabla^2 \varphi = div\ grad\varphi = -div\vec{E}, \qquad (7.24)$$

$$div(\vec{E} + 4\pi \cdot \vec{P}) = 4\pi\rho \qquad (7.25)$$

and

$$\vec{E} + 4\pi \cdot \vec{P} = \vec{D} \qquad (7.26)$$

is the vector of electrostatic induction.

By making use of the relation $\vec{P} = \alpha \cdot \vec{E}$, we find

$$\vec{D} = \vec{E} + 4\pi \cdot \alpha \cdot \vec{E} = \vec{E}(1 + 4\pi \cdot \alpha) = \varepsilon \cdot \vec{E}, \qquad (7.27)$$

where α is dielectric susceptibility, ε is dielectric constant called relative permittivity or simply dielectric permittivity.

In electrostatics $\vec{E} = -grad\varphi$ and $rot\vec{E} = 0$. In view of that and the relationships (7.25), (7.27) one obtains Maxwell's equations for dielectrics:

$$rot\vec{E} = 0;$$

$$div\vec{D} = 4\pi\rho;$$

$$\vec{D} = \varepsilon \cdot \vec{E}. \qquad (7.28)$$

8. Direct Current

8.1. Potential and Field in the presence of conductors

The following conductor model can be presented as the simplest one: there is an ionic crystalline core with slightly bound electrons inside it. Electrons, not ions, carry electricity. Here are the results of somel experiments.

Rike's experiment. During one year a current is conducted through three cylinders put one over other. Cylinders are copper, aluminium and copper again. After the experiment the penetration of one metal into another was not detected.

Tallman and Stuart's experiment. Current excitation in metals caused by inertia forces.

A coil made of metal wire is rapidly rotated around its axis. Terrestrial magnetic field in the coil was carefully compensated by means of fixed coils with current, so if the motion is uniform a current does not appear. But in case of rapid deceleration there is an electric current caused by movement of negatively charged particles with a specific charge e/m.

Within the errors of measurement it is similar to the charge of the electrons, which appear in experiments with cathode rays.

In constant field current stops existing, in conductor $\vec{E}=0$, that is the additional field from the redistribution of charges compensates the external field.

If $\vec{E}=0$, then $\oint(\vec{E}\vec{n})dS=0$ (integration is over all internal surfaces) and $q=0$. There are no compensated charges inside the conductor. The charge is distributed over the surface. The same is true for free charges.

$\vec{E}=0$, it means $\varphi_{in}(\vec{r})=const=V$. Parameter V is conductor potential. The redistribution of the free charge, caused by the field, is called electrostatic induction.

It's necessary to find $\varphi_{out}(\vec{r})$ (in view of electrostatic induction). In electric field electrons are redistributed, so the changed electric field should be found.

Problem statement. There is a charge q (on conductor including); there are conductors, potential of some of them is specified as V. It's necessary to find φ_{out} of all conductors.

Solution. The Poisson's equation $\nabla^2\varphi=-4\pi\rho$ is solved under the boundary condition $\varphi|_{r\in S}=V$. One finds φ, then $\vec{E}=-\nabla\varphi$.

We limit ourselves to a simple case, when charges are only on a conductor, or a conductor is connected to a battery. If $\varphi_{out}(\vec{r})$ is found, then one can find $\vec{E}(\vec{r})$ and $\vec{\sigma}(\vec{r})$ – surface charge density.

Let's select an area ΔS on the surface of the conductor and put it into a closed parallelepiped surface (Fig. 8.1). By the Gauss theorem $\oint \vec{E}\vec{n} dS = 4\pi q$.

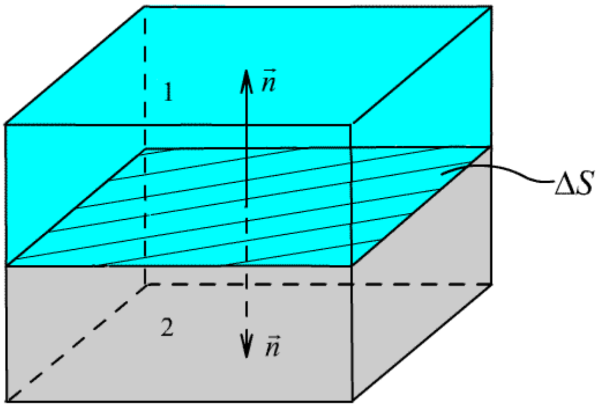


Fig. 8.1. To the solution Poisson`s equation with the boundary condition for the potential of φ on the conductor surface. The area ΔS is selected on the conductor surfaces and is surrounded by the closed surface of a rectangular parallelepiped

If parallelepiped height vanishes, then

$$\oint E_n dS = \left(E_n(1) + E_{n'}(2)\right)\Delta S = E_n(1)\Delta S, \tag{8.1}$$

since $\vec{E}_{in} \equiv \vec{E}_{n'}(2) = 0$.

On the other hand, according to the Ostrogradskiy-Gauss theorem

$$\oint E_n dS = 4\pi q = 4\pi\sigma\Delta S,$$

where σ is the surface charge density. It means that $E_n(\vec{r}) = 4\pi\sigma(\vec{r})$, with point $\vec{r}$ being on the conductor surface.

From the equation $E = -\nabla\varphi$ we obtain that $E_n = -(\nabla\varphi)_n = -\frac{\partial\varphi}{\partial n}$ that is,

$$\sigma(\vec{r}) = -\frac{1}{4\pi}\frac{\partial\varphi(\vec{r})}{\partial n} \tag{8.2}$$

and

$$q = -\frac{1}{4\pi}\oint \frac{\partial\varphi}{\partial n} dS. \tag{8.3}$$

Instead of the boundary condition $\varphi|_{r\in S} = V$ they use sometimes

$$-\frac{1}{4\pi}\oint \frac{\partial\varphi}{\partial n} dS = q. \tag{8.4}$$

8.2. Current in Metals. The Integral Form of Ohm's Law and Joule-Lenz's Law. Voltage

If the potential difference between the ends of a conductor is supported in a source, then a current flows through the conductor. Current intensity (amount of electricity per second) is directly proportional to the potential difference:

$$I = \frac{\varphi_1 - \varphi_2}{R}, \tag{8.5}$$

where R is the resistance.

On the other hand, the potential difference is found as

$$\varphi_1 - \varphi_2 = -\int_1^2 d\varphi = -\int_1^2 (\nabla\varphi, d\vec{r}) = \int_1^2 \vec{E}d\vec{r} = \varepsilon_{12}, \tag{8.6}$$

where ε_{12} is voltage.

The equation

$$IR = \varepsilon_{12} \tag{8.7}$$

is Ohm's law, it is also true for a non-stationary case, when there is no such a concept as the potential.

Let's consider a conductor with a current.

It's necessary to calculate the work on charge transferring in field $\vec{E}$ from point (1) to point (2):

$$A = \int_1^2 q\vec{E}d\vec{r} = -q\int_1^2 (\nabla\varphi d\vec{r}) = -q\int_1^2 d\varphi = (\varphi_1 - \varphi_2)q. \tag{8.8}$$

From mechanics it's known that if a force $\vec{F}$ acts on a body, it accelerates. The speed υ changes, and consequently, kinetic energy also changes. In a conductor this energy converts into thermal energy:

$$Q = I(\varphi_1 - \varphi_2) = I\varepsilon_{12} = I^2R. \tag{8.9}$$

This is the integral form of Joule-Lenz Law.

8.3. Current Density. The Differential Form of Ohm's Law and Joule-Lenz's Law

Let's divide the conductor into thin unions with current (Fig. 8.2). In this case it can be written for a current union

$$IR = \int_1^2 \vec{E}d\vec{r}.$$

Magnitude R is called resistance; it's connected with specific resistance ρ. Then we obtain

$$j\Delta S \cdot \rho \frac{\Delta l}{\Delta S} = E\Delta l, \; j\rho = E, \; \vec{j} = \frac{1}{\rho}\vec{E}.$$

Introduce also $\frac{1}{\rho} = \sigma$ – the conductivity.

I can be expressed through current density:

$$I = j\Delta S, \; R = \rho \frac{\Delta l}{\Delta S}.$$

As a result one gets

$$\vec{j} = \sigma\vec{E}. \tag{8.10}$$

The differential form of the Ohm's law, that is in a fixed space point.

For current union

$$Q = I^2 R = j^2 \Delta S^2 \rho \frac{\Delta l}{\Delta S} = j^2 \rho \Delta V. \tag{8.11}$$

Denote $\frac{Q}{\Delta V} = \tilde{q}$ as quantity of heat in unit volume. From equation (8.11) it follows that

$$\tilde{q} = j^2 \rho = \frac{j^2}{\sigma} = jE = \vec{j} \cdot \vec{E}. \tag{8.12}$$

This is the differential form of the Joule-Lenz Law.

The difference between stationary field of direct current from electrostatic field is that, for maintaining currents one needs constant power supply which is intended to compensate the energy loss in form of joule heat at the expense of other kinds of energy – mechanical power (dynamo), chemical (galvanic elements, accumulators), thermal energy. It's necessary that EMF (electromotive force) of non-electrostatic origin should act in certain sections of the circuit.

In the cell (Fig. 8.3) with water and acid, the diffusion takes place in solution $HCl \rightarrow H^+ + Cl^-$. Mobility of ions H^+ is higher, they move upward faster, that is a current appears without electric field (no $\vec{j} = \sigma\vec{E}$), caused by forces of non-electric nature. Let denote the field $\vec{E}_{ext}$, which can cause the same current, which appears due to the diffusion:

$$\vec{E}_{ext} = \frac{1}{\sigma}\vec{j}_{ext}. \tag{8.13}$$

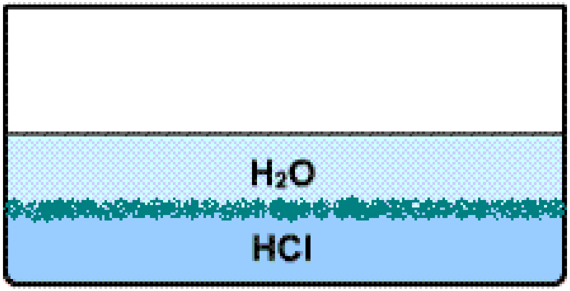


Fig. 8.2. In a vessel containing water and an acid, there is a diffusion. In a solution $HCl \to H^+ + Cl^-$*. Because mobility of ions* H^+ *is more, than mobility of ions* CL^-*, arises a current not the electric nature*

This implies

$$\vec{j} = \sigma\left(\vec{E} + \vec{E}_{ext}\right). \qquad (8.14)$$

The field $\vec{E}_{ext}$ arises at metal-electrolyte contact because of the different solubility of positive and negative ions at the contact of two metals with different electronic structure.

Let's write down the Ohm's law in the integral form:

$$I = \frac{1}{R}\left\{\int_1^2 \vec{E}d\vec{r} + \int_1^2 \vec{E}_{ext}d\vec{r}\right\}. \qquad (8.15)$$

For closed contour $\int_1^2 \to \oint$. In this case the first integral is equal to zero, since $\vec{E}d\vec{r} = -d\varphi$, $\oint d\varphi = 0$, that's why $IR = \oint \vec{E}_{ext}d\vec{r}$ is electromotive force, EMF $= \varepsilon$, and the Ohm's law takes the form:

$$IR = \varepsilon\,. \qquad (8.16)$$

9. Magnetostatic Field in the Magnetic

A substance placed in a field $\vec{B}$ magnetizes, that is each element dV gains a magnetic moment. There are three types of substances, which have different magnetizing mechanisms: diamagnetics, para- and ferromagnetics. If $\vec{m}_i$ is an atom magnetic moment, then

$$\lim_{\Delta V \to 0} \frac{\sum \vec{m}_i}{\Delta V} = \vec{M}$$

is a magnetization vector.

A magnetized body generates its own magnetic field, which adds to the external field: the presence of the substance changes the field. This is the field to be found.

9.1. Larmor's theorem. The magnetizing mechanism for diamagnetics

The electron in the orbit generates circular current, it has the magnetic moment

$$\vec{m} = \frac{1}{2c}\int[\vec{r}\vec{j}]dV = -\frac{e}{2c}[\vec{r}_e\vec{v}_e]\frac{m_e}{m_e} = -\frac{e}{2m_e c}\vec{L} = g\vec{L},$$

where $\vec{j} = \rho\vec{v}$, $\rho = -e\delta(\vec{r} - \vec{r}_e(t))$, vector $\vec{r}_e(t)$ characterizes the electron position at time t.

A magnetic moment of a rotating electron is directly proportional to a mechanistic moment, and proportional factor g is called the gyromagnetic ratio:

$$g = -\frac{e}{2m_e c}. \tag{9.1}$$

The magnetic moments of single electrons are summed to get an atom magnetic moment, in this case the total magnetic moment is either equal to zero, or is not equal to zero. If $\sum_i \vec{m}_i = \vec{m}_{at} = 0$ (without field), then this substance is a diamagnetic. For paramagnetics and ferromagnetics without field $\vec{m}_{at} \neq 0$. If there is an external field the case is different. In field $\vec{B}$ each magnetic moment $\vec{m}$ is effected by the moment $\vec{N} = [\vec{m}\vec{B}]$, which tends to turn the magnetic moment along the field.

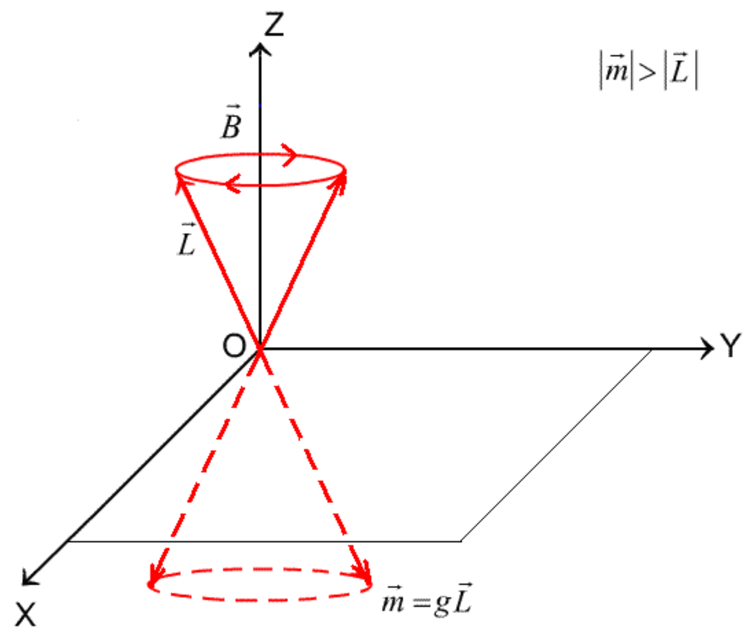


Fig. 9.1. Precession of a magnetic moment around the direction of magnetic field

But $\vec{m} = g\vec{L}$ and $\frac{d\vec{L}}{dt} = \vec{N}$, that's why

$$\frac{d\vec{L}}{dt} = g\left[\vec{L}\vec{B}\right]. \qquad (9.2)$$

Let $0Z$ axis coincide with $\vec{B}$. Then

$$\frac{d\vec{L}}{dt} = g\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ L_1 & L_2 & L_3 \\ 0 & 0 & B \end{vmatrix}, \qquad (9.3)$$

$$\dot{L}_1 = gBL_2,\ \dot{L}_2 = -gBL_1,\ \dot{L}_3 = 0.$$

Denote $gB = -\omega_L$. Then $\dot{L}_1 = -\omega_L L_2$, $\dot{L}_2 = \omega_L L_1$, $\dot{L}_3 = const$,

$$\ddot{L}_1 = -\omega_L \dot{L}_2 = -\omega_L^2 L_1.$$

It means that

$$\ddot{L}_1 + \omega_L^2 L_1 = 0. \qquad (9.5)$$

The solution of this equation is

$$L_1 = A\cos(\omega_L t - \alpha). \qquad (9.6)$$

From the first equation

$$L_2 = -\frac{1}{\omega_L}\dot{L}_1 = A\sin(\omega_L t - \alpha).$$

Consequently, $L_1^2 + L_2^2 = const$, and noting that $L_3 = const$, that is $L^2 = L_1^2 + L_2^2 + L_3^2 = const$. As the length remains constant, then the rotation direction changes, and as the projection onto $0Z$-axis is constant, then vector $\vec{L}$ precesses round $\vec{B}$ with angle velocity ω_L, where ω_L is Larmor's precessional frequency.

The precession is an additional rotation. There arises an additional moment. The magnetic moment is parallel or antiparallel to $\vec{B}$ for each atom electron. $\vec{m}_{am} = 0$ without field, in field an atom magnetic moment is $\vec{m}_{am} = \sum_{i=1}^{z} \Delta\vec{m}_i$ and it's oriented either in the direction of $\vec{B}$, or opposite $\vec{B}$.

The generation of an atom magnetic moment due to the precession in the external field explains the phenomenon of diamagnetism.

If all atom electrons preccess, then $\vec{v}_L(\vec{r}) = [\vec{\omega}_L \vec{r}]$, and the current $\vec{j}_L = \rho\vec{v}_L$ arises.

The value of atom magnetic moment is determined by the current density $\vec{j}_L$:

$$\vec{m}_{at} = \frac{1}{2c}\int\left[\vec{r}\vec{j}_L\right]dV \,. \tag{9.7}$$

As a result

$$\vec{m}_{at} = \alpha\vec{B} \tag{9.8}$$

for diamagnetics $\alpha < 0$.

9.2. Paramagnetic Properties

If in the absence of a magnetic field $\vec{m}_{at} \neq 0$, then such substances are called paramagnetics.

In this case magnetic moment $\vec{m}$ in field $\vec{B}$ has potential energy $W = -\left(\vec{m}\vec{B}\right)$ and tends to be orientated along field $\vec{B}$, but the thermal motion prevents it. In thermal equilibrium (as well as for polar molecules) the number of atoms with a magnetic moment orientated within the solid angle $d\Omega$ is equal to

$$dN = \frac{N}{2\pi\left(e^{a} - e^{-a}\right)} e^{a\cdot\cos\theta} d\Omega \,, \tag{9.9}$$

where

$$a = \frac{mB}{kT} \,. \tag{9.10}$$

It means that it’s Bolzmann’s distribution.

Again, as well as in electrostatics, we have

$$\vec{M} = \lim_{\Delta V \to 0}\frac{\Delta\vec{M}}{\Delta V} \,, \tag{9.11}$$

$$\Delta\vec{M} = m\int\vec{n}\frac{dN}{d\Lambda}d\Omega \,, \tag{9.12}$$

$$\Delta\vec{M} = \vec{e}_3 M \,, \tag{9.13}$$

$$\Delta\vec{M} = m\int\cos\theta\frac{dN}{d\Omega}\sin\theta d\theta d\varphi, \tag{9.14}$$

where

$$N = n_0 \Delta V \,. \tag{9.15}$$

That is why the calculation of the integral in the expression for ΔM (9.14) taking into account (9.9) gives

$$M = n_0 m\left(\frac{e^{a} + e^{-a}}{e^{a} - e^{-a}} - \frac{1}{a}\right) \equiv n_o m L_\infty(a). \tag{9.16}$$

With the increase of field $\vec{B}$ occurs the saturation:

$$(B \to \infty, a \to \infty, M \to n_0 m \equiv M_0) \,.$$

If $\vec{B}$ is small:

$$\vec{M} = \alpha\vec{B}, \tag{9.17}$$

the expression

$$\alpha = \frac{n_0 m^2}{3kT} \tag{9.18}$$

is Curie's Law.

Here vector $\vec{M}$ is parallel to field $\vec{B}$ and low diamagnetic effect is suppressed. Paramagnetic salts are used in achieving low temperatures T: the salt is cooled and magnetized up to the saturation, and then a slow adiabatic demagnetization is performed.

If field $\vec{B}$ changes its direction, then $\vec{M}$ also changes its direction, that is the magnetic moment $\vec{m}$ turns, but in this case the energy W increases. If the energy is constant, then T_{kin} decreases. In this way one can achieve temperatures $T \sim 10^{-3} K$. Then one gets $T \sim 10^{-6} K$ due to the adiabatic demagnetization of magnetic moments of nuclei.

9.3. Ferromagnetism

For ferromagnetics $\vec{m}_{at} \neq 0$, but in this case $\vec{m}$ of neighboring atoms strongly interact. The real ferromagnetic properties is explained by quantum mechanics, which proves the necessity of taking into account the forces of interchange between electrons of the atom. However, a lot of ferromagnetic properties can be explained in classical mechanics, if we assume the field affects the atom magnetic moment $\vec{m}$:

$$\vec{B}_{eff} = \vec{H} + b\vec{M}, \tag{9.19}$$

where $\vec{H}$ is the external field and b is a parameter, characterizing the properties of a given ferromagnetic, which is called Weiss's constant.

In this case the potential energy is equal to

$$W = -\left(\vec{m}, \vec{H} + b\vec{M}\right). \tag{9.20}$$

And even without external magnetic field for atom magnetic moments it is advantageous to line up along field $\vec{M}$, that is a spontaneous magnetization takes place (domains).

The further calculation is the same as for paramagnetics: setting in (9.16):

$$a = \frac{m(H + Mb)}{kT}, \tag{9.21}$$

we obtain

$$M = n_0 m \left\{ \frac{e^a + e^{-a}}{e^a - e^{-a}} - \frac{1}{a} \right\}, \tag{9.22}$$

but $a = a(M)$ and this formula can't be written as $M = f(H)$.

Further we consider the conditions under which the solution exists.

Let's assume, that the magnetic field H is small in comparison with a molecular field bM:

$$a \approx \frac{mbM}{kT}\frac{n_0}{n_0} = \frac{M_\infty bM}{n_0 kT} = \frac{M_\infty bM}{n_0 kT}\frac{M_\infty}{M_\infty} = \left(\frac{bM_\infty^2}{n_0 kT}\right)\frac{M}{M_\infty} = \frac{3\theta}{T}\frac{M}{M_\infty}, \qquad (9.23)$$

where we introduce the notations:

$$\theta = \frac{bM_0^2}{3n_0 k}$$

and

$$M_\infty = mn_o. \qquad (9.24)$$

From (9.23) it follows that

$$\frac{M}{M_\infty} == a\frac{T}{3\theta}. \qquad (9.25)$$

The equations (9.25) and (9.26) allow to find the graphic solution, that is to determine the $\frac{M}{M_\infty}$.

From (9.22) one finds

$$\frac{M}{M_\infty} == ctha - \frac{1}{a} \equiv L_\infty(a), \qquad (9.26)$$

where $L_\infty(a)$ is Langevene's function in a classical limit.

The equations (9.25) and (9.26) allow to find the graphic solution, that is to determine the $\frac{M}{M_\infty}$.

Equating the right sides of equations (9.25) and (9.26), we'll get

$$a\frac{T}{3\theta} = ctha - \frac{1}{a}. \qquad (9.27)$$

The plots intersect if

$$\frac{d}{da}\left(ctha - \frac{1}{a}\right)_{a=0} > \frac{T}{3\theta}. \qquad (9.28)$$

Dependence $\frac{M(a)}{M_\infty}$ is given in Fig. 9.2.

Expanding the function $\left(ctha - \frac{1}{a}\right)$ into series, and retaining 4 terms in the expansion yield

$$\frac{d}{da}\left(ctha - \frac{1}{a}\right) \cong \frac{d}{da}(\ldots) = \frac{1}{3}. \tag{9.29}$$

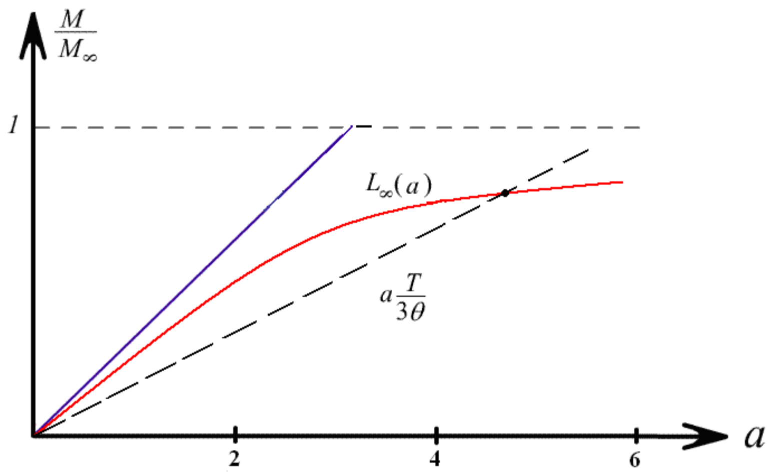


Fig. 9.2. Graphic finding magnetization M at $H = 0$ (that is spontaneous magnetization)

The intersection occurs if the inequality $\frac{1}{3} > \frac{T}{3\theta}$ is satisfied, that is $T < \theta$; θ is Curie temperature.

If $T < \theta$, then the substance is a ferromagnetic, if $T > \theta$, it's a paramagnetic: $\theta = 372^0$ for Ni and $\theta = 758^0$ for Fe.

9.4. Maxwell's equations in magnetics

Let's take some volume dV' in the magnetics.

In magnetic field $\vec{B}$ the substance is magnetized, magnetization vector $\vec{M}(r)$ arises and generates it's own field, which adds to $\vec{B}$. If "poor magnet" is in the coordinate origin, then $\vec{A}(\vec{r}) = \left[\vec{m}\frac{\vec{r}}{r^3}\right]$, if it is in point $\vec{r}\,'$, then $\vec{A}(\vec{r}) = \left[\vec{m}\frac{\vec{R}}{R^3}\right]$.

Let's calculate the vector potential caused by magnetization vector $\vec{M}(r)$:

$$\vec{A}_M(\vec{r}) = \int\left[M(\vec{r}\,')\frac{\vec{R}'}{R^3}\right]dV'. \tag{9.30}$$

Noting that

$$\frac{\vec{R}'}{R^3} = \nabla'\frac{1}{R} \tag{9.31}$$

and using the relation

$$rot'\frac{1}{R}\vec{M} = \frac{1}{R}rot'\vec{M} + \left[\nabla'\frac{1}{R}, \vec{M}\right], \tag{9.32}$$

we'll get

$$\left[\vec{M}\frac{\vec{R}}{R^3}\right]=\frac{1}{R}rot'\vec{M}-rot'\frac{\vec{M}}{R}. \tag{9.33}$$

There is Gauss's theorem a counterpart of Ostrogradskiy-Gauss's theorem:

$$\int_V rot\vec{A}dV=\oint\left[\vec{n}\vec{A}\right]dS, \tag{9.34}$$

where $\vec{n}dS=d\vec{S}$ and $\vec{n}$ is a normal vector to surface dS.

For infinite surface

$$\oint\left[\vec{n}\vec{A}\right]dS=0 \tag{9.35}$$

and

$$\vec{A}_M(\vec{r})=\int\frac{rotM}{R}dV', \tag{9.36}$$

at the same time

$$\vec{A}_j=\frac{1}{c}\int\frac{\vec{j}(\vec{r}')}{R}dV'. \tag{9.37}$$

Total potential is presented by the formula:

$$\vec{A}(\vec{r})=\frac{1}{c}\int\frac{\vec{j}+crotM}{R}dV', \tag{9.38}$$

where

$$\vec{B}=rot\vec{A}.$$

As is known the formula for $\vec{A}_j$ follows from the set of Maxwell's equations:

$$\begin{cases} rot\vec{B}=\dfrac{4\pi}{c}\vec{j} \\ div\vec{B}=0 \end{cases},$$

consequently, in the magnetic

$$\begin{cases} rot\vec{B}=\dfrac{4\pi}{c}\left(\vec{j}+c\cdot rot\vec{M}\right) \\ div\vec{B}=0 \end{cases} \tag{9.39}$$

or

$$\begin{cases} rot\left(\vec{B}-4\pi\vec{M}\right)=\dfrac{4\pi}{c}\vec{j} \\ div\vec{B}=0 \end{cases}, \tag{9.40}$$

where $\vec{B}-4\pi\vec{M}=\vec{H}$ is a magnetizing force:

$$rot\vec{H}=\frac{4\pi}{c}\vec{j}. \tag{9.41}$$

Earlier it was shown that $\vec{M} \sim \vec{B}$, that's why $\vec{H} \sim \vec{B}$, consequently,

$$\vec{B} = \mu\vec{H}\,, \tag{9.42}$$

where μ is a magnetic permeability.

One can write the equation, relating fields $\vec{M}$ and $\vec{H}$:

$$\vec{M} = \kappa\vec{H}\,, \tag{9.43}$$

where κ is called a magnetic susceptibility.

The final form of the set of Maxwell's equations in magneto static's for magnetic substances is as follows:

$$\begin{cases} rot\vec{H} = \dfrac{4\pi}{c}\vec{j}, \\ div\vec{B} = 0, \end{cases}, \tag{9.44}$$

where

$$\vec{B} = \mu\vec{H}\,, \tag{9.45}$$

is called the constraint equation.

If $\vec{j}$, μ are known, then one can calculate $\vec{B}$, $\vec{H}$.

9.5. The classification of Magnetics

Media with a magnetic conductivity $\mu \neq 1$ are called magnetics. In such media magnetic field $\vec{\mathrm{H}}$ and magnetic induction $\vec{B}$ do not coincide, and the relation between them can be nonlinear, as in ferromagnetics. Except magnetic conductivity μ one uses another property of magnetics – magnetic susceptibility κ. It relates the magnetic moment density of a substance $\vec{M}$ (magnetization vector) to the field strength $\vec{H}$:

$$\vec{M} = \chi\vec{H}\,.$$

The magnetic susceptibility is related to μ by the formula:

$$\mu = 1 + 4\pi\kappa\,. \tag{9.46}$$

The value κ can be positive or negative. If $\kappa > 0$, then the substance is called a paramagnetic. Among paramagnetics there is, for example, oxygen, aluminium, platinum, alkali elements. In the course of the value of the molar magnetic susceptibility

$$\kappa \approx 10^{-3} \ldots 10^{-6}\,. \tag{9.46}$$

Paramagnetism is caused by the orientation of magnetic moments of individual structure elements, which constitute the substance under external magnetic field. These magnetic moments do not depend on the external magnetic field, they can exist without it.

The substances with $\kappa > 0$ are called diamagnetics. For them $\kappa \sim 10^{-6}$. The examples of diamagnetics can be inert gasses. All bodies have diamagnetic properties. The table shows the values of the molar magnetic susceptibility for inert gasses and some ions in crystals.

Table

The values of the molar magnetic susceptibility for inert gasses and some ions in crystals

Substance	He	Ne	Ar	Cr	Xe	F^-	Li^+	Ca^{2+}	Ba^{2+}	I^-
$\kappa \cdot 10^{-6}$ $sm^3/\ mole$	–1,9	–7,2	–19,4	–28	–43	–9,4	–0,7	–10,7	–29	–50

Paramagnetism is stronger, and it predominates diamagnetism, that's why diamagnetism can be directly observed in the molecules with $\vec{H} = 0$. Among ferromagnetics there is iron, cobalt, nickel and a lot of their alloys. At low temperatures ferromagnetic elements can be observed in some rare earth elements (gadolinium, terbium, dysprosium, holmium, erbium and thulium).

10. Macroscopic Electrodynamics (The electrodynamics of Polarizing and Magnetizing Media)

10.1. Microscopic and Macroscopic Approaches to the Description of Electromagnetic Phenomena in Media

In the microscopic approach for the calculation of fields one needs to write down the equations for electromagnetic fields, generated by individual moving charges – electrons and nuclei, which constitute the substance, then to add the quantum-mechanical equations of motion for micro particles. Thus, the microscopic approach takes into consideration the processes on an atomic scale.

Maxwell's microscopic equations formulated above are also true when along with the electromagnetic field and the charges (currents), generating it, there is a substance, interacting with the field. Under the influence of the electromagnetic field the charges, which are in the medium, start moving, there appear currents in the medium (let denote the charge and current densities depending on the field strength as ρ and $\vec{j}$ be the correspondingly). There are charges and currents, which don't depend on the strength of the fields, they are called off-side currents (charges): ρ_{ext} и $\vec{j}_{ext}$. So, the bases of the electrodynamics of media are the Maxwell's microscopic equations:

$$rot\vec{e} = -\frac{1}{c}\frac{\partial \vec{h}}{\partial t},$$
$$rot\vec{h} = \frac{4\pi}{c}\left(\vec{j} + \vec{j}_{ext}\right) + \frac{1}{c}\frac{\partial \vec{e}}{\partial t},$$
$$div\vec{h} = 0,$$
$$div\vec{e} = 4\pi\left(\rho + \rho_{ext}\right). \qquad (10.1)$$

In the microscopic approach they should be supplemented with the mechanic equations, for example, Newton's equation:

$$m\ddot{\vec{r}} = e\left(\vec{e} + \frac{1}{c}\vec{v}\cdot\vec{h}\right) + \vec{f}_{ext}, \qquad (10.2)$$

where $\vec{f}_{ext}$ is a force of non-electromagnetic origin.

Such a detailed approach can't be fulfilled because of a great number of equations of type (2), as these equations should be written for each particle. But such a detailed description is not necessary.

Substance properties, exhibited in a specific form ρ and $\vec{j}$, should be described statically that is one should average the Maxwell's equations by the statistic assembly).Sometimes it proves to be sufficient, but more often one has to average (1) by physically small volumes ΔV and time intervals Δt.

If the medium consists of neutral atoms, then the electric microscopic field (measured or calculated) will be the strongest ($\sim 10^{16}$ in Gauss's system) in some points at fixed moments in regions of $\sim 3\cdot 10^{-13}$ cm, occupied with atom nuclei. Away from the nucleus the field gets weaker and becomes very weak at the distances, longer then the typical dimensions of the electron shell. Then in the region, occupied with another atom, the field increases (abruptly). Besides, rapid field fluctuations (temporary) are caused by the atoms motion. These fluctuations take place in plasma, crystals and other media at distances, correlated with distances between neighboring particles.

10.2. Maxwell's Equation Averaging

Let's take a physically small volume of medium ΔV, in which there is a great number of particles, but the averaged region is small in comparison with, for example, the electromagnetic wave-length (if there are any waves in the medium). Let's take time average Δt, which is much longer then a typical period of the charge motion in ΔV.

Let's average equations (1) by ΔV and Δt.

Let's find an average value of any component electromagnetic field $g(\vec{r},t)$:

$$g(\vec{r},t) = \frac{1}{\Delta V\Delta t}\int_{(\Delta V)} d^3\xi \int_{-\Delta t/2}^{\Delta t/2} d\tau g\left(\vec{r} + \vec{\xi}, t + \tau\right), \qquad (10.3)$$

where $d^3\xi$ is a volume element ΔV with the center in point with radius-vector $\vec{r}$:

$$\frac{\partial g}{\partial x} = \frac{1}{\Delta V \Delta t} \int_{(\Delta V)} d^3\xi \int_{-\Delta t/2}^{\Delta t/2} d\tau \frac{dg\left(\vec{r}+\vec{\xi}, t+\tau\right)}{\partial x}, \qquad (10.4)$$

that is

$$\frac{\partial \overline{g}}{\partial x} = \overline{\frac{\partial g}{\partial x}}, \qquad (10.5)$$

where x is a corresponding variable $\{x : x, \mathrm{y}, \mathrm{z}, \mathrm{t}\}$.

The same average values can be found as the result of the assembly average

$$g(\vec{r},t) = \lim_{N\to\infty} \frac{g'(\vec{r},t) + g''(\vec{r},t) + \ldots}{N}, \qquad (10.6)$$

where g', g'' are microscopic field values, which correspond to different assembly systems.

Again

$$\frac{\partial \overline{g}}{\partial x} = \overline{\frac{\partial g}{\partial x}}.$$

Thus,

$$\overline{rot\vec{e}} = rot\overline{\vec{e}}, \quad \overline{div\vec{h}} = div\overline{\vec{h}}, \quad \overline{\frac{\partial \vec{h}}{\partial t}} = \frac{\partial \overline{\vec{h}}}{\partial t}. \qquad (10.7)$$

The result of averaging is

$$rot\overline{\vec{e}} = -\frac{1}{c}\frac{\partial \overline{\vec{h}}}{\partial t}, \qquad rot\overline{\vec{h}} = \frac{4\pi}{c}\left(\overline{\vec{j}} + \overline{\vec{j}}_{ext}\right) + \frac{\partial \overline{\vec{e}}}{\partial t},$$

$$div\overline{\vec{h}} = 0, \qquad div\overline{\vec{e}} = 4\pi\left(\overline{\vec{\rho}} + \overline{\vec{\rho}}_{ext}\right). \qquad (10.8)$$

In microscopic electrodynamics $\overline{\vec{e}} = \vec{E}$, $\overline{\vec{h}} = \vec{B}$ (the vector of magnetic induction). Usually the bare in parameters $\overline{\vec{j}}, \overline{\vec{j}}_{ext}$, $\overline{\vec{\rho}}, \overline{\vec{\rho}}_{ext}$ is omitted, $\vec{j}, j_{ext}, \vec{\rho}, \rho_{ext}$ are considered as average (macroscopic) values of these parameters:

$$rot\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \qquad rot\vec{B} = \frac{4\pi}{c}\left(\vec{j} + \vec{j}_{ext}\right) + \frac{1}{c}\frac{\partial \vec{E}}{\partial t},$$

$$div\vec{B} = 0, \qquad div\vec{E} = 4\pi\left(\rho + \rho_{ext}\right). \qquad (10.9)$$

Equations (10.9) preserve their form despite the values of average region, whose choice depends on characteristics of a problem, experimental technique of field averaging and etc.

But the equation set is not complete, until the relationships are established

$$\left(\vec{E}, \vec{H}\right) \Leftrightarrow \left(\rho, \vec{j}\right) \text{ or } \left(\rho, \vec{j}\right) \Leftrightarrow \left(\rho_{ext}, \vec{j}_{ext}\right).$$

To determine these relations one can apply statistical notions about the motion of particles in a substance – the distribution function $f(\vec{r},\vec{p},t)$ of particles in the studied electromagnetic field is introduced, for a quantum description, the density matrix is introduced.

Thus, $f(\vec{r},\vec{p},t)d^3rd^3p$ is the average number of a part with coordinates within physically small volume $dV = d^3r$, and momentum components are in volume d^3p of the momentum space.

Let's find induced charges and currents:

$$\rho(\vec{r},t) = e\int f(\vec{r},\vec{p})d^3p\,, \tag{10.10}$$

$$\vec{j}(\vec{r},t) = e\int \vec{v} f(\vec{r},\vec{p},t)d^3p\,. \tag{10.11}$$

The distribution function is found as a solution of a kinetic equation.

The kinetic approach to the description of medium electromagnetic properties is the most detailed but it's too rather complicated, because one needs the detailed information about a medium structure.

Due to the lack of such information one has either to introduce phenomenological models or to use experimental relations between $(\rho,\vec{j})$ and $(\vec{E},\vec{B})$.

Usually one uses the Ohm's law in differential form:

$$\vec{j} = \sigma\vec{E} \text{ – for isotropic media} \tag{10.12}$$

or

$$\vec{j}_\alpha = \sigma_{\alpha\beta}\vec{E}_\beta \text{ – for anisotropic media} \tag{10.13}$$

(for fields, which are weak in comparison with atomic ones), where σ is a macroscopic characteristic of the substance, and does not depend on $\vec{E}$.

For strong fields and in anisotropic media

$$\vec{j}_\alpha = \sigma_{\alpha\beta}\vec{E}_\beta + \gamma_{\alpha\beta\gamma}\vec{E}_\beta\vec{E}_\gamma + \xi_{\alpha\beta\gamma}\vec{E}_\beta\vec{E}_\gamma\vec{E}_\delta\,. \tag{10.14}$$

The density of the induced current can be defined not only by $\vec{E}$, but also by a field gradient:

$$\vec{j}_\alpha = \sigma_{\alpha\beta}\vec{E}_\beta + \xi_{\alpha\beta\gamma}\frac{\partial\vec{E}_\beta}{\partial x_\gamma}\,. \tag{10.15}$$

In superconductors

$$rot\vec{j} = -\Lambda\vec{B}\,. \tag{10.16}$$

The solutions of electrodynamics equations in media are remarkable for their variety. It is conditioned by the difference in properties of media (laboratory and cosmic plasmas; metals similar in certain respects, semimetals and semiconductors; non-conducting media: solid and liquid dielectrics; different magnetics, and etc.).

11. Electromagnetic Field Equation for Different Media

Let's establish the relation of induced charges and currents to specific dipole moments.

In equations (10.3) $\rho_{ext}, \vec{j}_{ext}$ are given by external conditions and can be considered as non-homogeneous terms of the equation. Parameters ρ and $\vec{j}$ are not known in advance and they should be expressed in terms of some macroscopic parameters of the substance, which depend on the field strength. They can be written by analogy with the relation $\vec{j} = \sigma\vec{E}$, but it's more convenient to express ρ and $\vec{j}$ in terms of densities $\vec{P}$ and $\vec{M}$ of macroscopic electric and magnetic dipole moments of the medium, defined as

$$\vec{P} = \left(\sum_{(\Delta V)} \vec{p}_i\right)\frac{1}{\Delta V}, \tag{11.1}$$

$$\vec{M} = \frac{1}{\Delta V}\sum_{(\Delta V)} \vec{m}_i , \tag{11.2}$$

where $\vec{p}_i$, $\vec{m}_i$ are dipole moments of individual micro particles, of which the substance consists, ΔV is a microscopic small volume.

Let's express ρ in terms of $\vec{P}$.

Consider any electroneutral body and require

$$\int \vec{P} dV = \int c\vec{r} dV . \tag{11.3}$$

The integration is done over the whole body volume.

Multiply both sides of (11.3) by vector $\vec{a}$ and use the identity

$$\vec{a}\cdot\vec{P} = \left(\vec{P}\cdot\vec{\nabla}\right)\left(\vec{a}\cdot\vec{r}\right), \tag{11.4}$$

$$\int \rho\left(\vec{a}\cdot\vec{r}\right)dV = \int\left(\vec{P}\cdot\vec{\nabla}\right)\left(\vec{a}\cdot\vec{r}\right)dV = \int_V \nabla\left[\vec{P}\left(\vec{a}\cdot\vec{r}\right)\right]dV - \int\left(\vec{a}\cdot\vec{r}\right)div\vec{P}dV, \tag{11.5}$$

where

$$\int_V \nabla\left[\vec{P}\left(\vec{a}\cdot\vec{r}\right)\right]dV = \int_S \vec{n}\left[\vec{P}\left(\vec{a}\cdot\vec{r}\right)\right]dS = 0, \tag{11.6}$$

because $\vec{P}$ is equal to zero outside the body.

From this due to arbitrary $\vec{a}$:

$$\int \rho\vec{r} dV = -\int \vec{r} div\vec{P} dV , \tag{11.7}$$

$$\rho = -div\vec{P} . \tag{11.8}$$

Here $\int \rho dV = 0$ is carried out, the condition of the electroneutrality.

Let's express $\vec{j}$ in terms of the vectors of electric and magnetic polarization. Use the continuity equation for induced charges and currents:

$$\frac{\partial p}{\partial t} + div\vec{j} = 0. \qquad (11.9)$$

Substituting $\rho = -div\vec{P}$ gives

$$div\left(\vec{j} - \frac{\partial \vec{P}}{\partial t}\right) = 0, \qquad (11.10)$$

$$\vec{j} - \frac{\partial \vec{P}}{\partial t} = rot\vec{M}. \qquad (11.11)$$

In this case $rot\vec{M}' \neq 0$, as well as $\vec{P}$, only in volume V.

The current $\vec{j}$ can be presented as the sum of two currents: polarization current $\frac{\partial \vec{P}}{\partial t}$, because this current is bound up with the flow of charges, which form an electrodynamics moment of the substance; the second current $rot\vec{M}'$ is caused by the closed micro currents in the substance, which are not due to the microscopic electric field.

Consider current $\vec{j}$ first in the absence of electric field $\frac{\partial \vec{P}}{\partial t} = 0$.

Then $\vec{j} = rot\vec{M}'$ possesses the property:

$$\int \vec{j} dS = 0. \qquad (11.12)$$

That is the full current through any medium cross section is equal to 0:

$$\frac{1}{2c}\int \vec{r} \cdot \vec{j} dV = \int \vec{M} dV, \qquad (11.13)$$

where $\int \vec{M} dV$ is the total magnetic moment of the body.

Let $\vec{a}$ be an arbitrary constant vector. Multiplying scalarly $\vec{a}$ by both sides of equation (11.13) one gets

$$\frac{1}{2c}\int \vec{a} \cdot \left[\vec{r} \cdot \vec{j}\right] dV = \int \vec{a}\vec{M} dV, \qquad (11.14)$$

$$\vec{a} \cdot \left(\vec{r} \cdot \vec{j}\right) = \vec{a}\left[\vec{r} \cdot rot\vec{M}'\right] = \left[\vec{a} \cdot \vec{r}\right] rot\vec{M}' = \left[\vec{a} \cdot \vec{r}\right]\left[\nabla \cdot \overset{\downarrow}{\vec{M}'}\right].$$

Denote $\vec{f} = \left[\vec{a} \cdot \vec{r}\right]$ and consider

$$\begin{aligned} div\left[\vec{M}' \cdot \vec{f}\right] &= \nabla\left[\vec{M}' \cdot \vec{f}\right] = \nabla\left[\overset{\downarrow}{\vec{M}'} \cdot \vec{f}\right] + \nabla\left[\vec{M}' \cdot \overset{\downarrow}{\vec{f}}\right] = \vec{f}\left[\nabla \vec{M}'\right] - \vec{M}'\left[\nabla \vec{f}\right] = \\ &= div\left[\vec{M}'\left[\vec{a}\vec{r}\right]\right] + \vec{M}' rot\left[\vec{a}\vec{r}\right] = div\left[\vec{M}'\left[\vec{a}\vec{r}\right]\right] + 2\left(\vec{a}\vec{M}'\right). \end{aligned} \qquad (11.15)$$

Substituting (11.16) for (11.15) and using Ostrogradsky-Gauss's theorem, taking into account the vector $\vec{a}$ arbitrariness we find:

$$\vec{M}' = c \cdot \vec{M}. \qquad (11.16)$$

Thus, without the electric field

$$\vec{j} = c \cdot rot\vec{M}\,, \tag{11.17}$$

besides, using Stock's theorem, one can show that

$$\int \vec{j} dS = 0\,. \tag{11.18}$$

Expression (11.17) is also true for static electric field. If there is an alternating electromagnetic field and $\frac{\partial \vec{P}}{\partial t} \neq 0$, then $\frac{\vec{M}'}{c}$ does not coincide with the magnetization vector $\vec{M}'$. However, the formula

$$\vec{j} = \frac{\partial \vec{p}}{\partial t} + c \cdot rot\vec{M} \tag{11.19}$$

is true, if $\vec{M}'$ is not considered as a magnetic moment density.

Substituting the relations (11.8) and (11.9) for the density of the induced charges and currents in the system of equations (10.9) gives:

$$rot\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\,,\ div\left(E + 4\pi\vec{p}\right) = 4\pi\rho_{ext}\,,$$

$$rot\left(\vec{B} - 4\pi\vec{M}\right) = \frac{4\pi}{c}\vec{j}_{ext} + \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{E} + 4\pi\vec{p}\right),$$

$$div\vec{B} = 0\,. \tag{11.20}$$

12. Maxwell's Equations for Media, Constraint Equations

The system (11.20) is similar to that for vacuum. If we introduce two new field vectors:

$$\vec{D} = \vec{E} + 4\pi\vec{P}\,, \tag{12.1}$$

$$\vec{H} = \vec{B} - 4\pi\vec{M}\,. \tag{12.2}$$

Then the system of equations (11.20) takes the form:

$$rot\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\,,\ div\vec{D} = 4\pi\rho_{ext}\,,$$

$$rot\vec{H} = \frac{4\pi}{c}\vec{j}_{ext} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}\,,\ div\vec{B} = 0\,. \tag{12.3}$$

In contrast to Maxwell's system of equations for vacuum the set of equations in medium (12.3) contains four vectors: $\vec{E}$, $\vec{B}$, $\vec{D}$ and $\vec{H}$, that's why set of equation (12.3) is not closed and it should be supplemented with the equations of additional relations between four field vectors.

Until now we consider Maxwell's equations for media were dividing the charges into bound charges ($\rho = \bar{\rho}_{bond}$), exterior charges (ρ_{ext}), and with the current density $\bar{\rho}_{bond} = \frac{\partial \vec{p}}{\partial t} + crot\vec{M}$ и ($\vec{j}_{ext}$), where $\vec{P}$ and $\vec{M}$ depend on $\vec{E}$ and $\vec{B}$.

If we consider conductors, where together with the bound charges there are also free ones, then in this case there is the macroscopic motion of free charges in them caused by the field, that is an electric current. Such a current is called conduction current:

$$\overline{\rho v_{bond}} = \vec{j} . \qquad (12.4)$$

In view of the bound charges and conduction current the average density of macroscopic current can be written as

$$\overline{\rho v} = \overline{\rho v_{bond}} + \overline{\rho v_{free}} = \vec{j} + \frac{\partial \vec{P}}{\partial t} + c\,rot\vec{M} . \qquad (12.5)$$

In this case the average density of microscopic charges (instead of (11.8) $\rho = -div\vec{P}$) is

$$\overline{\rho} = \rho - div\vec{P} , \qquad (12.6)$$

where

$$\rho = \overline{\rho_{bond}} + \overline{\rho_{free}} . \qquad (12.7)$$

One can verify that

$$\frac{\partial \rho}{\partial t} + div\vec{j} = 0 , \qquad (12.8)$$

Then the equations (10.9) take the form:

$$\begin{cases} rot\vec{E} = -\dfrac{1}{c}\dfrac{\partial \vec{B}}{\partial t}, & rot\vec{H} = \dfrac{4\pi}{c}\left(\vec{j} + \vec{j}_{ext}\right) + \dfrac{1}{c}\dfrac{\partial \vec{D}}{\partial t}, \\ div\vec{B} = 0, & div\vec{D} = 4\pi\left(\rho + \rho_{ext}\right), \end{cases} \qquad (12.9)$$

where

$$\vec{D} = \vec{E} + 4\pi\vec{P} , \qquad (12.10)$$

$$\vec{H} = \vec{B} - 4\pi\vec{M} . \qquad (12.11)$$

In contrast to the system (12.11) the constraint equations are not universal; they are determined by the concrete medium properties. Further we will obtain the constraint equations for the media with the simplest properties. Now we suppose that

$$\vec{D} = \varepsilon \ \vec{E} , \qquad (12.12)$$

$$\vec{B} = \mu \ \vec{H} , \qquad (12.13)$$

$$\vec{j} = \sigma \vec{E} . \qquad (12.14)$$

By ε, μ and σ we mean some operators: they can be tensors of the second rank, differential or integral operators, and for fields ε and μ, slowly changing in space and time they reduce to multiplication of $\vec{E}$, $\vec{H}$ and $\vec{\sigma}$ by algebraic (or tensor) values.

The relation $\vec{j} = \sigma\vec{E}$ expresses the differential Ohm's law, where σ is a metal (substance) conductivity.

Operators ε and μ are the operators of electric and magnetic permeability. Macroscopic vectors of field strength $\vec{E}$ and field density $\vec{B}$ are very close to microscopic field strength $\vec{E}$ and magnetic inductivity $\vec{H}$ by their properties. It is just they that determine the force, which work on the charge put in the medium:

$$\vec{F} = q(\vec{E} + \frac{1}{c}[\vec{v}\vec{B}]). \quad (12.15)$$

On the surface of the medium, where the medium properties change stepwise, differential equations (12.9) are not true, and are replaced by the boundary conditions:

$$B_{2n} = B_{1n},\ D_{2n} - D_{1n} = 4\pi\sigma,\ E_{2\tau} = E_{1\tau},\ H_{1\tau} - H_{2\tau} = \frac{4\pi}{c} i_{\nu}, \quad (12.16)$$

where $\vec{\tau}$ and $\vec{n}$ are tangent and normal vector components, current i_{ν} is determined as $\lim_{h\to 0} \vec{\nu}\vec{h}$ and it's a projection onto the direction $\vec{\nu}$ of the surface current. Current i is in the tangent plane to surface Σ and not equal to zero, if the current of a finite force flows in a thin surface layer. $(\vec{\tau}, \vec{n}, \vec{\nu})$ is a triple of mutually orthogonal orts. Numbers 1 and 2 show the side of the surface, where the vectors are taken.

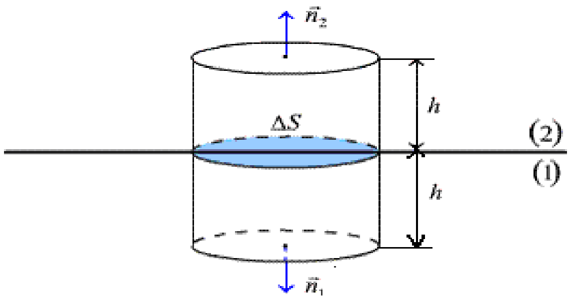


Fig. 12.1a. To the derivation of boundary conditions.
Let's extract a cylinder with base ΔS and height h on the bound of two media

Let's calculate i_{ν} and σ – the density of the surface current and charge, respectively

$$\lim_{h\to 0} \int_{\Delta v} (\rho + \rho_{ext}) dv = \int \sigma dS,\ \lim_{h\to 0} \int (\vec{j} + \vec{j}_{ext}) d\vec{S} = i_{\nu}.$$

In a particular case of anisotropic media $\varepsilon_{ik} = \varepsilon\delta_{ik}$, $\mu_{ik} = \mu\delta_{ik}$, $\sigma_{ik} = \sigma\delta_{ik}$ and the equations (12.12 – 12.14) take the form:

$$\vec{D} = \varepsilon\vec{E}\,,\ \vec{B} = \mu\vec{H}\,,\ \vec{j} = \sigma\vec{E}\,. \qquad (12.17)$$

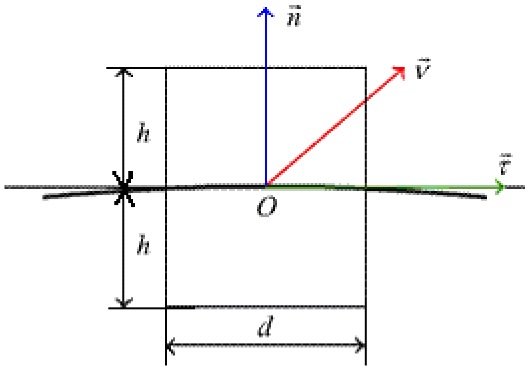


Fig. 12.1b. The same that on the Fig. 12.1a.
Side view, τ, n, σ is a triple of mutually orthogonal orts

Fig. 12.2 shows the behavior of some physical quantity during the transition from medium (1) to medium (2).

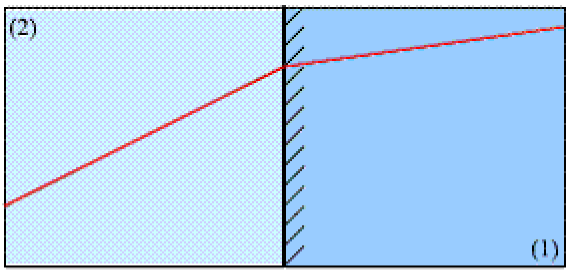


Fig. 12.2. The example of behavior of some physical value
at transition from medium (1) into medium (2)

13. The Energy Conservation Law in Macroscopic Electrodynamics

Let's write down the macroscopic dynamics equations, assuming that there are no exterior currents and charges:

$$rot\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \qquad (13.1a)$$

$$div\vec{D} = 4\pi\rho, \qquad (13.1b)$$

$$rot\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}, \qquad (13.1c)$$

$$div\vec{B} = 0. \qquad (13.1d)$$

From equations 13.1c and 13.1a it follows that

$$\vec{E}rot\vec{H} - \vec{H}rot\vec{E} = -div\left[\vec{E}\cdot\vec{H}\right].$$

Besides, let's take into account the constitutive equations

$$\vec{D} = \varepsilon\vec{E}, \tag{13.2}$$

$$\vec{B} = \mu\vec{H} \,. \tag{13.3}$$

As a result we obtain

$$-div\left[\vec{E}\cdot\vec{H}\right] = \frac{1}{c}\left\{\vec{E}\frac{\partial\vec{D}}{\partial t} + \vec{H}\frac{\partial\vec{B}}{\partial t}\right\} + \frac{4\pi}{c}\vec{j} \,. \tag{13.4}$$

Consider the expression

$$\frac{\partial}{\partial t}\left\{\vec{E}\vec{D} + \vec{H}\vec{B}\right\} = \vec{E}\frac{\partial\vec{D}}{\partial t} + \vec{D}\frac{\partial\vec{E}}{\partial t} + \vec{H}\frac{\partial\vec{B}}{\partial t} + \vec{B}\frac{\partial\vec{H}}{\partial t} \,. \tag{13.5}$$

Let's express $\vec{D}\frac{\partial\vec{E}}{\partial t} + \vec{B}\frac{\partial\vec{H}}{\partial t}$ in terms of $\vec{E}$, $\vec{H}$ and ε, μ:

$$\begin{aligned}\vec{D}\frac{\partial\vec{E}}{\partial t} + \vec{B}\frac{\partial\vec{H}}{\partial t} &= \vec{D}\cdot\frac{\partial}{\partial t}\left(\frac{\vec{D}}{\varepsilon}\right) + \vec{B}\cdot\frac{\partial}{\partial t}\left(\frac{\vec{B}}{\mu}\right) = \frac{\vec{D}}{\varepsilon}\frac{\partial\vec{D}}{\partial t} + \frac{\vec{B}}{\mu}\frac{\partial\vec{B}}{\partial t} + \\ &+ \left[-\frac{\vec{D}^2}{\varepsilon^2}\frac{\partial\varepsilon}{\partial t} - \frac{\vec{B}^2}{\mu^2}\frac{\partial\mu}{\partial t}\right] = \vec{E}\frac{\partial\vec{D}}{\partial t} + \vec{H}\frac{\partial\vec{B}}{\partial t} - \left[\vec{E}^2\frac{\partial\varepsilon}{\partial t} + \vec{H}^2\frac{\partial\mu}{\partial t}\right].\end{aligned} \tag{13.6}$$

Thus, from (13.5) and (13.6) one has

$$\frac{\partial}{\partial t}\left\{\vec{E}\vec{D} + \vec{H}\vec{B}\right\} = 2\left(\vec{E}\frac{\partial\vec{D}}{\partial t} + \vec{D}\frac{\partial\vec{E}}{\partial t}\right) + \left[\vec{E}^2\frac{\partial\varepsilon}{\partial t} + \vec{H}^2\frac{\partial\mu}{\partial t}\right].$$

It gives

$$\begin{aligned}\vec{E}\frac{\partial\vec{D}}{\partial t} + \vec{D}\frac{\partial\vec{E}}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial t}\left\{\vec{E}\vec{D} + \vec{H}\vec{B}\right\} + \frac{1}{2}\left[\vec{E}^2\frac{\partial\varepsilon}{\partial t} + \vec{H}^2\frac{\partial\mu}{\partial t}\right] - div\left[\vec{E}\vec{H}\right] = \\ &= \frac{1}{2c}\frac{\partial}{\partial t}\left\{\vec{E}\vec{D} + \vec{H}\vec{B}\right\} + \frac{1}{2c}\left[\vec{E}^2\frac{\partial\varepsilon}{\partial t} + \vec{H}^2\frac{\partial\mu}{\partial t}\right] + \frac{4\pi}{c}\vec{j}\vec{E} \,.\end{aligned} \tag{13.7}$$

If ε and μ do not depend on time, then we get

$$-div\left[\vec{E}\vec{H}\right] = \frac{1}{2c}\frac{\partial}{\partial t}\left\{\vec{E}\vec{D} + \vec{H}\vec{B}\right\} + \frac{4\pi}{c}\vec{j}\vec{E} \,,$$

$$-div\frac{c\left[\vec{E}\vec{H}\right]}{4\pi} + \frac{\partial}{\partial t}\frac{\left\{\vec{E}\vec{D} + \vec{H}\vec{B}\right\}}{8\pi} + \vec{j}\vec{E} = 0 \,. \tag{13.8}$$

Let's introduce the notations:

$$\omega = \frac{1}{8\pi}\left\{\vec{E}\vec{D} + \vec{H}\vec{B}\right\} = \frac{1}{8\pi}\left\{\varepsilon\vec{E}^2 + \mu\vec{H}^2\right\}, \tag{13.9}$$

$$\vec{S} = \frac{c}{4\pi}\left[\vec{E}\vec{H}\right], \tag{13.10}$$

where ω is the energy density of the electromagnetic field in a substance; $\vec{S}$ is Umov-Pointing's vector. Equation (13.8) takes the form:

$$\frac{\partial \omega}{\partial t} + div\vec{S} + \vec{j}\vec{E} = 0 . \tag{13.11}$$

It means, that the decreasing of energy density with time in the electromagnetic field in some point by the law of energy conservation (13.11) gives rise to the divergence of Pointing's vector, which is not equal to zero, and to the work of the field on free charges:

$$-\frac{\partial}{\partial t}\int \omega dV = \int dV div\vec{S} + \int dV \vec{j}\vec{E} . \tag{13.12}$$

Taking into account that for a substance at rest $\frac{\partial}{\partial t} = \frac{d}{dt}$, we write down the integral conservation law in macroscopic electrodynamics:

$$-\frac{d}{dt}\int \omega dV = \oint \vec{n}\vec{S}dS + \int_V \vec{j}\vec{E}dV . \tag{13.13}$$

The decreasing of the electromagnetic field energy in some volume V is equal to the sum of the energy flow through the surface, which bounds this volume, and field work, preformed on free charges in volume V in a time unit.

14. Quick-alternating Field in a Substance

14.1. Frequency Dispersion of Dielectric Permittivity

Consider the fields

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{E}(\vec{r},\omega)e^{-i\omega t}, \\ \vec{B}(\vec{r},t) &= \vec{B}(\vec{r},\omega)e^{-i\omega t} . \end{aligned} \tag{14.1}$$

In this case frequency ω is not small in comparison with frequencies typical for this medium. Let's suppose that there are no exterior charges and currents, and neglect the magnetic properties of the medium $\mu = 1$:

$$\vec{B}(\vec{r},t) = \mu\vec{H}(\vec{r},t) = \vec{H}(\vec{r},t), \tag{14.2}$$

$$\vec{B}(\vec{r},t) = \varepsilon \cdot \vec{E}(\vec{r},t). \tag{14.3}$$

Then the Maxwell's system of equations takes the form:

$$\begin{cases} rot\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \frac{i\omega}{c}\vec{H}, \\ div\vec{D} = 4\pi\rho, \\ rot\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{j} - \frac{i\omega\varepsilon}{c}\vec{E}(\vec{r},t) = \frac{4\pi}{c}\vec{j} - \frac{i\omega}{c}\vec{D}, \\ div\vec{B} = div\vec{H} = 0, \end{cases} \tag{14.4}$$

and the constraint equations will be written as

$$\vec{D} = \varepsilon\vec{E}, \tag{14.5}$$

$$\vec{j} = \sigma\vec{E}\,, \tag{14.6}$$

where σ and ε depend on ω:

$$\begin{cases} rot\vec{E} = \dfrac{i\omega}{c}\vec{H}, \\ div\vec{D} = 4\pi\rho, \\ rot\vec{H} = \dfrac{4\pi}{c}\vec{j} - i\dfrac{\omega}{c}\vec{D}, \\ div\vec{H} = 0. \end{cases} \tag{14.7}$$

The equation

$$rot\vec{H} = \frac{4\pi}{c}\vec{j} - i\frac{\omega}{c}\vec{D}$$

can be rewritten as

$$rot\vec{H} = \frac{4\pi}{c}\sigma\vec{E} - i\frac{\omega}{c}\varepsilon\vec{E} = -i\frac{\omega}{c}\left(\varepsilon + \frac{4\pi\sigma}{c}i\right)\vec{E}\,. \tag{14.7}$$

There's no sense to divide the charges and currents into free and bound, as under the influence of the quick alternating field both types of charges and currents oscillate in the space and hence they can be described in the same way. One term in the right side describes the contribution of bound electrons, and the second one $\left(\frac{4\pi\sigma}{c}i\right)$ describes conductivity electrons.

So it's convenient to introduce the total effective dielectric permittivity of the medium:

$$\tilde{\varepsilon} = \varepsilon + i\frac{4\pi\sigma}{\omega}. \tag{14.8}$$

Introducing the induction

$$\tilde{\vec{D}} = \tilde{\varepsilon}\vec{E}\,, \tag{14.9}$$

we write

$$rot\vec{H} = -\frac{i\omega}{c}\tilde{\vec{D}}\,. \tag{14.10}$$

The value ε, entering $\tilde{\varepsilon}$ can be complex. Further subscript ~ should be omitted, because both free and bound electrons are taken into account:

$$rot\vec{E} = \frac{i\omega}{c}\vec{H}\,, \tag{14.11}$$

$$rot\vec{H} = -i\frac{\omega}{c}\vec{D}, \quad (14.12)$$

$$divrot\widetilde{H} = -\frac{\omega}{c}div\vec{D} = 0, \quad (14.13)$$

that is $div\vec{D} = 0$ and $\vec{D} = \varepsilon\vec{E}$.

Thus, the equations, describing quick alternating fields will be written as follows:

$$rot\vec{E} = i\frac{\omega}{c}\vec{H},$$

$$rot\vec{H} = i\frac{\omega}{c}\vec{D},$$

$$div\vec{D} = 0. \quad (14.14)$$

And the constraint equation as

$$\vec{D} = \varepsilon\,\vec{E}. \quad (14.15)$$

Formally, due to the harmonic dependence of the fields on time, operator $\partial/\partial t$ is reduced to the substitution in the equations: $\partial/\partial t \to -i\cdot\omega$.

If in (14.4) one makes a change: $\omega \to i\frac{\partial}{\partial t}$, then the equations will be true for arbitrary dependence of the field strength on time, but they will be complicated operator equations, not differential in general case.

Actually, the constraint equation

$$\vec{D} = \varepsilon\,\vec{E}$$

for an arbitrary dependence on time means the integral relation between induction $\vec{D}$ at moment t and strength $\vec{E}$ at all previous moments:

$$\vec{D}(\vec{r},t) = \vec{E}(\vec{r},t) + \int_{-\infty}^{t} f(t-t')\vec{E}(\vec{r},t)dt = E(\vec{r},t) + \int_{0}^{\infty} f(\tau)\vec{E}(\vec{r},t-\tau)d\tau. \quad (14.17)$$

Here the causality principle is used: the induction can depend only on the field strength at previous moments. Physically it means that for quick alternating fields the stabilization of medium polarization can't follow the change immediately. Function $f(\tau)$ determines the field "memory" about the field, which existed at previous moments. Formally

$$\vec{D} = \hat{\varepsilon}\cdot\vec{E},$$

where $\hat{\varepsilon}$ is a linear integral operator.

Let's expand $\vec{D}(t)$ and $\vec{E}(t)$ into Fourier's integral:

$$\int\vec{D}(\vec{r},\omega)e^{-i\omega t}dt = \int\vec{E}(\vec{r},\omega)e^{-i\omega t}dt + \int_{0}^{\infty} f(\tau)\int E(\vec{r},\omega)e^{-i\omega(t-\tau)}dt.$$

It gives the relation:

$$\vec{D}(\vec{r},\omega)=\vec{E}(\vec{r},\omega)+\vec{E}(\vec{r},\omega)\int_0^\infty f(\tau)e^{i\omega\tau}d\tau=\vec{E}(\vec{r},\omega)\left[1+\int_0^\infty f(\tau)e^{i\omega\tau}d\tau\right]=\varepsilon\vec{E}(\vec{r},\omega),$$

that is

$$\vec{D}=\varepsilon\vec{E}=\varepsilon(\omega)\vec{E}\,, \tag{14.18}$$

where

$$\hat{\varepsilon}\cong\varepsilon(\omega)=1+\int_0^\infty f(\tau)e^{i\omega\tau}d\tau\,. \tag{14.19}$$

Let's write ε as a complex value:

$$\varepsilon_1(\omega)+i\varepsilon_2(\omega)=\varepsilon(\omega). \tag{14.20}$$

From (14.19) and (14.20) it follows

$$\varepsilon_1(\omega)=1+\int_0^\infty f(\tau)\cos\omega\tau\,, \tag{14.21}$$

$$\varepsilon_2(\omega)=\int_0^\infty d\tau f(\tau)\sin\omega\tau\,, \tag{14.22}$$

where

$$\varepsilon_1(-\omega)=1+\int_0^\infty f(\tau)\cos(-\omega\tau)=\varepsilon_1(\omega), \tag{14.23}$$

$$\varepsilon_2(-\omega)=-\varepsilon_2(\omega). \tag{14.24}$$

The relations of parity for

$$\varepsilon_{1,2}(-\omega)=\pm\varepsilon_{1,2}(\omega) \tag{14.25}$$

can be combined, to write

$$\varepsilon(-\omega)=\varepsilon(\omega)^*. \tag{14.26}$$

The dependence of the dielectric permittivity on the frequency is called a frequency dispersion of the dielectric permittivity.

15. Dielectric Permittivity Characteristics (Low and High Frequencies)

For low frequencies $\varepsilon(\omega)$ can be expanded into a series in powers ω. The expansion $\varepsilon_1(\omega)$ has only even degrees of ω, and that of $\varepsilon_2(\omega)$ has only uneven degrees. In the limit $\omega\to 0$ in a dielectric $\varepsilon(\omega)$ tends to $\varepsilon(0)$, which is a static value of the dielectric permittivity:

$$\lim_{\omega\to 0}\varepsilon(\omega)=\varepsilon(0). \tag{15.1}$$

Let's expand $\varepsilon(\omega)$ into Maclaurin's series:

$$\varepsilon(\omega) = \varepsilon_1(0) + \frac{\partial\varepsilon(\omega)}{\partial\omega^2}\omega^2 + \ldots + i\left(\varepsilon'(0)\omega + \frac{\varepsilon'''(\omega)\omega^3}{3!} + \ldots\right), \qquad (15.2)$$

$$\lim_{\omega\to 0}\varepsilon(\omega) = \varepsilon_1(0) \equiv \varepsilon(0).$$

In conductors, taking into consideration that now conductivity σ is also included into ε, as $\omega \to 0$ the dielectric permittivity has the pole:

$$\varepsilon_2(\omega) = \frac{4\pi\sigma}{\omega}, \qquad (15.3)$$

where σ is a normal conductivity for continuous current.

There arises a question whether there can exist a frequency range for which the dispersion phenomena are substantial, but still it's admissible to use a macroscopic approach for their description. One needs to verify if the conditions of establishment of electric polarization – dispersion, substantial for $\omega \geq \frac{1}{t_{rel}}$, t_{rel} are compatible (electronic mechanism is the quickest).

It's obvious that

$$t_{rel} \sim \frac{a}{v},\ \omega \geq \frac{v}{a}, \qquad (15.4)$$

where a is atomic sizes; v is a characteristic electron velocity.

Let's formulate the condition at which one can apply the macroscopic description: wave length λ, at which the field strength substantially changes, should be more then a. Cyclic frequency ω relates to linear frequency ν by the relationship:

$$\omega = 2\pi\nu,$$

that's why the condition

$$\lambda = cT >> a \text{ means } \frac{c}{\nu} >> a \text{ or } \frac{2\pi c}{\omega} >> a$$

or

$$\omega << \frac{c}{a}. \qquad (15.5)$$

Since $\frac{v}{c} \sim \frac{1}{137}$, then the inequalities $\omega \geq \frac{v}{a}$ and $\omega << \frac{c}{a}$ can be satisfied simultaneously, so the above-mentioned frequency range exists.

At high frequencies for all substances $\varepsilon \to 1$.

It follows from the fact that if the strength changes quickly, the processes establishing induction $\vec{D}$, different from $\vec{E}$, have no time for realization. If ω is more then all characteristic medium frequencies, then one can find the limiting form of function $\varepsilon(\omega)$, which is true for all substances.

15.1. Limiting Form Definition $\varepsilon(\omega)$. Plasma Frequency

Thus, let ω be high enough, that is it exceeds all characteristic medium frequencies. Then in the calculation of the substance dipole moment all electrons can be considered to be free, because we consider the times equivalent to $\frac{1}{\omega}$. The distance, covered by an electron during this time, is $\frac{v}{\omega} < \frac{c}{\omega}$, that is it's less then the distance, where the field strength is substantially changed. That's why this field can be considered as homogeneous in the definition of the electron displacement, caused by the field.

Let's write down the motion equation

$$m\ddot{\vec{r}} = e\vec{E}, \tag{15.6}$$

where

$$\vec{E} = \vec{E}(\omega)e^{-i\omega t}. \tag{15.7}$$

A solution is sought in the form

$$\vec{r}(t) = \vec{a}e^{-i\omega t}. \tag{15.8}$$

Substituting (15.8) in (15.6) we obtain the relation

$$\ddot{\vec{r}} = -\vec{a}\omega^2 e^{-i\omega t} = \frac{e}{m}\vec{E}(\omega)e^{-i\omega t}, \tag{15.9}$$

from which it follows that

$$\vec{a} = -\frac{e}{m\omega^2}\vec{E}(\omega), \tag{15.10}$$

$$\vec{r}(t) = -\frac{e}{m\omega^2}\vec{E}.$$

The multiplication of $\vec{r}(t)$ by $N \cdot e$, where N is an electron concentration, gives:

$$\vec{P} = -\frac{Ne^2}{m\omega^2}\vec{E}. \tag{15.11}$$

Since the induction $\vec{D} = \vec{E} + 4\pi\vec{P}$, then

$$\vec{D} = \left(1 - \frac{4\pi Ne^2}{m\omega^2}\right)\vec{E} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\vec{E}, \tag{15.12}$$

where

$$\omega_p = \sqrt{\frac{4\pi Ne^2}{m}}. \tag{15.13}$$

Parameter ω_p is called a plasma frequency.

On the other hand, $\vec{D}=\varepsilon(\omega)\vec{E}$, that is

$$\varepsilon(\omega)=1-\frac{\omega_p^2}{\omega^2}. \tag{15.14}$$

Thus, for the description of the quick alternating fields in the substance for arbitrary relation between strength and time the following form of the constraint equation is used

$$\vec{D}(\vec{r},t)=E(\vec{r},t)+\int_0^\infty f(\tau)\vec{E}(\vec{r},t-\tau)d\tau, \tag{15.15}$$

and instead of the equations

$$\begin{cases} rot\vec{E}=\dfrac{i\omega}{c}\vec{H}, \\ rot\vec{H}=-\dfrac{i\omega}{c}\vec{D}, \end{cases} \tag{15.16}$$

one can use the equations

$$\begin{cases} rot\vec{E}=-\dfrac{1}{c}\dfrac{\partial\vec{H}}{\partial t}, \\ rot\vec{H}=-\dfrac{1}{c}\dfrac{\partial\vec{D}}{\partial t}. \end{cases} \tag{15.17}$$

16. Complex Dielectric Permittivity of Rarefied Neutral Gas

Consider the problem of wave propagation in rarefied gas. The field strength changes according to the harmonic law:

$$\vec{E}=\tilde{\vec{E}}_0 e^{-i(\omega t-\vec{\kappa}\vec{r})},$$

$$\vec{H}=\tilde{\vec{H}}_0 e^{-i(\omega t-\vec{\kappa}\vec{r})}. \tag{16.1}$$

Under the action of an electromagnetic wave there arises the alternating dipole moment $\vec{P}=\vec{P}(t)$, which functionally depends on the fields $\vec{E}$ and $\vec{H}$ of the wave and atom properties of the medium.

Denote an electric dipole moment of one atom by $\vec{d}(t)$. If a volume unit contains N atoms, then

$$\vec{P}=N\vec{d}(t). \tag{16.2}$$

That's why for monochrome wave (without dispersion)

$$\vec{D}=\varepsilon(\omega)\vec{E}$$

and

$$\vec{D} = \varepsilon(\omega)\vec{E} = \vec{E} + 4\pi\vec{P} = \vec{E} + 4\pi N\vec{d}(t). \qquad (16.3)$$

Thus, to find $\varepsilon(\omega)$ it's necessary to know an explicit relation between $\vec{d}$ and $\vec{E}$. In classical theory this relation can be found on the basis of the oscillating atom model, in which an atom is considered as a motionless nucleus with point electrons, moving non-relativistically round it.

Let m, e be the electron mass and the charge respectively, γ is a coefficient, characterizing attenuation if $\gamma > 0$ (or in other words, dissipative properties of an oscillator), if $\gamma < 0$ antidissipative properties of an oscillator, $\vec{R}_0$ is an electron radius-vector without disturbing force (in this case the disturbing force is an electromagnetic wave), $\vec{R}(t)$ is an electron radius-vector in presence of disturbance, ω_0 is an oscillator frequency.

Quasi-elastic force $= -\kappa\left(\vec{R} - \vec{R}_0\right) = -\frac{\kappa}{m} m\left(\vec{R} - \vec{R}_0\right) = -\omega_0^2 m\left(\vec{R} - \vec{R}_0\right)$.

The motion equation for an oscillator under the action of disturbing force can be written as

$$m\ddot{\vec{R}} + m\gamma\dot{\vec{R}} + m\omega_0^2\left(\vec{R} - \vec{R}_0\right) = \vec{F}, \qquad (16.4)$$

where

$$\vec{F} = e\left\{\tilde{\vec{E}}_0 + \frac{1}{c}\left[\dot{\vec{R}}\tilde{\vec{H}}_0\right]\right\} e^{-i\left(\omega t - \vec{\kappa}\vec{R}\right)}. \qquad (16.5)$$

Is called Lorenz's force of electromagnetic wave.

Since the electron displacement vector with respect to the equilibrium position is equal to $\vec{r} = \left(\vec{R} - \vec{R}_0\right)$, then

$$m\ddot{\vec{r}} + m\gamma\dot{\vec{r}} + m\omega_0^2\vec{r} = e\left\{\vec{E}_0 + \frac{1}{c}\left[\dot{\vec{R}}\vec{H}_0\right]\right\} e^{-(\omega t - \vec{k}\vec{r}}), \qquad (16.6)$$

where we introduced the notations:

$$\vec{E}_0 = \tilde{\vec{E}}_0 e^{i\vec{\kappa}\vec{R}_0}, \ \vec{H}_0 = \tilde{\vec{H}}_0 e^{i\vec{\kappa}\vec{R}_0}. \qquad (16.7)$$

Equation (16.6) is nonlinear and can't be solved in general form. It's linearized by making use of the smallest of its parameters. In the non-relative case (electron in the atom has a non-relativistic motion) $\frac{\vartheta}{c} << 1$, besides, in electromagnetic wave $\left|\vec{E}_0\right| \sim \left|\vec{H}_0\right|$, that's why the magnetic part of the Lorenz's force can be neglected. Let's also take into consideration that displacement $\left|\vec{r}\right| << a$, where a is an interatomic distance and in view of this $\vec{\kappa}\vec{r} = 0$ can be considered in respect to microscopic electrodynamics and $e^{i\vec{\kappa}\vec{r}} = 1$. Then

$$m\ddot{\vec{r}} + m\gamma\dot{\vec{r}} + m\omega_0^2\vec{r} = e\vec{E}_0 e^{-i\omega t}. \qquad (16.8)$$

Multiply equations (16.8) by e and take into account that $\vec{d} = e\vec{r}$, where $\vec{d}$ is a dipole moment:

$$\ddot{\vec{d}} + \gamma\dot{\vec{d}} + \omega_0^2\vec{d} = \frac{e^2}{m}\vec{E}_0 e^{-i\omega t}. \qquad (16.9)$$

Generally speaking, the solution of equation (16.9) is a combination of the general solution of the homogeneous equation and particular solution of the heterogeneous equation. The solution of the homogeneous equation is a linear combination of two independent solutions, where the constants are determined by the initial conditions. It's necessary to take into account that for any initial conditions these solutions tend to zero because $e^{-\gamma t}$ as $t \to \infty$. This implies that a solutions of equation (16.9) should be sought as a particular solution of the heterogeneous equation.

Thus, we seek the solution in the form

$$\vec{d} = \vec{d}_0 e^{-i\omega t}. \qquad (16.10)$$

Substituting the expression for $\vec{d}$ in (16.9):

$$\left[\omega_0^2 - \omega^2 - i\gamma\omega\right]\vec{d} = \frac{e^2}{m}\vec{E}_0 e^{-i\omega t} = \frac{e^2}{m}\vec{E},$$

yields

$$\vec{d} = \frac{e^2\vec{E}}{m\left[\omega_0^2 - \omega^2 - i\gamma\omega\right]} = \frac{e^2}{m}\frac{\vec{E}\left[\omega_0^2 - \omega^2 + i\gamma\omega\right]}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}. \qquad (16.11)$$

The oscillation of vectors $\vec{d}$ and $\vec{E}$ proceed out of phase. Actually, let's represent the value

$$\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} = \frac{\omega_0^2 - \omega^2 + i\gamma\omega}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2} = \rho e^{i\psi} \qquad (16.12)$$

as

$$z = a + ib = |z|e^{i\psi}, \qquad (16.13)$$

where

$$z = \sqrt{a^2 + b^2}, \ \sin\psi = \frac{b}{\sqrt{a^2 + b}}, \ \cos\psi = \frac{a}{\sqrt{a^2 + b^2}}, \qquad (16.14)$$

as is obvious from Fig. 16.1.

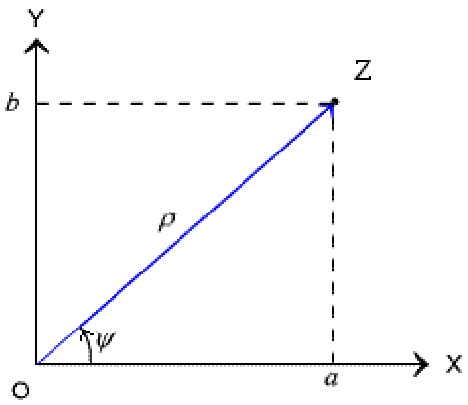


Fig. 16.1. The forms of complex number presentation: $z = a + ib = \rho e^{i\Psi}$

As a result we obtain:

$$\cos\psi = \frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2} \frac{1}{\sqrt{\frac{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^2}}} = \tag{16.15}$$

$$= \frac{\omega_0^2 - \omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}}$$

and

$$\sin\psi = \frac{\gamma\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}}.$$

From the expression for d (16.11) and formulas (16.12-16-15), it's obvious that

$$\vec{d} = \frac{e^2\vec{E}}{m\left(\omega_0^2 - \omega^2 - i\gamma\omega\right)} = \frac{e^2\vec{E}_0 e^{-i(\omega t - \psi)}}{m\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}}. \tag{16.16}$$

Comparing it with $\vec{E} = \vec{E}_0 e^{-i\omega t}$, we determine that phase difference ψ between the oscillation of vectors $\vec{d}$ and $\vec{E}$ depends both on the frequency of an incident wave and on atom characteristics γ and ω_0.

Let's study the relation between the phase difference ψ and the frequency

$$\psi = arctg\frac{\gamma\omega}{\omega_0^2 - \omega^2}.$$

It's obvious that

$$\begin{cases} 0 \le \psi \le \dfrac{\pi}{2} \text{ - if } \omega < \omega_0 \\ \psi = \dfrac{\pi}{2} \qquad \text{- in resonance} \\ \dfrac{\pi}{2} < \psi \le \pi \text{ - if } \omega_0 < \omega < \infty. \end{cases} \tag{16.17}$$

Let's find the polarization vector of the medium in view of (16.11):

$$\vec{P} = N\vec{d} = \frac{e^2 N\vec{E}\left[\omega_0^2 - \omega^2 + i\gamma\omega\right]}{m\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}. \qquad (16.18)$$

For a known polarization vector, let's find the expression for the induction vector:

$$\vec{D} = \varepsilon(\omega)\vec{E} = \vec{E} + 4\pi\vec{P} = \vec{E}\left[1 + \frac{4\pi e^2 N\left[\omega_0^2 - \omega^2 + i\gamma\omega\right]}{m\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}\right]. \qquad (16.19)$$

It is common to denote

$$\omega_p^2 = \frac{4\pi e^2 N}{m}, \qquad (16.20)$$

where ω_p is a plasma frequency [the frequency of plasma deviations or Langmuir's frequency], that's why in the constraint equation

$$\vec{D} = \varepsilon(\omega)\vec{E}.$$

The dielectric permittivity ε depends on the frequency and has the form

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2\left[\omega_0^2 - \omega^2 + i\gamma\omega\right]}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}. \qquad (16.21)$$

As it follows from formula (16.21), $\varepsilon(\omega)$ depends not only on the medium (in terms of γ and ω_0), but also on the frequency ω of incident electromagnetic radiation and in general case is a complex value:

$$\varepsilon = \varepsilon' + i\varepsilon'', \qquad (16.22)$$

$$\varepsilon' = 1 + \frac{\omega_p^2\left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}, \qquad (16.23)$$

$$\varepsilon'' = \frac{\omega_p^2\gamma\omega}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}. \qquad (16.24)$$

The oscillation of induction vector $\vec{D}$ proceed out of phase with $\vec{E}$ due to complexity of $\varepsilon(\omega)$:

$$\vec{D} = \left(\varepsilon' + i\varepsilon''\right)\vec{E} = \sqrt{\varepsilon'^2 + \varepsilon''^2}\,\vec{E}_0 e^{-i(\omega t - \eta)}, \qquad (16.25)$$

$$\eta = arctg\frac{\varepsilon''}{\varepsilon'} = arctg\frac{\omega_p^2\gamma\omega}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right] + \omega_p^2\left(\omega_0^2 - \omega^2\right)}. \qquad (16.26)$$

Let's analyze the dependences of ε' and ε'' on the frequency. The whole of frequency region can be divided into three parts, depending on $\frac{\partial\varepsilon'}{\partial\omega}$ (Fig. (16.2)):

I and III, where $\frac{\partial \varepsilon'}{\partial \omega} > 0$ is a range of normal dispersion;

II, where $\frac{\partial \varepsilon'}{\partial \omega} < 0$ is a range of anomalous dispersion;

$\varepsilon''(\omega)$ is much higher in the second range than in I and III;

ε'' is bound up with the dissipative term $(\sim \gamma)$, and that's why the energy absorption of an electromagnetic wave by the medium in II is much higher than in I and III.

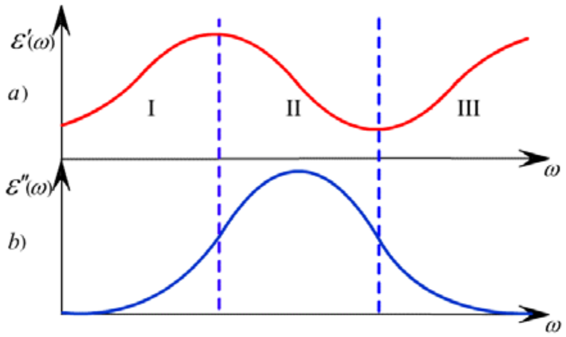


Fig. 16.2. The diagram of dependence of real and imaginary part of dielectric permittivity on the frequency: a) real part; b) imaginary part

Relatively high values of ε'' correspond to the opaque region, and low values – to the transparent region.

A important particular case of expression (16.21) is dielectric permittivity of plasma and plasma-like media. In such media electrons and nuclei are not bound into nuclei, consequently, quasi-elastic (recovery) force is equal to 0 $(\omega_0 = 0)$. Then if there is no absorption $(\gamma = 0)$:

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2}.$$

It means that if the frequency of forced oscillation coincides with plasma frequency, then the dielectric permittivity of plasma-like media is equal to 0.

17. Physical Meaning of Imaginary Part ε

From Maxwell's equations for macroscopic electrodynamics

$$rot\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t},$$

$$div\vec{D} = 4\pi\rho,$$

$$rot\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}, \quad div\vec{B} = 0 \qquad (17.1)$$

and from material equations

$$\vec{D} = \varepsilon\vec{E}\,, \qquad (17.2)$$

$$\vec{B} = \mu\vec{H}\,, \qquad (17.3)$$

one can deduce the law of energy conservation in differential form:

$$\frac{\partial\omega}{\partial t} + div\vec{S} + \vec{j}\vec{E} = 0\,, \qquad (17.4)$$

where

$$\omega = \frac{1}{8\pi}\left\{\vec{E}\vec{D} + \vec{H}\vec{B}\right\} = \frac{1}{8\pi}\left\{\varepsilon\vec{E}^2 + \mu\vec{H}^2\right\} \qquad (17.5)$$

is the energy density of electromagnetic field in the substance, and $\vec{S} = \frac{c}{4\pi}\left[\vec{E}\vec{H}\right]$ is Umov-Pointing's vector.

The integral law of conservation of energy has the form

$$-\frac{d}{dt}\int\omega dV = \oint\vec{n}\vec{S}dS + \int\vec{j}\vec{E}dV\,. \qquad (17.6)$$

And if there are no free charges, the law of conservation of energy can be written as

$$-\frac{d\varepsilon}{dt} = \oint\limits_S\vec{n}\vec{S}dS\,. \qquad (17.7)$$

And it means that the energy decrease of electromagnetic field (electromagnetic waves) in some medium volume V is equal to the total energy flow through the surface, bounding this volume V.

In previous section we established the relation between the imaginary part of dielectric permittivility $\varepsilon(\omega)$ and dissipation (or anti-dissipation) of the energy of electromagnetic wave, which propagates in neutral rarefied gas:

$$\varepsilon''(\omega) = \frac{\omega_p^2\gamma\omega}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}\,. \qquad (17.8)$$

This expression implies that:

$\gamma > 0$ corresponds to the attenuation or dissipative properties of the oscillator;

$\gamma < 0$ corresponds to the anti-dissipation, for example, for laser media.

This property is typical for other material media as well. If complex dielectric permittivility $\varepsilon(\omega)$ has $\varepsilon''(\omega) \neq 0$, it means that the medium either absorbs the energy of electromagnetic field, converting it to other types of energy (heat, for example), or transmits the stored energy to electromagnetic wave (anti-dissipating media, laser media).

If $\Pi \equiv \oint\vec{S}\vec{n}dS < 0$, then the dissipation of energy takes place, if $\Pi > 0$, then the energy of, for example, luminous electrons, in metastable atom levels of laser media, coming through electromagnetic radiation, converts into electromagnetic energy, which increases the energy flow of outflow electromagnetic waves.

Let's consider the problem: find how the sign Π relates to $Jm\varepsilon(\omega) \equiv \varepsilon^{n}(\omega)$. By the definition

$$\Pi = \oint \vec{S}\vec{n}dS = \int div\vec{S}dV, \qquad (17.9)$$

where the second side of the equation is written according to Ostrogradskiy-Gauss's theorem, where

$$div\vec{S} = \frac{c}{4\pi}div\left[\vec{E}\vec{H}\right] = \frac{c}{4\pi}\left\{\vec{H}rot\vec{E} - \vec{E}rot\vec{H}\right\}. \qquad (17.10)$$

From Maxwell's equations:

$$rot\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \text{ и } rot\vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}, \qquad (17.11)$$

it follows that

$$div\vec{S} = \frac{c}{4\pi c}\left\{-\vec{H}\frac{\partial \vec{B}}{\partial t} - \vec{E}\frac{\partial \vec{D}}{\partial t}\right\} = -\frac{1}{4\pi}\left\{\vec{H}\frac{\partial \vec{B}}{\partial t} + \vec{E}\frac{\partial \vec{D}}{\partial t}\right\}. \qquad (17.12)$$

Generally speaking, vectors $\vec{H}$, $\vec{B}$, $\vec{E}$, $\vec{D}$ should be real, if we want to leave a complex formulation of the vectors, then in $div\vec{S}$ it's necessary to take their Re parts:

$$div\vec{S} = -\frac{1}{4\pi}\left\{\mathrm{Re}\vec{H}\frac{\partial}{\partial t}\mathrm{Re}\vec{B} + \mathrm{Re}\vec{E}\frac{\partial}{\partial t}\mathrm{Re}\vec{D}\right\}. \qquad (17.13)$$

Since $\mathrm{Re}\vec{A} = \frac{1}{2}\left(\vec{A} + \vec{A}^*\right)$, then

$$div\vec{S} = -\frac{1}{16\pi}\left\{\left(\vec{H} + \vec{H}^*\right)\frac{\partial}{\partial t}\left(\vec{B} + \vec{B}^*\right) + \left(\vec{E} + \vec{E}^*\right)\frac{\partial}{\partial t}\left(\vec{D} + \vec{D}^*\right)\right\}. \qquad (17.14)$$

Let's assume that the wave is monochromatic, that is $\vec{E} = \vec{E}_0 e^{-i\omega t}$. The same refers to the fields $\vec{H}$, $\vec{B}$ and $\vec{D}$.

For the derivatives of fields with respect to we obtain

$$\frac{\partial \vec{B}^*}{\partial t} = +i\omega\vec{B}^*, \text{ and etc.}$$

The expression of the divergence of Umov-Pointing's vector takes the form

$$div\vec{S} = -\frac{i\omega}{16\pi}\left\{\left(\vec{H} + \vec{H}^*\right)\left(\vec{B} - \vec{B}^*\right) + \left(\vec{E} + \vec{E}^*\right)\left(\vec{D} - \vec{D}^*\right)\right\}. \qquad (17.15)$$

Let's use the equations

$$\vec{D}(\omega) = \tilde{\varepsilon}(\omega)\vec{E} = (\varepsilon' + i\varepsilon'')\vec{E}, \; \vec{B} = \mu\vec{H},$$

$$\vec{D}^* = (\varepsilon' - i\varepsilon'')\vec{E}^*,\ \vec{B}^* = \mu\vec{H}^*, \tag{17.16}$$

then formula (17.15) will transform as:

$$\begin{aligned} div\vec{S} &= \frac{i\omega}{16\pi}\left\{\mu\left(\vec{H}^2 - \vec{H}^{*2}\right) + \left(\vec{E} + \vec{E}^*\right)\left[\left(\varepsilon' + i\varepsilon''\right)\vec{E} - \left(\varepsilon' - i\varepsilon''\right)\vec{E}^*\right]\right\} = \\ &= \frac{i\omega}{16\pi}\left\{\mu\left(\vec{H}^2 - \vec{H}^{*2}\right) + \varepsilon'\left(\vec{E}^2 + \vec{E}^{*2}\right) + i\varepsilon''\left(\vec{E}^2 + 2\vec{E}\vec{E}^* + \vec{E}^{*2}\right)\right\}. \end{aligned} \tag{17.17}$$

This expression is not quite convenient for the analysis, because it contains the quick oscillating terms $\vec{H}^2$, $\vec{E}^2$, $\vec{H}^{*2}$, $\vec{E}^{*2}$ with frequency 2ω:

$$\vec{H}^2 = \left(\vec{H}_0 e^{-i\omega t}\right)^2 = \vec{H}_0^2 e^{-2i\omega t}. \tag{17.18}$$

Let's average the expression $\overline{\Pi}$ over a wave period: $T = \frac{2\pi}{\omega} = \frac{\lambda}{c}$,

$$\overline{\Pi} = \frac{1}{T}\int_0^T \Pi dt = \frac{1}{T}\int_V\int_0^T div\vec{S}dVdt = \int dV\,\overline{div\vec{S}}, \tag{17.19}$$

where

$$\overline{div\vec{S}} = \frac{1}{T}\int_0^T div\vec{S}dt. \tag{17.20}$$

When averaging the field squares there arises the integral:

$$\int_0^T dte^{-2i\omega t} = \int_0^T dte^{2i\omega t} = 0,$$

and the values $\vec{H}^2$, $\vec{E}^2$, $\vec{H}^{*2}$, $\vec{E}^{*2} \sim e^{-2i\omega t}$ don not occur in the expression for $\overline{div\vec{S}}$.

As a result one gets

$$\overline{div\vec{S}} = -\frac{\omega\varepsilon''}{8\pi}\vec{E}\vec{E}^* = -\frac{\omega\varepsilon''}{8\pi}\left|\vec{E}\right|^2. \tag{17.21}$$

Since $\omega\left|\vec{E}\right|^2 > 0$, then sign $\overline{div\vec{S}}$, and, consequently, sign $\overline{\Pi}$ depend on sign ε''.

If $\varepsilon'' > 0$, then $\overline{div\vec{S}} < 0$ in every point of volume V, and, consequently, $\overline{\Pi} < 0$.

Thus, $\varepsilon'' \neq 0$ is a direct consequence of dissipative properties of the media.

By the definition $\Pi = \oint_S \vec{S}\vec{n}dS$ is an energy flow through the surface, bounding the medium volume V.

Oscillating components describe the energy interchange between the field and the medium: the field, accelerating the charges, transmits them its energy, the charges while accelerating (retarding) radiate and as a result the energy is transmitted to the field.

On the average this process does not influence the energy interchange between the field and the medium. The interchange is determined only by components, which do not depend on time.

In the state of thermodynamic equilibrium the medium always absorbs and $\varepsilon'' > 0$, However, it's possible to generate such medium condition, at which $\varepsilon'' < 0$ and the increase of an electromagnetic wave occurs.

For some frequency values $\varepsilon'' = 0$ the medium becomes translucent for the radiation with such frequency. The frequency regions, where ε'' values are low enough, are called transmission band of the substance.

18. Dispersion Relations of Kramers–Kronig

Even for the elementary material medium of neutral rarefied gas the vector oscillation $\vec{P}$, caused by external electromagnetic wave, lags behind in phase as compared with $\vec{E}$ oscillations:

$$\vec{P} = N\vec{d} = \frac{Ne^2\vec{E}_0 e^{-i(\omega t - \psi)}}{m\sqrt{\left(\omega_0 - \omega_0^2\right)^2 + \gamma^2\omega^2}}, \qquad (18.1)$$

$$\psi = arctg\frac{\gamma\omega}{\omega_0^2 - \omega^2}, \qquad (18.2)$$

that is at every fixed moment $\vec{D} = \vec{E} + 4\pi\vec{P}$ is determined not only by $\vec{E}$ value at the same moment t, but also by the field values at the previous moments:

$$D(t) = \vec{E}(t) + 4\pi\int_0^T f(\tau)\vec{E}(t-\tau)d\tau . \qquad (18.3)$$

Where function $f(\tau)$ depends on the medium properties and demonstrates the influence of field $\vec{E}$ at previous moments on the state $\vec{D}$ at given moment *t*. It's obvious that $f(\tau)$ is a bounded function for all τ and quickly and smoothly tends to zero as $\tau \to \infty$. Let's expand (18.3) into Fourier's integral in time:

$$\vec{D}(t) = \int_{-\infty}^{+\infty} d\omega\vec{D}(\omega)e^{-i\omega t}, \qquad (18.4)$$

$$\vec{E}(t) = \int_{-\infty}^{+\infty} d\omega\vec{E}(\omega)e^{-i\omega t}. \qquad (18.5)$$

Substituting (18.4) and (18.5) into (18.3), gets

$$\vec{D}(\omega) = \vec{E}(\omega)\left\{1 + 4\pi\int_0^\infty f(\tau)e^{i\omega\tau}d\tau\right\}. \qquad (18.6)$$

For isotropic media the relation between $\vec{D}$ and t for each fixed value of ω has the form:

$$\vec{D}(\omega) = \varepsilon(\omega)\vec{E}(\omega). \tag{18.7}$$

Comparison of (18.6) and (18.7) gives:

$$\varepsilon(\omega) = 1 + 4\pi \int_0^\infty f(\tau) e^{i\omega\tau} d\tau . \tag{18.8}$$

This relation has been obtained without any detailed description of medium properties, proceeding only from the causality principle, therefore this expression should be true for wide of material media.

Using relation (18.8), let's study the properties of $\varepsilon(\omega)$.

Since the function is real, and

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega), \tag{18.9}$$

then

$$\varepsilon'(\omega) = 1 + 4\pi \int_0^\infty f(\tau) \cos \omega\tau d\tau , \tag{18.10}$$

$$\varepsilon''(\omega) = 4\pi \int_0^\infty f(\tau) \sin \omega\tau d\tau . \tag{18.11}$$

From relations (18.10) and (18.11) it follows:

$$\varepsilon'(-\omega) = \varepsilon'(\omega), \tag{18.12}$$

$$\varepsilon''(-\omega) = -\varepsilon''(\omega). \tag{18.13}$$

In relation (18.8) let's change over complex values

$$\omega \to z = x + iy , \tag{18.14}$$

where z is a complex frequency:

$$\varepsilon(z) = 1 + \int_0^\infty f(\tau) e^{i\tau z} d\tau . \tag{18.15}$$

In contras to (18.8) we define the function $f(\tau)$ as

$$4\pi f(\tau) \to f(\tau). \tag{18.16}$$

It obviously does not influence ε properties.

Let's write (18.5) in the form

$$\varepsilon(z) \equiv \varepsilon(x + iy) = 1 + \int_0^\infty f(\tau) e^{i\tau x} e^{-\tau y} d\tau . \tag{18.17}$$

Integral (18.17) converges for any $y > 0$, since the function $f(\tau)$ is bounded. Besides, for dielectric media function $f(\tau)$ tends fast to zero as $\tau \to \infty$, that's why the integral also converges at $y = 0$. It means that function $\varepsilon(z)$ has no peculiarities in the upper half plane, including the real axis, that is for $y=0$, except for the coordinate origin, where there is a simple pole for conductors.

Let's consider dielectrics as a medium.

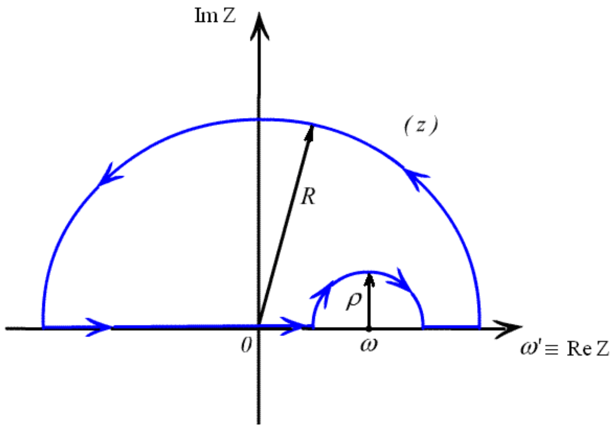


Fig. 18.1. The view of closed contour Γ in the complex plane of argument z while calculating integral 18.18

In view of the mentioned above, integral I over closed contour C (see Fig. 18.1) is equal to 0:

$$I = \oint_{\Gamma} \frac{\varepsilon(z)-1}{z-\omega} dz \equiv \oint_{\Gamma} \frac{f(z)}{z-\omega} dz = 0, \tag{18.18}$$

where

$$f(z) = \varepsilon(z) - 1. \tag{18.19}$$

According to the Cauchy integral formula the integral over closed contour Γ, bounding the area, inside of which there is an analytic function $f(z)$, reduces to function f, taken in some point ω:

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)dz}{z-\omega} = \begin{cases} f(\omega), \text{if } \omega \text{ is inside the contour,} \\ 0, \text{if } \omega \text{ is outside the contour .} \end{cases} \tag{18.20}$$

If point ω lies on an integration contour, then the Cauchy formula has the form:

$$f(\omega) = \frac{1}{\pi i} \int \frac{f(z)dz}{z-\omega}. \tag{18.21}$$

The integral over real axis with the excluded singular point ω is calculated in the principle value sense:

$$\lim_{\rho\to 0} \left\{ \int_{-\infty}^{\omega-\rho} \frac{f(x)}{x-\omega} dx + \int_{\omega+\rho}^{\infty} \frac{f(x)}{x-\omega} dx \right\} = P \int_{-\infty}^{+\infty} \frac{f(x)}{x-\omega} dx, \tag{18.22}$$

$$I = I_1 + I_2 + I_3 + I_4 = 0, \tag{18.23}$$

where

$$I_1 = \int_{-\infty}^{\omega-\rho} \frac{\varepsilon(x)-1}{x-\omega}dx\,,\ I_3 = \int_{\omega+\rho}^{\infty} \frac{\varepsilon(x)-1}{x-\omega}dx\,,$$

$$I_2 = \int_{C_\rho} \frac{\varepsilon(z)-1}{z-\omega}dz\,,\ I_4 = \int_{C_R} \frac{\varepsilon(z)-1}{z-\omega}dz\,. \tag{18.24}$$

Integral I_4 over contour C_R is equal to 0, since on the great-radius arc the integral element tends to zero faster then $\frac{1}{|z|}$ as $|z| \to \infty$. It follows from the form of function $\varepsilon(\omega)$ for high frequencies: $\varepsilon(\omega)-1 \sim -\frac{\omega_p^{\ 2}}{\omega^2}$, and integral element on the section of contour C_R at $|z| \to \infty$ is as $\frac{1}{|z|^3}$:

$$I_4 = 0\,, \tag{18.25}$$

$$\lim_{\rho\to 0}\left(I_1 + I_3\right) = P\int_{-\infty}^{+\infty} \frac{\varepsilon(x)-1}{x-\omega}dx\,, \tag{18.26}$$

where symbol P before the integral means that the integral is taken in the principle value sense.

Let's calculate I_2.

Let's pass from argument z to angle φ:

$$z = \omega + \rho e^{i\varphi}\,,\ dz = i\rho e^{i\phi} d\varphi,$$

$$z - \omega = \rho e^{i\varphi},$$

$$I_2 = \int_{C_\rho \to 0} \frac{\left[\varepsilon\left(\omega + \rho e^{i\varphi}\right)-1\right]}{\rho e^{i\varphi}} i\rho e^{i\phi} d\varphi = i\lim_{\rho\to 0}\int_{\pi}^{0}\left[\varepsilon\left(\omega+\rho e^{i\varphi}\right)-1\right]d\varphi =$$

$$i\int_{\pi}^{0}\left[\varepsilon\left(\omega\right)-1\right]d\varphi = -\pi i\left[\varepsilon\left(\omega\right)-1\right],$$

that is we obtain

$$I_2 = -\pi i[\varepsilon(\omega)-1]. \tag{18.27}$$

Since $I = I_1 + I_2 + I_3 + I_4 = 0$, then

$$-\pi i[\varepsilon(\omega)-1] + P\int_{-\infty}^{+\infty} \frac{\varepsilon(x)-1}{x-\omega}dx\,,$$

$$\varepsilon(\omega)-1 = \frac{1}{\pi i}P\int_{-\infty}^{+\infty} \frac{\varepsilon(x)-1}{x-\omega}dx\,. \tag{18.28}$$

Under the integral $x = \omega'$, covering all values of a real axis, except ω, it means:

$$\varepsilon(\omega) - 1 = \frac{1}{\pi i} P \int_{-\infty}^{+\infty} \frac{\varepsilon(\omega') - 1}{\omega' - \omega} d\omega', \tag{18.29}$$

$$\varepsilon(\omega) - 1 = -\frac{i}{\pi} P \int_{-\infty}^{+\infty} \frac{\varepsilon(\omega') - 1}{\omega' - \omega} d\omega', \tag{18.30}$$

$$\varepsilon'(\omega) + i\varepsilon''(\omega) - 1 = -\frac{i}{\pi} P \int_{-\infty}^{+\infty} \frac{\varepsilon'(\omega') + i\varepsilon''(\omega') - 1}{\omega' - \omega} d\omega'. \tag{18.31}$$

Let's isolate the real and imaginary parts in (18.31):

$$\varepsilon'(\omega) - 1 = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\varepsilon''(\omega')}{\omega' - \omega} d\omega',$$

$$\varepsilon''(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\varepsilon'(\omega') - 1}{\omega' - \omega} d\omega'. \tag{18.32}$$

Formula (18.32) is called Kramers–Kronig relations.

Knowing $\varepsilon''(\omega)$ in a wide interval ω (by measuring the electromagnetic wave absorption in the medium, because the absorption is directly related to $\varepsilon''(\omega)$ the imaginary part $\varepsilon(\omega)$), one can integrate in the first expression (18.32) and determine $\varepsilon'(\omega)$.

For conductors Kramers–Kroning formulas contain the additional term $i\frac{4\pi\sigma}{\omega}$ in $\varepsilon(\omega)$, that is $\varepsilon(\omega)$ has the singularity in point 0. That's why in integral I it's necessary to encircle the point $\omega = 0$ in the upper half plane. It leads to the relation

$$\varepsilon''(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\varepsilon'(\omega') - 1}{\omega' - \omega} d\omega' + \frac{4\pi\sigma}{\omega}. \tag{18.33}$$

The Kramers–Kronig relations can be written in different form.

Let's take into account the properties:

$$\varepsilon'(-\omega) = \varepsilon'(\omega),$$
$$\varepsilon''(-\omega) = -\varepsilon''(\omega), \tag{18.34}$$

1)

$$\varepsilon'(\omega) - 1 = \frac{1}{\pi} P \left[\int_{-\infty}^{0} \frac{\varepsilon''(\omega')}{\omega' - \omega} d\omega' + \int_{0}^{\infty} \frac{\varepsilon''(\omega')}{\omega' - \omega} d\omega' \right] =$$
$$= \frac{1}{\pi} P \int_{0}^{\infty} d\omega' \left[-\frac{\varepsilon''(-\omega')}{\omega' + \omega} d\omega' + \frac{\varepsilon''(-\omega')}{\omega' - \omega} d\omega' \right] = \tag{18.35}$$
$$= \frac{1}{\pi} P \int_{0}^{\infty} d\omega' \varepsilon''(\omega') \left[\frac{1}{\omega' + \omega} + \frac{1}{\omega' - \omega} \right].$$

Thus,

$$\varepsilon'(\omega)-1=\frac{2}{\pi}P\int_0^{\infty}\frac{\omega'\varepsilon''(\omega')d\omega'}{\omega'^2-\omega^2} \tag{18.36}$$

or

$$\varepsilon'(\omega)=1+\frac{2}{\pi}P\int_0^{\infty}\frac{\omega'\varepsilon''(\omega')}{\omega'^2-\omega^2}d\omega'; \tag{18.37}$$

or

2)

$$\begin{aligned}\varepsilon''(\omega)&=-\frac{P}{\pi}\left[\int_{-\infty}^{0}\frac{\varepsilon'(\omega')-1}{\omega'-\omega}d\omega'+\int_0^{\infty}\frac{\varepsilon'(\omega')-1}{\omega'-\omega}\right]=\\&=-\frac{P}{\pi}\left[-\int_0^{\infty}\frac{\varepsilon'(-\omega')-1}{\omega'+\omega}+\int_0^{\infty}\frac{\varepsilon'(\omega')-1}{\omega'-\omega}\right]d\omega'=\\&=\frac{P}{\pi}\left[\int_0^{\infty}\left(\varepsilon'(\omega')-1\right)\left[\frac{1}{\omega'-\omega}+\frac{1}{\omega'+\omega}\right]\right]d\omega'=\\&=-\frac{2\omega}{\pi}P\int_0^{\infty}\frac{\varepsilon'(\omega')-1}{\omega'^2-\omega^2}d\omega',\end{aligned} \tag{18.37}$$

where

$$\omega'\to-\omega,$$
$$\varepsilon'(-\omega')=\varepsilon'(\omega'),$$

$$\varepsilon''(\omega)=-\frac{2\omega}{\pi}P\int_0^{\infty}\frac{\varepsilon'(\omega')-1}{\omega'^2-\omega^2}d\omega'. \tag{18.38}$$

For the majority of material media, function $\varepsilon''(\omega)$ has one or several sharp maximums in the neighborhood of some characteristic frequencies and tends to zero rather quickly as $\omega\to\infty$ and $\omega\to 0$.

This property of $\varepsilon''(\omega)$ enables one to study the behavior of $\varepsilon'(\omega)$ in the range of low and high frequencies. Let's assume that $\varepsilon''(\omega')$ takes sufficiently large values only in the region $0<\omega_1<\omega'<\omega_2$ and tends to к 0 $(\varepsilon''(\omega')\to 0)$ out of this interval.

Then one can write

$$\varepsilon'(\omega)\approx 1+\frac{2}{\pi}\int_{\omega_1}^{\omega_2}\frac{\omega'\varepsilon''(\omega')d\omega'}{\omega'^2-\omega^2}. \tag{18.39}$$

If $\omega\gg\omega_2$ in the integration element and $\omega\gg\omega'$

$$\frac{1}{\omega'^2-\omega^2}=-\frac{1}{\omega^2}\frac{1}{1-\frac{\omega'^2}{\omega^2}}=-\frac{1}{\omega^2}\left[1+\frac{\omega'^2}{\omega^2}\right], \tag{18.40}$$

$$\varepsilon'(\omega) \approx 1 - \frac{A_1}{\omega^2} - \frac{A_2}{\omega^4}, \tag{18.41}$$

where

$$A_1 = \frac{2}{\pi} \int_{\omega_1}^{\omega_2} \omega' \varepsilon''(\omega') d\omega',$$

$$A_2 = \frac{2}{\pi} \int_{\omega_1}^{\omega_2} \omega'^3 \varepsilon''(\omega') d\omega'. \tag{18.42}$$

Experimental data show that this formula describes well the qualitative behavior of $\varepsilon'(\omega)$ in the domain of high frequencies.

Another limiting case is $\omega << \omega_1$.

In this case $\omega << \omega'$, since $\omega' > \omega_1$:

$$\frac{1}{\omega'^2 - \omega^2} = \frac{1}{\omega'^2} \frac{1}{1 - \frac{\omega^2}{\omega'^2}} = \frac{1}{\omega'^2}\left[1 + \frac{\omega^2}{\omega'^2}\right], \tag{18.43}$$

$$\varepsilon'(\omega) = 1 + B_1 + B_2\omega^2, \tag{18.44}$$

where

$$B_2 = \frac{2}{\pi} \int_{\omega_1}^{\omega_2} \frac{d\omega' \varepsilon''(\omega')}{\omega'^3}. \tag{18.45}$$

Using the relations, obtained above, one can make sure that expressions (18.41) and (18.44) correctly reproduce the qualitative behavior of $\varepsilon'(\omega')$ in the presence of any elementary medium (in this case it’s rarefied neutral gas).

19. Some Dielectric Permittivity Properties $\varepsilon(\omega)$

Dielectric permittivity in general case is a complex value

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega), \tag{19.1}$$

where

$$\varepsilon'(\omega) = \varepsilon'(-\omega); \; \varepsilon''(\omega) = -\varepsilon''(-\omega). \tag{19.2}$$

In isotropic and non-magnetic medium $(\mu = 1)$, which is in the thermodynamic equilibrium, parameter $\varepsilon''(\omega)$ has the properties:

$$\varepsilon''(\omega > 0) > 0, \; \varepsilon''(\omega < 0) < 0. \tag{19.3}$$

In dielectrics at low frequencies

$$\lim_{\omega \to 0} \varepsilon(\omega) = \varepsilon_0. \tag{19.4}$$

In conductive media the generalized induction vector:

$$\vec{D}(\omega) = \vec{E}(\omega)\left(1 + \frac{4\pi i\sigma}{\omega}\right) = \varepsilon(\omega)\vec{E}(\omega). \qquad (19.5)$$

As $\omega \to 0$, parameter $\frac{4\pi\sigma}{\omega} >> 1$, that's why

$$\lim_{\omega\to 0} \varepsilon(\omega) = \frac{4\pi i\sigma}{\omega}. \qquad (19.6)$$

Where σ is a conductivity, which coincides with the static conductivity at low frequencies.

High frequency asymptotic at $\varepsilon(\omega)$ (unlike asymptotic $\varepsilon(\omega)$ at low ω, which changes in different media) is the same for all media:

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \qquad (19.7)$$

where $\omega_p = \sqrt{4\pi Ne^2/m}$ is a plasma frequency, N is a concentration of charged particles

Fig. 19.1 shows the form of function (19.7).

Get the evaluation as a typical for solids electron concentration $N \sim 10^{22} \ldots 10^{23}\ \frac{1}{\text{см}^3}$ for plasma frequency:

$\omega_p \sim 10^{16}\ \frac{1}{\text{c}}$ is an ultra-violet part of spectrum.

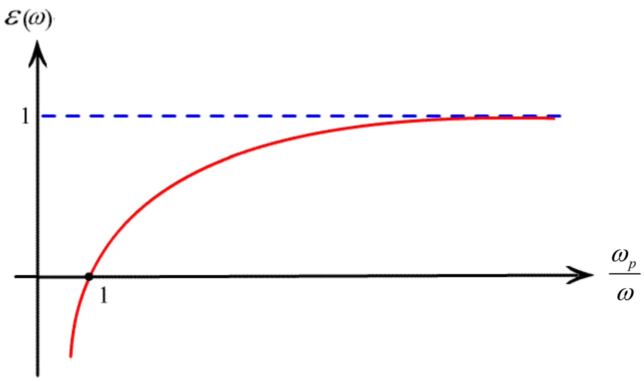


Fig. 19.1. High frequency asymptotic of dielectric permittivity $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$ *and* $\omega' = \omega$

It can be much lower for other media.

ω_p lies in the infrared region for the typical for semiconductors concentrations of the charge supports in the conduction region. ω_p is in the microwave range for the laboratory plasma.

Let's consider the steps to obtain the formula $\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$ evaluation.

We write down the equation of motion for a charge in the high frequency field:

$$m\ddot{\vec{r}} = e\vec{E}_0 e^{-i\omega t}. \qquad (19.8)$$

From here we find:

$$\vec{\vartheta} = i\frac{e\vec{E}_0}{m\omega}e^{-i\omega t}. \qquad (19.9)$$

Knowing the concentration and the velocity of charged particles we can find the current density:

$$\vec{j} = eN\vec{\vartheta} = i\frac{e^2 NE_0 e^{-i\omega t}}{m\omega}. \qquad (19.10)$$

Since the denominator contains mass, then it suffices to take into account only light particles (e^-).

Further one can find the electric induction vector:

$$\vec{D} = \vec{E} + 4\pi\vec{\mathrm{P}}, \qquad (19.11)$$

where

$$\vec{\mathrm{P}} = \int_0^t \vec{j}\,dt, \qquad (19.12)$$

$$\vec{\mathrm{P}} = -\frac{e^2 N}{m\omega^2}\vec{E}, \qquad (19.13)$$

$$\vec{D} = \vec{E} + 4\pi\vec{\mathrm{P}} = \left(1 - \frac{4\pi Ne^2}{m\omega^2}\right)\vec{E}. \qquad (19.14)$$

From here $\vec{D} = \varepsilon(\omega)\vec{E}$ we find the form of $\varepsilon(\omega)$

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}. \qquad (19.15)$$

Calculation of the dissipation (that is the energy loss due to the collisions of charges), which means the additional friction losses in the motion equation:

$$m\ddot{\vec{r}} = e\vec{E}_0 e^{-i\omega t} - \gamma \cdot \dot{\vec{r}}, \qquad (19.16)$$

from which we get the velocity. As a result we obtain the expression for the current density:

$$\vec{j} = \frac{ie^2\omega N\vec{E}}{m\left(\omega^2 + i\gamma\omega\right)}. \qquad (19.17)$$

And the expression for the dielectric permittivity

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\left(\omega + i\frac{\gamma}{2}\right)}. \qquad (19.18)$$

The analytical properties of function $\varepsilon(\omega)$ were discussed earlier: this function is characterized by the absence of zeros in the upper half plane of complex variable $\omega = \omega' + i\omega''$ (that is if $\omega'' > 0$). At $\omega = 0$ function $\varepsilon(\omega)$ has the pole, which is bypassed along the contour, shown in Fig. 19.2, while calculating integral $\oint \frac{\varepsilon(z)-1}{z-\omega} dz$.

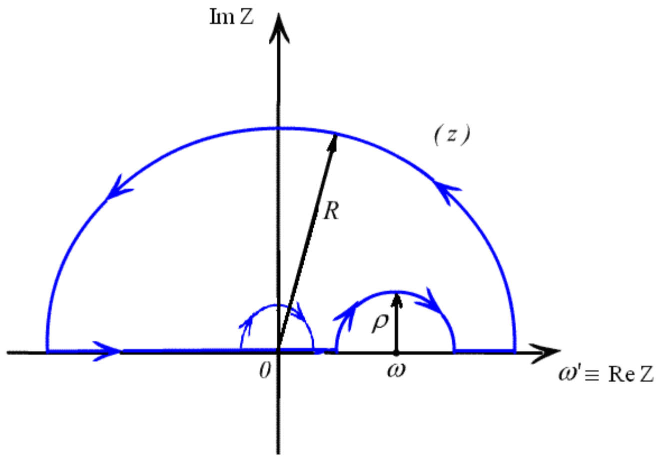


Fig. 19.2. The bypass of the contour at calculating integral 18.18 when function $\varepsilon(z)$ has a pole at $\omega' = 0$

Temporary dispersion of the dielectric permittivity is caused by the lag of the medium reaction and this reaction is determined by the field not only at a given moment t, but also at all the preceeding moments. The parameter, which allows evaluating the importance of temporary dispersion, is the relation $\frac{\omega}{\omega_0}$, that is the relation of the field frequency to the characteristic frequencies of the charge motion in the medium. The medium can have several frequencies ω_0. The most important among them is the frequency of electron motion at the length of the atomic scale

$$\omega_0 \sim \left(10^{15} \ldots 10^{16}\right) \frac{1}{\text{c}}. \tag{19.19}$$

Dispersion is important for $\frac{\omega}{\omega_0} >> 1$ (these are optical frequencies and ultra-violet radiation).

Further we consider the characteristic motion frequencies of atom nuclei in the crystal lattice

$$\omega_0 \sim 10^{13} \frac{1}{\text{c}} \text{ is the infra-red region.} \tag{19.20}$$

The frequency of electron collective motions also can be characteristic. Besides, the order of characteristic frequencies depends on the substance concentration (in plasma, for example) and can change within a wide range.

Natural frequencies describing the typical motion in good magnetics are the lowest.

Dielectric permittivity is often expressed in terms of dielectric susceptibility:

$$\varepsilon(\omega) = 1 + \chi' + i\chi'' \equiv \varepsilon'(\omega) + i\varepsilon''(\omega) \equiv 1 + \chi(\omega), \qquad (19.21)$$

$$1 + \chi'(\omega) = \varepsilon'(\omega), \qquad (19.22)$$

$$\chi''(\omega) = \varepsilon''(\omega). \qquad (19.23)$$

According to the results of quantum theory of photon scattering, dielectric susceptibility of homogeneous (isotropic) substance is expressed in terms of amplitude of scattering on the zero angle by an atom of the substance $f(0)$:

$$\chi(\omega) = \frac{4\pi N_0}{\omega} f(0), \qquad (19.24)$$

where N_0 is the number of atoms in unit volume.

The imaginary part of the susceptibility $\chi''(\omega)$ determines photon absorption and can be expressed according to the optic theorem in terms of absorption cross-section $\sigma(\omega)$:

$$\chi''(\omega) = \frac{N_0}{\omega}\sigma(\omega). \qquad (19.25)$$

That's why, using Kramers-Kronig dispersion relation, one can get the relation of the real part of the susceptibility $\chi'(\omega)$ with the imaginary part $\chi''(\omega)$:

$$\chi'(\omega) = \frac{2}{\pi} P\int_0^{\infty} \frac{\omega'\chi''(\omega')d\omega'}{\omega'^2 - \omega^2}. \qquad (19.26)$$

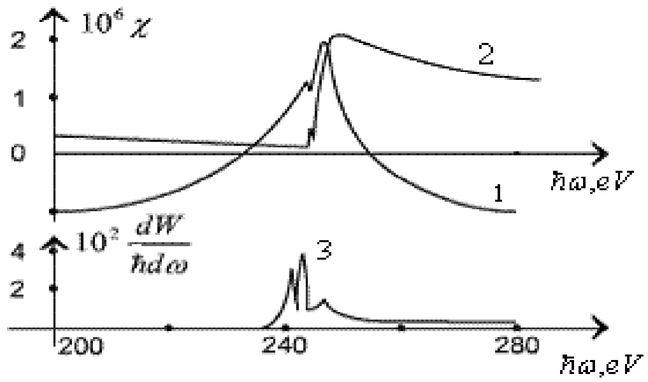


Fig. 19.3. The dependence on the frequency of dielectric susceptibility and on spectral density of radiation energy of an electron in argon near the L-edge of photo absorption: 1 – $\chi'(\omega)$; 2 – $\chi''(\omega)$; 3 – details of a spectrum of $\chi'(\omega)$

Thus, using detailed experimental data about absorption cross-section as the function of photon energy, one can calculate $\chi'(\omega)$ for a number of substances. These calculations were done in several works, where the integration was performed from the first ionization threshold (photo effect) to the threshold of e^+e^- pair production (for example: V.A. Basylev, N.K. Zhevago. The radiation of quick particles in the substance and in the external fields, §3.8 Roentgen Cherenkov's radiation).

Maximum values of $\chi'(\omega)$ are not too big for gaseous substances at normal pressure and it can be from $5\cdot10^{-7}$ for Ne $(\omega = 870\,\text{эВ})$ to 10^{-5} for Ar $(\omega = 250\,\text{эВ})$, and for solids – from 10^{-2} for carbon with density $\rho = 1{,}5\ \text{г}\cdot\text{см}^{-3}$ $(\omega = 284\,\text{эВ})$ to $5\cdot10^{-5}$ for Al $(\omega = 73\,\text{эВ})$.

20. Electromagnetic Waves in Isotropic Media

Consider first the electromagnetic wave in vacuum:

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp\left(i\vec{k}\vec{r} - i\omega t\right),\ \vec{H}(\vec{r},t) = \vec{H}_0 \exp\left(i\vec{k}\vec{r} - i\omega t\right), \tag{20.1}$$

If electric and magnetic fields satisfy the equations

$$\square \vec{E} = 0,\ \square \vec{H} = 0, \tag{20.2}$$

$$\square = \vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}, \tag{20.3}$$

then as a result we obtain

$$\left(-\vec{k}^2 + \frac{\omega^2}{c^2}\right)\vec{E}_0 \exp\left(i\vec{k}\vec{r} - i\omega t\right) = 0. \tag{20.4}$$

Thus, for vacuum $\omega = c\left|\vec{k}\right|$, and amplitudes $\vec{E}$ and $\vec{H}$ of electromagnetic field are mutually orthogonal to vector $\vec{k}$. It's seen from the substitution of $\vec{E}(\vec{r},t) = \vec{E}_0 \exp\left(i\vec{k}\vec{r} - i\omega t\right)$ and $\vec{H}(\vec{r},t) = \vec{H}_0 \exp\left(i\vec{k}\vec{r} - i\omega t\right)$ in the Maxwell's system of equations with $\rho = 0$, $j = 0$:

$$\begin{cases} rot\vec{E} = -\dfrac{1}{c}\dfrac{\partial \vec{H}}{\partial t}, & rot\vec{H} = \dfrac{1}{c}\dfrac{\partial \vec{E}}{\partial t}, \\ div\vec{E} = 0, & div\vec{H} = 0, \end{cases} \tag{20.5}$$

$$rot\vec{E} = \left[\vec{\nabla}\cdot\vec{E}\right] = \left[\vec{\nabla}\cdot\vec{E}_0\right]\exp\left(i\vec{k}\vec{r} - i\omega t\right) = i\left[\vec{k}\cdot\vec{E}_0\right]\exp\left(i\vec{k}\vec{r} - i\omega t\right), \tag{20.6}$$

$$-\frac{1}{c}\frac{\partial \vec{H}}{\partial t} = i\frac{\omega}{c}\vec{H}_0 \exp\left(i\vec{k}\vec{r} - i\omega t\right). \tag{20.7}$$

From the equation

$$rot\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t},$$

it follows:

$$i\left[\vec{k}\cdot\vec{E}_0\right]\exp\left(i\vec{k}\vec{r}-i\omega t\right)=i\frac{\omega}{c}\vec{H}_0\exp\left(i\vec{k}\vec{r}-i\omega t\right)=i\left|\vec{k}\right|\vec{H}_0\exp\left(i\vec{k}\vec{r}-i\omega t\right), \qquad (20.8)$$

that is

$$\vec{H}_0=\left[\frac{\vec{k}}{k}\cdot\vec{E}_0\right]=\left[\vec{n}\cdot\vec{E}_0\right], \qquad (20.9)$$

$$\vec{H}_0=\left[\vec{n}\cdot\vec{E}_0\right], \qquad (20.10)$$

where

$$\vec{n}=\frac{\vec{k}}{\left|\vec{k}\right|}=\frac{c\vec{k}}{\omega}. \qquad (20.11)$$

From $rot\vec{H}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$ it follows that $i\left[\vec{k}\cdot\vec{H}_0\right]=-i\left|\vec{k}\right|\vec{E}_0$ or

$$\left[\vec{H}_0\cdot\vec{n}\right]=\vec{E}_0. \qquad (20.12)$$

From equation $div\vec{E}=0$ it follows $\vec{n}\vec{E}_0=0$, (20.13)

from equation $div\vec{H}=0$ it follows $\vec{n}\vec{H}_0=0$, (20.14)

where $\vec{n}$ is a unit vector in the direction of wave propagation.

Vectors $\vec{E}_0$, $\vec{H}_0$, $\vec{n}$ are three mutually perpendicular vectors (Fig. 20.1), with $\left|\vec{E}_0\right|=\left|\vec{H}_0\right|$.

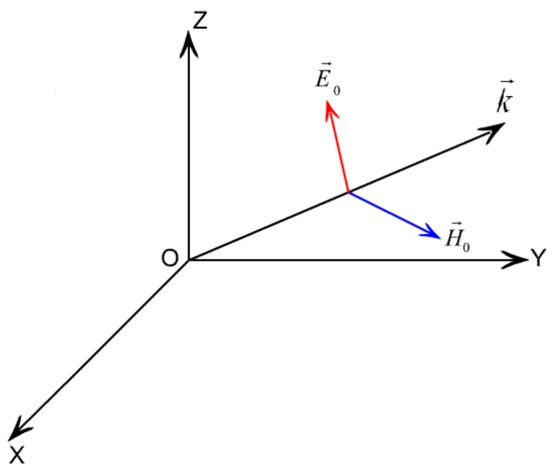


Fig. 20.1. The propagation of the flat electromagnetic wave in space

What relations appear between the characteristics of a plane electromagnetic wave during its propagation in the substance? Let's assume that $\mu=1$, the substance is isotropic and homogeneous.

In this case for the vector of electric induction one has

$$\vec{D}(\vec{r},t) = \vec{D}_0 \exp\left(i\vec{k}\vec{r} - i\omega t\right),$$

where there is the relation between $\vec{D}$ and $\vec{E}$ are related as

$$\vec{D} = \varepsilon(\omega)\vec{E},$$

where $\varepsilon(\omega)$ is a complex dielectric permittivity.

Because the time and coordinates of vectors $\vec{E}$, $\vec{H}$ and $\vec{D}$ are only in the exponent, then it's easy to establish the equivalence of the following operations:

$$rot\vec{H} \quad \to \quad i\left[\vec{k}\cdot\vec{H}\right], \tag{20.15}$$

$$div\vec{D} \quad \to \quad i\left(\vec{k}\vec{D}\right) = i\varepsilon(\omega)\left(\vec{k}\vec{E}\right), \tag{20.16}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\vec{D} \quad \to \quad -i\frac{\omega}{c}\vec{D} = -i\frac{\omega}{c}\varepsilon(\omega)\vec{E}. \tag{20.17}$$

Then the equations for fast-alternating fields are transformed as:

$$\begin{cases} rot\vec{H} = \dfrac{1}{c}\dfrac{\partial\vec{D}}{\partial t} & \to \quad \left[\vec{k}\cdot\vec{H}\right] = -\dfrac{\omega}{c}\varepsilon(\omega)\vec{E}, \\ div\vec{D} = 0 & \to \quad \varepsilon(\omega)\left(\vec{k}\vec{E}\right) = 0, \\ rot\vec{E} = -\dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t} & \to \quad i\dfrac{\omega}{c}\mu\vec{H} = \dfrac{\omega}{c}\vec{H} = i\left[\vec{k}\cdot\vec{E}\right], \\ div\vec{B} = 0 & \to \quad i\vec{k}\mu\vec{H} = 0, \qquad \left(\vec{k}\cdot\vec{H}\right) = 0. \end{cases}$$

(20.18)

Let's write down the resulting set of equations:

$$\begin{cases} \left[\vec{k},\vec{H}\right] = -\dfrac{\omega}{c}\varepsilon(\omega)\vec{E}, \\ \varepsilon(\omega)\left(\vec{k},\vec{E}\right) = 0, \\ \left[\vec{k},\vec{E}\right] = \dfrac{\omega}{c}\vec{H}, \\ \left(\vec{k}\cdot\vec{H}\right) = 0, \end{cases} \tag{20.19}$$

since vectors $\vec{k}$ and $\vec{H}$ are mutually orthogonal.

Assume that $\varepsilon(\omega) \neq 0$. Then vectors $\vec{k}$ and $\vec{E}$ are also mutually perpendicular and from the first two equations it follows that $\vec{E}$ and $\vec{H}$ are orthogonal. Thus, as well as in vacuum, electromagnetic waves are transverse.

Substituting $\vec{H} = \frac{c}{\omega}\left[\vec{k},\vec{E}\right]$ in the expression $\left[\vec{k},\vec{H}\right] = -\frac{\omega}{c}\varepsilon(\omega)\vec{E}$, gives

$$\frac{c}{\omega}\left[\vec{k}\left[\vec{\kappa}\vec{E}\right]\right] = \frac{c}{\omega}\left[\vec{k}\left(\vec{k}\vec{E}\right) - \vec{E}\vec{k}^2\right] = -\frac{\omega}{c}\varepsilon(\omega)\vec{E}, \tag{20.20}$$

that is

$$\left(\vec{k}^2 - \frac{\omega^2}{c^2}\varepsilon(\omega)\right)\vec{E} = 0. \tag{20.21}$$

From equation 20.21 it follows that electromagnetic waves can propagate in the medium, if their propagation vector is related to the frequency by the formula

$$k^2 = \frac{\omega^2}{c^2}\varepsilon(\omega), \tag{20.22}$$

where $k = \left|\vec{k}\right|$.

In the substance, transparent to the wave, when

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega) = \varepsilon'(\omega) \tag{20.23}$$

with $i\varepsilon''(\omega) = 0$, we get

$$k = \frac{\omega}{c}\sqrt{\varepsilon(\omega)}. \tag{20.24}$$

For the waves of optical range the value $\sqrt{\varepsilon} = \sqrt{\varepsilon'}$ is a index of refraction. If $\varepsilon'' \neq 0$ the propagation vector $\vec{k}$ becomes complex:

$$\vec{k} = \vec{k}' + i\vec{k}'', \tag{20.25}$$

where $\vec{k}'$ and $\vec{k}''$ are real values.

Squaring both parts of equation (20.25) and taking into consideration (20.24) and (20.23), we obtain

$$\vec{k}^2 = \vec{k}^2 - \vec{k}^2 + 2i\vec{k}'\vec{k}'' = \frac{\omega^2}{c^2}\left(\varepsilon'(\omega) + i\varepsilon''(\omega)\right), \tag{20.26}$$

$$\begin{cases} k'^2 - k''^2 = \dfrac{\omega^2}{c^2}\varepsilon'(\omega), \\ 2\vec{k}'\vec{k}'' = \varepsilon''(\omega)\dfrac{\omega^2}{c^2}. \end{cases} \tag{20.27}$$

In general, for two vectors there are only two equations, and there is the freedom in choosing these vectors. If vectors $\vec{k}'$ and $\vec{k}''$ have the same direction, then relations (20.27) determine uniquely their moduli.

Let

$$\vec{k}' = \vec{e}k', \ \vec{k}'' = \vec{e}k'', \tag{20.28}$$

then we find

$$2\vec{k}'\vec{k}'' = 2\vec{k}'\vec{k}''\varepsilon''(\omega)\frac{\omega^2}{c^2}. \tag{20.29}$$

Let

$$\vec{k} = \vec{e}k, \tag{20.30}$$

where

$$k = \frac{\omega}{c}(n + ix). \tag{20.31}$$

On the other hand

$$\vec{k} = \vec{e}(k' + ik''). \tag{20.32}$$

From the equality

$$\vec{k}^2 = k^2, \text{ где } k^2 = \left[\frac{\omega}{c}(n + ik)\right],$$

we find

$$\frac{\omega^2}{c^2}\left(n^2 - x^2 + 2inx\right) = k'^2 - k''^2 + 2ik'k'' = \frac{\omega^2}{c^2}\varepsilon'(\omega) + 2i\frac{\omega^2}{c^2}\varepsilon''(\omega). \tag{20.33}$$

From here we obtain

$$n = \sqrt{\frac{|\varepsilon| + \varepsilon'}{2}}, \tag{20.34}$$

$$x = \sqrt{\frac{|\varepsilon| - \varepsilon'}{2}}, \tag{20.35}$$

where

$$|\varepsilon| = \sqrt{\varepsilon'^2 + \varepsilon''^2},$$

$$n = \sqrt{\varepsilon} \equiv \sqrt{\varepsilon'}.$$

If $\varepsilon(\omega)$ is a complex value, then k is an absorption ratio of the medium. What will happen if at some value of $\omega = \omega_0$ dielectric permittivity is $\varepsilon(\omega_0) = 0$?

From the system of equations 20.19 there is a solution at which $\vec{H} = 0$ and $\vec{E} \parallel \vec{k}$, if $\varepsilon(\omega_0) = 0$.

Consequently, there can be pure electric alternating fields, called longitudinal waves, in the medium. Under our consideration there is no relation between wave vectors $\vec{k}$ and frequency ω_0, but there can be such relation, when we account not only for the frequent dispersion of dielectric permittivity but also for the spatial dispersion. Then

$$\varepsilon = \varepsilon(\vec{k}, \vec{\omega}). \tag{20.36}$$

And the condition $\varepsilon(\omega_0) = 0$ will turns into

$$\varepsilon(\vec{k}, \omega_0) = 0, \tag{20.37}$$

hence the relation between the frequency and the wave vector in longitudinal waves follows.

Chapter 3
RADIATION OF CHARGED PARTICLES IN VACUUM, MEDIA AND PERIODIC STRUCTURES

21. Electromagnetic field of a fast charged particle in medium. Vavilov and Cherenkov's radiation

Let a fast particle with a charge e move at a constant velocity approaching to the velocity of light in a transparent isotropic medium. While moving the particle loses the energy, of course (for the disturbance, ionization of the medium atoms, radiation, etc.), but in the first approximation its motion can be regarded as uniform. Thus, electromagnetic field is made by a uniformly moving outside charge:

$$\rho_{ext}(\vec{r},t) = e\delta(\vec{r}-\vec{v}t),$$

$$\vec{j}_{ext}(\vec{r},t) = e\vec{v}\delta(\vec{r}-\vec{v}t).$$

Maxwell's equations have the following form:

$$\begin{cases} rot\vec{E} = -\dfrac{1}{c}\dfrac{\partial\vec{H}}{\partial t}, \\ div\vec{E} = 4\pi\rho_{ext}, \\ rot\vec{H} = \dfrac{1}{c}\dfrac{\partial\vec{D}}{\partial t} + \dfrac{4\pi}{c}\vec{j}_{ext}, \\ div\vec{H} = 0, \end{cases} \tag{21.1}$$

and

$$\vec{B} = \mu\vec{H}, \ \mu = 1, \ \vec{D} = \varepsilon\vec{E}.$$

To find the characteristics of the electromagnetic field let's expand the intensities $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ and other values in Fourier integral:

$$\vec{E}(\vec{r},t) = \int\frac{d\vec{\kappa}}{(2\pi)^3}\int\frac{d\omega}{2\pi}e^{i(\vec{\kappa}\vec{r}-\omega t)}\vec{E}(\vec{\kappa},\omega),$$

$$\vec{H}(\vec{r},t) = \int\frac{d\vec{\kappa}}{(2\pi)^3}\int\frac{d\omega}{2\pi}e^{i(\vec{\kappa}\vec{r}-\omega t)}\vec{H}(\vec{\kappa},\omega),$$

$$\vec{D}(\vec{r},t) = \int\frac{d\vec{\kappa}}{(2\pi)^3}\int\frac{d\omega}{2\pi}e^{i(\vec{\kappa}\vec{r}-\omega t)}\vec{D}(\vec{\kappa},\omega),$$

$$\rho_{ext}(\vec{r},t)=\int\frac{d\vec{\kappa}}{(2\pi)^3}\int\frac{d\omega}{2\pi}e^{i(\vec{\kappa}\vec{r}-\omega t)}\rho_{ext}(\vec{\kappa},\omega),$$

$$\vec{j}_{ext}(\vec{r},t)=\int\frac{d\vec{\kappa}}{(2\pi)^3}\int\frac{d\omega}{2\pi}e^{i(\vec{\kappa}\vec{r}-\omega t)}\vec{j}_{ext}(\vec{\kappa},\omega).$$

For Fourier-images of the densities of the outside charges and currents we get

$$\rho_{ext}(\vec{\kappa},\omega)=\int d\vec{r}\int dte^{-i(\vec{\kappa}\vec{r}-\omega t)}\rho_{ext}(\vec{r},t)=e\int d\vec{r}\int dte^{-i(\vec{\kappa}\vec{r}-\omega t)}\delta(\vec{r}-\vec{v}t)=$$

$$=e\int dte^{-i(\vec{\kappa}\vec{v}-\omega)t}=2\pi e\delta(\omega-\vec{\kappa}\vec{v}), \tag{21.2}$$

$$\vec{j}_{ext}(\vec{\kappa},\omega)=2\pi e\vec{v}\,\delta(\omega-\vec{\kappa}\vec{v}). \tag{21.3}$$

The relation

$$\vec{H}(\vec{\kappa},\omega)=\frac{c}{\omega}\left[\vec{\kappa}\cdot\vec{E}(\vec{\kappa},\omega)\right] \tag{21.4}$$

follows from the first equation of the set of equations (21.1).

Taking the Fourier-image of the third equation of set (21.1) and taking into account expression (21.3), we get

$$i\left[\vec{\kappa}\cdot\vec{H}(\vec{\kappa},\omega)\right]=-\frac{i\omega}{c}\vec{D}(\vec{\kappa},\omega)+\frac{8\pi^2 e\vec{v}}{c}\delta(\omega-\vec{\kappa}\cdot\vec{v}) \tag{21.4'}$$

or

$$\omega\vec{D}(\vec{\kappa},\omega)=-c\left[\vec{\kappa}\cdot\vec{H}(\vec{\kappa},\omega)\right]-8\pi^2 ei\vec{v}\delta(\omega-\vec{\kappa}\vec{v}). \tag{21.5}$$

Using the constraint equation

$$\vec{D}=\varepsilon\vec{E},$$

we get for the Fourier-images

$$\vec{D}(\vec{\kappa},\omega)=\varepsilon\vec{E}(\vec{\kappa},\omega), \tag{21.6}$$

and from equation $div\vec{D}=4\pi\rho$ we have

$$\varepsilon\left(\vec{\kappa},\vec{E}(\vec{\kappa},\omega)\right)=-8\pi^2 ei\delta(\omega-\vec{\kappa}\vec{v})$$

or

$$\left(\vec{\kappa},\vec{D}(\vec{\kappa},\omega)\right)=-8\pi^2 ei\delta(\omega-\vec{\kappa}\vec{v}).$$

Let's transform the first addend in the right part (21.5):

$$\left[\vec{\kappa}\cdot\vec{H}(\vec{\kappa},\omega)\right]=\frac{c}{\omega}\left[\vec{\kappa}\cdot\left[\vec{\kappa}\cdot\vec{E}(\vec{\kappa},\omega)\right]\right]=\frac{c}{\omega}\left\{\vec{\kappa}\left(\vec{\kappa}\vec{E}\right)-\vec{E}\vec{\kappa}^2\right\}. \tag{21.7}$$

The scalar product of vectors $(\vec{k}\,\vec{E})$, containing (21.7), has the form

$$\vec{\kappa}\vec{E}=\frac{1}{\varepsilon}\left(\vec{\kappa}\vec{D}\right)=-\frac{8\pi^2 ei\delta(\omega-\vec{\kappa}\vec{v})}{\varepsilon}. \tag{21.8}$$

From equations (21.4) – (21.8) it directly follows

$$\varepsilon\omega\vec{E} = -\frac{c^2}{\omega}\left\{\vec{\kappa}\left(\vec{\kappa}\vec{E}\right) - \vec{E}\kappa^2\right\} = -\frac{c^2}{\omega}\left\{-\frac{8\pi^2 ei\delta\left(\omega - \vec{\kappa}\vec{v}\right)}{\varepsilon}\vec{\kappa} - \vec{E}\kappa^2\right\} - 8\pi^2 ei\vec{v}\delta\left(\omega - \vec{\kappa}\vec{v}\right).$$

After some simple manipulations

$$\varepsilon\omega\vec{E} = \frac{c^2}{\omega}\vec{\kappa}^2\vec{E} + 8\pi^2 ei\delta(\omega - \vec{\kappa}\vec{v})\left[-\vec{v} + \frac{c^2}{\omega\varepsilon}\vec{\kappa}\right],$$

$$\left(\varepsilon\omega - \frac{c^2\vec{\kappa}^2}{\omega}\right)\vec{E} = -8\pi^2 ei\delta(\omega - \vec{\kappa}\vec{v})\left[\frac{\omega\vec{v}}{c^2} - \frac{\vec{\kappa}}{\varepsilon}\right]\frac{c^2}{\omega},$$

$$\frac{c^2}{\omega}\left(\vec{\kappa}^2 - \frac{\omega^2}{c^2}\varepsilon\right)\vec{E} = -\frac{c^2}{\omega}8\pi^2 ei\delta(\omega - \vec{\kappa}\vec{v})\left[\frac{\omega\vec{v}}{c^2} - \frac{\vec{\kappa}}{\varepsilon}\right].$$

We finally get

$$\vec{E}(\vec{\kappa},\omega) = 8\pi^2 ei\delta(\omega - \vec{\kappa}\vec{v})\left[\frac{\omega\vec{v}}{c^2} - \frac{\vec{\kappa}}{\varepsilon(\omega)}\right]\frac{1}{\kappa^2 - \frac{\omega^2}{c^2}\varepsilon(\omega)}. \qquad (21.9)$$

The Fourier-image of the intensity of the magnetic field is connected with $\vec{E}(\vec{\kappa},\omega)$ according to formula (21.4):

$$\vec{H}(\vec{\kappa},\omega) = \frac{c}{\omega}\left[\vec{\kappa}, \vec{E}(\vec{\kappa},\omega)\right].$$

We find the coordinate dependences of the fields using the Fourier transformations.

The obtained expressions will be used to find the loss of energy spent on Cherenkov's radiation. The Vavilov-Cherenkov's radiation arises when a fast charged particle passes in a transparent medium at speed $v > \frac{c}{n}$, where $n = \sqrt{\varepsilon}$ is the index of medium refraction. This radiation is nothing but the radiation of the medium atoms, polarized by a passing particle. It isn't connected with the particle acceleration, which takes place for example by bremsstrahlung. Thus it doesn't depend on the particle mass and is conditioned by its speed $\vec{v}$, charge e and dielectric permittivity ε (that is by the properties of the medium).

Usually, the asymptotic expressions for $\vec{E}(\vec{r},t)$, $\vec{H}(\vec{r},t)$ are calculated then Undo- a Pointing's vector is drawn, and with its help the intensity of electromagnetic radiation (Cherenkov's radiation) is found.

But we will give a different derivation.

The energy loss of a particle is conditioned by the works of the force acting on the particle from the field induced by the particle. Let's choose the axis Z in the direction $\vec{v}$.

Then the energy loss per a way unit

$$\frac{dW}{dz} = -eE_z(\vec{r} = \vec{v}t, t), \tag{21.10}$$

with the intensity taken in the point of the particle location.

From the Fourier transformation

$$\vec{E}(\vec{r}, t) = \int \frac{d\vec{\kappa}}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{i(\vec{\kappa}\vec{r} - \omega t)} \vec{E}(\vec{\kappa}, \omega),$$

and from the expression for the Fourier-image

$$\vec{E}(\vec{\kappa}, \omega) = 8\pi^2 ie \left(\frac{\omega\vec{v}}{c^2} - \frac{\vec{\kappa}}{\varepsilon(\omega)} \right) \frac{\delta(\omega - \vec{\kappa}\vec{v})}{\vec{\kappa}^2 - \frac{\omega^2}{c^2}\varepsilon(\omega)},$$

we find

$$\frac{dW}{dz} = -8\pi^2 ie^2 \int \frac{d\omega}{2\pi} \int \frac{d\vec{\kappa}}{(2\pi)^3} \left[\frac{\omega v}{c^2} - \frac{\kappa_z}{\varepsilon(\omega)} \right] \frac{\delta(\omega - \kappa_z v)}{\vec{\kappa}^2 - \frac{\omega^2}{c^2}\varepsilon(\omega)}. \tag{21.11}$$

While calculating the integrals in the three-dimensional space let's employ cylindrical coordinates:

$$\kappa_x = q\cos\varphi,$$

$$\kappa_y = q\sin\varphi,$$

$$\kappa_z = \kappa_z,$$

as a result we get

$$d\vec{\kappa} \equiv d^3\kappa = d\kappa_z q dq d\varphi,$$

$$\kappa^2 = \kappa_x^2 + \kappa_y^2 + \kappa_z^2 = \kappa_z^2 + q^2.$$

Formula (21.11) will take the form

$$\frac{dW}{dz} = -8\pi^2 ie^2 \int \frac{d\omega}{2\pi} \int_0^{2\pi} d\varphi \int q dq \int \frac{d\kappa_z}{(2\pi)^3} \left[\frac{\omega v}{c^2} - \frac{\kappa_z}{\varepsilon(\omega)} \right] \frac{\delta(\omega - \kappa_z v)}{\left(q^2 + \kappa_z^2 - \frac{\omega^2}{c^2}\varepsilon(\omega) \right)} =$$

$$= -\frac{ie^2}{\pi} \int d\omega \int q dq \left[\frac{\omega v}{c^2} - \frac{\omega}{v\varepsilon} \right] \frac{1}{v} \frac{1}{\left(q^2 - \frac{\omega^2}{c^2}\varepsilon(\omega) + \frac{\omega^2}{v^2} \right)} =$$

$$= \frac{ie^2}{\pi} \int_{-\infty}^{+\infty} d\omega \int q dq \left[\frac{\omega}{v^2\varepsilon(\omega)} - \frac{\omega}{c^2} \right] \frac{1}{\left(q^2 - \frac{\omega^2}{c^2}\varepsilon(\omega) - \frac{c^2}{v^2} \right)}. \tag{21.12}$$

Here we used the property of δ-function

$$\int f(x)\delta(ax)dx = \frac{1}{|a|} f(0),$$

which in other form means

$$\int f(x)\delta(ax-b)dx = \frac{f(b)}{|a|}.$$

The latter relation allows us to calculate the integral with the integration variable k_z:

$$\int f(\kappa_z)\delta(\kappa_z v-\omega)d\kappa_z = \frac{f(\kappa_z)}{v}.$$

We are interested not only in the energy loss per a way unit due to Cherenkov's radiation but also in the energy loss due to Cherenkov's radiation in a unit interval of frequencies in the vicinity of a given frequency value ω:

$$\frac{d^2W(\omega)}{d\omega dz},$$

and

$$\frac{dW}{dz} = \int_0^\infty d\omega \frac{d^2W(\omega)}{d\omega dz} d\omega; \qquad (21.13)$$

$$\int_0^\infty \frac{d^2W(\omega)}{d\omega dz} d\omega = \frac{ie^2}{\pi}\left\{ \int_{-\infty}^0 d\omega\cdot\omega\left(\frac{1}{c^2}\right)\left(1-\frac{c^2}{v^2\varepsilon(\omega)}\right)\int\frac{qdq}{\left[q^2-\frac{\omega^2}{c^2}\left(\varepsilon(\omega)-\frac{c^2}{v^2}\right)\right]} + \right.$$

$$\left. +\int_0^\infty d\omega\cdot\omega\left(-\frac{1}{c^2}\right)\left(1-\frac{c^2}{v^2\varepsilon(\omega)}\right)\frac{qdq}{\left[q^2-\frac{\omega^2}{c^2}\left(\varepsilon(\omega)-\frac{c^2}{v^2}\right)\right]}\right\} =$$

$$= -\frac{ie^2}{\pi c^2}\left\{\int_0^\infty d\omega\cdot\omega\left(1-\frac{c^2}{v^2\varepsilon(\omega)}\right)\int\frac{qdq}{\left[q^2-\frac{\omega^2}{c^2}\left(\varepsilon(\omega)-\frac{c^2}{v^2}\right)\right]} - \right.$$

$$\left. -\int_0^\infty d\omega\cdot\omega\left(1-\frac{c^2}{v^2\varepsilon(-\omega)}\right)\int\frac{qdq}{\left[q^2-\frac{\omega^2}{c^2}\left(\varepsilon(-\omega)-\frac{c^2}{v^2}\right)\right]}\right\}.$$

Thus,

$$\int_0^\infty \frac{d^2W(\omega)}{d\omega dz} = -\frac{ie^2}{\pi c^2}\left\{\left(1-\frac{c^2}{v^2\varepsilon(\omega)}\right)\int\frac{qdq}{\left[q^2-\frac{\omega^2}{c^2}\left(\varepsilon(\omega)-\frac{c^2}{v^2}\right)\right]}-\right.$$

$$\left.-\left(1-\frac{c^2}{v^2\varepsilon(-\omega)}\right)\int\frac{qdq}{\left[q^2-\frac{\omega^2}{c^2}\left(\varepsilon(-\omega)-\frac{c^2}{v^2}\right)\right]}\right\}. \qquad (21.14)$$

The medium is thought to be transparent, that is, $Jm\varepsilon(\omega)$ is rather small. One can put $\operatorname{Re}\varepsilon(\omega)$ in the factors before the integrals instead of $\varepsilon(\omega)$ and to use the relation

$$\operatorname{Re}\varepsilon(\omega) = \operatorname{Re}\varepsilon(-\omega) = n^2(\omega).$$

As for the integrand expression, it will have a singular point if we neglect $Jm\varepsilon$,

$$q^2 = \frac{\omega^2}{c^2}\left(n^2 - \frac{c^2}{v^2}\right), \qquad (21.15)$$

with

$$\varepsilon(\omega) = n^2 > \frac{c^2}{v^2}. \qquad (21.16)$$

If the condition $n^2 > \frac{c^2}{v^2}$ is not fulfilled, there is no singular point, the addends in the curly brackets are reduced and $\frac{d^2W}{d\omega dz} = 0$, that is, there is no Cherenkov's radiation.

The condition

$$\varepsilon^2(\omega) = n^2 > \frac{c^2}{v^2} \qquad (21.17)$$

or its equivalent

$$v > \frac{c}{n} \qquad (21.18)$$

acts as a threshold condition for Cherenkov's radiation to appear.

In the state of thermodynamic equilibrium the medium always absorbs and $Jm\varepsilon(\omega) \equiv \varepsilon''(\omega) > 0$ (remember that $div\vec{S} = -\frac{\omega\varepsilon''}{8\pi}\left|\vec{E}^2\right| < 0$ for $\varepsilon'' > 0$).

Let the condition $n^2 > \frac{c^2}{v^2}$ be fulfilled.

The result depends on the way the singular point, which is on the way of integration, is bypassed. In our case the singular point of the first term of the subintegral function is not on the real axis but is shifted upwards (as $Jm\varepsilon(\omega) > 0$), therefore, one can put

$$q^2 = \frac{\omega^2}{c^2}\left(n^2 - \frac{c^2}{v^2}\right) + i\delta,$$

where

$$\delta = \frac{\omega^2}{c^2}\varepsilon''(\omega).$$

Let's introduce a new variable x instead of q:

$$x = q^2 - \frac{\omega^2}{c^2}\left(n^2 - \frac{c^2}{v^2}\right), \tag{21.19}$$

$$qdq = \frac{dx}{2},$$

$$\frac{d^2W}{d\omega dz} = -\frac{ie^2\omega}{2\pi c^2}\left(1 - \frac{c^2}{v^2n^2}\right)\left\{\int\frac{dx}{x - i\delta} - \int\frac{dx}{x + i\delta}\right\}. \tag{21.20}$$

Let's use the well-known relation

$$\lim_{\delta\to 0}\frac{1}{x - a \pm i\delta} = \mp i\pi\delta(x - a) + \frac{P}{x - a}. \tag{21.21}$$

This operator identity should be understood as:

$$\int\frac{1}{x - a \pm i\delta} = P\int\frac{1}{x - a} \mp i\pi\int dx\delta(x - a) = P\int\frac{dx}{x - a} \mp i\pi. \tag{21.22}$$

At $a = 0$ we get from (21.22)

$$\int\frac{1}{x \pm i\delta} = P\int\frac{dx}{x} \mp i\pi. \tag{21.23}$$

Here $P\int\frac{dx}{x - a}$ is an integral in the sense of the principal value.

The expression in the curly brackets (21.20) is easily calculated

$$\left\{\int\frac{dx}{x - i\delta} - \int\frac{dx}{x + i\delta}\right\} = \left(P\int\frac{dx}{x} + i\pi\right) - \left(P\int\frac{dx}{x} - i\pi\right) = 2i\pi,$$

and we finally get

$$\frac{d^2W}{d\omega dz} = \frac{e^2\omega}{c^2}\left(1 - \frac{c^2}{v^2n^2}\right). \tag{21.24}$$

It follows from formula (21.24) that the intensity of radiation doesn't depend on the mass of the transiting particle and is conditioned only by its speed and charge as well as by the optical properties of the medium. The radiation, which appears at a given point of the gildepath, propagates at angle θ with respect to the speed of the particle, where

$$\cos\theta = \frac{c}{nv}. \tag{21.25}$$

It follows from the fact that absolute magnitude $|\vec{\kappa}|$ in the case when the expression (21.16) $\varepsilon(\omega) = n^2 > \frac{c^2}{v^2}$ is defined by a singular point of the denominator in expression (21.11)

$$\kappa^2 = \frac{\omega^2}{c^2}\varepsilon(\omega) = \frac{\omega^2}{c^2}n^2,$$

$$\kappa = \frac{\omega}{c}n.$$

Expression $\kappa_z = \kappa \cdot \cos\theta$ is found by δ-function

$$\delta(\omega - \vec{\kappa}\vec{v}) = \delta(\omega - \kappa_z v)$$

and is equal to

$$\kappa_z = \frac{\omega}{v}.$$

That's why $\frac{c}{nv}$ is calculated from the proportions:

$$\frac{c}{n} = \frac{\omega}{\kappa},\ v = \frac{\omega}{\kappa_z},\ \frac{c}{nv} = \frac{\omega}{\kappa}\cdot\frac{\kappa_z}{\omega} = \frac{\kappa_z}{\kappa} = \cos\theta.$$

The radiation intensity due to a way unit takes the form

$$\frac{d^2W}{d\omega dz} = \frac{e^2\omega}{c^2}\left(1 - \frac{c^2}{v^2n^2}\right) = \frac{e^2\omega}{c^2}\left(1 - \cos^2\theta\right).$$

Condition $v > \frac{c}{n}$ means that the particle speed exceeds the phase velocity of the propagation of electromagnetic disturbance in the medium. The particle "detaches itself" from the field it has created, making free electromagnetic field-radiation. The condition for Cherenkov's radiation to appear can be fulfilled if $\varepsilon(\omega) = n^2 > \frac{c^2}{v^2}$ or $n(\omega) > 1$, that is why it falls at the visible and ultra-violet parts of spectrum, as namely for these frequencies $n(\omega) > 1$.

The particular properties of Cherenkov's radiation led to the creation of the fast-particle detector (Cherenkov's indicators) which allows defining the absolute magnitudes and the directed speed of fast particles and their charge.

If particles pass the medium at a given speed with known ε, the light will be radiated at the Cherenkov's angle

$$\cos\theta_c = \frac{c}{nv} = \frac{1}{\beta\sqrt{\varepsilon}}.$$

Measuring of angle θ_c allows us to define v.

Since dielectric permittivity depends on frequency $\varepsilon \equiv \varepsilon(\omega)$, the light radiation of different wave length (frequency) will be radiated at slightly different angles.

Let's draw a typical dispersion curve $\varepsilon(\omega)$ with a range of anomalous dispersion in the upper end of the frequency interval.

The conditions for Cherenkov's radiation to appear are

$$\varepsilon(\omega) = n^2 > \frac{c^2}{v^2}$$

or

$$\varepsilon(\omega) > \frac{1}{\beta^2}.$$

The fact is that the media have a strong absorbing property in the anomalous dispersion range. That is why, the maximum of Cherenkov's radiation is lower than the resonance frequency ω_0. To select a small frequency interval and to increase the accuracy in speed measuring a narrow-band filter is applied. For the narrow part $\beta \le 1$ a gas can be used as a medium, then the dielectric permittivity differs from 1 slightly, and the value $(\varepsilon - 1)$ can vary within wide limits changing the gas pressure. The counters using Cherenkov's radiation are widely used in experimental physics. In the particle physics they serve as speed meters. They work as mass-spectrometers in combination with a device for impulse determination; they also serve as discriminators to detect undesirable slow particles.

22. Lienard-Wiehert's potentials and the field of a point charge

The task of finding an alternating field in vacuum by given charge distribution $\rho(\vec{r}, t)$ and current distribution is solved by calculating the retarded potentials:

$$\vec{A}(\vec{r}, t) = \frac{1}{c}\int \frac{\vec{j}\left(\vec{r}\,', t - \frac{R}{c}\right)}{R} dV', \tag{22.1}$$

$$\varphi(\vec{r},t)=\int\frac{\rho\left(\vec{r}',t-\frac{R}{c}\right)}{R}dV', \tag{22.2}$$

where $\vec{R}=|\vec{r}-\vec{r}'|$; $\vec{r}$ is a radius vector of the observation point, $\vec{r}'$ is a radius vector of the field source, dV' is the volume element of the field source.

One can introduce the notations:

$$j_\mu=(\vec{j},ic\rho), \tag{22.3}$$

$$A_\mu=\left(\vec{A},i\varphi\right),\ \mu=1,2,3,4, \tag{22.4}$$

$$c\beta_\mu=(\vec{v},ic)=c\left(\vec{\beta},i\right). \tag{22.5}$$

In case of point charge e in point $\vec{z}(t')$ moving at speed $c\vec{\beta}\ (t')$, it corresponds to

$$\vec{j}_\mu(\vec{r}',t')=ec\beta_\mu(t')\delta(\vec{r}'-\vec{r}(t')). \tag{22.6}$$

The four-dimensional potential

$$A_\mu(\vec{r},t)=\frac{1}{c}\iint\frac{j_\mu(\vec{r}',t')}{R}\delta\left(t'+\frac{R}{c}-t\right)d\vec{r}'dt', \tag{22.7}$$

where

$$\vec{R}=\vec{r}(t)-\vec{r}'(t'). \tag{22.8}$$

In this case after taking the volume integral with the help of $\delta(\vec{r}'-\vec{r}(t'))$, one can show that

$$A_\mu(\vec{r},t)=e\int\frac{\beta_\mu(t')}{R(t')}\delta\left(t'+\frac{R(t')}{c}-t\right)dt', \tag{22.9}$$

where

$$R(t')=|\vec{r}(t)-\vec{r}(t')|. \tag{22.10}$$

Integration over dt' can be taken using the properties of δ-function

$$\int g(x)\delta[f(x)-\alpha]dx=\left[\frac{g(x)}{\left|\frac{df}{dx}\right|}\right]_{f(x)=\alpha}. \tag{22.11}$$

Let's introduce the function

$$f(t')\equiv t'+\frac{R(t')}{c}. \tag{22.12}$$

Then

$$\frac{df}{dt'} \equiv \kappa = 1 + \frac{1}{c}\frac{dR}{dt'} = 1 - \vec{n}\vec{\beta}, \tag{22.13}$$

where $c\vec{\beta}(t')$ is an instantaneous speed of the particle; $\vec{n} = \dfrac{\vec{R}(t')}{R(t')}$.

As a result we get

$$A_\mu(\vec{r},t) = \left[\frac{e\beta_\mu}{\kappa R}\right]_{t'+R(t')/c=t} \equiv \left[\frac{e\beta_\mu}{\kappa R}\right]_{ret}. \tag{22.14}$$

It follows from formula (22.14) that

$$\varphi(\vec{r},t) = e\left[\frac{1}{\kappa R}\right]_{ret},$$

$$\vec{A}(\vec{r},t) = e\left[\frac{\vec{\beta}}{\kappa R}\right]_{ret}. \tag{22.15}$$

The potentials (22.15) are called Lienard-Wiehert's potentials.

For the non-relativistic case $\beta \to 0$ and $\kappa \to 1$ we get common non-relativistic formulas.

To calculate fields $\vec{E}$ and $\vec{B}$ we use formula (22.9):

$$\vec{E} = -grad\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \tag{22.16}$$

$$\vec{B} = rot\vec{A}. \tag{22.17}$$

While calculating the fields, the coordinates of the observation point are included only in R, that is why

$$grad = gradR\frac{\partial}{\partial R} = \vec{n}\frac{\partial}{\partial R}. \tag{22.18}$$

As a result we get

$$\vec{E}(\vec{r},t) = e\int\left[\frac{\vec{n}}{R^2}\delta(f-t) + \frac{1}{cR}\left(\vec{\beta}-\vec{n}\right)\delta'_{f-t}(f-t)\right]dt', \tag{22.19}$$

$$\vec{B}(\vec{r},t) = e\int\left[\vec{n}\vec{\beta}\right]\left\{-\frac{\delta(f-t)}{R^2} + \frac{1}{cR}\delta'_{f-t}(f-t)\right\}dt'. \tag{22.20}$$

Integrating by parts the terms with δ'_{f-t}, we get

$$\vec{E}(\vec{r},t) = e\left[\frac{\vec{n}}{\kappa R^2} + \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\vec{n}-\vec{\beta}}{\kappa R}\right)\right]_{ret}, \tag{22.21}$$

$$\vec{B}(\vec{r},t) = e\left[\frac{\left[\vec{\beta}\vec{n}\right]}{\kappa R^2} + \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\left[\vec{\beta}\vec{n}\right]}{\kappa R}\right)\right]_{ret}. \tag{22.22}$$

Noting that

$$\frac{1}{c}\frac{d\vec{n}}{dt'}=\frac{\left[\vec{n}\left[\vec{n}\vec{\beta}\right]\right]}{R}, \tag{22.23}$$

and making dt' differentiation of vector $\vec{n}$ in (22.21) and (22.22) where it is included, we get

$$\vec{E}(\vec{r},t)=e\left[\frac{\vec{n}}{\kappa^2R^2}+\frac{\vec{n}}{c\kappa}\frac{d}{dt'}\left(\frac{1}{\kappa R}\right)-\frac{\vec{\beta}}{\kappa^2R^2}-\frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\vec{\beta}}{\kappa R}\right)\right]_{ret}, \tag{22.24}$$

$$\vec{B}(\vec{r},t)=e\left[\left\{\frac{\vec{\beta}}{\kappa^2R^2}+\frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\vec{\beta}}{\kappa R}\right)\right\}\cdot\vec{n}\right]_{ret}. \tag{22.25}$$

It follows from relations (22.24) and (22.25), that

$$\vec{B}=\left[\vec{n}\vec{E}\right] \tag{22.26}$$

and note, that both parts are taken taking into account the delay.

Using

$$\frac{d}{dt'}\vec{\beta}=\dot{\vec{\beta}}, \tag{22.27}$$

$$\frac{1}{c}\frac{d}{dt'}(\kappa R)=\beta^2-\vec{\beta}\vec{n}-\frac{R}{c}\vec{n}\dot{\vec{\beta}}. \tag{22.28}$$

After some transformations we obtain

$$\vec{E}(\vec{r},t)=e\left[\frac{(\vec{n}-\vec{\beta})(1-\beta^2)}{\kappa^3R^2}\right]_{ret}+\frac{e}{c}\left[\frac{\vec{n}}{\kappa^3R}\times\left\{(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\right\}\right]_{ret}, \tag{22.29}$$

$$\vec{B}=\left[\vec{n}\vec{E}\right]. \tag{22.30}$$

The expression for $\vec{E}$ consists only of the part containing speed $\vec{\beta}$ and of the part depending on acceleration. The first part is static, decreases $\sim\frac{1}{R^2}$, the second part is the radiation field and $\vec{E}\perp\vec{B}$, and this part $\sim\frac{1}{R}$.

23. Full power radiated by an accelerating charge. Larmor's formula and its relativistic generalization

Let an accelerating charge be observed in a system where its velocity is mach smaller than velocity of light. If $\beta<<1$ then according to the formula (22.29) the electric field intensity becomes

$$\vec{E}_a=\frac{e}{c}\left[\frac{\vec{n}\times\left(\vec{n}\times\dot{\vec{\beta}}\right)}{R}\right]_{ret}, \tag{23.1}$$

since $\kappa \approx 1$.

When the radiation angular distribution is studied in the relativistic case, one should distinguish between "intensity" and "power" of radiation. The radiation intensity dI in a solid angle $d\Omega$ can be calculated by means of Umov-Poynting's vector

$$\vec{S} = \frac{c}{4\pi}\left[\vec{E}\vec{B}\right] = \frac{c}{4\pi}\left[\vec{E}_a\left[\vec{n}\vec{E}_a\right]\right] = \frac{c}{4\pi}\left|\vec{E}_a\right|^2 \tag{23.2}$$

and equals

$$dI = R^2(\vec{S}\cdot\vec{n})d\Omega = \frac{c}{4\pi}R^2\left|\vec{E}_a\right|^2 d\Omega . \tag{23.3}$$

The quantity $dI/d\Omega$ presents the electromagnetic energy flux inside of the unit solid angle measured in a fixed system. One can easily show that it does not equal to the speed of the particle energy loss because of radiating in the unit solid angle in a chosen direction $-\frac{d^2\varepsilon}{dt'd\Omega}$, where dt' is an interval of the delayed time. During the time dt an observer would register the energy $\left(\frac{dI}{d\Omega}\right)dt$ which obviously equals to an energy emitted by the particle during the time dt':

$$-(\frac{d^2\varepsilon}{dt'd\Omega})dt' = (\frac{dI}{d\Omega})dt. \tag{23.4}$$

An amount of the particle energy loss in a unit time is called power of radiation

$$W = -\frac{d\varepsilon}{dt'}. \tag{23.5}$$

So, the intensity and the power of radiation in a unit solid angle are coupled by the following relation

$$\frac{dW}{d\Omega} = \frac{dI}{d\Omega}\frac{dt}{dt'} = (1-\vec{n}\cdot\vec{\beta}(t'))\frac{dI}{d\Omega}. \tag{23.6}$$

In contrast to the intensity the power is an invariant quantity, i.e. it has the same form in different inertial systems.

From the invariance of $-\frac{d\varepsilon}{dt'}$ follows

$$-\frac{d\varepsilon}{dt'} = -\frac{d\varepsilon_0}{d\tau}, \tag{23.7}$$

where $d\varepsilon_0$ represents an energy in the concomitant coordinate system.

In a general case time intervals dt' and $d\tau$ are coupled by the following relation

$$dt' = \frac{d\tau}{(1-\frac{v^2}{c^2})^{1/2}} \tag{23.8}$$

and since in the concomitant system $v=0$ then $dt' = d\tau$.

Using formulas (23.6) and (23.7) we find

$$W \equiv -\frac{d\varepsilon}{dt'} = \int \frac{dI_0}{d\Omega} d\Omega . \tag{23.9}$$

The expression (23.9) also means that

$$\frac{dW}{d\Omega} = \frac{dI_0}{d\Omega} .$$

From the formula (23.3) we obtain the following one

$$\frac{dI_0}{d\Omega} = \frac{c}{4\pi} R^2 \left|\vec{E}_a\right|^2 , \tag{23.10}$$

where the electric field intensity in the concomitant system is defined by (23.1) and depends only on the acceleration. Denoting an angle between $\dot{\vec{v}}$ and $\vec{n}$ as θ from (23.10) we obtain the following formula for power of radiation in a unit solid angle

$$\frac{dW}{d\Omega} = \frac{e^2}{4\pi c^3} \dot{\vec{v}}^2 \sin^2\theta . \tag{23.11}$$

By the way, as it follows from (23.1), the radiation is polarized in the plain of vectors $\dot{\vec{v}}$ and $\vec{n}$.

Integrating (23.9) over all solid angles we obtain the full instantaneous power

$$W = \frac{2}{3}\frac{e^2 \dot{v}^2}{c^3} = \frac{2}{3}\frac{e^2 a^2}{c^3} . \tag{23.12}$$

So, famous Larmor's formula for non-relativistic accelerating charge is derived.

Larmor's formula can also be written as follows:

$$W = \frac{2}{3}\frac{e^2}{m^2 c^3}\left(\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt}\right), \tag{23.13}$$

where m is the mass, $\vec{p}$ is the impulse of a charged particle.

If $d\tau = \frac{dt'}{\gamma}$ is an increment of the proper time, then the relativistic generalization of Larmor's formula is

$$W = \frac{2}{3}\frac{e^2}{m^2 c^3}\left(\frac{dp_\mu}{d\tau}\frac{dp_\mu}{d\tau}\right), \tag{23.14}$$

where

$$\left(\frac{dp_\mu}{d\tau}\frac{dp_\mu}{d\tau}\right) = \left(\frac{d\vec{p}}{d\tau}\right)^2 - \frac{1}{c}\left(\frac{d\varepsilon}{d\tau}\right)^2 = \left(\frac{d\vec{p}}{d\tau}\right)^2 - \beta^2\left(\frac{dp}{d\tau}\right)^2 . \tag{23.15}$$

Here the following relations were used $p = mc\beta\gamma$, $\varepsilon = \gamma mc$ and $\varepsilon^2 = m^2c^4 + \vec{p}^2c^2$.

After calculating the derivates

$$\frac{d\vec{p}}{d\tau} = \frac{d}{d\tau}(\gamma m\vec{v}) = mc\frac{d}{d\tau}(\gamma\vec{\beta}) \tag{23.16}$$

and

$$\frac{dp}{d\tau} = mc\frac{d}{d\tau}(\gamma\beta), \tag{23.17}$$

taking into account that $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta^2 = \vec{\beta}^2$ and $\beta\dot{\beta} = \vec{\beta}\cdot\dot{\vec{\beta}}$, $\frac{d\vec{p}}{d\tau} = \gamma\frac{d\vec{p}}{dt'}$ and the relation

$$\left[\vec{\beta}\times\dot{\vec{\beta}}\right]^2 = \left[\vec{\beta}\times\dot{\vec{\beta}}\right]\cdot\left[\vec{\beta}\times\dot{\vec{\beta}}\right] = \vec{\beta}^2\dot{\vec{\beta}}^2 - \left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2, \tag{23.18}$$

we get

$$\left(\frac{dp_\mu}{d\tau}\frac{dp_\mu}{d\tau}\right) = \gamma^6m^2c^2\left\{\dot{\vec{\beta}}^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right\}, \tag{23.19}$$

$$W = \frac{2}{3}\frac{e^2}{m^2c^3}\gamma^6m^2c^2\left\{\dot{\vec{\beta}}^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right\} = \frac{2}{3}\frac{e^2\gamma^6}{c}\left[\dot{\vec{\beta}}^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right]. \tag{23.20}$$

This is the relativistic generalization of Larmor's formula discovered by Lienard in 1898.

Let's consider the task about the motion of a charged particle with charge e and mass m in the external electric and magnetic field. The particle moves with acceleration under the influence of the force

$$\vec{F} = e\left\{\vec{E} + \frac{1}{c}\left[\vec{\upsilon}\times\vec{B}\right]\right\} \tag{23.21}$$

and as a result of radiation it loses the energy

$$\Delta\varepsilon = \int_{-\infty}^{+\infty} W dt, \tag{23.22}$$

where radiation power W is given by the formula:

$$W = \frac{2}{3}\frac{e^2\gamma^6}{c}\left[\dot{\vec{\beta}}^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right] = \frac{2}{3}\frac{e^2}{c^3}\gamma^6\left\{\dot{\vec{\upsilon}}^2 - \frac{\left[\vec{\upsilon}\times\dot{\vec{\upsilon}}\right]^2}{c^2}\right\}. \tag{23.23}$$

It is convenient to transform the formula for power so that it includes external fields $\vec{E}$ and $\vec{B}$.

For this purpose let's write down the law of motion of a relativistic particle under the action of force (23.14):

$$\frac{d\vec{p}}{dt} = e\left\{\vec{E} + \frac{1}{c}\left[\vec{\upsilon} \times \vec{B}\right]\right\}. \quad (23.24)$$

In the further calculations we will need the relativistic relations for full energy and particle impulse:

$$\varepsilon = \frac{mc^2}{\sqrt{1 - \upsilon^2 / c^2}} = \gamma\, mc^2, \quad (23.25)$$

$$\vec{p} = \frac{m\vec{\upsilon}}{\sqrt{1 - \upsilon^2 / c^2}} = \gamma\, m\vec{\upsilon}, \quad (23.26)$$

and

$$\varepsilon^2 = \vec{p}^{\,2} c^2 + m^2 c^4, \quad (23.27)$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \upsilon / c$.

Multiplying scalarly velocity vector $\vec{\upsilon}$ by both parts of (23.24), we get

$$\vec{\upsilon}\,\frac{d\vec{p}}{dt} = e\left(\vec{E}\,\vec{\upsilon}\right). \quad (23.28)$$

Differentiating both parts of the relation (23.27) with respect to time, we obtain

$$\varepsilon\,\frac{d\varepsilon}{dt} = c^2\,\vec{p}\,\frac{d\vec{p}}{dt} \quad (23.29)$$

or

$$\frac{d\varepsilon}{dt} = \frac{1}{\gamma\, m}\left(\vec{p}\,\frac{d\vec{p}}{dt}\right) = e\left(\vec{E}\,\vec{\upsilon}\right), \quad (23.30)$$

where the last part of the equality is obtained by using the motion law (23.24), formula $\vec{p} = \gamma\, m\vec{\upsilon}$ and equality (23.28).

Let's square both parts of equality (23.24):

$$\left(\frac{d\vec{p}}{dt}\right)^2 = e^2\left\{\vec{E} + \frac{1}{c}\left[\vec{\upsilon} \times \vec{B}\right]\right\}^2 = \left(\frac{d}{dt}\frac{m\vec{\upsilon}}{\sqrt{1 - \upsilon^2 / c^2}}\right)^2 =$$

$$= m^2\gamma^4\left\{\dot{\vec{\upsilon}}^2\left(1-\beta^2\right) + \frac{\left(\vec{\upsilon}\,\dot{\vec{\upsilon}}\right)^2}{c^2} + \gamma^2\,\frac{\left(\vec{\upsilon}\,\dot{\vec{\upsilon}}\right)^2}{c^2}\right\}. \quad (23.31)$$

We get one more useful relation.

From formula (23.25) after time differentiation we find

$$\frac{d\varepsilon}{dt} = \frac{d}{dt}\left[mc^2\left(1-\frac{\upsilon^2}{c^2}\right)^{-1/2}\right] = \gamma^3 m\left(\vec{\upsilon}\cdot\dot{\vec{\upsilon}}\right). \tag{23.32}$$

By comparing (23.30) and (23.32) we conclude that

$$\left(\vec{\upsilon}\cdot\dot{\vec{\upsilon}}\right) = \frac{e\left(\vec{E}\cdot\vec{\upsilon}\right)}{\gamma^3 m}. \tag{23.33}$$

According to formula (23.31) we write

$$\dot{\vec{\upsilon}}^2\left(1-\beta^2\right) + \frac{\left(\vec{\upsilon}\cdot\dot{\vec{\upsilon}}\right)^2}{c^2} = \frac{e^2}{m^2\gamma^4}\left\{\left(\vec{E}+\frac{1}{c}\left[\vec{\upsilon}\times\vec{B}\right]\right)^2 - \frac{\left(\vec{E}\cdot\vec{\upsilon}\right)^2}{c^2}\right\}. \tag{23.34}$$

It is easy to check that the left-hand side of (23.34) is exactly the expression in braces

$$\left\{\dot{\vec{\upsilon}}^2 - \frac{\left[\vec{\upsilon}\times\dot{\vec{\upsilon}}\right]^2}{c^2}\right\},$$

in formula (23.23) for power W, that is why

$$W = \frac{2}{3}\frac{e^4\gamma^2}{m^2c^3}\left\{\left(\vec{E}+\frac{1}{c}\left[\vec{\upsilon}\times\vec{B}\right]\right)^2 - \frac{\left(\vec{E}\cdot\vec{\upsilon}\right)^2}{c^2}\right\}. \tag{23.35}$$

Let a particle move in the cyclic accelerator in a constant magnetic field B. In this case $\vec{E}=0$, $\vec{\upsilon}\perp\vec{B}$ and one can write

$$W = \frac{2}{3}\frac{e^4\gamma^2\upsilon^2B^2}{m^2c^5}. \tag{23.36}$$

In the non-relativistic case $(\gamma\approx 1)$ the pulsatance equals

$$\omega = \frac{eB}{mc} \tag{23.37}$$

and radius $\rho\equiv R_0$ is equal to

$$R_0 = \upsilon/\omega = \frac{mc\upsilon}{Be} = \frac{mc^2\beta}{Be}. \tag{23.38}$$

Expressing from (23.38) the magnetic field through the radius of the orbit R_0, we get a classical Larmor's formula:

$$W_{Larmor} = \frac{2}{3}\frac{e^2c\beta^4}{R_0^2} = W_{non-rel}. \tag{23.39}$$

In the relativistic case

$$\omega = \frac{eB}{mc}\sqrt{1-\beta^2} = \frac{ecB}{\varepsilon} = \frac{eB}{\gamma mc}, \tag{23.40}$$

$$R_0 = \frac{\gamma mc^2 \beta}{Be}. \qquad (23.41)$$

As a result we get a relativistic Lienard's formula for the intensity of radiation of a fast particle moving in a constant magnetic field:

$$W_{Lienard} = \frac{2}{3}\frac{e^2 c\beta^4 \gamma^4}{R_0^2} \equiv W_{non-rel}\left(\frac{\varepsilon}{mc^2}\right)^4 = W_{non-rel}\gamma^4 \equiv W_{cl}. \qquad (23.42)$$

It follows from the Lienard's formula that there occurs rapid energy waste of the electron moving along the circle: radiating power W is proportional to the fourth degree of the energy.

24. The application of relativistic formula for radiation power to the calculation of energy loss in the accelerators of charged particles

From the relation (23.14) it follows that at a given external force (a given speed of impulse change) $W \sim \frac{1}{m^2}$ that is the influence of radiation effects is most considerable for e^-. Further we will consider only the radiation of electrons.

The radiation in a linear accelerator

The basic diagram of the acceleration of charged particles in linear accelerator is shown in Fig. 24.1a.

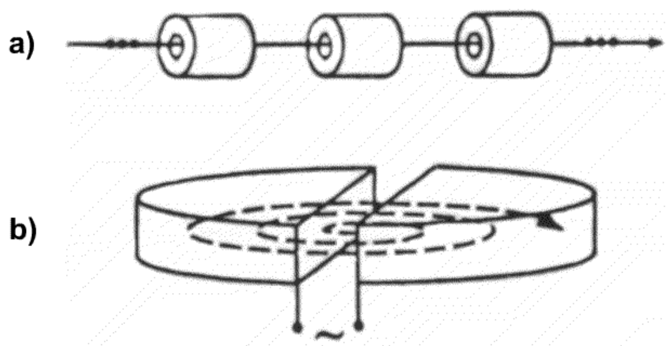


Fig. 24.1. Linear and circular accelerators:
a) linear accelerator; b) cyclotron

Motion is one-dimensional and from formulas (23.14) and (23.15) it follows that:

$$W = \frac{2}{3}\frac{e^2}{m^2c^3}\left[\left(\frac{d\vec{p}}{d\tau}\right)^2 - \beta^2\left(\frac{dp}{d\tau}\right)^2\right] = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\vec{p}}{d\tau}\right)^2\left(1-\beta^2\right) = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\vec{p}}{dt'}\right)^2. \qquad (24.1)$$

As the speed of an impulse change is equal to the energy change per a way unit, then

$$\frac{dE}{dx}=\frac{dp}{dt'}.$$

It is easy to show if we write

$$\frac{dE}{dx}=\frac{dE}{dp}\frac{dp}{dx}=\frac{dE}{dp}\frac{dp}{dt'}\cdot\frac{dt'}{dx}=\frac{1}{v}\frac{dE}{dp}\frac{dp}{dt'}=\frac{dp}{dt'}.$$

Thus, in the one-dimensional case

$$W=\frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2. \qquad (24.2)$$

But $\frac{dE}{dx}$ depends only on the external forces, W doesn't depend on E or p. Let's find the ratio of radiation power to the power from the external sources:

$$\frac{W}{dE/dt'}=\frac{2}{3}\frac{e^2}{m^2c^3}\frac{dE}{dx}\frac{dE}{dx}\cdot\frac{1}{dE/dt'}=\frac{2}{3}\frac{e^2}{m^2c^3}\frac{1}{v}\frac{dE}{dx}=$$

$$=\frac{2}{3}\frac{\left(e^2/mc^2\right)}{mc^2}\frac{1}{\beta}\frac{dE}{dx}\xrightarrow[\beta\to1]{}\frac{2}{3}\frac{r_e}{mc^2}\frac{dE}{dx}, \qquad (24.3)$$

where $r_e=2{,}82\cdot10^{-13}\,cm$.

The losses of radiation are inessential, if energy gain ΔE at distance $\Delta x=r_e$ doesn't exceed $mc^2=0{,}511\,MeV$, that is if the gain is less than $2\cdot10^{14}\,MeV/m$. Usually, energy gain in a linear accelerator is $\Delta E\le10\,MeV/m$.

Thus, the energy losses in a linear accelerator are negligible.

The radiation in cylindrical accelerators

The basic diagram of the acceleration of charged particles in a cylindrical accelerator is shown in Fig. 24.1b, 24.2a, 24.2b.

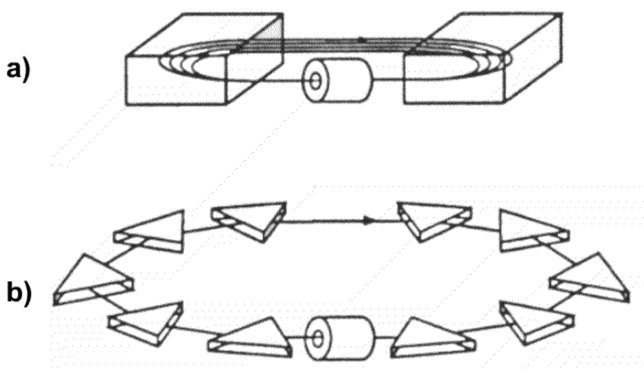


Fig. 24.2. Charged particle accelerators:
a) microtron; b) synchrotron

In these installations, the direction of impulse $\vec{p}$ changes fast during the particle motion in an orbit, and the energy change per one turn is small:

$$\left|\frac{d\vec{p}}{d\tau}\right| = \gamma\omega|\vec{p}| >> \frac{1}{c}\frac{dE}{d\tau}, \qquad (24.4)$$

$$W = \frac{2}{3}\frac{e^2}{m^2c^3}\gamma^2\omega^2|\vec{p}|^2 = \frac{2}{3}\frac{e^2c}{\rho^2}\beta^4\gamma^4 \qquad (24.5)$$

with $\omega = \frac{c\beta}{\rho}$, ρ is an orbital radius.

The losses of energy for the radiation per one turn is

$$\delta E = \frac{2\pi\rho}{c\beta}W = \frac{4\pi}{3}\frac{e^2}{\rho}\beta^3\gamma^4. \qquad (24.6)$$

At $\beta \approx 1$ one can use the formula

$$\delta E(MeV) = 8{,}85\cdot 10^{-2}\frac{[E(GeV)]^4}{\rho(m)}.$$

For calculating the radiation power in cylindrical accelerators it is convenient to use the formula

$$W(Watt) = \frac{10^6}{2\pi}\frac{\delta E(MeV)}{\rho(m)}J(A). \qquad (24.7)$$

In the biggest electronic accelerators radiation power is 0,1 W per 1 μA of beam current. The radiation can be detected easily(synchrotron radiation).

25. Angular distribution of radiation of an accelerating particle

In the non-relativistic case for an accelerating charged particle the electric field depends on the acceleration:

$$\vec{E}_a = \frac{e}{c}\left[\frac{\vec{n}\times\left(\vec{n}\times\dot{\vec{\beta}}\right)}{R}\right]_{ret}, \qquad (25.1)$$

where index *ret* by the bracket shows that the field is calculated at the moment $t' = t - \frac{R(t')}{c}$. The energy flow, determined by the Umov-Pointing's vector

$$\vec{S} = \frac{c}{4\pi}\left|\vec{E}_a\right|^2\vec{n}, \qquad (25.2)$$

inside space angle $d\Omega$ at far distances from the particle is equal to

$$\frac{dI}{d\Omega} = \vec{S} \cdot \vec{n} R^2 = \frac{c}{4\pi} \left| \vec{E}_a \right|^2 = \frac{e^2}{4\pi c^3} \dot{\upsilon}^2 \sin^2 \theta, \qquad (25.3)$$

where θ is the angle between acceleration $\dot{\upsilon}$ and vector $\vec{n}$, directed to the point of observation. Thus, the angle dependence of radiation is described by simple dependence $\sin^2 \theta$.

Let's consider the field of a relativistic particle at far distances from an accelerating charge. In this case in the expression for electric field intensity

$$\vec{E}(\vec{r},t) = e\left[\frac{(\vec{n}-\vec{\beta})(1-\vec{\beta}^2)}{\text{k}^3 R^2}\right]_{ret} + \frac{e}{c}\left[\frac{\vec{n}}{k^3 R} \times \left\{(\vec{n}-\vec{\beta}) \times \dot{\vec{\beta}}\right\}\right]_{ret}, \qquad (25.4)$$

one can leave only the second term.

The first term presents a quasistationary field, which doesn't contain acceleration $\dot{\vec{\beta}}$ and is proportional to R^2. The energy flow in this quasistationary field within the solid angle $d\Omega$ decreases if R increases as $1/R^2$.

The energy flow per unit solid angle for the second term is equal to

$$\frac{dI}{d\Omega} = (\vec{S}\,\vec{n}) R^2 = \frac{e^2}{4\pi c}\left[\frac{1}{k^6}\left|\vec{n} \times \left[(\vec{n}-\vec{\beta}) \times \dot{\vec{\beta}}\right]\right|^2\right]_{ret} \qquad (25.5)$$

and depends on R only through the time argument $t' = t - R(t')/c$. In other words, the energy flows through the area elements $R^2 d\Omega$ within the chosen space angle $d\Omega$, which are at different distances from the particle, is the same at the corresponding moments of time (taking into account the terminal velocity of the energy transfer). And the electromagnetic field propagates from the particle that induced it to infinity.

Under these conditions there arises the radiation field which detaches from its source. In the case of quasistationary fields it doesn't occur. The quasistationary field always remains connected with the particle and doesn't produce the flow propagating to infinity.

Analyzing formula (25.5) we note that the relativistic effects are due to two factors. The first is connected with the mutual position of vectors $\vec{\beta}$ and $\dot{\vec{\beta}}$; the second one is bound up with the transition from the coordinate frame where the particle is at rest to the laboratory coordinate frame, as a result there appear exponential orders in the denominator:

$$\kappa = 1 + \frac{1}{c}\frac{dR(t')}{dt} = 1 - \vec{n} \cdot \vec{\beta}.$$

For the ultra-relativistic particles the angle distribution is defined just by the dependence on κ.

Let's transform formula (25.5). Squaring

$$\left|\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]\right|$$

and calculating by means of the vector algebra, we get

$$\frac{dI}{d\Omega}=\frac{e^2}{4\pi c}\left\{\frac{2\left(\vec{n}\dot{\vec{\beta}}\right)\left(\vec{\beta}\dot{\vec{\beta}}\right)}{\kappa^5}+\frac{\dot{\vec{\beta}}^2}{\kappa^4}-\frac{\left(1-\beta^2\right)\left(\vec{n}\dot{\vec{\beta}}\right)}{\kappa^6}\right\}. \tag{25.6}$$

Let's consider some special cases

1. The non-relativistic particle, $\beta << 1$. Neglecting the terms of order β, we get

$$\frac{dI}{d\Omega}=\frac{e^2}{4\pi c}\dot{\vec{\beta}}^2\sin^2\theta=\frac{e^2}{4\pi c^3}\dot{\upsilon}^2\sin^2\theta, \tag{25.7}$$

where θ is the angle between the acceleration and the direction $\vec{n}$ to the point of observation. The radiation is distributed symmetrically with respect to the direction $\dot{\vec{\upsilon}}$ and is maximal in the direction perpendicular to the direction $\dot{\vec{\upsilon}}$.

2. An ultra-relativistic particle, $\gamma=\frac{1}{\sqrt{1-\beta^2}}>>1$, the acceleration $\dot{\vec{\upsilon}}$ and the velocity are parallel: $\dot{\vec{\upsilon}}\parallel\vec{\upsilon}$. Let's denote the angle between $\vec{n}$ and $\vec{\upsilon}$ by θ. From (25.6) we get in this approximation

$$\frac{dI}{d\Omega}=\frac{e^2\dot{\vec{\upsilon}}^2\sin^2\theta}{4\pi c^3\left(1-\beta\cos\theta\right)^6}. \tag{25.8}$$

If $\cos\theta\approx 1$, then the denominator is small under this condition, and almost all the radiation is concentrated in the area of small θ, though $\frac{dI}{d\Omega}(\theta=0)=0$.

If at small angles we restrict ourselves to the expansions $\sin^2\theta\approx\theta^2$, $\cos\theta\approx 1-\frac{\theta^2}{2}$ and if we take into account that $\upsilon\sim c$, we get

$$\frac{dI}{d\Omega}=\frac{16e^2\dot{\upsilon}^2\gamma^{10}\left(\gamma\theta\right)^2}{\pi c^3\left(1+\gamma^2\theta^2\right)^6}. \tag{25.9}$$

The radiation is in the cone with the resolution of some $1/\gamma$.

3. The ultra-relativistic particle, the acceleration is perpendicular to velocity: $\dot{\vec{\upsilon}}\perp\vec{\upsilon}$. From formula (25.6) we get

$$\frac{dI}{d\Omega}=\frac{e^2}{4\pi c}\left\{\frac{\dot{\vec{\beta}}^2}{\kappa^4}-\frac{\left(1-\beta^2\right)\left(\vec{n}\dot{\vec{\beta}}\right)}{\kappa^6}\right\}=$$

$$=\frac{e^2\dot{\vec{\upsilon}}^2}{4\pi c^3}\left\{\frac{1}{\left(1-\frac{\upsilon}{c}\cos\theta\right)^4}-\frac{\left(1-\frac{\upsilon^2}{c^2}\right)\sin^2\theta\cos^2\varphi}{\left(1-\frac{\upsilon}{c}\cos\theta\right)^6}\right\}. \tag{25.10}$$

Here θ is the angle between $\vec{n}$ and $\vec{\upsilon}$, φ is the horizontal angle of vector $\vec{n}$ with the plane where the vectors $\vec{\upsilon}$ and $\dot{\vec{\upsilon}}$ are. The intensity is symmetrical only with the respect to plane $\vec{\upsilon}\cdot\vec{\upsilon}$ and it turns to zero in two directions of this plane which make the angle $\theta = \arccos(v/c)$ with the velocity.

The distribution (25.10), as well as (25.8), is focused in the forward direction and for small θ has the form:

$$\frac{dI}{d\Omega} = \frac{4e^2\dot{\upsilon}^2\gamma^8}{\pi c^3\left(1+\gamma^2\theta^2\right)^4}\left[1-\frac{4\gamma^2\theta^2\cos^2\varphi}{\left(1+\gamma^2\theta^2\right)^2}\right]. \tag{25.11}$$

The product $\vec{S}\cdot\vec{n}$ is an energy flow per unit of time through a unit area element, and this energy flow is produced by the charge radiation in moment $t' = t - R(t')/c$. It is registered in the point of observation in moment t. If it is necessary to define the energy radiated for a finite time of the acceleration from $t' = T_1$ to $t' = T_2$, it is needed to take the integral:

$$W = \int\limits_{t=T_1+\frac{R(T_1)}{c}}^{t=T_2+\frac{R(T_2)}{c}} \left[\vec{S}\,\vec{n}\right]_{t'} dt = \int\limits_{t'=T_1}^{t'=T_2} (S\,n)\frac{dt}{dt'}dt' . \tag{25.12}$$

In this connection it is interesting to know the value

$$(S\,n)\frac{dt}{dt'}, \tag{25.13}$$

which presents the intensity of the energy radiated through a unit surface per unit of the proper time of the charge. The power radiated per unit solid angle is defined by the relation

$$\begin{aligned}\frac{dW}{d\Omega} &= R^2\left(S\,\vec{n}\right)\frac{dt}{dt'} = \kappa R^2\left(\vec{S}\,\vec{n}\right) \equiv R^2(\vec{S}\cdot\vec{n})(1-\vec{n}\cdot\vec{\beta}) = \\ &= (1-\vec{n}\cdot\vec{\beta})\frac{dI}{d\Omega}.\end{aligned} \tag{25.14}$$

If the charge accelerates only in the interval during which $\vec{\beta}$ and $\dot{\vec{\beta}}$ don't change essentially by the direction and the magnitude, and the observation point is so far that $\vec{n}$ and R change non-essentially, value $\frac{dI(t')}{d\Omega}$ will be proportional to the angular distribution of the radiated energy:

$$\frac{dI(t')}{d\Omega} = \frac{e^2}{4\pi c}\frac{\left|\vec{n}\times\left\{\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}\right|^2}{\kappa^5}. \tag{25.15}$$

a) Consider a special case of rectilinear motion $\left(\dot{\vec{\upsilon}} \parallel \vec{\upsilon}\right)$. From the formula (25.15) we get

$$\frac{dI(t')}{d\Omega} = \frac{e^2}{4\pi c^3} \frac{\dot{\upsilon}^2 \sin^2\theta}{(1-\beta\cos\theta)^5}. \qquad (25.16)$$

At $\beta << 1$ this relation reduces to Larmor's formula:

$$\frac{dI(t')}{d\Omega} = \frac{e^2}{4\pi c^3} \dot{\upsilon}^2 \sin^2\theta. \qquad (25.7)$$

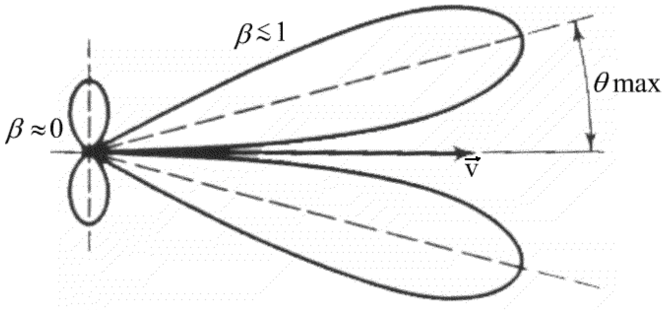


Fig. 25.1. Radiation distribution of a charge accelerating in the direction of motion

The dependence of radiation intensity on the angle is shown in Fig. 25.1, for the non-relativistic case $(\beta \approx 0)$ and for the ultra-relativistic case $(\beta \leq 1)$.

In Fig. 25.2 it is shown schematically the radiation distribution for a slow non-relativistic particle, which moves in the linear accelerator. There is no radiation propagated along the direction of acceleration.

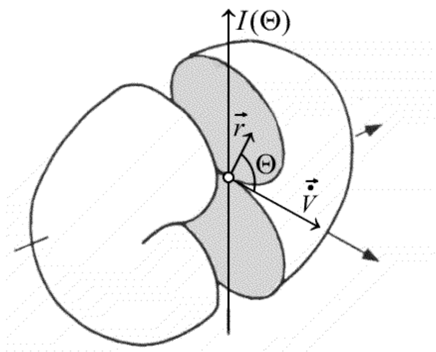


Fig. 25.2. Radiation distribution for a slow no relativistic particle moving in the linear accelerator

In Fig. 25.3 it is shown schematically synchrotron radiation for a relativistic particles, which moves in the circular acceleration.

The angle for which the radiation intensity is maximal is equal to

$$\theta_{\max} = \arccos\left[\frac{1}{3\beta}(\sqrt{1+15\beta^2}-1)\right].$$

The angle for which the radiation intensity is maximal is equal to

$$\theta_{\max} = \arccos\left[\frac{1}{3\beta}(\sqrt{1+15\beta^2}-1)\right].$$

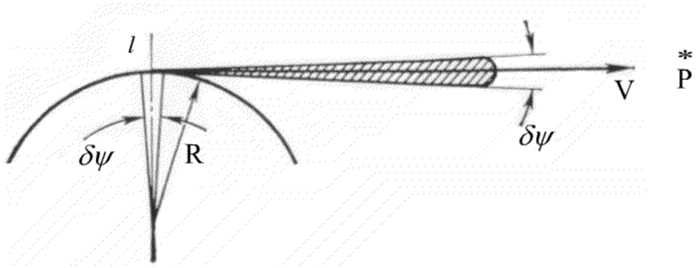


Fig. 25.3. The scheme of generation and registration of synchrotron radiation

In the limit $\beta \to 0$ $\theta_{\max} \to \dfrac{1}{2\gamma}$ and intensity $\dfrac{dI}{d\Omega}\Big|_{\max} \sim \gamma^8$. At $\beta = 0{,}5$ (it corresponds to $T_e \sim 80$ keV we get $\theta_{\max} = 38{,}2°$. For the relativistic particle angle $\theta_{\max} \sim \dfrac{mc^2}{E}$, that is it is very small. That is why all radiation is concentrated in a narrow cone along the direction of motion. In this case we get from (16) an approximate formula (using the expansion $\sin^2\theta \approx \theta^2$, $\left(1 - \dfrac{\upsilon}{c}\cos\theta\right)^5 \approx \dfrac{\gamma - 10}{32}\left[1 + \theta^2\gamma^2\right]^5$):

$$\frac{dI(t')}{d\Omega} \approx \frac{8}{\pi}\frac{e^2\dot{\upsilon}^2}{c^3}\gamma^8 \frac{(\gamma\theta)^2}{\left(1 + \gamma^2\theta^2\right)^5}. \tag{25.17}$$

The value γ^{-1} is used as a natural unit for measuring angles.

Fig. 25.4 shows the angle dependence of radiation intensity (25.17), for which the angles are taken in units of γ^{-1}. The maximum of distribution is $\theta = \dfrac{1}{2}\gamma^{-1}$, and a half of intensity corresponds to $\theta = 0{,}23\gamma^{-1}$ and $\theta = 0{,}91\gamma^{-1}$.

In the relativistic limiting case $(\beta \to 1)$ the mean-square value of the angle is

$$\sqrt{\langle\theta\rangle^2} = \gamma^{-1} = {mc^2}/{E}. \tag{25.18}$$

The intensity distribution in angle shown in Fig. 25.4 is typical and independent of the positioning of vectors $\vec{\beta}$ and $\dot{\vec{\beta}}$.
calculate the total radiation power by the formula:

$$\begin{gathered} I = \int\left(\frac{dI(t')}{d\Omega}\right)d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi}\left[\frac{e^2\dot{\upsilon}^2}{4\pi c^3}\frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}\right]\sin\theta\, d\theta = \\ = \frac{e^2\dot{\upsilon}^2}{2c^3}\int_0^{\pi}\frac{\sin^2\theta\, d\theta}{(1 - \beta\cos\theta)^5} = \frac{2}{3}\frac{e^2}{c^3}\dot{\upsilon}^2\gamma^6. \end{gathered} \tag{25.19}$$

б) Let’s derivate the formula for the angular distribution of radiation at a momentary motion of the charge along the circle. $\left(\dot{\vec{\beta}} \perp \vec{\beta}\right)$.

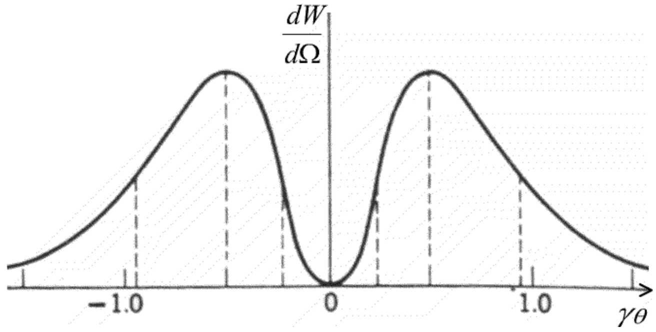


Fig. 25.4. Angular radiation distribution for the relativistic particle

Let's bind up the coordinate origin with the moving particle with the direction of instant speed $\vec{\beta}$ coinciding with the direction of axis z, and the direction of acceleration $\dot{\vec{\beta}}$ with the direction of axis x (see Fig. 25.5).

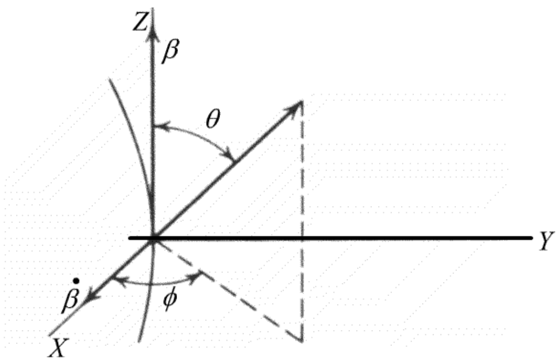


Fig. 25.5. The position of a moving particle in the chosen coordinate fram

Unit vector $\vec{n}$, directed to the observation point is characterized by angles θ and φ in the spherical coordinates.

From formula (25.10) taking into account that $\frac{dI(t')}{d\Omega} = (1 - \vec{n}\,\vec{\beta})\frac{dI}{d\Omega}$, we get for the angular distribution

$$\frac{dI(t')}{d\Omega} = \frac{e^2\dot{\upsilon}^2}{4\pi\, c^3}\frac{1}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\theta}{\gamma^2(1-\beta\cos\theta)^2}\right]. \tag{25.20}$$

Like in the previous case studied in point a), there is a typical relativistic radiation concentration in the direction of motion.

Considering angle θ to be small and $\gamma^{-1} << 1$, one can expand into Taylor series by θ and γ^{-2}:

$$\left(1-\frac{\upsilon}{c}\cos\theta\right)^3 \approx \left[1-\left(1-\frac{\gamma^{-2}}{2}\right)\left(1-\frac{\theta^2}{2}\right)\right]^3 \approx \left[\frac{\gamma^{-2}}{2}+\frac{\theta^2}{2}\right]^3 \approx \frac{\gamma^{-6}}{8}\left[1+\theta^2\gamma^2\right]^4,$$

$$\left(1-\frac{\upsilon}{c}\cos\theta\right)^2 \approx \frac{\gamma^{-4}}{4}\left[1+\theta^2\gamma^2\right]^2,$$

and to present approximately the angle distribution of the radiation intensity:

$$\frac{dI(t')}{d\Omega} \approx \frac{2}{\pi}\frac{e^2\dot{\upsilon}^2}{c^3}\gamma^6 \frac{1}{\left(1+\gamma^2\theta^2\right)^3}\left[1-\frac{4\gamma^2\theta^2\cos^2\varphi}{\left(1+\gamma^2\theta^2\right)^2}\right].. \qquad (25.21)$$

The total intensity is obtained if one integrates over all the angles (25.20):

$$I(t') = \frac{2}{3}\frac{e^2\dot{\upsilon}^2}{c^3}\gamma^4 . \qquad (25.22)$$

At the motion in a circle (when the velocity is perpendicular to acceleration) $\frac{d\vec{p}}{dt} = \gamma m\dot{\vec{\upsilon}}$. That is why formula (25.22) will take the following form

$$Icircular(t') = \frac{2}{3}\frac{e^2\gamma^2}{m^2c^3}\left(\frac{d\vec{p}}{dt}\right)^2 .$$

The motion in a linear accelerator is rectilinear and the radiation intensity is equal to

$$Irectilinear(t') = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\vec{p}}{dt}\right)^2 .$$

As the impulse change is equal to the force given the value of the applied force we find, that the radiation at a transverse acceleration is γ^2 times more than at a longitudinal acceleration.

26. The charge radiation at arbitrary ultra-relativistic motion

The radiation of an accelerating ultra-relativistic particle at an arbitrary moment of time can be thought to be a coherent superposition of radiations which are due to the components of the acceleration vector, parallel and perpendicular velocity:

$$\frac{dI}{d\Omega} = \frac{dI_\perp}{d\Omega} + \frac{dI_\parallel}{d\Omega}, \qquad (26.1)$$

where $\frac{dI_\perp}{d\Omega} \sim \dot{\vec{\upsilon}}_\perp^2$, $\frac{dI_\parallel}{d\Omega} \sim \dot{\vec{\upsilon}}_\parallel^2$ and $\dot{\vec{\upsilon}} = \dot{\vec{\upsilon}}_\perp + \dot{\vec{\upsilon}}_\parallel$.

But, as it was stated above, if the longitudinal and transverse forces which cause the accelerations $\dot{\vec{\upsilon}}_\parallel$ and $\dot{\vec{\upsilon}}_\perp$ correspondingly are of the same order, the radiation intensity is defined only by the transverse component of acceleration $\dot{\vec{\upsilon}}_\perp$.

One can say, that at every moment the radiation coincides with that of the charge that moves along the arc of the circle having the instant radius of curvature:

$$\rho = \frac{\upsilon^2}{\dot{\upsilon}_\perp} \approx \frac{c^2}{\dot{\upsilon}_\perp}. \qquad (26.2)$$

The angle distribution of radiation is described in this case by formulas and at an arbitrary moment it presents the radiation stretched in the shape of a needle-shaped cone with the axis of symmetry which is both a tangent to the path and the vector of the instant speed in the point where the particle is at this moment.

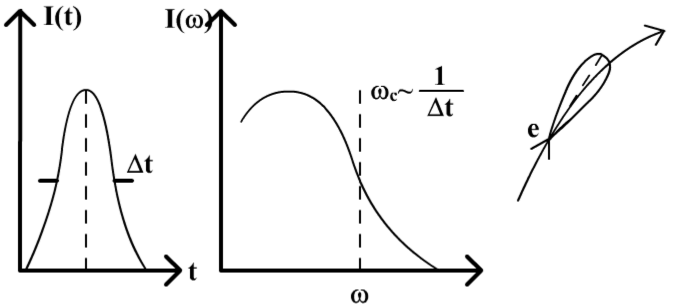


Fig. 26.1. Time and frequency characteristics of a moving particle

At the detector D the particle radiation falls during Δt, that is why all the frequencies will be presented in the radiation spectrum up to the critical frequency $\omega_c \sim \dfrac{1}{\Delta t}$.

The observer will register the radiation as a flash when the vector of the instant speed is directed to the detector, or as a succession of flashes if the particle motion is periodical like in the synchrotron (Fig. 26.1).

At the angle resolution of the ray of order γ^{-1} the way passed by the particle for the time $\Delta t'$, is $\upsilon \Delta t' \sim c\Delta t' = \rho\gamma^{-1}$ (Fig. 26.2), where ρ is an instant radius of curvature and consequently:

$$\Delta t' \sim \rho / c\gamma . \qquad (26.3)$$

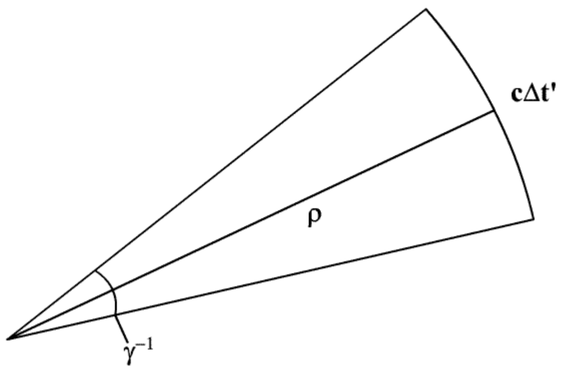


Fig. 26.2. To the derivation of formula (26.3)

And for the observer the time interval of observation will be

$$\Delta t \sim \left\langle \frac{dt}{dt'} \right\rangle \Delta t . \qquad (26.4)$$

As

$$\left\langle \frac{dt}{dt'} \right\rangle = \langle \kappa \rangle \sim \frac{1}{\gamma^2}, \tag{26.5}$$

the duration of the flash of radiation registered by the detector is

$$\Delta t \sim \gamma^{-3} \frac{\rho}{c}. \tag{26.6}$$

From the Fourier integral used to analyze impulses or wave packets of finite sizes, it follows that will be presented spectral components up to critical frequency ϖ_c in the impulse of duration Δt (Fig. 26.1):

$$\omega_c \sim \frac{1}{\Delta t} \sim \left(\frac{c}{\rho} \right) \gamma^3. \tag{26.7}$$

If a particle makes a circular motion like in the cyclic accelerator, then the value c/ρ is equal to pulsatance ω_0. At an arbitrary motion it is also a characteristic frequency of motion. At the ultra-relativistic motion of a particle $E >> mc^2$ the spectrum of radiation spreads up to the frequency, which is equal to

$$\omega_c \cong \omega_0 \gamma^3, \tag{26.8}$$

that is much bigger (γ^3 times) than basic frequency Thus, for the synchrotron for 200 MeV $\gamma_{\max} \sim 400$ and $\omega_c = 6 \cdot 10^7\ \omega_0$. The basic frequency is $\omega_0 = 3 \cdot 10^8$ Hz or the corresponding wavelength is $1000\,A$. That is why, though the basic frequency is in the area of ~ 100 MHz, the radiation spectrum spreads out in the visible region. Further we will study the nature of the angle distribution of radiation for different spectral components and the frequency dependence of the total energy of radiation.

27. Spectral and angular distribution of the energy radiated by accelerating charges

The total energy radiated per unit of the solid angle is defined as

$$\frac{dQ}{d\Omega} = \int_{-\infty}^{+\infty} \left| \vec{f}(t) \right|^2 dt, \tag{27.1}$$

where

$$\left| \vec{f}(t) \right|^2 = \left| \sqrt{\frac{c}{4\pi}} (R\vec{E}) \right|^2_{ret} = \frac{dI(t)}{d\Omega},\ Q = \int_{-\infty}^{+\infty} I(t) dt \tag{27.2}$$

and $\vec{E}$ is the intensity of the electric field, measured at far distances from the accelerating charge in so that the area of acceleration is seen at a small solid angle. Here we consider the instant power or strength of radiation per a unit of the solid angle, which depends on time in the laboratory reference frame. That is $\frac{dI(t)}{d\Omega}$, in contrast to $dI(t')/d\Omega$, studied in section 25. This necessity is caused by a frequent need to know the spectrum of radiation from observer's point of view. Besides, let's assume that the total radiated energy is finite. The supposition about the observation at far distances allows us to take into account only the part connected with acceleration and to drop the part dealing with a uniform motion of the charge in the expression for intensity $\vec{E}$:

$$\vec{E} \approx \vec{E}_a = \frac{e}{c}\left[\frac{\vec{n}}{\kappa^3 R}\times\left\{\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}\right]_{ret}, \tag{27.3}$$

where index *ret* by bracket means that the value is calculated for the moment

$$t = t' + R(t')/c\,.$$

The expression (27.1) can be written with the help of Fourier transformation as an integral over frequencies.

To do it we introduce the amplitude $\vec{f}(\omega)$ of function $\vec{f}(t)$:

$$\vec{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\vec{f}(t)e^{i\omega t}dt\,. \tag{27.4}$$

And the inversed transformation

$$\vec{f}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\vec{f}(\omega)e^{-i\varpi t}d\omega. \tag{27.5}$$

Taking into account all the transformations, formula (27.1) can be presented in the form

$$\frac{dW}{d\Omega} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}d\omega\int_{-\infty}^{+\infty}d\omega'\vec{f}^{*}(\omega')\cdot\vec{f}(\omega)e^{i(\omega'-\omega)t}\,. \tag{27.6}$$

Using the integral representation of Dirac's δ-function

$$\delta(x-x') = \frac{1}{2\pi}\int e^{i(\omega'-\omega)t}dt = \delta(\omega'-\omega), \tag{27.7}$$

we transform the expression (27.6) to the form

$$\frac{dW}{d\Omega} = \int_{-\infty}^{+\infty}\left|\vec{f}(\omega)\right|^2 d\omega = \int_{0}^{\infty}\frac{dI(\omega)}{d\Omega}d\omega\,. \tag{27.8}$$

Negative frequencies having no physical sense, it is necessary to take positive frequencies integration in (27.8). It means that

$$\frac{dI(\omega)}{d\Omega} = \left|\vec{f}(\omega)\right|^2 + \left|\vec{f}(-\omega)\right|^2. \tag{27.9}$$

If $\vec{f}(t)$ is a real value, then

$$\vec{f}(-\omega)=\vec{f}^{*}(\omega), \tag{27.10}$$

as it follows from (27.2) and (27.5), therefore

$$\frac{dI(\omega)}{d\Omega}=2\left|\vec{f}(\omega)\right|^{2}. \tag{27.11}$$

The value $\frac{dI(\omega)}{d\Omega}$ is equal to the energy radiated per unit of the space angle in a unit interval of frequencies or to spectral intensity of radiation per unit of the space angle. Thus, the relationship between the measurement of the radiation energy in time and its frequency spectrum is established.

From formulas (27.2)–(27.4) we find

$$\vec{f}(\omega)=\sqrt{\frac{e^{2}}{8\pi^{2}c}}\int_{-\infty}^{+\infty}e^{i\varpi t}\left(\frac{\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]}{\kappa^{3}}\right)_{t'}dt. \tag{27.12}$$

Let's change to variable dt':

$$t=t'+\frac{R(t')}{c}.$$

In this case

$$dt=\left(\frac{dt}{dt'}\right)dt'=\left(1+\frac{1}{c}\frac{dR(t')}{dt'}\right)dt'=\left(1-\vec{n}\vec{\beta}\right)dt'=\kappa dt' \tag{27.13}$$

and formula (27.12) takes the form

$$\vec{f}(\omega)=\sqrt{\frac{e^{2}}{8\pi^{2}c}}\int_{-\infty}^{+\infty}e^{i\omega\left(t'+\frac{R(t')}{c}\right)}\left(\frac{\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]}{\kappa^{3}}\right)_{t'}\frac{dt}{dt'}dt'. \tag{27.14}$$

As it is supposed that the observation point is quite far from the area where the charged particle moves (see Fig. 27.1), it is possible to consider that unit vector $\vec{n}$ doesn't change in due course and

$$R(t')\cong r\left(1-\vec{n}\cdot\vec{r}'(t')\right). \tag{27.15}$$

If we drop constant phase factor $e^{ir/c}$ in (27.14), which appears after the substitution of (27.15) into (27.14) and doesn't play any role, as $dI(\omega)/d\Omega\sim\left|\vec{f}(\omega)\right|^{2}$, then, taking into account the relations above and dropping the dashes to simplify the formula, we get

$$\vec{f}(\omega)=\sqrt{\frac{e^{2}}{8\pi^{2}c}}\int_{-\infty}^{+\infty}e^{i\omega(t-n\cdot r(t)/c}\frac{\vec{n}\times\left[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\right]}{\kappa^{2}}dt. \tag{27.16}$$

If the law of the charge motion is set, then the dependence $\vec{r}(t)$ is known and one can find $\vec{\beta}(t)$ and $\dot{\vec{\beta}}(t)$, that is why the integral in (27.16) is calculated as a function of the frequency ϖ and the function of direction to the observation point $\vec{n}$.

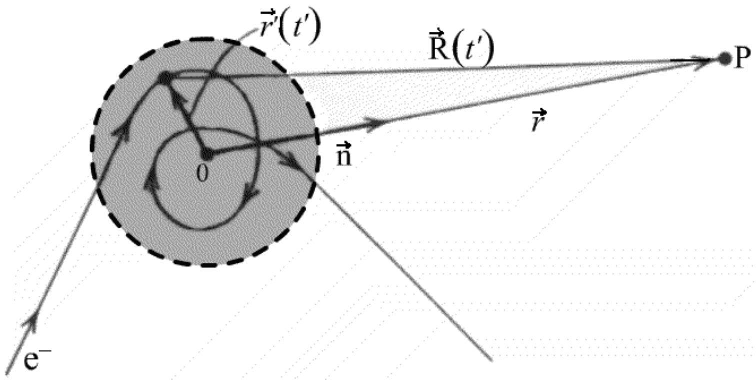


Fig. 27.1. The position of a radiating particle and the observation point P with respect to the coordinate origin O

Substituting (27.16) into (27.11), we find

$$\frac{dI(\omega)}{d\Omega}=\frac{e^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty}\frac{\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]}{\left(1-\vec{\beta}\vec{n}\right)^2}e^{i\omega\left\{t-\vec{n}\vec{r}(t)/c\right\}}dt\right|^2 . \qquad (27.17)$$

The convenience of expression (27.17) is that the integral is taken only over the time interval where acceleration $\dot{\vec{\beta}}\neq 0$.

The expression becomes simpler if one uses the relation

$$\frac{\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]}{\kappa^2}=\frac{d}{dt}\left[\frac{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)}{\kappa}\right], \qquad (27.18)$$

which is directly proved by the time value differentiation

$$\frac{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)}{\kappa}. \qquad (27.19)$$

In this case the multiplier before the exponent in (27.17) is the exact differential. Taking integral by parts and assuming that $\dot{\vec{\beta}}$ vanishes at the beginning and at the end of the integration we get for the spectral intensity

$$\frac{dI(\omega)}{d\Omega}=\frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty}\vec{n}\times\left[\vec{n}\times\vec{\beta}\right]exp\left\{i\omega\left[t-\frac{\vec{n}\cdot\vec{r}(t)}{c}\right]\right\}dt\right|^2 . \qquad (27.20)$$

We note that in expressions (27.17) and (27.20) the radiation polarization is defined by the direction of vector integral. To define the radiation intensity with a given polarization it is necessary to find the scalar product of the integral per unit vector of a given polarization before computing the squared module.

28. Spectral and angular distribution of radiation for continuous distribution of moving charges and moments

In the above section we got the expression for the spectral and angle distributions of the energy radiated by the accelerated charges:

a) for a single charge:

$$\frac{dI(\omega)}{d\Omega}=2\left|\vec{f}(\omega)\right|^2=\frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty} e^{i\omega\left(t-\frac{\vec{n}\vec{r}(t)}{c}\right)}\left\{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)\right\}dt\right|^2; \tag{28.1}$$

b) for a group of N of accelerated charges (with substitution):

$$e\vec{\beta}e^{-i\frac{\omega}{c}\vec{n}\vec{r}(t)}\to\sum_{j=1}^{N}e_j\beta_j e^{-i\frac{\omega}{c}\vec{n}\vec{r}_j(t)}; \tag{28.2}$$

c) in the limiting case of continuous distribution of charges:

$$\sum_{j}^{N}e_j\beta_j e^{-i\frac{\omega}{c}\vec{n}\vec{r}_j(t)}\to\frac{1}{c}\int d^3r\vec{j}(\vec{r},t)e^{-\frac{\omega}{c}\vec{n}\vec{r}}, \tag{28.3}$$

where $\frac{\omega\vec{n}}{c}=\vec{k}$ is a wave vector, $\vec{j}(\vec{r},t)$ is a current density.

As a result we get

$$\frac{dI(\omega)}{d\Omega}=\frac{\omega^2}{4\pi^2c^3}\left|\int_{-\infty}^{+\infty}dt\int d^3r\left\{\vec{n}\times\left(\vec{n}\times\vec{j}(\vec{r},t)\right)\right\}e^{i\omega\left(t-\frac{\vec{n}\cdot\vec{r}}{c}\right)}\right|^2. \tag{28.4}$$

The radiation of the moving moment

Let's use the fact that the rotor of magnetization vector $\vec{M}(\vec{r},t)$ is related to the current:

$$\vec{j}_M=c\,rot\,\vec{M}(\vec{r},t). \tag{28.5}$$

Let's show that after the substitution of (28.5) into (28.4) it is possible to obtain the relation for $\frac{dI(\varpi)}{d\Omega}$ in the form

$$\frac{dI_M(\omega)}{d\Omega}=\frac{\omega^4}{4\pi^2c^3}\left|\int_{-\infty}^{+\infty}dt\int d^3r\left[\vec{n}\times\vec{M}(\vec{r},t)\right]e^{i\omega\left(t-\frac{\vec{n}\cdot\vec{r}}{c}\right)}\right|^2. \tag{28.6}$$

For that we notice that

$$\vec{n}\times\left[\vec{n}\times\vec{J}_M(\vec{x},t)\right]e^{i\omega\left(t-\frac{\vec{n}\cdot\vec{x}}{c}\right)}=c\left\{\vec{n}\times\left[\vec{n}\times\left[\vec{\nabla}\times\overset{\downarrow}{\vec{M}}\right]\right]e^{i\omega\left(t-\frac{\vec{n}\cdot\vec{x}}{c}\right)}\right\}.$$

Here the operator $\vec{\nabla}$ acts on $\vec{M}$.

If we use tensor notations to denote the vector product of two vectors:

$$\left[\vec{A}\vec{B}\right]=e_i A_j B_k \varepsilon_{ijk},$$

where e_i are the unit vectors, ε_{ijk} is a antisymmetrical tensor of the 3rd rank, the expression (28.6) can be written in the form

$$\frac{dI_M(\omega)}{d\Omega}=\frac{\omega^2}{4\pi^2c^3}\left|c\int_{-\infty}^{+\infty}dt\prod_{r=1}^{3}\int dx_r R_{jS}\left(\nabla_j\overset{\downarrow}{M_S}\right)e^{i\omega\left(t-\frac{n_r x}{c}\right)}\right|. \quad (28.7)$$

By virtue of the identity

$$\nabla_j\left(R_{jS}M_S\,e^{i\omega\left(t-\frac{n_r x_r}{c}\right)}\right)=R_{jS}\left(\nabla_j\overset{\downarrow}{M_S}\right)e^{i\omega\left(t-\frac{n_r x_r}{c}\right)}+$$

$$+R_{jS}M_S\left(\nabla_j e^{i\omega\left(t-\frac{\overset{\downarrow}{n_r x_r}}{c}\right)}\right). \quad (28.8)$$

And the general Gauss's theorem:

$$\prod_{r=1}^{3}\int dx_r\nabla_j\left\{R_{jS}M_S\,e^{i\omega\left(t-\frac{n_r x_r}{c}\right)}\right\}=$$

$$=\prod_{r=1}^{3}\int dx_r\frac{\partial}{\partial x_j}\left\{R_{jS}M_S\,e^{i\omega\left(t-\frac{n_r x_r}{c}\right)}\right\}=$$

$$=\prod_{r=1}^{3}\delta_{rj}R_{jS}M_S\,e^{i\omega\left(t-\frac{n_r x_r}{c}\right)}\Big|_{x_r=-\infty}^{+\infty}=0 \quad (28.9)$$

as

$$\nabla_j\left(R_{jS}M_S\,e^{i\omega\left(t-\frac{n_r x_r}{c}\right)}\right)=R_{jS}\left(\nabla_j\overset{\downarrow}{M_S}\right)e^{i\omega\left(t-\frac{n_r x_r}{c}\right)}+R_{jS}M_S\left(\nabla_j e^{i\omega\left(t-\frac{\overset{\downarrow}{n_r x_r}}{c}\right)}\right), \quad (28.10)$$

we get

$$R_{jS}\left(\nabla_j \overset{\downarrow}{M}_S\right)e^{i\omega\left(t-\frac{n_r x_r}{c}\right)} = -R_{jS}M_S\left(\nabla_j e^{i\omega\left(t-\frac{n_r x_r}{c}\right)}\right) = -iR_{jS}M_S\frac{n_j\omega}{c}e^{i\omega\left(t-\frac{n_r x_r}{c}\right)}. \quad (28.11)$$

In the above formulas tensor R_{jS} has form $\varepsilon_\alpha n_p n_m \varepsilon_{\alpha\beta e}\varepsilon_{emn}\varepsilon_{njS}$ and doesn't change in due course at sufficiently great distances of the observation point from the area of the particles having magnetic moment $\vec{M}$. Noting that

$$R_{jS}M_S n_j = \vec{n}\times[\vec{n}\times[\vec{n}\times M]] = -[\vec{n}\times\vec{M}], \quad (28.11)$$

we get

$$R_{jS}M_S n_j = i\frac{\omega}{c}\left[\vec{n}\times\vec{M}(\vec{r},t)e^{i\omega\left(t-\frac{\vec{n}\vec{x}}{c}\right)}\right] \quad (28.12)$$

and formula (28.7) taking into account (28.8)–(28.12) takes the form

$$\frac{dI_M(\omega)}{d\Omega} = \frac{\omega^4}{4\pi^2c^3}\left|\int_{-\infty}^{+\infty}dt\int d^3\vec{r}\left[\vec{n}\times\vec{M}(\vec{r},t)\right]e^{i\omega\left(t-\frac{\vec{n}\vec{r}}{c}\right)}\right|^2. \quad (28.6)$$

Thus, the relations (28.4) and (28.6) are equivalent if

$$\vec{j} \equiv \vec{j}_M = c\, rot\, \vec{M}(\vec{r},t).$$

29. Radiation spectrum of the relativistic particle at an instant motion along the circle

As was established above, the ultra-relativistic particle at arbitrary acceleration radiates in the same way as the charge moving at constant speed υ along the circle with radius ρ equal to the instant radius of curvature:

$$\rho = \frac{\upsilon^2}{\dot{\upsilon}}.$$

To compare let's remember the known formula for the centripetal acceleration of a particle which moves in a circle of constant radius ρ:

$$a = \dot{\upsilon} = \frac{\upsilon^2}{\rho},$$

when vector $\vec{\upsilon}$ changes its direction but doesn't change in magnitude.

The radiation propagates in the narrow cone the axis of which is directed along vector $\vec{\upsilon}$ and is registered by the observer as a short impulse of radiation, which appears while a needle-shape ray passes the observation point. To define the frequency and angular distribution of energy it is necessary to compute the integral in the expression for spectral intensity

$$\frac{dI(\omega)}{d\Omega}=\frac{e^2\omega^2}{4\pi^2 c}\left|\int\limits_{-\infty}^{+\infty}\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)e^{i\omega\left(t-\frac{\vec{n}\cdot\vec{r}}{c}\right)}dt\right|^2 .$$

As the duration of radiation impulse given by the formulae

$$c\Delta t' \sim \rho\sin\theta \sim \rho\gamma^{-1},$$

$$\Delta t' \sim \rho/c\gamma$$

is very small, it is necessary to know speed $\vec{\beta}$ and position $\vec{r}(t)$ of the particle on the small curve of the path where the tangent is approximately directed to the observation point.

Fig. 29.1 shows the chosen reference system.

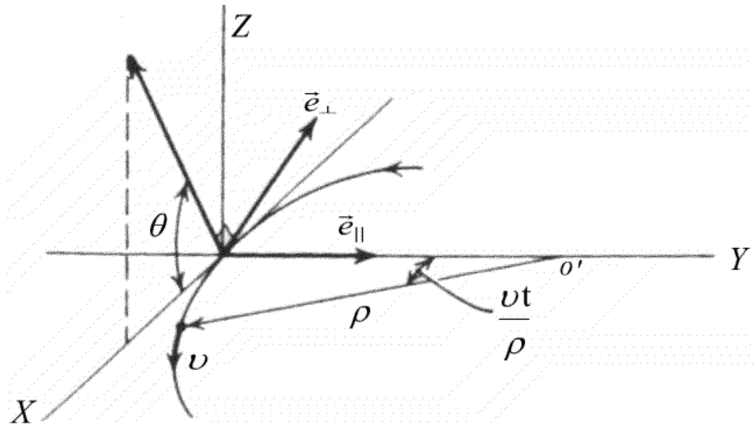


Fig. 29.1. The position of piece of trajectory and of the instant radius of the curve of the accelerating relativistic charged particle velosity of prticle $\vec{\upsilon}$

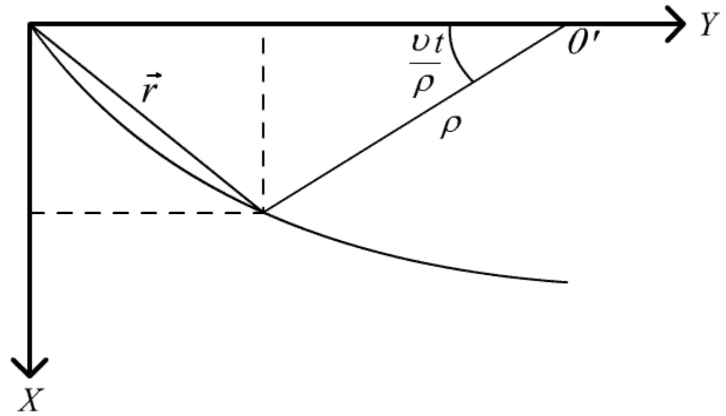


Fig. 29.2. The position of piece of trajectory of the particle in plane XOY

The segment of the path and the instant radius of curvature ρ lie in plane *xy* (Fig. 29.2), vector $\vec{n}$ is in plane *xz*, $\vec{e}_{||}$ is a unit vector in the direction of axis Oy $\left(\vec{e}_{\parallel} = \vec{e}_2\right)$, $\vec{e}_{\perp} = \vec{n} \times \vec{e}_2$ is the vector of normal polarization approximately corresponding to the polarization perpendicular to the orbital plane.

The latter statement is valid only for small angles θ, but the radiation intensity has a noticeable value namely for these angles. Let's remind that vector $\vec{n} = \frac{\vec{R}}{R}$ is directed to the observation point from the point where the charge is. In the well-known book "Particle Accelerator Physics" by Helmut Wiedemann, in the chapter devoted to the synchrotron radiation, vector $\vec{R}$ is directed from the observer to the point where the point is. That is why vector $\vec{n}$ in that book has the opposite sign.

Suppose that at the moment of time $t = 0$ the particle was in the coordinate origin. Computing the double vector product in the integration element we get

$$\vec{n} \times \left(\vec{n} \times \vec{\beta}\right) = \beta\left\{-\vec{e}_{\parallel} \sin\left(\upsilon t/\rho\right) + \vec{e}_{\perp} \cos\left(\upsilon t/\rho\right)\sin\theta\right\}. \tag{29.1}$$

Here the following expressions are used

$$\vec{n} = \vec{e}_1 n_x + \vec{e}_3 n_z, \tag{29.2}$$

$$\vec{\beta} = \vec{e}_1 \beta \cos\left(\upsilon t/\rho\right) + \vec{e}_2 \beta \sin\left(\upsilon t/\rho\right) \tag{29.3}$$

with

$$n_x = \cos\theta, \quad n_z = \sin\theta, \tag{29.4}$$

$$\vec{e}_2 = \vec{e}_{\parallel}, \quad \vec{e}_{\perp} = \vec{n} \times \vec{e}_2. \tag{29.5}$$

Radius vector $\vec{r}(t)$ is drawn from the coordinate origin in plane (x, y) to the point where the particle is (Fig. 29.2), and is equal to

$$\vec{r}(t) = \vec{e}_1 x(t) + \vec{e}_2 y(t). \tag{29.6}$$

The exponent in the integration element turns out to be equal to

$$\omega\left(t - \frac{\vec{n}\vec{r}(t)}{c}\right) = \omega\left[t - \frac{\rho}{c}\sin\left(\upsilon t/\rho\right)\cos\theta\right]. \tag{29.7}$$

As we study the instant motion it is possible to limit oneself to the time interval close to $t = 0$ and to small angles θ. Making the expansion of the trigonometric functions in a power series in small parameters $\upsilon t/\rho$ and θ, we obtain:

$$\sin\left(\upsilon t/\rho\right) \approx \frac{\upsilon t}{\rho} - \frac{\upsilon^3 t^3}{3!\rho^3}, \tag{29.8}$$

$$\cos\theta \approx 1 - \theta^2/2. \tag{29.9}$$

Assuming that

$$\beta \approx 1 - \frac{\gamma^{-2}}{2}, \tag{29.10}$$

we get the following expression

$$\omega\left(t - \frac{\vec{n}\cdot\vec{r}(t)}{c}\right) \approx \frac{\omega}{2}\left[t - \left(\gamma^{-2} + \theta^2\right) + \frac{\upsilon^2 t^3}{3\rho^2}\right]. \tag{29.11}$$

Using the evaluation relations

$$t \sim \rho/c\gamma \text{ and } \theta \sim \left\langle\theta^2\right\rangle^{1/2},$$

one can show that the neglected terms of the expansion with respect to the others $\sim \gamma^{-2}$.

Further we write

$$\vec{n}\times\left(\vec{n}\times\vec{\beta}\right) = \beta\left[-\vec{e}_{\parallel}\left(\upsilon t/\rho\right) + \vec{e}_{\perp}\theta\right] \approx \left[\left(c/\rho\right)\left(-\vec{e}_{\parallel}t\right) + \vec{e}_{\perp}\theta\right]. \tag{29.12}$$

Substituting (29.11) into (29.12) in the integration element and introducing the amplitudes

$$A_{\parallel}(\omega) \approx \frac{\upsilon}{\rho}\int_{-\infty}^{+\infty} t\exp\left\{i\frac{\omega}{2}\left[\left(\gamma^{-2} + \theta^2\right)t + \frac{\upsilon^2 t^3}{3\rho^2}\right]\right\}dt, \tag{29.13}$$

$$A_{\perp}(\omega) \approx \theta\int_{-\infty}^{+\infty} \exp\left\{i\frac{\omega}{2}\left[\left(\gamma^{-2} + \theta^2\right)t + \frac{\upsilon^2 t^3}{3\rho^2}\right]\right\}dt, \tag{29.14}$$

we find the expression for the angle distribution of the spectral intensity presented in the form of expansion by unit vectors of polarization:

$$\frac{dI(\omega)}{d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|-\vec{e}_{\parallel}A_{\parallel}(\varpi) + \vec{e}_{\perp}A_{\perp}(\varpi)\right|^2. \tag{29.15}$$

We made expansion in a series in small time intervals, but the limits of integration spread up to $\pm\infty$. It may seem to contradict to the accepted approximation, but it is necessary to take into account that for most of frequencies the phase is a quickly oscillating function. That is why the integration element differs from zero only within the time interval that is considerably less than the interval necessary to justify the assumptions at the trigonometric functions expansions in a series in time. That is why the upper and lower limits of integration are taken correspondingly $\pm\infty$, without making an essential error. The accepted approximation fails to fulfill only at frequencies $\omega \sim \omega_0 = \frac{c}{\rho}$, but for the relativistic particles practically the whole radiation spectrum corresponds to considerably higher frequencies.

In expressions (29.13)–(29.14) let's change to new variable $x = \left(\upsilon t/\rho\right)\left[\gamma^{-2} + \theta^2\right]^{1/2}$ and introduce the parameter

$$\frac{dI(\omega)}{d\Omega}=\frac{e^2\omega^2}{4\pi^2 c}\left|-\vec{e}_{\parallel}A_{\parallel}(\omega)+\vec{e}_{\perp}A_{\perp}(\omega)\right|^2 . \qquad (29.16)$$

The expressions with these notations are

$$A_{\parallel}(\omega)=\left(\rho/\upsilon\right)\left(\gamma^{-2}+\theta^2\right)\int_{-\infty}^{+\infty}x\exp\left\{i\frac{3}{2}\zeta\left(x+x^3/3\right)\right\}dx , \qquad (29.17)$$

$$A_{\perp}(\omega)=\left(\rho/\upsilon\right)\theta\left(\gamma^{-2}+\theta^2\right)^{1/2}\int_{-\infty}^{+\infty}\exp\left\{i\frac{3}{2}\zeta\left(x+x^3/3\right)\right\}dx . \qquad (29.18)$$

The integral included in (29.17) and (29.18), are expressed through the modified Bessel functions $K_{1/3}$ and $K_{2/3}$:

$$\int_0^{\infty}x\sin\left[\frac{3}{2}\zeta\left(x+x^3/3\right)\right]dx=\frac{1}{\sqrt{3}}K_{2/3}(\zeta), \qquad (29.19)$$

$$\int\cos\left[\frac{3}{2}\zeta\left(x+x^3/3\right)\right]dx=\frac{1}{\sqrt{3}}K_{1/3}(\zeta). \qquad (29.20)$$

The behavior of these functions depends on the parameter

$$\zeta=\frac{1}{3}\frac{\omega}{\omega_L}\frac{1}{\gamma^3}\left(1+\gamma^2\theta^2\right)^{3/2}\approx\frac{1}{2}\frac{\omega}{\omega_C}\left(1+\gamma^2\theta^2\right)^{3/2}. \qquad (29.21)$$

In formula (29.11) parameter ζ is expressed through Larmor frequency $\omega_L=\upsilon/\rho$, which is related to the characteristic parameter – the critical photon frequency

$$\omega_C=\frac{3}{2}\frac{c\gamma^3}{\rho}. \qquad (29.22)$$

Intensity of radiation in any direction at motion in area of the big frequencies sharply falls.

Functions $K_{1/3}$ and $K_{2/3}$ are finite at small values of the argument and exponentially decrease at large values of the argument.

29.1. Spectral distribution of synchrotron radiation

On the basis of formulas (29.15)–(29.22) we get the following expression for the energy of the synchrotron radiation propagating per unit of solid angle and falling to a unit interval of frequencies:

$$\frac{dI(\omega)}{d\Omega}=\frac{3r_C\cdot mc^2\gamma^2}{4\pi^2 c}\frac{\omega^2}{\omega_C^2}\left(1+\gamma^2\theta^2\right)^2\left[K_{2/3}(\zeta)+\frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}K_{1/3}^2(\zeta)\right]=$$

$$=\frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2\left(\gamma^{-2}+\theta^2\right)^2\left[K_{2/3}^2(\zeta)+\frac{\theta^2}{\gamma^{-2}+\theta^2}K_{1/3}^2(\zeta)\right], \qquad (29.23)$$

where in the first part of the equation the classical particle radius is used

$$r_C = \frac{e^2}{mc^2}, \tag{29.24}$$

with the rest energy mc^2 and the cutoff frequency of the radiation spectrum is ω_C (29.22).

From the formula (29.15), the radial spectrum is conditioned by contributions of two components of orthogonal polarization, one of which lies in the plane of the particle path, and the other is almost parallel to the deflecting magnetic field.

If we denote the unit vector of polarization $\vec{e}_{\parallel} = \vec{e}_{\sigma}$ and $\vec{e}_{\perp} = \vec{e}_{\pi}$, as it is accepted in the literature, the first addend in (29.23), proportional to $K^2_{2/3}(\zeta)$, is conditioned by σ-mode, and the second by π-polarization mode. σ-mode corresponds to the radiation polarized in the orbit plane, π-mode corresponds to the radiation polarized perpendicular to this plane. For these two modes not only the contributions into the spectral intensity are different but also the distributions in space. Fig. 29.3 shows the diagrams of direction for σ- and π-modes of polarization.

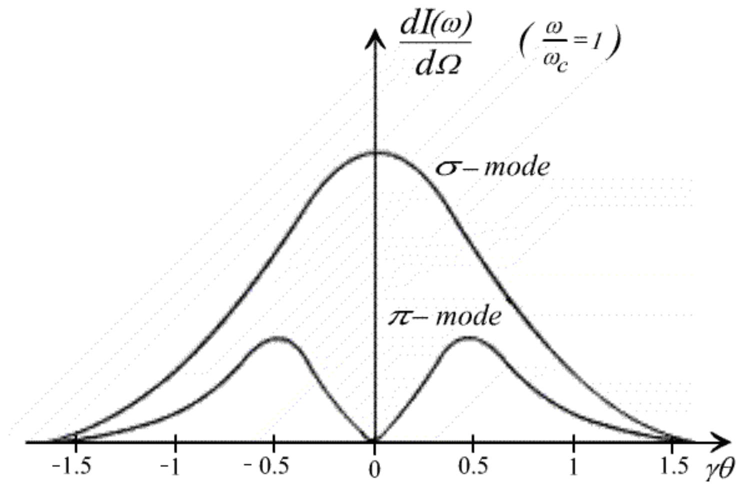


Fig. 29.3. σ- and π-modes of polarization

29.2. Angular distribution of synchrotron radiation

Let's return to the expression (29.23):

$$\frac{dI(\omega)}{d\Omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega\rho}{c}\right)^2 \left(\gamma^{-2} + \theta^2\right)^2 \left[K^2_{2/3}(\zeta) + \frac{\theta^2}{\gamma^{-2} + \theta^2} K^2_{1/3}(\zeta)\right].$$

As was noted, the first addend in the square brackets corresponds to the radiation polarized in the orbit plane, and the second component is perpendicular to this plane.

Let's calculate the angular distribution by the energy:

$$\frac{dW}{d\Omega} = \int_0^{\infty} \frac{dI(\omega)}{d\Omega} d\omega = \frac{7}{16} \frac{e^2}{\rho} \frac{\left[1 + \frac{5}{7} \frac{\theta^2}{\gamma^{-2} + \theta^2}\right]}{\left[\gamma^{-2} + \theta^2\right]^{5/2}}. \quad (29.25)$$

If we take the integral by all the angles, then one can note that the energy of radiation with polarization parallel to the orbit plane is 7 times greater than the energy of the radiation with the perpendicular polarization.

Thus, the radiation of the relativistic moving charge is mostly though not fully polarized in the plane of motion. In the case of the non-relativistic motion, the radiation is fully polarized in the plane of motion.

Let's give the angular distribution of the radiation intensity at different frequencies. At frequencies of order ω_C the radiation is concentrated in angle area $\sim \gamma^{-1}$. For $\omega << \omega_C$ the sizes of the angle area are big and for high frequencies they are less.

If we integrate expression (29.23) by the angles $I(\omega) = \int \frac{dI(\omega)}{d\Omega} d\Omega$, then we get the radiation power

$$I(\omega) \cong 2\sqrt{3} \frac{e^2}{c} \gamma \frac{\omega}{\omega_C} \int_{2\omega/\omega_c}^{\infty} K_{5/3}(x) dx. \quad (29.26)$$

The radiation described by the relations (29.23) and (29.26), is called synchrotron radiation (it was first observed in the electronic synchrotron in 1948). The Fig. 29.4 shows the dependence of the synchrotron radiation intensity (in units $e^2\gamma/c$ on frequency ω (in units ω_C)).

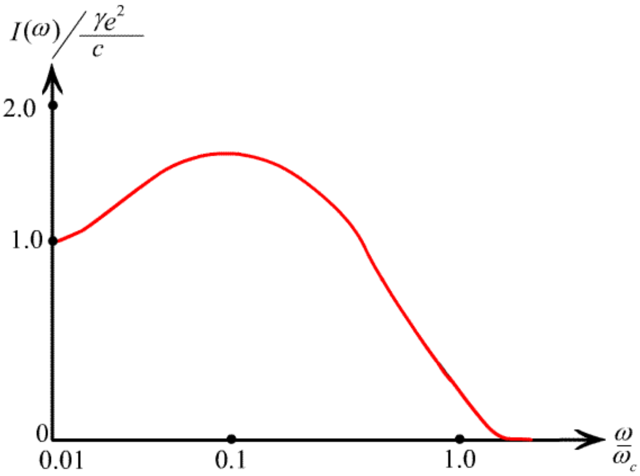


Fig. 29.4. The dependence of the synchrotron radiation intensity (in units $e^2\gamma/c$) on frequency ω (in units ω_C)

At the periodic motion in a circle the radiation spectrum is in fact discrete and consists of the set of frequencies multiple of the basic frequency $\omega_0 = \frac{c}{\rho}$. The charged particle periodically repeats its motion with the frequency $\nu = \frac{c}{2\pi\rho}$ turns per second. That is why it is more simpler to talk about the angular distribution of the radiation power on the n-harmonic but not about the radiation energy in a unit interval of frequencies at the particle pass.

The corresponding formulas have the form:

$$\frac{dP_n}{d\Omega} = \frac{1}{2\pi}\left(\frac{c}{\rho}\right)^2 \frac{dI(\omega)}{d\Omega}\Big|_{\omega = n\omega_0}, \qquad (29.27)$$

$$P_n = \frac{1}{2\pi}\left(\frac{c}{\rho}\right)^2 I(\omega = n\omega_0). \qquad (29.28)$$

The theoretical relations were compared with the experimental data. For that the radiation spectrums were averaged by the period of acceleration cycle, as the energy of the electrons constantly increases. At maximal energy 80 MeV the radiation spectrum is in the range from the basic frequency $\omega_0 \cong 10^9$ Hertz to $\omega_C \approx 10^{16}$ Hertz ($\lambda = 1700\,\mathring{A}$). The radiation has a blue-white color in the visible area. The results of measurements are in agreement with the theory.

The synchrotron radiation was observed while investigating the sunspots and the Crab nebula whose radiation spectrum is in the space beginning in the area of radio frequencies up to the far ultra-violet area, and the radiation is strongly polarized. Such radiation may be given by the electrons with $E = 10^{12}$ eV at the motion in a circular or spiral orbit in the magnetic field $\sim 10^{-4}$ gauss.

30. The theory of synchrotron radiation (SR)

30.1. Spectral-angular distribution of power. Shott's formula

Let's introduce the function characterizing the distribution of the spectral intensity at frequency ω and within the space angle $\frac{dI(\omega,\theta,\phi)}{d\Omega} \equiv W(\omega,\theta,\varphi)$, where θ and φ are spherical angles of a unit vector, which are directed in the direction of electromagnetic wave propagation. The total radiation power, that is the energy radiated by an electron per unit of time, equals

$$W = \int_0^\infty d\omega \oint d\Omega\, W(\theta,\varphi,\omega). \qquad (30.1)$$

Due to its quasidiscreteness it is convenient to characterize the synchrotron radiation by the number of harmonic of radiation frequency.

$$\omega = \nu\omega_0 = \nu ecB/\varepsilon\,, \tag{30.2}$$

where $\varepsilon = \dfrac{mc^2}{\sqrt{1-\beta^2}}$ is the total energy of a relativistic electron, ω_0 is the cyclotron frequency equal to the frequency of electron motion in magnetic field. Then the expression for total radiation power can be presented as a sum by the number of harmonics ν:

$$W = \sum_{\nu} \oint d\Omega W_\nu(\theta,\varphi). \tag{30.3}$$

Let an electron move in a constant magnetic field directed along axis z (Fig. 30.1), and the motion occurs in plane (*x, y*).

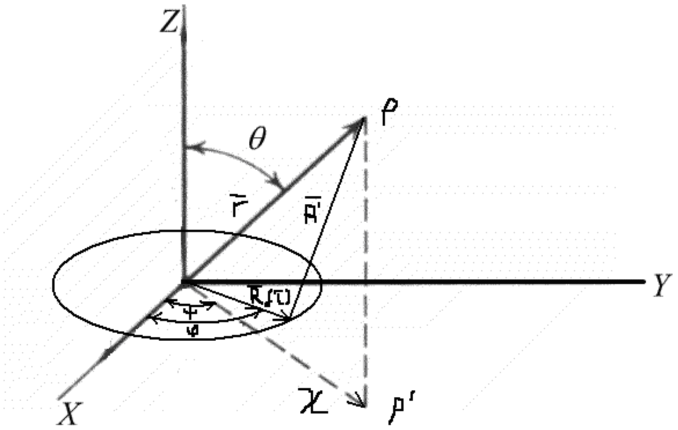


Fig. 30.1. To calculation of the radiation of an electron moving along the circle

The path of the electron is a circle:

$$x = R_0 \cos\psi, \quad y = R_0 \sin\psi, \quad \psi = \omega\tau\,, \tag{30.4}$$

where $R_0(\tau)$ defines the position of an electron in the orbit of radius R_0, $\omega = \upsilon/R$ is the angular speed of the electron, τ is time.

If we neglect the force of radiation friction, considering that the radiation slightly influences on the electron motion, then the power of radiation can be written through –the Umov-Pointing's vector that includes the electromagnetic fields of radiation induced by the charge:

$$\vec{E}_{rad} \cong \vec{E}, \quad \vec{B}_{rad} \cong \vec{B}:$$

$$W = \frac{c}{4\pi}\oint \left[\vec{E}\,\vec{B}\right] dS\,. \tag{30.5}$$

Let's define the radiation fields at moment t at point p of the wave zone with coordinates r,θ,φ. Usually the wave zone is considered to be an area the size of which is much bigger of the effective sizes of the frame (for example, the radius of orbit of the electron gyrating in the magnetic field). As we saw before, physically it means the isolation of the transverse part of the electromagnetic field from the general expressions for the fields of the moving charge. Actually SR is also observed near the plane of the electron orbit, which has macroscopic sizes. The analysis of the notion "wave zone" in this case shows that the wave zone starts with the distances that exceed the effective length of radiation

$$r >> \lambda^{eff} = R_0\left(1-\beta^2\right)^{3/2} = R_0\gamma^{-3}, \tag{30.6}$$

that is why SR can be observed (when $\gamma << 1$) at the distances less than the radius of the orbit.

The electric and magnetic fields are defined through vector potential $\vec{A}$:

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = rot\,\vec{A}, \quad \left|\vec{E}\right| = \left|\vec{B}\right|. \tag{30.7}$$

The boundary condition in the point of observation is specified as a common condition of Sommerfild's radiation, which describes the waves propagating from the source, that is, the delayed potential is taken as a solution of the wave equation:

$$\vec{A} = \frac{e}{c}\cdot\int\frac{\vec{\upsilon}(\tau)}{R'}\delta\left(\tau - t + \frac{R'}{c}\right)d\tau, \tag{30.8}$$

$$\vec{\upsilon} = \frac{d\vec{R}_0(r)}{d\tau}, \ \vec{R}' = \vec{r} - \vec{R}_0(\tau).$$

In the wave zone

$$\left|\vec{R}'\right| = r\cdot\sqrt{1 - \frac{2\vec{r}\vec{R}_0}{r^2} + \frac{R_0^2}{r^2}} \cong r - \frac{\vec{r}\vec{R}_0}{r} = r - R_0\cdot\sin\theta\cos\chi, \tag{30.9}$$

where $\chi = \omega r / c - \varphi$.

For vector potential the approximate expression is got:

$$\vec{A} = \frac{e}{cr}\int\vec{\upsilon}(r)\delta\left(\tau - t + \frac{r}{c} - \frac{R_0\sin\theta\cos\chi}{R_0}\right)d\tau. \tag{30.10}$$

Let's expand δ – function in Fourier series

$$\delta(\tau) = \frac{\omega}{2\pi}\sum_{\nu=-\infty}^{+\infty}e^{i\nu\omega\tau}. \tag{30.11}$$

Taking into account that the integration should be limited by the period of electron motion, we get

$$\vec{A} = \sum_{\nu=-\infty}^{+\infty}\vec{A}(\nu)e^{-i\nu\phi}. \tag{30.12}$$

This expression takes the following form with Fourier-component $\vec{A}(\nu)$:

$$\vec{A}(\nu)=\frac{e}{cr}\int_{-\pi}^{\pi}\vec{\upsilon}e^{i(\nu\alpha-\nu\beta\sin\theta\sin\alpha)}d\alpha\,. \tag{30.13}$$

In expressions (30.12) and (30.13) the following expressions are used

$$\varphi=\omega t-\frac{\omega r}{c}-\varphi+\frac{\pi}{2}\,,\ \alpha=\chi+\frac{\pi}{2}\,. \tag{30.14}$$

Writing the speed projections in the spherical coordinate system, we get

$$A_\phi(\nu)=\frac{e}{r}i\beta J'_\nu(\nu\beta\sin\theta),$$
$$A_\theta(\nu)=-ctg\,J_\nu(\nu\beta\sin\theta), \tag{30.15}$$

where $J_\nu(x), J'_\nu(x)$ is Bessel's function and its *x* derivative, with

$$J_\nu(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(\nu\alpha-x\sin\alpha)}d\alpha\,. \tag{30.16}$$

As a result the field components in the spherical coordinate system are defined:

$$E_\theta=H_\phi=\frac{1}{r}\frac{\partial}{\partial r}(r\,A_\theta)=-\frac{2\beta e}{rR_0}ctg\theta\sum_{\nu=1}^{\infty}\nu J_\nu(\nu\beta\sin\theta)\sin\nu\varphi,$$
$$-E_\phi=H_\theta=\frac{1}{r}\frac{\partial}{\partial r}(r\,A_\phi)=\frac{2\beta^2 e}{rR_0}\sum_{\nu=1}^{\infty}\nu J'_\nu(\nu\beta\sin\theta)\cos\nu\varphi\,. \tag{30.17}$$

Let's calculate the angular distribution of the radiation power by Pointing's vector integral, which follows from the definition of power

$$\frac{dW}{d\Omega}=r^2\frac{c}{4\pi}\left(E_\varphi^2+E_\theta^2\right). \tag{30.18}$$

Let's average this expression over the period of revolution of the electron using the relations:

$$\frac{1}{T}\int_0^T\cos\nu\varphi\,\cos\nu'\varphi\,dt=\frac{1}{T}\int_0^T\sin\nu\varphi\,\sin\nu'\varphi\,dt=\frac{1}{2}\delta_{\nu,\nu''}\,,$$
$$\frac{1}{T}\int_0^T\sin\nu\varphi\,\cos\nu'\varphi\,dt=0. \tag{30.19}$$

As a result we get Shott's formula:

$$W(\nu,\theta)=\frac{e^2c\beta^2}{2\pi R_0^2}\nu^2\left\{ctg^2\theta\,J_\nu^2(\nu\beta\sin\theta)+\beta^2J_\nu'^2(\nu\beta\sin\theta)\right\}, \tag{30.20}$$

which presents the spectral and angular distribution of power of SI system. The total power is found in the form

$$W = \sum_{\nu=1}^{\infty} \oint d\Omega\, W(\nu,\theta). \tag{30.21}$$

30.2. Polarization characteristics of SR

Shott's formula was obtained as an exact solution of the task about the radiation of a charge moving in a circle by methods of classical electrodynamics. The classical electrodynamics considers the radiation to be a continuous emission of electromagnetic waves by an accelerating charge. in quantum theory of SR one should replace Shott's formula by another one which takes into account quantum amendments in super-strong magnetic fields and for big values of the electron energy. The polarization characteristics of SR were taken into account later while creating the quantum theory of SR. To describe two independent states of the linear polarization of SR the vector potential $\vec{A}$ is expanded in unit vectors of polarization $\vec{e}_\sigma$ and $\vec{e}_\pi$, which are orthogonal to each other and in unit wave vector $\hat{\vec{k}}$, coinciding in the direction with Umov-Pointing's vector.

The total power of radiation can be presented in the form

$$W = W_\sigma + W_\pi . \tag{30.22}$$

To describe the circular polarization of SR the vector potential is written in the following way:

$$\vec{A} = A_1 \vec{e}_{(1)} + A_{-1} \vec{e}_{(-1)}, \tag{30.23}$$

where complex vectors $\vec{e}_{(l)} (l = \pm 1)$ are related to unit vectors $\vec{e}_\sigma$ and $\vec{e}_\pi$ by the formulas

$$\vec{e}_{(l)} = \frac{1}{\sqrt{2}} \left(\vec{e}_\sigma + il\vec{e}_\pi \right). \tag{30.24}$$

Value $l = 1$ corresponds to the left polarization and $l = -1$ corresponds to the right one. The total radiation power equals

$$W = W_1 + W_{-1} . \tag{30.25}$$

The generalization of Shott's formula, taking into account the polarization characteristics of SR, is given by the formula:

$$W_i(\nu,\theta) = \frac{e^2 c \beta^2}{2\pi R^2} \nu^2 \left\{ l_\sigma \beta J'_\nu (\nu\beta \sin\theta) + l_\pi ctg\theta\, J_\nu (\nu\beta \sin\theta) \right\}^2 . \tag{30.26}$$

The power of σ-component of linear polarization with $l_\sigma = 1$, $l_\pi = 0$ is

$$W_\sigma(\nu,\theta)=\frac{e^2 c\beta^2}{2\pi R^2}\nu^2\beta^2 J'^2(\nu\beta\sin\theta). \tag{30.27}$$

And the power of π-component with $l_\sigma=0$, $l_\pi=1$ is

$$W_\pi(\nu,\theta)=\frac{e^2 c\beta^2}{2\pi R^2}\nu^2 ctg^2\theta\, J_\nu^2(\nu\beta\sin\theta). \tag{30.28}$$

To find the power of the circularly polarized radiation it is necessary to take $l_\sigma=l_\pi=\frac{1}{\sqrt{2}}$ for the right circular polarization and $l_\sigma=-l_\pi=\frac{1}{\sqrt{2}}$ for the left one.

At a fixed number of harmonic, the proportion of the amplitudes characterizing linear polarization equals to

$$W_\sigma / W_\pi=\frac{ctg^2\theta\, J_\nu^2(\nu\beta\sin\theta)}{\beta^2 J_\nu'^2(\nu\beta\sin\theta)}. \tag{30.29}$$

At $\theta=\pi/2$ the radiation is completely linearly polarized and vector of electric field $\vec{E}$ oscillates in the orbital plane and W_π-component equals 0.

The scheme of the position of vectors $\vec{E}_\sigma$ and $\vec{E}_\pi$, which are components of linear polarization, is presented in Fig. 30.2.

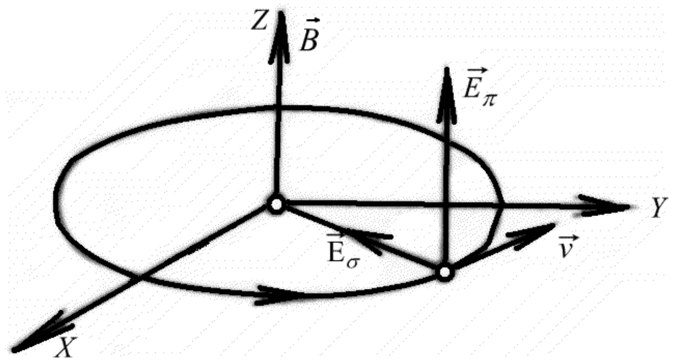


Fig. 30.2. The scheme of vector E_σ and E_π disposition, which are components of linear polarization

Above the orbital plane (as $0<\theta<\pi/2$) the synchrotron radiation has the left elliptic polarization, and under the orbital plane (as $\pi/2<\theta<\pi$) it has the right one. The total power of radiation can be found if the summation in (30.27) and (30.28) is made by the number of harmonics and the integration is by the space angle:

$$W_{\sigma,\pi}=\sum_{\nu=1}^{\infty}\oint d\Omega\, W_{\sigma,\pi}(\nu,\theta). \tag{30.30}$$

As a result we get

$$W_\sigma = \frac{6+\beta^2}{8} W,$$
$$W_\pi = \frac{2-\beta^2}{8} W, \tag{30.31}$$

with

$$W = W_\sigma + W_\pi = \frac{2}{3} \frac{e^2 c \beta^4}{R^2 \left(1-\beta^2\right)^2} = \frac{2}{3} \frac{e^2 c \beta^4}{R^2} \gamma^4 . \tag{30.32}$$

In the ultra-relativistic case of electron motion SR has a strongly expressed linear polarization. If $\beta \to 1$

$$W_\sigma = \frac{7}{8} W , \quad W_\pi = \frac{1}{8} W . \tag{30.33}$$

In the non-relativistic case as $\beta \to 0$ SR is also polarized:

$$W_\sigma^{nonrel} = \frac{3}{4} W^{nonrel} , \quad W_\pi^{nonrel} = \frac{1}{4} W^{\text{nonrel}} , \tag{30.34}$$

where

$$W^{nonrel} = \frac{2}{3} \frac{e^2 c \beta^4}{R^2} = \frac{2}{3} \frac{e^2}{c^3} \dot{\upsilon}^2 , \tag{30.35}$$

which is the well-known Larmor's formula for the dipole radiation.

30.3. The properties of angular distribution of SR

Let's sum in formulas (30.26) over the number of harmonic ν. It can be done using the relations

$$\sum_{\nu=1}^{\infty} \nu^2 J_\nu^2 \left(\nu x\right) = \frac{x^2 \left(4+x^2\right)}{16 \left(1-x^2\right)^{7/2}},$$
$$\sum_{\nu=1}^{\infty} \nu^2 J_\nu'^2 \left(\nu x\right) = \frac{4+3x^2}{16 \left(1-x^2\right)^{5/2}}. \tag{30.36}$$

As a result we get

$$W_i(\theta) = \frac{e^2 c \beta^4}{32\pi R^2} F_i(\theta), \tag{30.37}$$

where

$$F_\sigma(\theta) = \frac{4 + 3\beta^2 \sin^2\theta}{\left(1-\beta^2 \sin\theta\right)^{5/2}}, \quad F_\pi(\theta) = \frac{\cos^2\theta \left(4 + \beta^2 \sin^2\theta\right)}{\left(1-\beta^2 \sin^2\theta\right)^{7/2}}, \tag{30.38}$$

and the total power equals

$$W = \sum_i \oint d\Omega W_i(\theta) = \oint d\Omega \frac{e^2 c \beta^4}{32\pi R^2} F_0(\theta), \tag{30.39}$$

where

$$F_0 = F_\sigma + F_\pi = 4\frac{1+\cos^2\theta - \left(1+3\beta^2\right)\sin^4\theta\ \beta^4/4}{\left(1-\beta^2\sin^2\theta\right)^{7/2}}. \qquad (30.40)$$

The angular distribution for the total power at the non-relativistic limit equals

$$W_0 = \frac{e^2 c\beta^4}{8\pi R^2}\left(1+\cos^2\theta\right). \qquad (30.41)$$

It is the characteristic distribution of the radiation power for non-relativistic particles. If $\beta \to 1$, $1-\beta^2 << 1$ the maximal value of the power is in the plane of the electron orbit:

$$\frac{W_0(\theta=\pi/2)}{W_0(\theta=0)} = \frac{4+3\beta^2}{8\left(1-\beta^2\right)^{5/2}}. \qquad (30.42)$$

Fig. (30.3) shows the scheme of angular distribution of SR in the non-relativistic and relativistic cases. In this case angle opening γ, inside of which the radiation takes place, is small:

$$\delta\psi \sim \sqrt{1-\beta^2} = \left(mc^2/\varepsilon\right) = \gamma^{-1}. \qquad (30.43)$$

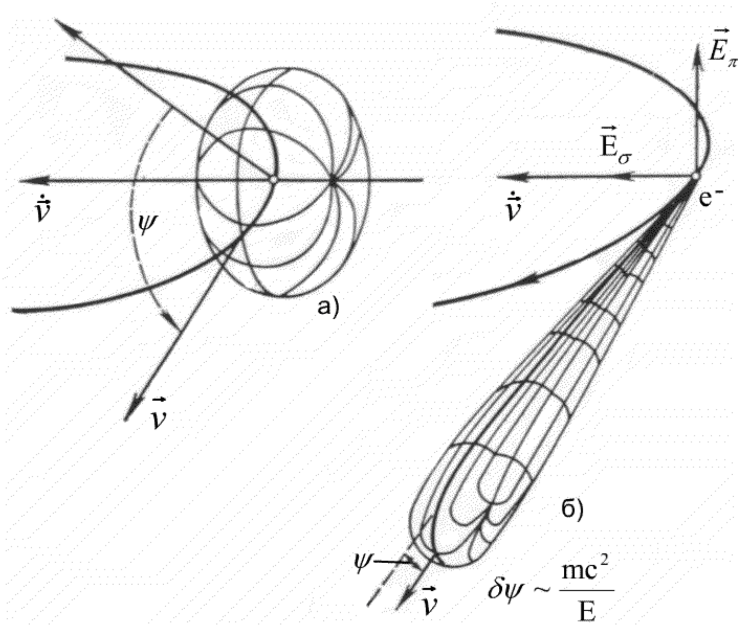


Fig. 30.3. The scheme of angular distribution of SR power: a) nonrelativistic motion; b) relativistic motion

This can be checked by getting in the denominator (17):

$$\theta = \pi/2 + \delta\psi. \qquad (30.44)$$

$$1-\beta^2\sin^2\theta = 1-\beta^2\cos^2\delta\psi \cong 1-\beta^2+(\delta\psi)^2. \qquad (30.45)$$

It means that

$$\delta\psi \sim \sqrt{1-\beta^2}\,. \qquad (30.46)$$

The form of angular distribution can be understood on the basis of the relativistic formula of angle transformation

$$\sin\psi = \frac{\sqrt{1-\beta^2}\sin\psi'}{1+\beta\cos\psi}, \qquad (30.47)$$

where ψ' is the angle between the direction of radiation and the vector of the particle speed in K' frame, where the particle is at rest, and ψ is the angle at which the radiation in the laboratory reference frame K is observed. For the maximum of dipole radiation $\psi' = \pi/2$. As a result we get

$$\sin\psi \sim \delta\psi = \sqrt{1-\beta^2} = \left(mc^2/E\right) = \gamma^{-1}. \qquad (30.48)$$

The angle of the cone in which the radiation is concentrated, is very small. For example, for a synchrotron with the energy 300 MeV the speed υ equals 0,9999987*c*, where *c* is the light velocity and $\delta\psi = 0{,}1^\circ$.

If we present the polarization components $W_{\sigma,\pi}$ as

$$W_i = \frac{e^2c\beta^4}{32\pi R^2\left(1-\beta^2\right)^{5/2}}\oint f_i(\zeta_1)\,d\Omega, \qquad (30.49)$$

where

$$\zeta_1 = \cos\theta\Big/\sqrt{1-\beta^2} \qquad (30.50)$$

and

$$f_i(\zeta_1) = 7l_\sigma^2\Big/\left(1+\zeta_1^2\right)^{5/2} + 5\zeta_1^2\Big/\left(1+\zeta_1^2\right)^{7/2} + 64\zeta_1 l_\sigma l_\pi\Big/\pi\sqrt{3}\left(1-\zeta_1^2\right)^3, \qquad (30.51)$$

the dependence of the linear and circular polarization of SR on the angle (on parameter ζ_1) has the form (Fig. 30.4).

Here the index i takes the values: $i=\sigma$ $(l_\sigma = 1,\ l_\pi = 0)$, $i=\pi$ $(l_\sigma = 0,\ l_\pi = 1)$ – the components of the linear polarization; $i=\pm 1$ $\left(l_\sigma = l_\pi = 1/\sqrt{2}\ \ l_\pi = -l_\sigma = 1/\sqrt{2}\right)$ – the components of the right and left circular polarization. The total power of radiation is obtained by summing

$$f_0 = f_\sigma + f_\pi = f_1 + f_{-1}.$$

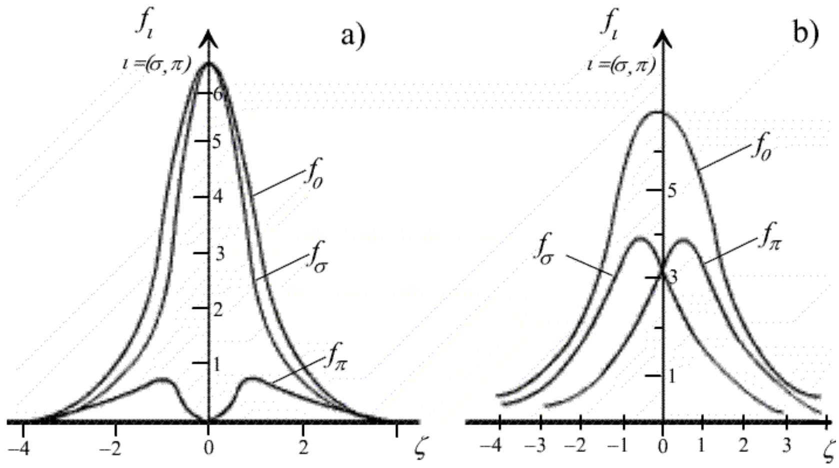


Fig. 30.4. The dependence of linear
(a) and circular; (b) polarization on the radiation angle

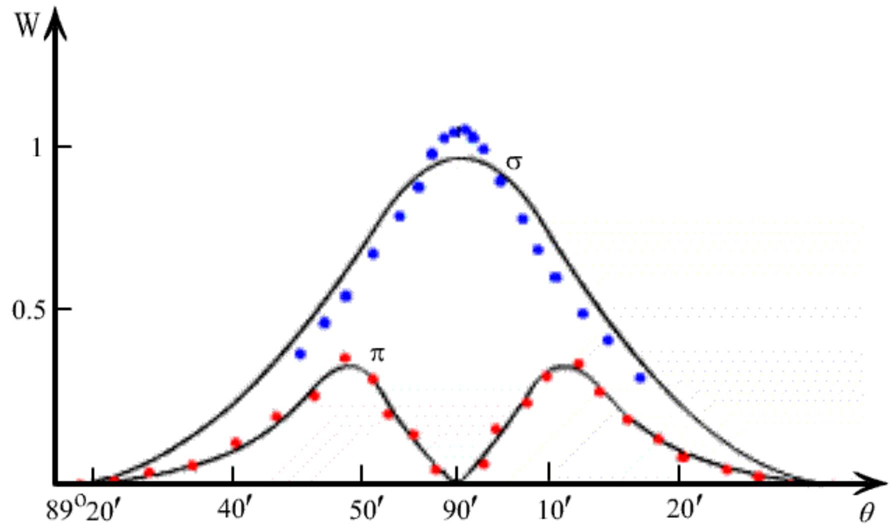


Fig. 30.5. The comparison of experimental and theoretical data for angular distribution of SR power W_i *(*$i = \sigma, \pi$*;* $\lambda = 408 nm$*,* $E = 250 MeV$ *)*

Fig. 30.6 and Fig. 30.7 show the instant distributions of the total power of SR (profile) and W_π.

According to Fig. 30.4, σ-component of the linear polarization has the maximum in the plane of the electron orbit (at $\zeta_1 = 0$), and π-component in this case turns to 0. π-component has the maximum at $\zeta \approx 2/5$. The components of the circular polarization have the maximum at $\zeta_1 = \pm 0{,}34$. Thus, in the plane of rotation orbit SR is fully polarized. The experiment made on the synchrotron FIAN (see Fig. 30.5), justifies good agreement with the theory of SR, though at $\theta = \pi/2$, the radiation power of π-component of SR doesn't vanish. It is conditioned by the fact that the vector of instant speed of the electron deviates from the orbital plane owing to the betatron oscillation.

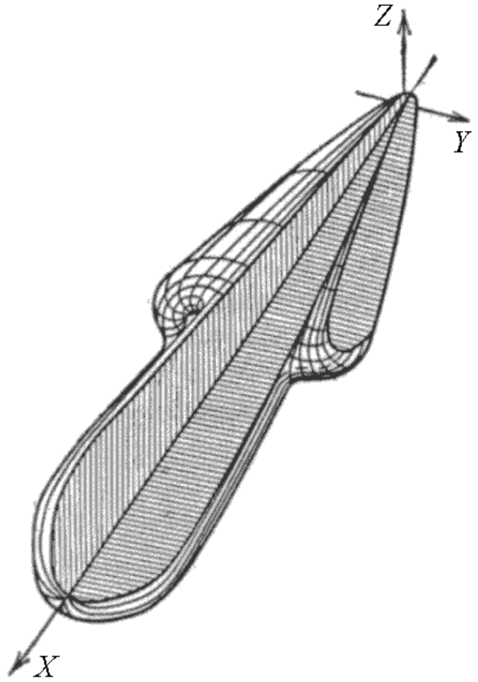


Fig. 30.6. Instant distribution of SR power (in profile)

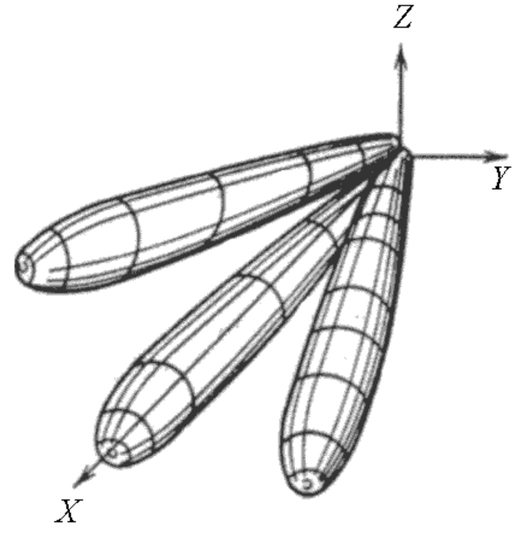


Fig. 30.7. Instant distribution of SR power of π-component of polarization (the fourth peak is not shown)

30.4. Spectral distribution of SR

Let's integrate Shott's formula $W(v,\theta)$ over angle θ. As a result we get

$$W(v)=\frac{e^2 c\beta v}{R^2}\left\{2\beta^2 J'_{2v}(2v\beta)-\left(1-\beta^2\right)\int_0^{2v\beta} J_{2v}(x)dx\right\}. \qquad (30.52)$$

This formula shows the dependence on the number of harmonic, that is, it corresponds to the observation of SR at the frequency $\omega = v\omega_0$.

In the non-relativistic case $\beta \to 0$ taking into account asymptotic expressions for Bessel's function

$$J_{2\nu}(2\nu\beta) \cong {(\nu\beta)^{2\nu}}\big/{(2\nu)!}, \quad J'_{2\nu}(2\nu\beta) \cong {(\nu\beta)^{2\nu-1}}\big/{(2\nu-1)!} \tag{30.53}$$

we get

$$W^{nonrel}(\nu) = 2\frac{e^2 c}{R^2}\beta^{2\nu+2}\frac{\nu^{2\nu+2}}{(2\nu+1)}\nu+1. \tag{30.54}$$

It means that the maximal radiation falls at the basic harmonic $\nu = 1$ (dipole radiation).

In relativistic case $\beta \to 1$ from the properties of Bessel's function $J_n(z)$ if $n >> 1$ and $n \sim z$ has the maximum at point $n = z + 0{,}8 z^{1/3}$, that is, the function $J_{2\nu}(2\nu\beta)$ is maximal in point

$$2\nu = 2\nu\beta + 0{,}8(2\nu\beta)^{1/3},$$

which corresponds to

$$2\nu(1-\beta) \approx (1-\beta^2) = 0{,}82^{1/3}\nu^{1/3}$$

or

$$\nu^{-2/3} \sim 1-\beta^2,$$

that is why

$$\nu_{\max} \approx (1-\beta^2)^{-3/2} = \left({\varepsilon}\big/{mc^2}\right) = \gamma^3. \tag{30.55}$$

This specific peculiarity of SR was first marked by L.A. Arzimovich and I.J. Pomeranchuk.

30.5. Coherence of SR

30.5.1. The coherence of SR at even distribution of electrons in a circle

The classical theory of SR, some results of which were described above, refers to the radiation of one electron moving in a circular orbit in magnetic field.

In practice there are $10^{12} - 10^{13}$ electrons in the accelerators and accumulating rings at the same time (see Fig. 30.8). They fill either the whole orbit as in the betatron or formed into separate bunches as in the synchrotron. In this case the power of radiation depends on the interference of the waves emitted by separate electrons and there arises a coherent synchrotron radiation, when the radiation power of *N* electrons doesn't equal the sum of radiation powers of every electron separately.

Let N electrons radiate, which are distributed in an orbit in an arbitrary way and move at the same speed. According to the formulas for the electric and magnetic vectors of radiation field of the charge in the wave zone, when there is a rotation of N electrons in a circle, we get

$$E_{\varphi} = H_{\theta} = \frac{2e\beta^2}{rR}\sum_{\nu=1}^{\infty}\sum_{j=1}^{N}\nu J'_{\nu}\left(\nu\beta\sin\theta\right)\cos\nu\left(\varphi+\psi_j\right), \tag{30.56}$$

where ψ_j is the initial phase of the j th electron.

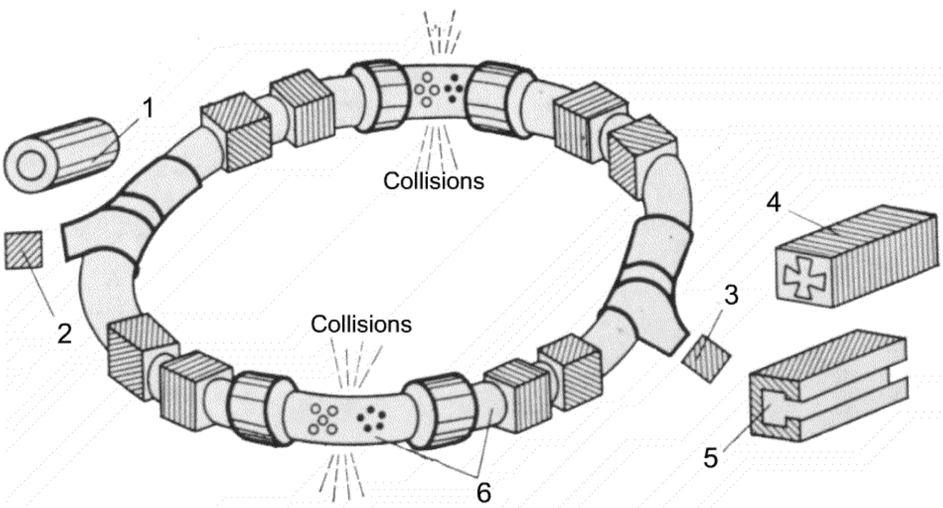


Fig. 30.8. Basic elements of accumulating ring:
1) HF-element; 2) injector e^+; 3) injector e^-; 4) focusing magnet;
5) defocusing magnet; 6) vacuum chamber

Then the radiation power of N electrons for the harmonic ν differs from $W(\nu)$ by factor of coherence S_N:

$$W_N(\nu) = W(\nu)S_N, \tag{30.57}$$

where

$$S_N = N + \sum_{j=1}^{N}\sum_{\substack{j'=1 \\ (j=j')}}^{N}\cos\nu\left(\psi_j - \psi_{j'}\right). \tag{30.58}$$

In case of a chaotic distribution of the electrons on the orbit the value of the sum equals zero and the radiation of the electron isn't coherent, that is, it doesn't depend on the radiation of the other particles. In this case

$$S_N = N, \tag{30.59}$$

and the radiation energy of N electrons equals the sum of the energy radiated by the separate particles:

$$W_N(\nu) = NW(\nu). \tag{30.60}$$

Suppose the particles are distributed uniformly on the circle, with the angle between the neighboring electrons being equal to $2\pi/N$. Then we get

$$S_\nu = N\left\{1+\sum_{j=1}^{N}\cos 2\pi j\frac{\nu}{N}\right\} = N(-1)^\nu \frac{\sin\nu\pi}{tg\, \nu\pi/N}. \qquad (30.61)$$

If $\nu/N = S$, the coherence factor doesn't equal to zero only when S is a whole number, that is, if the number of harmonics is multiple of the number of the electrons. For S_N we get

$$S_N = \lim_{\varepsilon\to 0}(-1)^\nu N\frac{\sin\nu\pi(1+\varepsilon)\cos S\pi(1+\varepsilon)}{\sin S\pi(1+\varepsilon)} = N^2. \qquad (30.62)$$

The radiation power in this case equals

$$W_N(\nu) = N^2 W(\nu). \qquad (30.63)$$

The total power is equal to

$$W = \sum_{S=1}^{\infty} W_S, \qquad (30.64)$$

where

$$W_S = \frac{e^2 c\beta}{R^2}N^3 S\left\{2\beta^2 J'_{2SN}(2SN\beta) - (1-\beta^2)\int_0^{2SN\beta} J_{2SN}(x)dx\right\}. \qquad (30.65)$$

The analysis of this formula, based on the approximation of Bessel's function for small values of the argument in the no relativistic case ($\beta\to 0$), leads to the formula:

$$W_{S=1} = \frac{2e^2 c\beta^2 N^3 (N+1)(N\beta)^{2N}}{R^2(2N+1)(2N)!}. \qquad (30.66)$$

According to this formula, the radiation is maximal only for the case of one electron motion ($N=1$). The contribution of other electrons leads to a strong decrease of the general power. Thus, in the non-relativistic case the coherent radiation is strongly suppressed in comparison with the radiation of one electron.

In the ultra-relativistic case where $1-\beta^2 << 1$, the analysis shows that the coherence can take place for not very large N ($N << \gamma^3$), that is, for the long-wave part of the spectrum.

In the limiting case of big concentrations of the electrons for $N\sim\gamma^3$, when the number of particles has the order of the critical harmonic number, all the radiation is suppressed.

It is appropriate to mention here that the circular current, which can be considered an electron motion in a circle with $N\to\infty$, does not radiate.

Does the phenomenon of coherence of SR arise in case of motion of separate bunches which don't fill in the whole orbit but only its part?

30.5.2. SR coherence in case of motion of separate electron bunches

Let the electrons fill only a part of the orbit, that is, we consider the motion of electrons forming in the space into some bunch.

Let's write the coherent factor S_N in the form

$$S_N = N + \sum_{j=1}^{N} \sum_{\substack{j'=1 \\ (j' \neq j)}}^{N} \cos \nu\left(\psi_j - \psi_{j'}\right) = N + N(N-1) f_\nu = N^2 f_\nu , \qquad (30.67)$$

where

$$f_\nu = \sum_{\substack{j,j'=1 \\ (j' \neq j)}} \cos \nu\left(\psi_j - \psi_{j'}\right). \qquad (30.68)$$

Supposing that all the electrons are independent and are distributed in the bunch symmetrically with respect to some average position (on the azimuth equal to zero), one can get

$$f_\nu = \left[\int_{-\infty}^{+\infty} \cos \nu \varphi \, \omega(\varphi) \, d\varphi \right]^2 . \qquad (30.69)$$

Here $\omega(\varphi)$ is the probability that the electrons are in the orbit in the interval of angles from φ to $\varphi + d\varphi$. Further one can consider the cases:

a) the uniform distribution in some interval α:

$$\omega(\varphi) = \begin{cases} 1/\alpha , \ -\alpha/2 \le \varphi j \le \alpha/2 \\ 0, \ \textit{when is outside the interval} \end{cases}; \qquad (30.70)$$

b) Gaussian distribution

$$\omega(\varphi) = \frac{1}{2\sqrt{\pi}} e^{-\phi^2 / \alpha^2} . \qquad (30.71)$$

For case a) we get

$$f_\nu = \left| \frac{\sin \alpha \nu/2}{\alpha \nu/2} \right|^2 . \qquad (30.72)$$

For case b) we get

$$f_\nu = e^{-2(\nu\alpha/2)^2} . \qquad (30.73)$$

In both cases the radiation power of the bunch equals the sum of coherent and non-coherent parts:

$$W_N(\nu) = W^{noncoh}(\nu) + W^{coh}(\nu) = W(\nu)N + W(\nu)N^2 f(\nu). \qquad (30.74)$$

The total power of radiation loss can be calculated by summing the expression for $W_N(\nu)$ by all harmonics ν.

One can conclude from the form of dependence of f_v on the harmonic number in the examples of uniform and Gaussian distributions, that the maximal coherent radiation of the bunch falls to the range of wave lengths of order of the bunch size (that is, the region of low harmonics). The total power, obtained by the integration over the spectrum for both studied cases of the uniform and Gaussian distribution of electrons by the azimuth within the bunch, is given by the following expressions:

$$W^{coh}_{uniform} = N^2 \frac{e^2 c}{R^2}\left(\frac{\sqrt{3}}{\alpha}\right)^{4/3} = 2{,}1 N^2 \frac{e^2}{R^2}\alpha^{-4/3}, \tag{30.75}$$

$$W^{coh}_{Gauss} = N^2 \frac{e^2 c}{R^2}\left(\frac{\sqrt{3}}{\alpha}\right)^{4/3} \frac{\Gamma\left(2/3\right)}{\pi\sqrt{3}\, 2^{1/3}} = 0{,}56 N^2 \frac{e^2}{R^2}\alpha^{-4/3}. \tag{30.76}$$

It turns out that the power of the coherent radiation is not very sensible to the kind of form-factors of coherence f_v, which describes the charge distribution in the bunch. In the long-wave range the coherent radiation power of the bunch, which has angular sizes α, is in proportional to $\alpha^{-4/3}$ and doesn't depend on the particle energy.

It should be noted that the coherent synchrotron radiation is greatly influenced by the conducting shielding surfaces, which are parallel to the bunch. Physically it is explained by the interference and mutual decay of the fields induced by the particle and its image in the conducting surfaces.

Fig. 30.9 shows the spectral distribution of SR power taking into account the coherent part of the radiation. It is seen that the radiation is non-coherent in the region of high frequencies and has the maximum near

$$\omega = \omega_{\parallel} p \cong \gamma^3 . \tag{30.77}$$

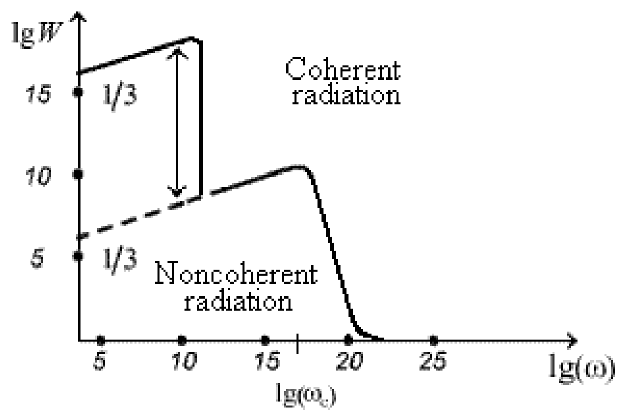


Fig. 30.9. Spectral distribution of SR power.
Coherent part of radiation is taken into account (coherent/ no coherent radiation)

The X-radiation with the wavelength ~ 0,2 nm corresponds to this frequency. In the long-wave part of the spectrum the wavelength becomes comparable with the bunch sizes and all the electrons radiate coherently. The radiation power, shown in the Fig. above, depending on the frequency corresponds to Gaussian case of distribution by the azimuth about $\varphi = 0$ in bunch of $N = 10^{11}$ electrons.

The coherent radiation phenomenon shows up not only in the synchrotron but also in other kinds of radiation. The common feature for all kinds of coherent radiation is the fact that an electron bunch with the length less than the radiating wavelength radiates coherently. Such a bunch radiates as a single whole and the radiation power is given by the formula:

$$W_{bunch}^{coh} = N_e^2 W , \qquad (30.78)$$

where W is the radiation power of one electron. If the expression for W contains the squared charge of electron e^2, then the expression for W_{bunch}^{coh} contains $(N_e)^2$. If the electrons are not grouped in bunches, they radiate independently, They have random phases, and the radiation fields mutually suppress each other because of the interference of separate electrons. In this case

$$W_{bunch}^{noncoh} = N_e W . \qquad (30.79)$$

If we take into account that $N_e \sim 10^{11}$ electrons participate in an acceleration cycle, a great increase in radiating power is possible owing to the coherence factor at the same current of accelerator. The indispensable condition should be the following: the particles should be grouped into bunches at the distances less than the length of the wave radiated by them. In this sense the hopes are set on the undulator as a source of great radiation power by making such bunches of electrons.

But it is a difficult task to make the coherent bunches for millimeter and submillimeter wave lengths. Moreover this task seemed to be impossible to solve for the optical wave lengths.

The investigations showed that there arises the phenomenon of self-modulation at the motion of electrons in the undulator, and these particles at the same time are influenced by the field of the light wave (by the laser). In other words, the self-modulation is a longitudinal grouping of the electrons leading to the coherent bunches with the length of order of optical wave. Moreover the electron grouping can occur when there is no influence of the external electromagnetic wave, that is, when there is no cavity resonator but there is a “inoculating” wave of spontaneous radiation, which is possible when using the single-pass laser.

The phenomenon of self-enhancement of the spontaneous radiation is due to the properties of undulator: when the electrons pass through a long

(~ tens of meters) undulator, first they radiate non-coherently, and the radiation power equals

$$W^{tot} = N_e \cdot W \, . \tag{30.80}$$

Due to the grouping mechanism there arises a phase correlation. It means that random initial fields of spontaneous radiation are intensified due to the interference, and the total radiation power becomes a coherent one:

$$W^{tot} = N_e^2 \cdot W \, . \tag{30.81}$$

The mechanism of self-enhancement of the spontaneous radiation underlies the strong source – the undulator of large length, which is in the special channel of the accumulating ring.

30.5.3. The coherent length of radiation

An important role in the theory of the relativistic particle radiation is played by the notion of the length of radiation formation or the coherent length. This notion is used in the qualitative analysis of different kinds of radiation. Spectral-angular energy density W, radiated by a particle with the charge e for all the period of motion in the path $\vec{r}(t)$ at speed $\vec{\upsilon}(t)$ in vacuum can be written in the form

$$\frac{d^2W}{d\omega \, d\Omega} = \left(\frac{\omega}{2\pi}\right)^2 \left|\vec{e} \cdot \vec{j}\left(\vec{k}, \omega\right)\right|^2 , \tag{30.82}$$

where ω and $\vec{k}$ are the frequency and the wave vector of the radiated electromagnetic waves $\left(k = \dfrac{\omega}{c}\right)$, $\vec{e} = \{\vec{e}_1, \vec{e}_2\}$ is the unit vector of polarization which has two components orthogonal to wave vector $\vec{k}$, $d\Omega$ is the solid angle, in which the radiation is directed and $\vec{j}\left(\vec{k}, \omega\right)$ is the Fourier-component of the particle current

$$\vec{j}\left(\vec{k}, \omega\right) = e \int_{-\infty}^{+\infty} \vec{\upsilon}(t) \exp\left[i\left(\omega t - \vec{k} \cdot \vec{r}\right)\right] dt \, . \tag{30.83}$$

The above given formulas can be obtained by computing the Pointing's vector flux through the sphere which is sufficiently far from the charge. This distance should be such that the charge field in every point of the sphere can be considered as a plane wave of the radiation. However, for the relativistic particles this distance should be greater than not only the wave length λ, but also greater than the quantity called the coherent length:

$$l_{coh} = \lambda \gamma^2 , \tag{30.84}$$

$$\gamma = \left(E/mc^2\right). \tag{30.85}$$

The expression for the coherent length can be derived using the following considerations. If the radiation of a relativistic particle occurs, it doesn't happen instantly: the particle and the wave should diverge at least at a distance equal to the wave length. As $\upsilon \sim c$, the radiation is mostly directed forwards in the direction of the particle motion. That is why the particle has time to pass the distance

$$l_{coh} \approx \upsilon t_{coh}. \tag{30.86}$$

The coherence time is defined in the following way. According to (30.83) the particle current is the source of the elementary wave with phase $\varphi = \omega t - \vec{k} \cdot \vec{r}(t)$. The radiation field can be considered the result of the interference of such waves. The difference in the phase of elementary waves does not exceed π, if they are emitted by the particle in the direction $\vec{k}$ for the time of coherence:

$$t_{coh} = \pi / \omega - \vec{k}\vec{\upsilon}, \tag{30.87}$$

which is found from the relation

$$\varphi = \omega t - \vec{k}\vec{r}(t) = \left(\omega - \vec{k}\vec{\upsilon}\right)t. \tag{30.88}$$

Setting in (30.88) $\varphi = \pi$ and $t = t_{coh}$, we get the formula (30.87). For time t_{coh} the particle passes the way according to (30.86) and (30.87)

$$l_{coh} = \frac{\upsilon\pi}{\omega - \vec{k}\vec{\upsilon}}. \tag{30.89}$$

Considering $\vec{k}$ close to the direction $\vec{\upsilon}$, we have

$$l_{coh} \approx \frac{\upsilon\pi}{\omega - k\upsilon} = \frac{\upsilon\pi}{2\pi c / \lambda - 2\pi / \lambda\,\upsilon} = \frac{\lambda\upsilon}{c}\frac{1}{1 - \upsilon / c} = \frac{\lambda\beta}{1-\beta} \cong \frac{\lambda\,\beta}{1 - 1 + 1 / 2\gamma^2} \cong \lambda\gamma^2. \tag{30.90}$$

Thus, the coherent length in the relativistic case equals

$$l_{coh} = \lambda\gamma^2. \tag{30.91}$$

31. The characteristics of radiation in the wiggler and the undulator

The monochromaticity and spectral-angular density of the energy of the synchrotron radiation can be essentially increased if one uses the magnetic field periodic in space. In this case the electron moves a periodic path, for example the sinusoid (curve). For SR the effective length of radiation formation is a small path section $l = R\delta\psi \cong R\gamma^{-1}$. The arc length of the radiation formation has the order of the field period at one electron pass in the undulator.

When the magnetic field changes according to the sinusoidal law the path of electron in the flat undulator is close to the sinusoid. And in the spiral undulator the electron moves in a spiral.

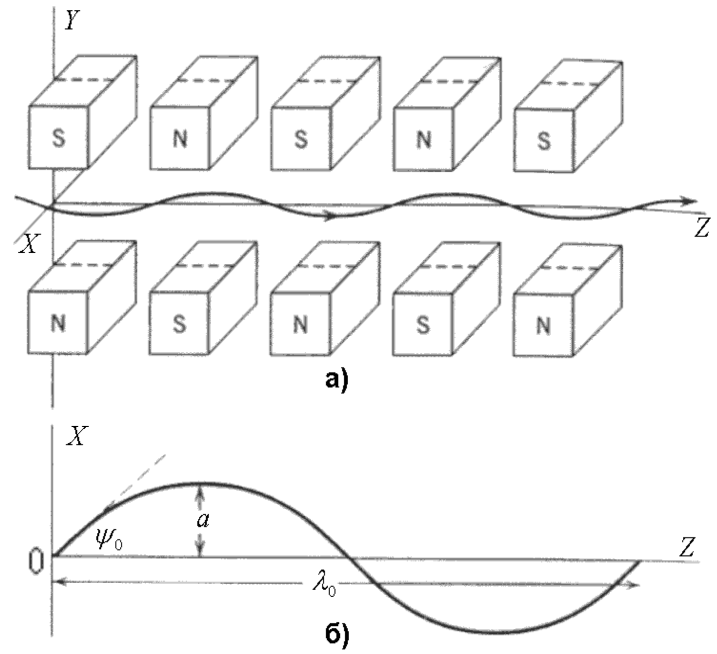


Fig. 31.1. The scheme of the flat undulator

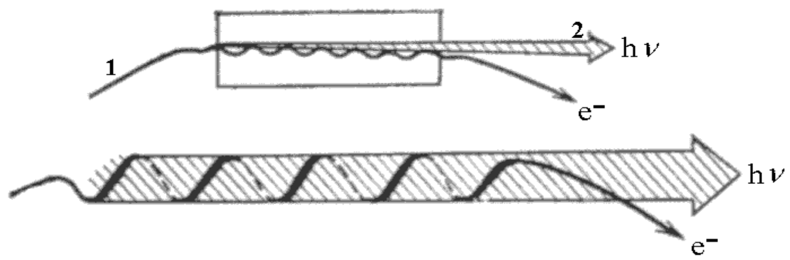


Fig. 31.2. The scheme of the spiral undulator:
1 – beam of electrons; 2 – undulator radiation

During the sinusoidal motion along the axis *Ox at* a longitudinal speed $\vec{\beta}_{\parallel}$ the electron radiates the filed $\vec{E}_1(\vec{k})$ on the first period, $\vec{E}_2(\vec{k})$ on the second one.

Let's denote the angle between wave vector $\vec{k}$ and velocity $\vec{\beta}_{\parallel}$ by θ. The travel time of the electron through the first period is

$$\Delta t_1 = \frac{d}{\beta c}, \tag{31.1}$$

and the travel time of the electromagnetic wave from the first period to the initial position of the wave front from the second period equals

$$\Delta t_2 = \frac{d\cos\theta}{c}. \tag{31.2}$$

The difference in phase of two wave packets is denoted by φ:

$$\varphi = \omega(\Delta t_1 - \Delta t_2) = \frac{2\pi}{\lambda}\beta c\left(\frac{d}{\beta c} - \frac{d\cos\theta}{c}\right) = 2\pi\frac{d}{\lambda}(1-\beta\cos\theta). \qquad (31.3)$$

That is why, one can write

$$\vec{E}_2(\vec{k}) = \vec{E}_1(\vec{k})\exp(i\varphi). \qquad (31.4)$$

Analogously one obtains

$$\vec{E}_3(\vec{k}) = \vec{E}_2(\vec{k})\exp(i\varphi),$$

$$\vec{E}_n(\vec{k}) = \vec{E}_{n-1}(\vec{k})\exp(i\varphi) = \vec{E}_1(\vec{k})\exp(i(n-1)\varphi). \qquad (31.5)$$

Let's find the total field of the periodical structure containing *N* elements

$$\vec{E}(\vec{k}) = \sum_{i=1}^{N}\vec{E}_i(\vec{k}) = \vec{E}_1(\vec{k})\left\{1+\exp(i\varphi)+\exp(i2\varphi)+\ldots+\exp\left[i(N-1)\varphi\right]\right\}, \qquad (31.6)$$

that is

$$\vec{E}(\vec{k}) = \vec{E}_1\frac{1-\exp(iN\varphi)}{1-\exp(i\varphi)}. \qquad (31.7)$$

The spectral-angular distribution of the radiation intensity of the fields, made at *N*-periods equals

$$\frac{d^2W_N}{d\omega d\Omega} = G\left|\vec{E}(\vec{k})\right|^2 = G\left|\vec{E}_1(\vec{k})\right|^2\frac{\left|1-\exp iN\varphi\right|^2}{\left|1-\exp i\varphi\right|^2} = \frac{d^2W}{d\omega d\Omega}F_N, \qquad (31.8)$$

where *G* is the proportionality factor in the expression for the intensity distribution.

It is obvious that

$$G\left|\vec{E}_1(\vec{k})\right|^2 = \frac{d^2W}{d\omega d\Omega} \qquad (31.9)$$

is the spectral-angular distribution of the radiation intensity made at one period.

Let's evaluate the resonance factor F_N:

$$F_N = \frac{\left|1-\exp iN\varphi\right|^2}{\left|1-\exp i\varphi\right|^2} = \frac{4\sin^2 N\varphi/2}{4\sin^2\varphi/2}. \qquad (31.10)$$

Radiation is maximal when $\varphi/2 \to 0$ and

$$\lim_{\varphi/2\to 0}\frac{\sin N\varphi/2}{\sin\varphi/2} = N. \qquad (31.11)$$

In this case we obtain

$$F_N \to N^2 . \tag{31.12}$$

Thus, when the resonance condition is satisfied

$$\varphi = 2\pi \frac{d}{\lambda}(1 - \beta\cos\theta) = 2\pi m , \tag{31.13}$$

where *m* is a whole number, the spectral-angular density of the radiation increases N^2 times in comparison with one period.

The fundamental role in the theory of the electron radiation in the periodical structures is played by parameter *K*, which presents the proportion of angle ψ to the effective radiation angle γ^{-1}:

$$K = \psi\gamma, \tag{31.14}$$

ψ is the maximal angle between a tangent to a trajectory and the average speed of a particle $\langle\vec{\beta}\rangle$.

For the undulator

$$K = \psi\gamma = \gamma\frac{d}{R} = \gamma\frac{\beta e}{\beta\varepsilon}d . \tag{31.15}$$

For a sinusoid path

$$\beta_z = \beta_\perp = \beta\frac{K}{\gamma}\sin\left(\frac{2\pi}{d}x\right), \tag{31.16}$$

$$\beta_x = \beta_\parallel = (\beta^2 - \beta^2{}_\perp)^{1/2} \tag{31.17}$$

with $\beta = const$.

Averaging over the period of oscillations, we find

$$\langle\beta_\perp\rangle = \frac{1}{2}\beta^2\psi^2 = \frac{K^2}{2\gamma^2}, \tag{31.18}$$

$$\langle\beta_\parallel\rangle = \beta^2 - \langle\beta_\perp^2\rangle = 1 - \gamma^{-2} - \frac{K\gamma^{-2}}{2} = 1 - \frac{1 + K^2/2}{\gamma^2} . \tag{31.19}$$

In the undulators in reference system K', where the period-average speed of a particle equals zero, the particle oscillates at non-relativistic $K << 1$ or slightly-relativistic $(K \sim 1)$ speed. That is why the particle in reference frame K' can radiate electromagnetic waves only with the frequency ω', equal to the frequency of oscillations ω_0'. Since all the processes in the reference frame K' proceed more slowly than in the laboratory frame

$$\omega_0' = \omega_0\gamma^{-1}, \tag{31.20}$$

where $\omega_0 = 2\pi/T$ is the frequency of oscillations in reference frame *K*.

The frequency of radiation registered due to the Doppler's effect depends on the angle θ between the direction of observation and axis *Ox*, along which the particle moves at speed $\langle\beta_{\|}\rangle$:

$$\omega_L = \frac{\omega_0}{1-\langle\beta_{\|}\rangle\cos\theta}. \qquad (31.21)$$

For the relativistic case

$$\omega_L(\theta) = \frac{\omega_0}{1-\left[1-\frac{1+K^2/2}{2\gamma^2}\right]\left(1-\frac{\theta^2}{2}\right)} = \frac{2\gamma^2\omega_0}{1+\gamma^2\theta^2+K^2/2}. \qquad (31.22)$$

Form this we obtain for the wave length at $\theta = 0$

$$\lambda = \frac{2\pi c}{\omega_L} = \frac{dK^2/2}{2\gamma^2}. \qquad (31.23)$$

At $K = 10, \gamma = 10^3$ and $d = 4$ cm, we have $\lambda = 10^{-4}$ cm, i. e. the electron emits the visual light in the undulator.

Depending on the value of parameter K ($K << 1$, $K \approx 1$ or $K >> 1$) the angular and spectral characteristics of the undulator radiation can essentially change.

The devices, in which $K \le 1$ and the number of periods $N \sim 10\ldots10^2$, are called undulators, and the devices in which $K >> 1$, and the number of periods $N = 1\ldots3$ are called wigglers. These names derive from the English words (undulate – "wavy" and to wiggle – to move in small side-to-side or turning movements).

Further we shall result some formulas and estimations of characteristics of radiation in the wiggler and undulators, taken of work Bazylev and Zhevago [15]. Thus relativistic units are used: $\hbar = 1$, $m = 1$, $c = 1$. In these units $e^2 \approx 1/137$, unit of length $\hbar/mc$, a time unit $\hbar/mc^2$, an energy unit mc^2, unit of frequency of radiation $mc^2/\hbar$, unit of a magnetic (electric) field $m^2c^{5/2}/\hbar^{3/2}$. For reception of formulas in usual units it is necessary to proceed preliminary to dimensionless variables

On the basis of the formula

$$\omega = \frac{\omega_0}{1-\langle\upsilon_x\rangle\cos\theta}. \qquad (31.24)$$

One can conclude that a certain frequency is radiated in the endless undulator with a harmonic field on condition $K << 1$ and at a fixed value of angle θ to the axis of the undulator.

For the real undulator with length L the radiation line has a finite width:

$$\Delta\omega \cong \omega\frac{d}{L} = \frac{\omega}{N}, \qquad (31.25)$$

where *d* is the period and *N* is the number of periods of the field.

The radiation in the undulator can be not only at frequency (1), but also at the other harmonics multiple of the basic frequency.

That is why the radiation spectrum consists not of one but of a series of equidistant lines.

At number of periods $N \approx 10^2$ it is possible to obtain the radiation in the undulator with a natural width of the line

$$\Delta\omega^{(1)} \cong \omega^{(1)}/N, \qquad (31.26)$$

where index 1 refers to the first harmonic.

This radiation is directed in the solid angle:

$$\Delta\Omega \cong \frac{2\pi}{\gamma^2 N}. \qquad (31.27)$$

For average number of photons dN_γ/dt, emitted per unit of time at frequencies from $\omega_{\max}^{(1)}/2$ to $\omega_{\max}^{(1)}$ one can obtain the approximate expression if the power of radiation

$$d\varepsilon/dt = -\frac{2}{3}e^2 \frac{1}{T}\int_0^T B_z^2(x)dx, \qquad (31.28)$$

where

$$B_z(x) = B\sin\frac{2\pi x}{T} \qquad (31.29)$$

is divided by the average energy of the photon:

$$\langle\omega\rangle \approx \frac{\omega_{\max}^{(1)}}{2}. \qquad (31.30)$$

We get the total number of the photons emitted by one electron in the undulator if we multiply the value dN_γ/dt by the time of the undulator transit $\tau = TN$:

$$N_\gamma \approx \frac{2}{3}\pi e^2 K^2 \left(1 + K^2/2\right) N. \qquad (31.31)$$

Having divided expression (7) by $\langle\omega\rangle$ we obtain the average spectral density of radiation dN_γ/dt. If we divide dN_γ/dt by the space angle (4), we obtain the following approximate expression for the spectral-angular density of the radiation energy

$$\frac{d^2W}{d\omega\, d\Omega} \sim \frac{1}{3}e^2 K^2 \gamma^2 N^2. \qquad (31.32)$$

At value $K >> 1$ the motion in reference frame K' becomes ultra-relativistic and the maximal radiation is shifted to a higher number of harmonic:

$$\frac{d^2W}{d\omega\, d\Omega} \sim \frac{1}{3} e^2 K^2 \gamma^2 N^2 . \qquad (31.33)$$

Frequency $\omega^{(1)}$ at $K >> 1$ is $K^2/2$ times less than the same frequency in the undulator at $K \approx 1$. One can draw a conclusion that the maximal density of the radiation intension in the wiggler corresponds to the frequencies:

$$\omega^{(1)} \sim \sqrt{2}\omega_0 \gamma^2 K = \frac{eB}{\sqrt{2}} \gamma^2 , \qquad (31.34)$$

which don't depend on the period of the magnetic structure and coincide with the frequencies of the synchrotron radiation in constant magnetic field which equals $\sim B/\sqrt{2}$. The fact is, that in the wiggler, as well as in the synchrotron, the radiation if formed at length l_{coh}, which is much (~ n_{eff} times) less than the period of the magnetic structure (and for SI system – the radius of the orbit curvature).

In the wiggler at certain angle θ a relatively large number of lines is radiated which correspond to such harmonics n, that

$$\omega^{(1)} \sim \sqrt{2}\omega_0 \gamma^2 K = \frac{eB}{\sqrt{2}} \gamma^2 . \qquad (31.35)$$

Every line has the width

$$\Delta\omega^{(n)} \approx \omega^{(n)} / nN ,$$

where

$$\omega(n) = \omega^{(1)} n . \qquad (31.36)$$

If we divide the power of radiation (31.28) by the characteristic energy of photon $\hbar\omega^{(n)}$ (9), we will obtain the average number of the photons emitted in unit of time, dN_γ/dt with energy in the range of $\hbar\Delta\omega^{(n)}$. Multiplying dN_γ/dt by time of the wiggler pass $\tau = TN$, we obtain the total number of the photons, N_γ, emitted by one electron in the wiggler with N periods in the interval:

$$N_\gamma \sim \frac{\sqrt{2}}{3} \pi e^2 K N . \qquad (31.37)$$

The relativistic particle radiates in the range of angle γ^{-1}. If we talk about the radiation in the wiggler in the plane perpendicular to the plane of the particle oscillation, the effective angle where the radiation is concentrated equals

$$\chi_{eff} \sim \gamma^{-1} . \qquad (31.38)$$

The radiation in the wiggler is within the angle in the plane of the oscillation

$$\psi = K\gamma^{-1}. \tag{31.39}$$

The spectral-angular distribution of the radiation energy of one electron in the wiggler is given by the formula, as it follows from the previous proportions:

$$\frac{d^2W}{d\omega\, d\Omega} \sim \frac{1}{3}\frac{1}{\sqrt{2}} e^2\gamma^2 N. \tag{31.40}$$

If we compare (31.40) with the corresponding expression for SI, the value given (31.40) turns out to be *2N* times larger. Such an increase is achieved because of the overlapping of the radiation cones from different sections of the particle orbit.

In the conclusion let's cite some formulas derived in the strict theory of the undulator radiation.

Only the odd harmonics at angle $\theta = 0$ to the undulator axis are radiated for the flat undulator with the sinusoid field. In this case the spectral-angular density of the intensity is given by the expressions

$$\frac{d^2I}{d\omega d\Omega}\Big|_{\theta=0} = \frac{e^2\omega_0\gamma^2}{2}\frac{K}{\left(1+K^2/2\right)^2}\sum_{n=1}^{\infty}{}' n^2 \times$$

$$\times\left[J_{\frac{n-1}{2}}\left(\frac{np^2}{4\left(1+p^2/2\right)}\right) - J_{\frac{n+1}{2}}\left(\frac{np^2}{4\left(1+p^2/2\right)}\right)\right]^2 \frac{\sin^2 N\bar{\zeta}_n}{\pi N\bar{\zeta}_n^2}, \tag{31.41}$$

where

$$\zeta_n = \frac{\omega T}{4\gamma^2}\left[1+(\theta\gamma)^2 + K^2\!\big/2\right] - \pi n; \tag{31.42}$$

$$\bar{\zeta}_n \equiv \zeta_n(\theta = 0) \tag{31.43}$$

and the prime by the sum means that the summation is done only by the odd harmonics n. The spectral distribution of the radiation in the sinusoid magnetic field has the form:

$$\frac{dI}{d\zeta} = \bar{I} f(\zeta), \tag{31.44}$$

where

$$f(\zeta) = \frac{3}{\pi}\int_1^{\infty}\frac{\sin^2 N\pi z\zeta}{\pi N(z\zeta - 1)}\frac{1}{z^2}\left(1 - \frac{2}{z} + \frac{2}{z^2}\right)dz, \tag{31.45}$$

$\zeta = \omega/2\gamma^2\omega_0$; $\bar{I} = e^4\gamma^2B^2/3$ is the power of radiation (see formulas (31.29)–(31.30)).

For the endless undulator $(N \to \infty)$

$$f(\zeta) = 3\left(1 - 2\zeta + 2\zeta^2\right). \tag{31.46}$$

The numerical integration of expression $d^2I/d\omega d\Omega$ by the solid angles leads to the expression

$$dI/d\omega = 2e^2\omega_0 F(\zeta), \tag{31.47}$$

$$\zeta = \frac{2\gamma^2\omega_0}{1 + K^2/2}. \tag{31.48}$$

At $K = 0{,}5$ the radiation spectrum slightly differs by the shape from the dipole approximation. For the increase of K the spectrum approaches the spectrum of the synchrotron radiation.

The spectral-angular density of the radiation intensity of electron in the spiral magnetic field has the form:

$$\frac{d^2I}{d\omega d\Omega} = \frac{e^2\omega^2}{\pi^2} \times$$

$$\times \sum_{n=1}^{\infty} \left[\frac{K^2}{2\gamma^2} J_n'^2(n\aleph) + \frac{\left(1 - (\theta\gamma)^2 + \frac{K^2}{2}\right) J_n^2(n\aleph)}{\left(2\theta\gamma^2\right)^2} \right] \frac{T \sin^2 NJ_n}{NJ_n^2}, \tag{31.49}$$

where the following notations are introduced

$$J_n = \omega \frac{T}{4\gamma^2}\left[1 + (\theta\gamma)^2 + \frac{K^2}{2}\right] - n\pi,$$

$$K = \frac{eBT}{\sqrt{2}\pi}, \quad \aleph = \sqrt{2}K\gamma\theta \Big/ \left[1 + (\theta\gamma)^2 + \frac{K^2}{2}\right]. \tag{31.50}$$

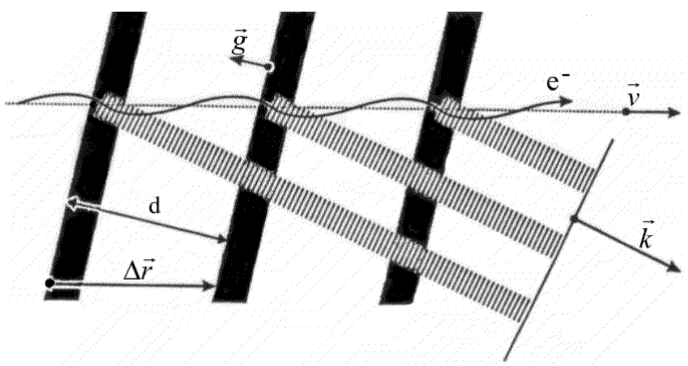


Fig. 31.3. The coherent waves radiated by the electron during its pass through a periodical structure (undulators, crystal and so forth)

Along the axis of the spiral undulator $(\theta = 0)$ only the first harmonic is radiated $(n = 1)$ with the spectral-angular density of intensity

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2 \omega_0 \gamma^2}{2} \frac{K^2}{\left(1 + K^2/2\right)^2} \frac{\sin^2 N\tilde{\zeta}_1}{\pi N \tilde{\zeta}_1^2}, \qquad (31.51)$$

where $\tilde{\zeta}_1 = \zeta_1(\theta = 0)$.

Let's mark the polarization characteristics of the radiation in the undulators. The radiation in the undulator with the sinusoid filed at fixed angle θ to the axis of the undulator is completely flat polarized. The polarization in the spiral undulator becomes elliptic in a general case and at $\theta = 0$ it becomes circular.

32. Short survey of radiation sources using relativistic electrons

When the synchrotron radiation was discovered one started to investigate other radiation effects caused by interaction of relativistic electrons with medium.

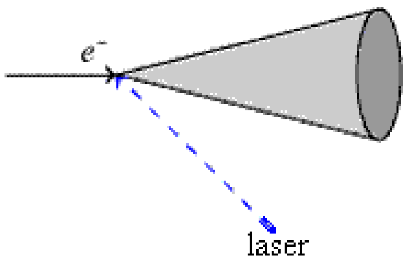


Fig. 32.1. Compton scattering

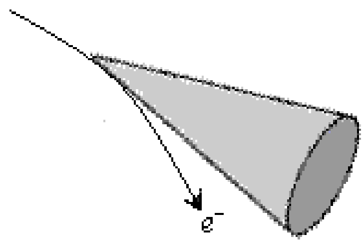


Fig. 32.2. The synchrotron radiation

In spite of detailed theoretical and experimental examinations of these phenomena they started to be regarded as radiation sources not long ago. It also refers to the well known Compton scattering shown schematically in Fig. 32.1. Thankl to the creation of electrons with high intensity and large-power lasers of the Compton scattering becomes a new interesting source of pulsating x-ray radiation. The other effects based on the interaction with medium can be conventionally divided into two classes. The first class includes such phenomena that can be investigated by analogy with synchrotron radiation Fig. 32.2 and undulator radiation, which is closely related to it.

Radiation arises when an electron passes through a crystal in the field of planes and axes of the crystal. In this case the periodical crossing of crystallographic planes by the electron leads to the path disturbance and to the emitting of coherent bremsstrahlung CB which is schematically shown in Fig. 32.3.

If a particle moves almost parallel to such axis or plane, it can be "captured" in the mode of stable undulator path along this axis or the plane. The result is emission of channeling radiation (CHR) shown in Fig. 32.4.

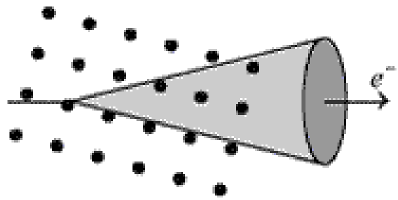


Fig. 32.3. The coherent bremsstrahlung

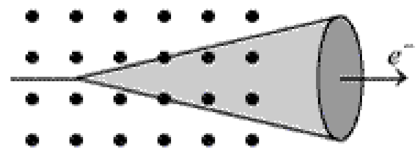


Fig. 32.4. Channeling radiation

The second class includes the radiations in medium caused by electrical field of the particle. In its nature they are close to the Vavilov-Cherenkov's effect.

Transition radiation (TR) arises when a moving charged particle crosses the boundaries of two media with different dielectric and magnetic properties (Fig. 32.5).

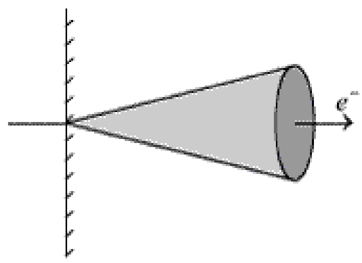


Fig. 32.5. Transition radiation

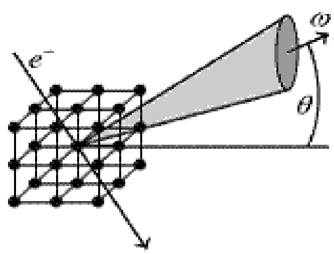


Fig. 32.6. Parametrical X-rays

In this way parametric X-rays (PXR) (Fig. 32.6) can be regarded as the combination of two radiations: 1) transition radiation which appears at particle cross of a crystal surface and reflected by crystallographic planes at Bragg's angles 2) the radiation emitted inside the crystal.

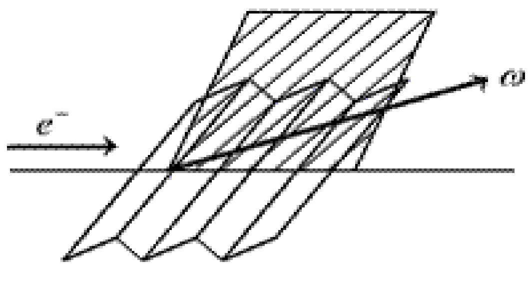


Fig. 32.7. Smith-Purcell radiation

Smith-Purcell's effect (Fig. 32.7) implies that the radiation arises when a charged particle passes near and parallel to the surface of a metal diffraction grating. This radiation also belongs to the second class though it can be explained by applying classical diffraction theory to the diffraction of the waves quickly decreasing as the distance extends. These waves relate to the coulomb field of a relativistic charged particle.

32.1. Channeling Radiation

The channeling radiation arises, when the charged particle passes through a crystal along a symmetry axis. The channeling can be axial (along a nuclear chain) or planar (between nuclear planes). The basic idea by theoretical consideration of the phenomenon of the channeling consists that the true potential of atoms of a crystal is replaced with the potential averaged on coordinates of atoms in the crystal axis or plane (speak accordingly about axial and planar channeling). Such approach is most comprehensible, if the particle falls on an axis or a plane under a small angle. As forces of an attraction (and for positively charged particle – repulsive force) are directed by atoms to one party there is a smooth change of a direction of an impulse of a particle because of collision with the big number of atoms of an axis or a plane. In a case of electrons the potential of a chain of atoms is drawing and a particle or crosses a chain, or (when energy of its transversal movement does not suffice for overcoming of a potential barrier) makes fluctuations about a chain and, hence, passes through area of thermal fluctuations of atoms of a chain. Thanks to possibility of collision of a particle with the atom which has deviated position of balance, there is a strong instability of its movement in continuous potential. If the particle has, for example, the initial orbital moment concerning an axis z it any time will be kept outside of area $\rho < u$ (because of action of a centrifugal barrier). Thus instability of movement in continuous potential is shown only at the account of not elastic scattering on the electrons of a crystal. For an ideal crystal (atoms – the motionless centre of a static force field located strictly periodically) the particle, moving under a small angle to a chain, "does not feel" influence of individual atoms and the chain potential can be averaged on longitudinal coordinate z.

The average potential looks like

$$V(\rho) = \frac{1}{d}\int_{-\infty}^{\infty} \varphi(\sqrt{\rho^2 + z^2}, \qquad (32.1a)$$

where $\varphi(r)$ – potential of separate atom, d – distance between atoms.

In the same way it is possible to arrive, when the particle falls under a small angle on crystal plane.

Positively charged particles are reflected by the potential of a plane averaged on co-ordinates y и z этой of a plane

$$V(x) = \frac{1}{S}\iint \varphi(\sqrt{x^2 + \rho^2})d^2\rho, \qquad (32.1b)$$

where S is the area falling to one atom in plane $\rho = \{y, z\}$.

Negatively charged particles makes oscillation about a plane and their movement on equilibrium trajectories in the potential of $V(x)$ менее is less steady in comparison with movement of positively charged particles which oscillate between the next planes. The matter is that positrons do not approach close to fluctuating atoms of a plane.

Movement channeling particles and their radiation is influenced by the form of continuous potentials, depth of the potential wells and height of the barriers limiting their transversal movement.

At the heart of calculations of continuous potentials lay known approach for the isolated atoms:

Thomas-Fermi, Thomas-Fermi-Dirak, Hartry-Fok with various analytical approximations (on Moliere, on Firsov, etc.).

All expressions, which turn out by means of specified above approximations appear, very difficult to use them in analytical calculations of spectra and other characteristics at channeling.

In their many cases it is possible with sufficient accuracy ≤ 25 % replace in simple modeling potentials

So for plane channeling positrons the continuous potential looks like a parabola almost everywhere within the channel

$$V(x) = m\Omega_0^2 x^2 / 2\,. \tag{32.2a}$$

For electrons the average potential of a plane can be presented in the form of Peshle-Teller potential.

$$V(x) = -V_0 ch^{-2} x / b\,, \tag{32.2b}$$

where crystal potential

$$V_0 = 10 \ldots 10^3\ eV.$$

In the classical approach the problem about radiation of electrons and positrons at their scattering on crystal axes is reduced to a problem about particle movement in two-dimensional Coulomb potential:

$$V(\rho) = \mp\alpha / \rho. \tag{32.2c}$$

On distances ρ from an axis, smaller amplitudes of thermal fluctuations, the average potential can be presented in parabolic potential

$$V(\rho) = \beta\rho^2. \tag{32.2d}$$

For channeling it is necessary, that particles have been focused in relation to an axis of a crystal within angle, smaller, than Lindhart angle

$$\theta_L = \sqrt{\frac{V_0}{E}}\,, \tag{32.3}$$

approximately in 10 times.

The critical angle is defined by value of potential barrier $V_0 \approx 10\ldots10^3 eV$, and it is limited because of thermal fluctuations. Usually size $V_0 \sim 100\, eV$ for middle and easy elements and $V_0 \sim 1000\, eV$ for the heavy elements.

That channeling it was observed at $E_e = 1\, MeV$, it is necessary that the crystal has been focused concerning a bunch with accuracy 1 %.

The channeling it is not periodical in relation to a lattice; periodicity depends on depth of potential.

Channeling radiation is similar to the undulator radiation with small values of *K* parameter and has the spectrum

$$\frac{dI}{d\omega} = \frac{3I_0\omega}{\omega_{\max}^2}\left(1 - 2\frac{\omega}{\omega_{\max}} + 2\frac{\omega^2}{\omega_{\max}^2}\right) \qquad (32.4)$$

with

$$\omega < \omega_{\max}, \qquad (32.5)$$

where

$$\omega_{\max} = 2\Omega_0 n\gamma^{3/.2}, \qquad (32.6)$$

$$\Omega_0 = \sqrt{\frac{2V_0}{m_e r_s^2}}$$

is the frequency of oscillations and n is the number of a harmonic, r_s is a minimal distance on which electron comes nearer to an axis, that is radius of shielding.

The quantity I_0 depends on the fact if there occurs the flat or axial channeling and also on the kind of a particle – a positron or an electron. The radiation intensity has the same angular dependence as the undulator radiation:

$$\omega(\theta) = \frac{2\gamma^2\omega_{\max}}{1+\theta^2\gamma^2}. \qquad (32.7)$$

Channeling radiation is promising as an object of the scientific research and must be taken into account while solving applied tasks.

- The radiation spectrum contains the information about the non-coherent scattering of electrons in the crystal lattice, which is of some interest for electronic microscopics, X-ray methods of the crystal research, etc.
- The interaction potentials of the electrons and positrons with the crystal lattice is defined (for the diamond, silicon and others) to within 1 % of the spectrum.
- It is possible to get information about the electronic density of different crystal structures, about the background spectrum.
- There were experimental studies of anisotropy of the thermal oscillations and their correlation with the help of channeling radiation.

- With the help of channeling radiation it is possible to study defects in crystals for example to define their location, concentration, kinds and forms. And this method is not destructive in comparison to the method of ion beams.
- The radiation of the x-ray range allows conducting the researches of the ultra-short processes ~ tens picoseconds.
- It is possible to create a high power beam source of radiation of x-ray and gamma ranges.
- At an optimum choice of the crystal thickness and at a certain energy the positron spectrum is highly monoenergetic, which presents a certain interest for nuclear spectroscopy and medicine.
- It is possible to irradiate different parts of a body at different depths by changing the photon energy, varying the positron energy and the crystal.
- The radiation can be applied for detection of the particles of ultra-high energies.

32.2. The coherent bremsstrahlung

If the medium is crystal-like, apart from the bremsstrahlung, caused by the scattering of the charged particle on the medium nucleus, the periodic grating can induce an additional coherent bremsstrahlung.

Then the cross-section for the bremsstrahlung is modified and has the form

$$\sigma_\gamma = \sigma_0(\gamma)e^{-g^2a^2} \cdot 8\pi^3 \frac{N}{\Delta}\sum_{\vec{g}} \left|S_{\vec{g}}\right|^2 \delta\left(\vec{k} - \vec{g}\right), \tag{32.8}$$

where $\sigma_0(\gamma)$ is the section of the bremsstrahlung.

The first exponent is Deby-Valler's factor which is responsible for the thermal motion of N atoms in the crystal, a^2 is the mean-square amplitude, $g = \frac{2\pi}{d}$ is the period of the orbit lattice, $\left|S_{\vec{g}}\right|^2$ is the factor of the crystal-like structure, Δ is the volume of a unit cell, $\vec{k}$ is the wave vector.

The radiation is highly polarized and has a sharp maximum near the axis of the electron bunch.

32.3. Transition radiation

Transition radiation arises when the particle moves uniformly and linearly in the non-homogeneous medium, or in the non-homogeneous medium and/or changing in time, or when such medium is spread near the path of the charged particle. In the general case transition radiation can exist simultaneously and interfere with Cherenkov's radiation and with the radiation which appears at the accelerating motion of the particle (the bremsstrahlung, synchrotron and others). If the particle moves at constant speed

$$\upsilon < c/n,$$

then the Vavilov and Cherenkov's radiation does not arise. But if $n = 1$, there should be no radiation at all. For the radiation to appear in vacuum it is necessary that the charge (or multipole) accelerate, i. e. parameter υ/c characterizing the radiation should change. If there is a transparent medium, then this parameter has the form $\upsilon/c_{ph} = \upsilon\, n/c$ and it equals the proportion of the particle velocity to phase velocity of light $c_{ph} = c/n(\omega)$. Thus, in the presence of the medium, parameter $\upsilon n/c$ can change not only as a result of speed change υ, but also at the expense of the change along the path of the index of refraction. Even at $\upsilon = const$ the radiation arises because index of refraction n changes. For the absorbing but not magnetic medium the role of he index of refraction n is played by $\sqrt{\varepsilon} = n + i\kappa$, where ε is the complex dielectric permittivity of the medium. The simplest task of such type is the crossing of the border of two media (or the border of vacuum and medium). It is possible to give an obvious explanation of the reason why transition radiation appears while the charge crosses the border. The electromagnetic field of the first medium can be presented as the field of the charge itself and as the field of its image moving in the second medium towards the charge. While crossing the border it is as if the particle and its image "are annihilated" or get transformed, which leads to the radiation. If the second medium is an ideal mirror, the charge and its image will be fully "annihilated" (see Fig. 32.8).

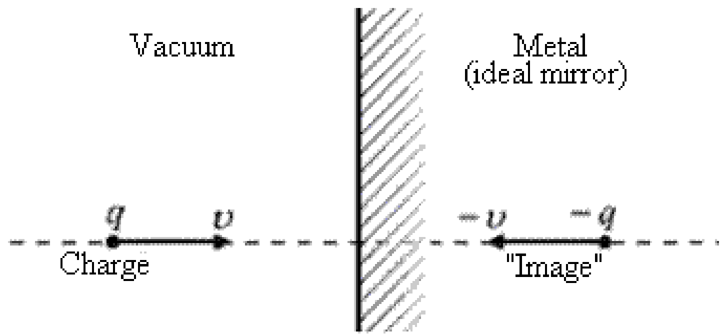


Fig. 32.8. The generation of transition radiation

Let the particle with charge e move from the medium with dielectric permittivity ε_1, into the medium with dielectric permittivity ε_2. At a normal fall to the flat border (Fig.) the task is solved quite easily. As usual, at first field $\vec{E}(\vec{r},t)$ is found, which presents the proper field of a particle and the field of radiation. The transition to Fourier-images of the fields and the current of the particle with the help of expansion in Fourier integrals allows us to calculate the spectral-angular distribution of radiation in a standard way $\frac{d^2W}{d\omega\, d\Omega}$. Without getting into details of the calculation let's give the expression for $\frac{d^2W}{d\omega\, d\Omega}$, which presents the energy emitted by the electron per unit intervals of frequencies and of the solid angle in the direction at angle θ_1, with respect to the direction of the particle motion, and the radiation is registered in area (1):

$$\frac{d^2W}{d\omega\, d\Omega} = \frac{r_e m_e c^2 \beta^2}{\pi^2 c} \sqrt{\varepsilon_1} \sin^2\theta_1 \cos^2\theta_1 \frac{F_n^2}{F_d^2}, \qquad (32.9)$$

where

$$F_n = (\varepsilon_2 - \varepsilon_1)\left(1 - \beta^2\varepsilon_1 + \beta\sqrt{\varepsilon_2 - \varepsilon_1 \sin^2\theta_1}\right);$$

$$F_d = \left(1 - \beta^2\varepsilon_1 \cos^2\theta_1\right)\left(1 + \beta\sqrt{\varepsilon_2 - \varepsilon_1 \sin^2\theta_1}\right)\left(\varepsilon_2 \cos\theta_1 + \sqrt{\varepsilon_1\varepsilon_2 - \varepsilon_1^2 \sin^2\theta_1}\right). \qquad (32.10)$$

The expression for $\frac{d^2W}{d\omega\, d\Omega}$, registrating the radiation in the forward direction (i. e. in the area 2) follows from the previous expression with a mutual exchange of the indexes $1 \leftrightarrow 2$ and the change of the speed β to $-\beta$. The frequency spectrum lies from the microwave range up to the frequency $\omega < \gamma c\sqrt{4\pi zN\tau_e}$, where N is the density of the atoms of the medium and z is the atomic number. For the transition from vacuum $(\varepsilon_1 = 1)$ into metal $(\varepsilon_2 \to \infty)$ spectral-angular distribution has the form:

$$\frac{d^2W}{d\omega\, d\Omega} = \frac{r_e m_e c^2}{\pi^2 c} \frac{\beta^2 \sin^2\theta}{\left(1 - \beta^2 \cos^2\theta\right)^2}. \qquad (32.11)$$

The radiation maximum falls to the angle $\theta = 1/\beta\gamma$. Integrating by the angles, we get spectral radiation distribution:

$$dW/d\omega = \frac{r_e m_e c^2}{2\pi\beta c}\left[\left(1 + \beta^2\right)\ln\left(\frac{1+\beta}{1-\beta}\right) - 2\beta\right] \approx \frac{2 r_e m_e c^2}{\pi\beta c} \ln\gamma \qquad (32.12)$$

for $\gamma >> 1$.

TASKS

Task № 1

Topics: Mathematical apparatus of the field theory Vector algebra and tensor algebra Vector analysis

1. Give the definitions of vector, second-rank tensor and S-rank tensor in three-dimensional space.
2. Give the definitions of polar axial vectors
3. Give the definition for a fully antisymmetric unit three-rank tensor ε_{ijk}.
4. Write the expression for the components of vector product $\left[\vec{A}\,\vec{B}\right]$ and rot $\vec{B}(\vec{r})$ with the help tensor ε_{ijk}. Point out the way these values are transformed at rotations and reflexions.
5. Prove the equalities:

   a) $\varepsilon_{ikl}\varepsilon_{lmn} = \delta_{im}\delta_{kn} - \delta_{in}\delta_{km}$;

   b) $\varepsilon_{ikl}\varepsilon_{klm} = 2\delta_{im}$;

   c) $\varepsilon_{ikl}\varepsilon_{ikl} = 6$.
6. Write in the invariant vector form:

   a) $\varepsilon_{inl}\varepsilon_{irs}\varepsilon_{lmp}\varepsilon_{stp}a_n a_r b_m c_t$;

   b) $\varepsilon_{inl} \quad \varepsilon_{krs} \quad \varepsilon_{lmp} \quad \varepsilon_{stp} \quad a_r \quad a'_n \quad b_k \quad b'_i \quad c_t \quad c'_m$.
7. Prove the identities with the help of operator $\vec{\nabla}$ and using the rules of derivation and vector multiplication without going to the projections onto reference axis. Functions $\varphi, \psi, \vec{A}, \vec{B}, \vec{C}$ are the functions of coordinates:

   a) $grad(\varphi\psi) = \varphi\, grad\psi + \psi\, grad\varphi$;

   b) $div\left(\varphi\vec{A}\right) = \varphi\, div\vec{A} + \vec{A}\cdot grad\,\varphi$;

   c) $rot\left(\varphi\vec{A}\right) = \varphi\, rot\vec{A} - \left[\vec{A}\ \ grad\ \varphi\right]$;

   d) $div\left[\vec{A}\,\vec{B}\right] = \vec{B}\ rot\vec{A} - \vec{A}\ rot\vec{B}$;

   e) $rot\left[\vec{A}\,\vec{B}\right] = \vec{A}\, div\vec{B} - \vec{B}\, div\vec{A} + \left(\vec{B}\cdot\vec{\nabla}\right)\cdot\vec{A} - \left(\vec{A}\cdot\vec{\nabla}\right)\cdot\vec{B}$;

   f) $grad\left(\vec{A}\vec{A}\cdot\vec{B}\right) = \left[\vec{A}\, rot\vec{B}\right] + \left[\vec{B}\, rot\vec{A}\right] + \left(\vec{B}\cdot\vec{\nabla}\right)\vec{A} + \left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}.$

8. Prove the identities:
   a) $\vec{C}\cdot grad\left(\vec{A}\cdot\vec{B}\right)=\vec{A}\cdot\left(\vec{C}\cdot\vec{\nabla}\right)\vec{B}+\vec{B}\cdot\left(\vec{C}\cdot\vec{\nabla}\right)\vec{A}$;
   b) $\left(\vec{C}\cdot\vec{\nabla}\right)\left[\vec{A}\,\vec{B}\right]=\vec{A}\times\left(\vec{C}\cdot\vec{\nabla}\right)\vec{B}-\vec{B}\times\left(\vec{C}\cdot\vec{\nabla}\right)\vec{A}$;
   c) $\left(\vec{\nabla}\cdot\vec{A}\right)\vec{B}=\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}+\vec{B}\,div\vec{A}$;
   d) $\left(\vec{A}\times\vec{B}\right)\cdot rot\vec{C}=\vec{B}\cdot\left(\vec{A}\cdot\vec{\nabla}\right)\vec{C}-\vec{A}\cdot\left(\vec{B}\cdot\vec{\nabla}\right)\vec{C}$;
   e) $\left(\vec{A}\times\vec{\nabla}\right)\times\vec{B}=\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}+\vec{A}\times rot\vec{B}-\vec{A}\cdot div\vec{B}$;
   f) $\left(\vec{\nabla}\times\vec{A}\right)\times\vec{B}=\vec{A}\cdot div\vec{B}-\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}-\vec{A}\times rot\vec{B}-\vec{B}\times rot\vec{A}$.
9. Calculate $grad\varphi(r)$, $div\varphi(r)\vec{r}$, $rot\varphi(r)\vec{r}$, $\left(\vec{a}\vec{\nabla}\right)\varphi(r)\vec{r}$, where $\vec{a}$ – a constant vector.
10. Calculate div and rot of vectors $(\vec{a}\cdot\vec{r})\vec{b}$, $(\vec{a}\cdot\vec{r})\vec{r}$, $[\vec{a}\,\vec{r}]$, $\varphi(\vec{r})[\vec{a}\times\vec{r}]$, $\vec{r}\times[\vec{a}\vec{r}]$, where $\vec{a}$ and $\vec{b}$ are constant vectors.
11. Calculate
    $grad\,\vec{A}(\vec{r})\cdot\vec{r}$, $grad\,\vec{A}(\vec{r})\cdot\vec{B}(r)$, $div\varphi(\vec{r})\vec{A}(r)$, $rot\varphi(\vec{r})\vec{A}(\vec{r})$, $\left(\vec{a}\vec{\nabla}\right)\varphi(\vec{r})\vec{A}(r)$.
12. Calculate $grad\dfrac{\vec{p}\,\vec{r}}{r^3}$ and $rot\dfrac{[\vec{p}\,\vec{r}]}{r^3}$, where $\vec{p}$ is a constant vector, using the expression for grad and rot in spherical coordinates.

Reference literature

1. Tamm I.E. Basic Theory of Electricity. – Moscow: Nauka, 1989. – 504 p.
2. Batygin V.V. and Toptygin I.N. Problems in Electrodynamics. – Moscow: Nauka, 1970. – 504 p.

Task № 2

Topic: Electrostatics and magnetostatics in vacuum

1. What is the main aim of electrostatics?
2. Write Coulomb's law for small charges.
3. Write the interaction force for two charge systems distributed with volume density $\rho_1(\vec{r})$ and $\rho_2(\vec{r})$. Present this force through the integral of electric field intensity $\vec{E}(\vec{r})$, made by system (2), and $\rho_1(\vec{r})$.
4. Derive Maxwell's equation for electrostatics, using the integral Gauss' law and Ostrogradsky-Gauss' theorem $\left(div\vec{E}=...\right)$ and Stoke theorem $\left(rot\vec{E}=...\right)$.
5. Show that Poisson's equation $\nabla^2\varphi=-4\pi\rho$ follows from Maxwell's equation, and $\vec{E}=-grad\,\varphi$, where $\varphi(\vec{r})=\int\dfrac{\rho(\vec{r}\,')dV'}{R}$, $\vec{R}=\vec{r}-\vec{r}\,'$.

6. Formulate Ampere and Bio-Savar's laws in differential form and explain their physical sense.
   Write the interaction force between two closed currents I_1 and I_2. Go to bulk currents.
7. Write Maxwell's equation for magnetostatics.
   Make sure that $\vec{B} = rot\vec{A}$,
   where

$$\vec{A} = \frac{1}{c}\int \vec{j}(\vec{r}\,')\frac{dV'}{R}$$

   and

$$\nabla^2 \vec{A} = -\frac{4\pi}{c}\vec{j}.$$

8. Write potentials $\varphi(\vec{r})$ and $\vec{A}(\vec{r})$ in dipole approximation.
9. Find the intensity $\vec{E}$ of electric field, whose potential φ is equal to:
   a) $\vec{a}\cdot\left[\vec{b}\,\vec{r}\right]$;
   b) $\left[\vec{a}\,\vec{r}\right]\left[\vec{k}\,\vec{r}\right]$;
   c) $\left(\vec{a}\,\vec{r}\right)\cos\left(\pi\,\vec{r}\right)$;
   d) $\frac{\vec{d}\,\vec{r}}{\vec{r}}$;
   vectors $\vec{a}$, $\vec{b}$, $\vec{k}$, $\vec{d}$ don not depend on coordinates and time.
10. Is it possible to create the electrostatic field in space with intensity
   $\vec{E} = \vec{a}\times\vec{b}$
   where $\vec{a}$ is a constant vector?
11. While calculating the rotor and divergence of magnetic field $\vec{B}$, make sure that the expression in magnetostatics

$$\vec{B}(\vec{r}) = \frac{1}{c}\int \frac{\vec{j}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')dV'}{\left|\vec{r}-\vec{r}\,'\right|^3}$$

   satisfies Maxwell's equations.

Reference literature

1. Tamm I.E. Basic Theory of Electricity. – Moscow: Nauka, 1989. – 504 p.
2. Sivuhin D.V. The General Course of Physics. – Moscow: Fizmatlit, 2002. – Vol 3. Electricity. – 656 p.

Task № 3

Topics: Alternating electromagnetic field. The electromagnetic induction law. Maxwell's equations for alternating electromagnetic field and their solution. Retarded potential

1. Explain the cause of e.m.f. occurrence in a moving conductor in magnetic field. Write the mathematic expression of the phenomenon of electromagnetic induction in integral and differential forms. At what expense does the e.m.f. appear in a closed motionless conductor in magnetic field?
2. Maxwell's displacement current. Its physical sense.
3. Write Maxwell's set of equations in differential form for alternating electric and magnetic fields. What laws experimentally proved underlie these equations?
4. Write Maxwell's set of equations for alternating fields in integral form.
5. How are Maxwell's equations solved? Write the equations which are satisfied by scalar and vector potentials of electromagnetic field with additional Lorenz' condition imposed on these potentials. How does the condition look like?
6. What physical principle is connected with the choice of electromagnetic potentials satisfying D`Alamber's equation in the form of retarded potentials? Explain the physical sence of such choice.
7. A conductor having a cusp form $y = kx^2$ is in a homogeneous magnetic field $\vec{B}$, perpendicular to plane XY. A shunt is transferred with a constant acceleration a from the vertex of the parabola, and the initial velocity is $\upsilon_0 = 0$. Find the e.m.f. of the induction in the developed contour as the function of coordinate y.

Direction

Let's denote $e.m.f. = \varepsilon_i$. Let's choose normal line $\vec{n}$ to the plane of the contour in the direction of vector $\vec{B}$. Then the change of magnetic flux $d\phi$ can be connected with change of vector flux $\vec{B}$ (from one side) through surface $d\vec{S}$, made by the shunt and the contour in the form of parabola, and from the other side ε_i.

Reference literature

1. Jackson J.D. Classical Electrodynamics. – 3rd ed. – New York: John Wiley & Sons, Inc., 1999.
2. Sivuhin D.V. The General Course of Physics. – Moscow: Nauka, 2002. – Vol 3. Electricity. – 656 p.

Task № 4

Topic: Relativistic mechanics and electrodynamics

1. System K' moves relative to system K at a velocity $\vec{V}$ along axis x, measured in system K. Write:
   a) direct and backward Lorentz transform for coordinates and time with the help of Lorentz matrix;
   b) relativistic law of composition of velocities;
   c) formulas of Lorentz transform at an arbitrary direction of velocity $\vec{V}$ with respect to the coordinate frame.
2. Prove (show) that a four-dimensional element of volume $dxdydzdt$ is invariant with respect to Lorentz transform.
3. A beam of light is contained in the element of a space angle $d\Omega$. Show that Lorentz transform leaves invariant value $\omega^2 d\Omega$.
4. Write Lorentz transform for impulse and energy.
5. Let a particle move at a velocity $\vec{\upsilon}$. Let's incorporate symbols $\vec{\beta} = \vec{\upsilon}/c$ and $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$. Write the expressions for a full energy of the particle, rest energy and particle impulse with the help of these symbols. What relation are the energy and the impulse of a relativistic particle connected with?
6. During the experiment the impulse and the energy of a particle were found. Find its speed and mass.
7. What is the path of a relativistic particle with mass m and charge q, incoming in a cross magnetic field with induction $\vec{B} = const$ at a velocity υ?
8. Write Lorentz transform for fields $\vec{E}$ and $\vec{B}$ (system K' moves respect to K at a velocity $\vec{\upsilon} = c\vec{\beta}$; a relativistic factor has form $\gamma = 1/\sqrt{1-\beta^2}$). Considering that speed $\vec{\upsilon}$ is directed in an arbitrary way, write Lorentz transform for $\vec{E}'_{\parallel}$, $\vec{B}'_{\parallel}$ and $\vec{E}_{\perp}$, $\vec{B}_{\perp}$ where indexes $\parallel$ and $\perp$ denote parallelism and perpendicularity to velocity $\vec{\upsilon}$.

References

1. Ugarov V.A. Special Theory of R elativity. – Moscow: Editorial URSS, 2005. – 384 p.
2. Bredov M.M., Rumjantsev V.V., Toptygin I.N. Classical Electrodynamics. – Moscow: Nauka, 1985. – 400 p.

Task № 5

Topic: Maxwell's equations in media

1. What is the difference of microscopic and macroscopic approaches to the description of electromagnetic phenomena in media?
2. Write the set of Maxwell's microscopic equations.
3. Give the definition of a physically small volume ΔV and physically small interval of time Δt. Give examples.
4. Describe the basic stages of small substance volume and time averaging of Maxwell's equations.
5. Write the expression for a central tendency of some component of electromagnetic field.
6. How do Maxwell's equations look like after the averaging in the presence of induced and offside densities of charges and currents?
7. State the connection of an induced charges and currents with specific dipole moments.

   a) Show that current $\vec{j}$ is the sum of 2 currents: polarization current $\frac{\partial \vec{P}}{\partial t}$ and current $rot\,\vec{M}$, conditioned by closed micro currents in substance.

   b) Using the formula for a full magnetic moment of a body

   $$\int \vec{M} dV = \frac{1}{2c} \int \vec{r} \times \vec{j} dV,$$

   where $\vec{j} = rot\,\vec{M}'$ (in the absence of induced charges), show that

   $$\vec{M}' = c\ \vec{M}.$$

   c) Using the formula

   $$\vec{j} = \frac{\partial \vec{P}}{\partial t} + c \cdot rot\ \vec{M},$$

   derive Maxwell's equations in the form

   $$rot\,\vec{E} = -c\frac{\partial \vec{B}}{\partial t},\quad div\left(\vec{E} + 4\pi\vec{P}\right) = 4\pi\rho_{ext},$$

   $$rot\left(\vec{B} - 4\pi\vec{M}\right) = \frac{4\pi}{c}\vec{j}_{ext} + \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{E} + 4\pi\vec{P}\right),$$

   $$div\,\vec{B} = 0.$$

   d) Write Maxwell's equations incorporating two new vectors

   $$\vec{D} = \ldots$$
   $$\vec{H} = \ldots$$

   e) Write Maxwell's equation after averaging in the presence of free bounded and offside densities of charges and currents.

f) Write the constraint equations which should be use to add Maxwell's equations.

8. Using Maxwell's equations with exterior currents and charges and constitutive equations (constraint equations), derive the equations

$$\frac{\partial \omega}{\partial t} + div\vec{S} + \vec{j}\,\vec{E} = 0,$$

where $\vec{S} = ...$ Pointing vector in macroscopic electrodynamics, $\omega = ...$ energy density of electromagnetic field in substance. Discuss the physical sense of this equation.

9. Write the energy conservation laws in macroscopic electrodynamics in integral form.

References

1. Bredov M.M., Rumyantsev V.V., Toptygin I.N. Classical Electrodynamics. – Moscow: Nauka, 1985. – P. 190–199.
2. Galitsky V.M., Ermachenko V.M. Macroscopic electrodynamics. – Moscow: Vysshaya Shkola, 1988. – 159 p.
3. Pamyatnyh E.A., Turov E.A. Basic Electrodynamics of Material Medium in Alternating and Non-uniform Fields. – Moscow: Nauka, Physmatlit, 2000. – 204 p.

Task № 6

Topic: Complex dielectric permittivity of a rarefied indifferent gas

1. Considering the interaction of an electromagnetic wave with atoms of indifferent gas it is possible to get units of volume for a dipole moment

$$\vec{P} = N\vec{d}\left(t\right),$$

where $\vec{d}(t)$ – dipole moment of atom. Using the connection

$$\vec{D} = \varepsilon(\omega)\vec{E} = \vec{E} + 4\pi\vec{P},$$

find $\varepsilon(\omega)$, having stated the dependence $\vec{P}$ on $\vec{E}$. For that use the oscillator model, on the basis of which it is possible to write the equation of motion for a dipole moment. The motion equation for a dipole moment has the form

$$\ddot{\vec{d}} + \gamma\dot{\vec{d}} + \omega_0^2 d = \frac{e^2}{m}\vec{E}_0 e^{-i\omega t}, \qquad (1)$$

where $\vec{d} = e\vec{r}(t)$ – dipole moment of atom and $\vec{r}(t)$ – vector of an electron bias relative to the field, ω_0 – oscillator frequency, component $\gamma\dot{\vec{d}}$ characterizes oscillation damping.

2. Discuss under what conditions it is possible to write equation (1).
3. Supposing that the solution of equation (1) has the form

$$\vec{d} = \vec{d}_0 \cdot e^{-i\omega t},$$

find this solution in the form

$$\vec{d} = (\ldots)\vec{E} = [\ldots]\cdot \vec{E}_0 \cdot e^{-i(\omega\cdot t-\psi)},$$

where ψ – phases difference between the oscillations of vectors $\vec{d}$ and $\vec{E}$, which depends on ω, γ, ω_0, charge e and mass m of an electron in an atom.

Define form $\vec{d}$ and ψ.

Draw the dependence diagram ψ on ω for the regions:

a) $0 \le \psi \le \frac{\pi}{2}, \quad \omega \le \omega_0$;

b) $\psi = \frac{\pi}{2}$ in resonance;

c) $\frac{\pi}{2} \le \psi \le \pi, \quad \omega_0 \le \omega \le \infty$.

4. Write the expressions for vectors

$$\vec{P} = N\vec{d},\ \vec{D} = \vec{E} + 4\pi\vec{P}$$

and find the expression for dielectric permittivity $\varepsilon(\omega)$, from the congruence c $\vec{D} = \varepsilon(\omega)\vec{E}$.

5. Supposing

$$\varepsilon(\omega) = \varepsilon' + i\varepsilon'',$$

find the form ε' и ε'' and draw the dependence diagram $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$ on the frequency.

Direction: incorporate the symbol

$$\omega_p{}^2 = \frac{4\pi Ne^2}{m},$$

where ω_P – plasma frequency.

6. Find the form $\varepsilon(\omega)$ at $\omega_0 = 0$ and $\gamma = 0$.
7. Using the integral law of energy conservation in the absence of free charges find as the sign of total flow of electromagnetic energy $\Pi = \int div\vec{S}dV$ is connected with

$$Jm\varepsilon(\omega) \equiv \varepsilon''(\omega),$$

and

$$\overline{\Pi} = \frac{1}{T}\int_0^T \Pi\, dt,$$

where T is a period of electromagnetic wave. Discuss the result, that is, reveal physical sense $Jm\varepsilon(\omega) \equiv \varepsilon''(\omega)$.

Using the integral law of energy conservation in the absence of free charges find as the sign of total flow of electromagnetic energy $\Pi = \int d\vec{w} \cdot \vec{S} \cdot dV$ is connected with

$$Jm\varepsilon(\omega) \equiv \varepsilon''(\omega),$$

and

$$\overline{\Pi} = \frac{1}{T}\int_0^T \Pi \, dt,$$

where T is a period of electromagnetic wave. Discuss the result, that is, reveal physical sense $Jm\varepsilon(\omega) \equiv \varepsilon''(\omega)$.

Reference literature

1. Ryazanov M.I. Introductory Electrodynamics of Condensed Matter. – Moscow: Physmatlit, 2002. – 320 p.

Task № 7

Topics: Dielectric permittivity of medium.
Dispersion relations of Kramers-Kronig

1. Write the integral relation between induction vector $\vec{D}(t)$ and field intensity $\vec{E}(t)$. Expanding the fields in Fourier integral, derive the constraint equation

$$\vec{D}(\omega) = \varepsilon(\omega)\vec{E}(\omega),$$

where $\varepsilon(\omega)$ – dielectric permittiivity.

2. Thinking that

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega),$$

state the kind of functions $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$, written through function $f(\tau)$. Discuss the physical sense of function $f(\tau)$ and its properties.

3. Study the function of a complex variable $\varepsilon(z)$. Write the integral Cauchy formula for function $\varepsilon(z)-1$ and explain the choice of integration contour for analytic function $\varepsilon(z)-1$.
4. Using the integral Cauchy formula for function $\varepsilon(z)-1$ and properties of parity of function $\varepsilon'(\omega)$ и $\varepsilon''(\omega)$, derive dispersion relation of Kramers-Kronig (including the media, which are conductors).
5. Discuss the way Kramers-Kronig relations are used to get the information about the dielectric permittivity of substances.
6. Dielectric permittivity of substance $\varepsilon(\omega)$ is connected with complex susceptibility by relation

$$\varepsilon(\omega) = 1 + \chi'(\omega) + i\chi''(\omega).$$

According to quantum theory of photon scattering, the susceptibility of an isotropic substance is expressed through the amplitude of photon scattering on the substance atoms at zero angle

$$\chi(\omega)=\frac{4\rho N}{\omega}\cdot f(0),$$

where N is the number of atoms in a unit of volume. Using the optic theorem, express the imaginary part of sensitivity through the photon-absorption cross-section $\sigma_{t0t}(\omega)$.

Write dispersion relations of Kramers-Kronig for susceptibility $\chi''(\omega)$ and $\chi'(\omega)$.

7. Read and make notes from the books cited further about the dependence of dielectric permeability ε (or susceptiability χ') on the frequency (or wave length).
8. Read and make notes from Ch. Kittel. Introduction to Solid State Physics. Ch.7 the following parts:

Dielectric polarizability (derive the formula of Clausius-Massoti), measuring of dielectric coefficient. Dipole relaxation and dielectric loss. Complex dielectric coefficient and the loss angle.

Reference literature

1. Bredov M.M., Rumyantsev V.V., Toptygin I.N. Classical Electrodynamics. – Moscow: Nauka, 1985. – 400 p.
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3. Bazylev V.A. and Zhevago N.K. Radiation of High Energy Particles in a Medium and External Fields. – Moscow: Nauka, 1987. – 272 p.
4. Galicky V.M., VYermachenko.M. Macroscopic electrodynamics. – Moscow: Higher school, 1988. – 159 p.
5. Denisov V.I. Introduction to Electrodynamics of Material Media. – Moscow: University Press, 1999. – 168 p.
6. Ryazanov M.I. Introductory Electrodynamics of Condensed Matter. – Moscow: Physmatlit, 2002. – 320 p.

Task № 8a

Topic: Lienard-Wiehert potentials and the field of a point charge

1. The task of finding the alternating electromagnetic field is solved by finding delayed potentials

$$\vec{A}(\vec{r},t)=\frac{1}{C}\int\frac{\vec{j}\left(\vec{r}',t'-\frac{R}{c}\right)dV}{R},$$

$$\vec{\varphi}(\vec{r},t)=\frac{1}{C}\int\frac{\rho\left(\vec{r}',t-\frac{R}{c}\right)dV'}{R},$$

where $R=\left|\vec{r}-\vec{r}'\right|$, $\vec{r}$ – radius-vector of the observation point, $\vec{r}'$ – radius-vector of the field source, dV' – the volume element of the field source.

Incorporate four-dimensional symbols for the potentials, current density and velocity. Write the expression of four-dimensional current density for a point charge *e*, which is in point $\vec{r}(t')$ and moves at a speed $c\vec{\beta}(t')$. Avoiding volume integration with the help of δ-function, show that the four-dimensional vector potential has the form

$$A_\mu(\vec{r}\ ,t)=e\int\frac{\beta_\mu(t')}{R(t')}\delta\left(t'+\frac{R(t')}{c}-t\right)dt'. \tag{1}$$

2. Using (1), write the expression for vector and scalar potentials $\vec{A}(\vec{r},t)$ and $\varphi(\vec{r},t)$ correspondingly.
3. Using the formulas

$$\vec{E}=-grad\varphi-\frac{1}{c}\frac{\partial A}{\partial t},$$

$$\vec{B}=rot\vec{A},$$

having made sure beforehand that

$$grad=\vec{n}\frac{\partial}{\partial R},$$

were

$$\vec{n}=\frac{\vec{R}}{R},$$

get the relations

$$\vec{E}\ \ (\vec{r},t)=e\int\left[\frac{\vec{n}}{R^2}\delta\ \ (f-t)+\frac{1}{cR}\left(\vec{\beta}-\vec{n}\right)\delta'_{f-t}(f-t)\right]dt', \tag{2}$$

$$\vec{B}\ \ (\vec{r},t)=e\int\left[\vec{n}\ \ \vec{\beta}\right]\left[-\frac{\delta\ \ (f-t)}{R^2}+\frac{1}{cR}\delta'_{f-t}(f-t)\right]dt'. \tag{3}$$

Here the symbol is incorporated

$$f(\ t')=t'+\frac{R(\ t')}{c}. \tag{4}$$

Let's denote also

$$\frac{df}{dt'}=1+\frac{1}{c}\frac{dR}{dt'}=1-\vec{n}\vec{\beta}=k. \tag{5}$$

4. Integrating by parts in the integral, containing $\delta'_{f-t}(f-t)$, and using the property of δ-function

$$\int g(x)\delta[f(x)-\alpha]dx=\left[\frac{g(x)}{df/dx}\right]_{f(x)=\alpha} \tag{6}$$

write (2) and (3) in the form

$$\vec{E}(\vec{r},t)=e\left[\frac{\vec{n}}{\kappa R^2}+\frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\vec{n}-\vec{\beta}}{\kappa R}\right)\right]_{ret}; \qquad (2')$$

$$\vec{B}(\vec{r},t)=e\left[\frac{\left[\vec{\beta}\vec{n}\right]}{\kappa R^2}+\frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\left[\vec{\beta}\vec{n}\right]}{\kappa R}\right)\right]_{ret}. \qquad (3')$$

5. Make sure that

$$\frac{1}{c}\frac{d\vec{n}}{dt'}=\frac{\left[\vec{n}\left[\vec{n}\,\vec{\beta}\right]\right]}{R}. \qquad (7)$$

And differentiating vector $\vec{n}$ by dt' in (2') and (3') where it is really included, get

$$\vec{E}(\vec{r},t)=e\left[\frac{\vec{n}}{\kappa^2 R^2}+\frac{\vec{n}}{c\kappa}\frac{d}{dt'}\left(\frac{1}{\kappa R}\right)-\frac{\vec{\beta}}{\kappa^2 R^2}-\frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\vec{\beta}}{\kappa R}\right)\right]_{ret}; \qquad (2'')$$

$$\vec{B}(\vec{r},t)=e\left[\left\{\frac{\vec{\beta}}{\kappa^2 R^2}+\frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\vec{\beta}}{\kappa R}\right)\right\}\cdot\vec{n}\right]_{ret}. \qquad (3'')$$

Show that

$$\vec{B}=\left[\vec{n}\cdot\vec{E}\right]\cdot$$

6. Using the expression

$$\frac{d}{dt'}\vec{\beta}=\dot{\vec{\beta}}$$

and having made sure beforehand that

$$\frac{1}{c}\frac{d}{dt'}\left(k\quad R\right)=\beta^2-\vec{\beta}\quad\vec{n}-\frac{R}{c}\vec{n}\quad\dot{\vec{\beta}},$$

after some transformations, carry formulas (2'') and (3'') to the form

$$\vec{E}(\vec{r},t)=e\left[\frac{\vec{n}}{\kappa^2 R^2}+\frac{\vec{n}}{c\kappa}\frac{d}{dt'}\left(\frac{1}{\kappa R}\right)-\frac{\vec{\beta}}{\kappa^2 R^2}-\frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\vec{\beta}}{\kappa R}\right)\right]_{ret} \qquad (8')$$

or

$$\vec{E}\left(\vec{r},t\right)=e\left[\frac{\left(\vec{n}-\vec{\beta}\right)\left(1-\beta^2\right)}{k^3R^2}\right]_{ret}+\frac{e}{c}\left[\frac{\vec{n}}{k^3R}\times\left\{\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}\right]_{ret} \qquad (8'')$$

and

$$\vec{B}=\left[\vec{n}\vec{E}\right].$$

Discuss the physical sense of the 1st and the 2nd addends in $(8'')$.

Reference literature

1. Jackson J.D. Classical Electrodynamics. – New York, London: Wiley, 1962. – 656 p.
2. Bredov M.M., Rumyantsev V.V., and Toptygin I.N. Classical Electrodynamics. – Moscow: Nauka, 1985. – 400 p.

Task № 8b

Topic: Total power, radiating by an accelerating charge Larmor's formula and its relativistic generalization. The radiation loss of energy in accelerators

1. Using the expression for electromagnetic field $\vec{E}(\vec{r},t)$, incorporated with (through) Lienard-Wiehert potentials (taking into account only the radiation field of an accelerating charge),

   Calculate the instant energy flux

$$\vec{S} = ...$$

and energy flux inside the space angle $d\Omega$ at a far distance from the particle

$$dI = \vec{S}\,\vec{n}\,R^2 d\Omega = \ldots$$

   Write the expression for dI in non-relativistic approximation $\beta << 1$ and $\kappa \sim 1$, incorporating the angle θ between $\vec{n}$ and $\vec{\beta}$.

   Derive Larmor's formula for the radiation intensity

$$I = \int \left(\frac{dI}{d\Omega} \right) d\Omega = \frac{2}{3} \frac{e^2 a^2}{c^3}. \qquad (1)$$

2. Having written formula (1) in the form

$$I = \frac{2}{3} \frac{e^2}{m^2 c^3} \left(\frac{d\vec{p}}{dt} \cdot \frac{d\vec{p}}{dt} \right). \qquad (2)$$

   It is possible to come to its relativistic generalization

$$I = \frac{2}{3} \frac{e^2}{m^2 c^3} \left(\frac{dp_\mu}{d\tau} \cdot \frac{dp_\mu}{d\tau} \right), \qquad (3)$$

   where

$$\left(\frac{dp_\mu}{d\tau} \cdot \frac{dp_\mu}{d\tau} \right) = inv.$$

   and τ – intrinsic time in the system, which is connected with the particle.

   Using the relativistic relation for impulse $\vec{p} = m\vec{\beta} c\gamma$, energy $E = \gamma mc^2$ and

$$E^2 = m^2 c^4 + \vec{p}^{\,2} c^2$$

   show that

$$\left(\frac{dp_\mu}{d\tau} \cdot \frac{dp_\mu}{d\tau} \right) = \left(\frac{d\vec{p}}{d\tau} \right)^2 - \frac{1}{c^2} \left(\frac{dE}{d\tau} \right)^2 = \left(\frac{d\vec{p}}{d\tau} \right)^2 - \beta^2 \left(\frac{dp}{d\tau} \right)^2.$$

After finding derivates

$$\frac{d\vec{p}}{d\tau} = \frac{d}{d\varepsilon}(\gamma m \vec{\upsilon}) = mc\frac{d}{d\tau}\left(\gamma\vec{\beta}\right)$$

and

$$\frac{dp}{d\tau} = mc\frac{d}{d\tau}\left(\gamma\beta\right),$$

taking into account that $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta^2 = \vec{\beta}^2$, $\beta\dot{\beta} = \vec{\beta}\dot{\vec{\beta}} <$

$$\left(\frac{d\vec{\beta}}{dt'} = \dot{\vec{\beta}} \ , \quad d\tau = \frac{dt'}{\gamma}\right) \ , \quad \frac{d\vec{p}}{d\tau} = \gamma\frac{d\vec{p}}{dt'}$$

and the relation

$$\left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2 = \left[\vec{\beta}\times\dot{\vec{\beta}}\right]\cdot\left[\vec{\beta}\times\dot{\vec{\beta}}\right] = \vec{\beta}^2\,\dot{\beta}^2 - \left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2,$$

derive the formula

$$\left(\frac{dp_\mu}{d\tau}\cdot\frac{dp_\mu}{d\tau}\right) = \gamma^6 m^2 c^2\left\{\dot{\vec{\beta}}^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right\}$$

and Lienard's formula

$$I = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[\dot{\vec{\beta}}^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right],$$

which is the relativistic generalization of Larmor's formula.

3. Show that the radiation intensity of electrons in the linear accelerator is determined by the formula

$$I = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2,$$

where E – energy, x is the pathway.

The radiation loss of energy per one turn in the cyclic accelerator is given by the formula

$$\delta E\left(МэВ\right) = 8{,}85\cdot 10^{-2}\,\frac{\left[E\left(ГэВ\right)\right]^4}{\rho\left(м\right)}.$$

Calculate δE for the synchrotron with $E_{\max} \sim 5$ GeV, ρ = 10 м.

References

1. Jackson J.D. Classical Electrodynamics. New York, London: Wiley, 1962. – 656 p.
2. Bredov M.M., Rumyantsev V.V., Toptygin I.N. Classical Electrodynamics. – Moscow: Nauka, 1985. – 400 p.

3. Widemann H. Particle Accelerator Phisics. Vol. 2. Nonlinear and Higher – Order Beam Dynamics. – Berlin – Heidelberg: Springer Verlag, 1999. – 479 p..

Task № 9

Topic: Angular distribution of an accelerating charge

1. Velocity $\vec{\upsilon}$ of a relativistic particle at some moment of time t' is parallel to its acceleration $\dot{\vec{\upsilon}}$. The instant distribution of radiation intensity is given by formula 25.8

$$\frac{dI}{d\Omega} = \frac{e^2 \dot{\vec{\upsilon}}^2 \sin^2\theta}{4\pi c^3 (1-\beta\cos\theta)^6},$$

where $\beta = \upsilon/c$, θ is the angle between the acceleration direction and the direction of radiation.

2. Draw angular (polar) diagram of radiation for the cases:
a) $\upsilon \approx 0$ и б) $\upsilon \neq 0$.

3. Find what angle θ_0 does the radiation maximum appear at.
What is the limit of the angle θ_0 a) at $\beta \to 0$; b) at $\beta \to 1$?
What angle to the velocity direction does the radiation in ultra-relativistic case appear at?
Thinking $\theta << 1$ show, that formula $\frac{dI}{d\Omega}$ takes the form

$$\frac{dI}{d\Omega} = \frac{8e^2 \dot{\vec{\upsilon}}^2 \theta^2}{\pi c^3 \left[\gamma^{-2} + \theta^2\right]^5},$$ where $\gamma^{-1} = mc^2/\varepsilon$, ε – total energy of a particle.

What cone is the radiation for an ultra-relativistic particle concentrated in?

4. Find what is the total intensity of radiation

$$I = \int \frac{dI}{d\Omega} d\Omega$$ and full speed of energy loss $-d\varepsilon/dt'$.

References

1. Jackson J.D. Classical Electrodynamics. – 3rd ed. – New York: John Wiley & Sons, Inc., 1999. – 795 p.
2. Bredov M.M., Rumyantsev V.V., Toptygin I.N. Classical Electrodynamics. – Moscow: Nauka, 1985. – 400 p.
3. Batygin V.V. and Toptygin I.N. Problems in Electrodynamics. – Moscow: Nauka, 1970. – 504 p.

Task № 10

Topic: Synchrotron radiation

1. Describe the basic features of the radiation of an ultra-relativistic particle moving at constant velocity υ along the circle with an instant radius of curvature ρ.

   Write the formula for the spectral intensity of radiation per a unit of space angle. How is the duration of radiation impulse $\Delta t'$ connected with the instant radius of curvature?

2. Let the plane of a particle path coincide with the plane XOY and at moment $t=0$ the particle pass through the coordinate origin. Incorporating unit vectors of polarization $\vec{e}_{\parallel}$ and $\vec{e}_{\perp}$, directed correspondingly along axis OY and perpendicular to the plane of the particle path, show that the double vector product included in the sub integral expression for $\frac{dI}{d\Omega}$ has the form

$$\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)=\beta\left\{-\bar{e}_{\parallel}\sin\left(\frac{\upsilon t}{\rho}\right)+\bar{e}_{\perp}\cos\left(\frac{\upsilon t}{\rho}\right)\sin\theta\right\}.$$

3. Make sure, that at a small interval of time near $t=0$ and small angles after expanding of trigonometric functions in series by small parameters $\upsilon t/\rho$ and θ, and limiting oneself to

$$\beta\approx 1-\gamma^{-2}/2,$$

   it is possible to get the expression for the radiation intensity in the form of expansion by polarization vectors

$$\frac{dI(\omega)}{d\Omega}=\frac{e^2\omega^2}{4\pi^2 c}\left|-\vec{e}_{\parallel}A(\omega)+\vec{e}_{\perp}A(\omega)\right|^2,$$

   where $A_{\parallel}(\omega)$ and $A_{\perp}(\omega)$ are the amplitudes, which correspond to radiation polarization in the path plane and perpendicular to the path plane correspondingly.

4. Write the expressions for amplitudes $A_{\parallel}(\omega)$ and $A_{\perp}(\omega)$ through modified Bessel's functions $K_{1/3}$ and $K_{2/3}$ (Macdonald's functions) and derive the expression for $dI(\omega)/d\Omega$:

$$dI/d\Omega=\frac{e^2}{3\pi^2 c}\left(\frac{\varpi\rho}{c}\right)\left(\gamma^{-2}+\theta^2\right)\left[K_{2/3}^2(\zeta)+\frac{\theta^2}{\gamma^{-2}+\theta^2}K_{1/3}^2(\zeta)\right],$$

   where

$$\zeta\cong\frac{1}{2}\frac{\omega}{\omega_C}\left(1+\gamma^2\theta^2\right)^{3/2},$$

and

$$\omega_C = \frac{3}{2}\frac{c\gamma^3}{\rho}$$ – critical frequency of a photon.

5. What can you tell about the radiation polarization in relativistic and non-relativistic cases? What area of angles is the radiation basically concentrated in at frequencies $\omega \sim \omega_C$?
6. Write the derived formulas at a periodic particle motion along the circle taking into account that the radiation frequency ω is aliquot to the basic frequency ω_0

$$\omega = n\,\omega_0,$$

in the form of radiation intensity on n-th harmonic

$$\frac{dI_n}{d\Omega} = \ldots,$$
$$I_n = \ldots$$

References

1. Jackson J.D. Classical Electrodynamics. – 3rd ed. – New York: John Wiley & Sons, Inc., 1999. – 795 p.
2. Radiation Theory of Relativistic Rarticles / ed. V.A. Bordovitsyn. – Moscow: Fizmatlit, 2002. – Ch. 1. – Bagrov V.G., Bordovitsyn V.A. Classical Theory of Synchrotron Radiation.
3. Widemann H. Particle Accelerator Phisics Vol. 2, Nonlinear and Higer – Order Dynamics. – Berlin – Heidelberg: Springer Verlag, 1999. – 479 p.

Task № 11

Topic: The theory of synchrotron radiation

There is a strict theory of synchrotron radiation), in particular, it is derived Shott's formula for spectral and angular distribution of power of synchrotron radiation $W(\nu,\theta)$, where ν is a number of harmonic, connecting radiation frequency ω with basic frequency ω_0.

1. Show that in the relativistic case the frequency of electron rotation in constant magnetic field *B* equals to $\omega_0 = ecB/\varepsilon$, where ε – total energy of electron.
2. Study section "Theory of synchrotron radiation" paying attention to the way Shott's formula is derived. Describe the basic stages of the calculations leading to this formula. Give its form and write the total power of radiation using Shott's formula.
3. What form does the total power of radiation *W* have (the relativistic generalization of Lienard's formula)? How is it connected with Larmor's non-relativistic formula?
4. Give the generalization of Shott's formula, which take into account the polarization properties of synchrotron radiation.

How do σ- and π-components of radiation power look like (components of linear polarization), written through the total power of radiation: a) in non-relativistic case; b) in relativistic case?

5. Discuss the features of dependence of linear and circular polarization SR on the radiation angle.
6. There are N electrons on a circular orbit at the same time. Study the influence of the interference of the fields, made by these electrons on the radiation intensity of n-th Fourier-harmonic. Study special cases:
   a) completely disordered arrangement of the electrons;
   b) regular array of the electrons at the angular distance $2\pi/N$ from each other;
   c) electrons are arranged in the form of a bunch, the dimensions of which are small in comparison with the orbit (in some cases, the result depends considerably on the relation of a wave length to the dimensions of the bunch.)

   Direction: while solving task 6*, it is necessary to find beforehand Fourier-components of radiation field $\vec{A}_n$, $\vec{B}_n$ of the charge e, which moves along a circular path of radius a at a relativistic speed υ and to study the character of polarization of Fourier-components.
7. The application of synchrotron radiation in science, medicine and engineering. Give your examples.

References

1. Jackson J.D. Classical Electrodynamics. – New York, London: Wiley, 1962. – 656 p.
2. Ternov I.M. and Mihajlin V.V. Synchrotron radiation. Theory and experiment. – Moscow: Energoatomizdat, 1986. – 296 p.
3. Batygin V.V. and Toptygin I.N. Problems in Electrodynamics. – 2nd ed. rev. sup. – Moscow: Nauka, 1970. – 504 p.
4. Kulipanov G. N., Skrinsky A N. U.Ph. Vol. 122, issue 3, 1977. – P. 369–418.

Task № 12

Topic: Undulator radiation and other kinds of the intensive electromagnetic radiation of relativistic electrons

1. Mode of functioning of the undulator and wiggler.

   Compare the effective length of SI formation with the corresponding length of formation of undulator radiation. What conclusion can be done?
2. How are the spectral-angular distributions of the radiation intensity from the field made on one and N periods of the undulator connected with each other? Do more detailed computations given in the lecture.
3. The effective parameter of the undulator is K. Its connection with the radiation frequency and the radiated wavelength λ. Do the corresponding calculations, given in the lecture, more in details.

4. The electron with impulse $p = 500 MeV/c$ moves in the plane undulator in field $B = 10\ kHz$.
   a) What is the instant radius of the path which the electron moves along? Make sure that $R_0 = p/0{,}3B$, where R is measured in *cm*, p – in *MeV/c and B* – in *kHz*.
   b) Find the synchrotron radiation loss, emitted by the electron per one turn on the path of radius R_0.
   c) Find the radiation loss, emitted by the electron on the curve $\psi \sim \dfrac{d}{2R_0}$, where the period of undulator d = 4 cm.
   d) Calculate the undulator radiation loss per one period $\Delta t = d/c$.
   e) Find the value of the undulator parameter $K = \gamma\psi$, where $\psi \sim d/R_0$ for the considered case.
   f) Evaluate the wavelength of the undulator radiation, emitted at angle $\theta = 0$ in the reference frame (θ is the angle between the direction of the longitudinal velocity $\vec{\beta}_{\|}$ and the wave vector $\vec{k}$ of the electromagnetic wave.)
5. Describe briefly the typical features of the following kinds of radiation:
   a) Compton scattering, which appear when the electrons with a great number of particles in the bunch scatter on the photons from a high-throughput laser
   b) coherent bremsstrahlung;
   c) channeling radiation;
   d) transition radiation;
   e) parametric X-radiation;
   f) the radiation called as Smith-Purcell's effect in scientific literature.

   Direction: see the corresponding descriptions of these kinds of radiation in the lecture and in the recommended books.

Reference literature

1. Ternov I.M. and Mihajlin V.V. Synchrotron radiation. Theory and experiment. – Moscow: Energoatomizdat, 1986. – 296 p.
2. Bazylev V.A. and Zhevago N.K. Radiation of High Energy Particles in a Medium and External Fields. – Moscow: Nauka, 1987. – 272 p.
3. Rullhusen P., Artru X., Dhez P. Novel radiation sources using relativistic elektrons. – Series on Synhrotron Radiation Techniques and Applications – Vol. 4, World Scientific Pubblishing, 1998. – 202 p.
4. Wuchao A., Tigner M. Handbook of Accelerator Phisics and Engineering. –, Singapur, New Yersey, London, Hong Kong: World Scientific.

TESTS

I. Electricity

1. Which of the further mentioned characteristics does the electric field have?

1. It influences charged particles and bodies.
2. It has energy and inertness.
3. It influences non-charged material bodies.
4. It influences magnetized bodies.
5. It is conditioned by the magnetic field changing in time.

2. Which of the further mentioned characteristics does the electrostatic field have?

1. It influences material bodies.
2. It influences charged particles or bodies.
3. It is conditioned by a magnetic filed changing in time.
4. It influences the conductors with current.
5. It has energy.

3. How should the distance between point charges be changed to reduce the interaction force between them 4 times?

1. To be increased 4 times.
2. To be reduced 4 times.
3. To be increased 16 times.
4. To be increased 2 times.
5. To be reduced 2 times.

4. What determines the numerical value of intensity in a given point of electric field?

1. The potential energy of a positive unit charge placed in a given point of the field.
2. The potential energy of an arbitrary "trial" charge placed in a given point of the field.
3. The force influencing a positive unit charge in a given point of the field.
4. The force influencing any trial charge placed in a given point of the field.
5. The work done while moving a positive unit charge from perpetuity to a given point of the field.

5. What determines a numerical value of potential in a given point of electrostatic field?

1. The potential energy of a positive unit charge placed in a given point of the field.
2. The potential energy of any trial charge placed in a given point of the field.
3. The work done while moving a positive unit charge from perpetuity to a given point of the field.
4. The force influencing a positive unit charge in a given point of the field.
5. The force influencing a trial charge placed in a given point of the field.

6. What determines the circulation of intensity vector of the electric field along the closed contour *L*?

1. The line integral of the kind $\oint E\,dl\cos\left(\widehat{\vec{E}\,\overrightarrow{dl}}\right)$.
2. The force influencing the positive unit charge moved along the given contour.
3. The line integral of the kind $\oint qE\,dl\cos\left(\widehat{\vec{E}\,\overrightarrow{dl}}\right)$.
4. The work done by the field while moving a positive unit charge along the given contour.
5. The work done by the field while moving an arbitrary electric charge along the given contour.

7. Which of the enumerated characteristics of the electrostatic field shows that this field id perpendicular?

1. The electrostatic field influences charged particles and bodies.
2. The work at moving a charge in the electrostatic field along the closed contour is equal to zero.
3. The charge placed in a given point of the field has the only value of potential energy.
4. The circulation of intensity vector of the electrostatic field along the closed contour equals to zero.
5. The electrostatic field has energy.

II. Electrostatics of dielectrics

1. How does the interaction force of two point charges change at their moving from the medium with a relative dielectric permeability ε into the vacuum at a fixed distance *r* between the charges? The task should be solved in SI system.

1. It increases ε times.
2. It reduces ε times.
3. It reduces $\varepsilon_0\varepsilon$ times.
4. It increases $\varepsilon_0\varepsilon$ times.
5. It increases $4\pi\varepsilon\varepsilon_0$ times.

2. How does the interaction force of two point charges change when they move from vacuum into medium if the distance between them is reduced three times?

1. It increases 27 times.
2. It reduces 27 times.
3. It increases 9 times.
4. It reduces 243 times.
5. It reduces 9 times.

3. Which of the following expressions determine the intensity of the electrostatic field of a point charge (in SI system)?

1. $q/4\pi\varepsilon_0\varepsilon r^2$; 2. $q/4\pi\varepsilon_0\varepsilon r$; 3. $q/4\pi r^2$;
4. $-q/4\pi\varepsilon_0\varepsilon r^2$; 5. $-q/4\pi r^2$.

4. What is the relation between the intensities in point's *A* and *C* of the field of a point charge $+q$?

(OA=AC)

1. $E_A = E_C$; 2. $E_A = 2E_C$; 3. $E_A = 4E_C$;
4. $E_A = \frac{1}{2}E_C$; 5. $E_A = \frac{1}{4}E_C$.

5. Point out the answers where the unit of electric field intensity, the unit of intensity circulation of electric field along the given contour, the unit of surface density of a charge and the unit of capacity and electric induction are in the following order:

1. N/C; J; C/m; F; V/m.
2. V/m; V; C/m^2; F; C/m^2.
3. N; V; C; C; C/m^2.
4. J; F; C/m^2; V; C/m.
5. N/C; V; C/m^2; F; C/m^2.

6. Point out the answer, which characterizes the process of dielectric polarization?

1. The bias of molecular dipoles of the external electric field.
2. The bias of molecular dipoles in the direction of potential gradient of external electric field.
3. The dielectric acquires some charge in the electric field.
4. The dielectric bias in the direction of external electric field.
5. The appearance of the preferred orientation of molecular dipoles in electric field.

7. What is the relation between the intensity $\vec{E}$ of electric field in dielectric and the intensity $\vec{E}_0$ of the external electric field?

1. $|\vec{E}| = |\vec{E}_0|$; 2. $|\vec{E}| > |\vec{E}_0|$; 3. $|\vec{E}| < |\vec{E}_0|$; 4. $\vec{E} = \vec{E}_0$.

8. What is the vector of dielectric polarization?

1. The dipole moment of dielectric molecule.
2. The vector sum of dipole moments of the molecules of the whole dielectric.
3. The surface charge, which appears under the dielectric polarization.
4. The vector sum of dipole moments of the dielectric molecules, which are divided by the unit of its volume.
5. The vector sum of dipole moments of the dielectric molecules, which is referred to its mass unit.

9. What happens in the polar dielectric when it is brought into the homogeneous electrostatic field?

1. Dielectric electrization.
2. The bias of molecular dipoles along the field.
3. The bias of molecular dipoles against the field.
4. The alignment of the electric moments of molecular dipoles against the field.
5. The alignment of the electric moments of molecular dipoles along the field.

10. Which of the following dielectric features is specific only for non-polar dielectrics in the absence of electric field?

1. The sum vector of electric moments of all the dielectric molecules is equal to 0.
2. The resultant vector of electric moments of molecules of the volume unit of dielectric equals to 0.
3. The electric moment of every molecule is equal to 0.
4. The electric moment of every molecule differs from 0.
5. The resultant vector of electric moments of the molecules included in the mass unit of dielectric is equal to 0.

11. Which of the following expressions (in SI system) define the cubic energy density of electric field?

1. $c\varphi^2/2$; 2. $\frac{1}{2}DE$; 5. $\varepsilon_0\varepsilon E$.

3. $\frac{c\Delta\varphi^2}{2}$; 4. $\frac{1}{2}\varepsilon_0\varepsilon E^2$;

III. Current in conductors. Magnetic field

1. Point out which of the following conditions determine the conductor resistance.

1. The emf of the source, which the conductor is connected with.
2. The current in the circuit.
3. The dimension and the conductor material.
4. The potential difference at the ends of the cord.
5. The cord voltage.

2. Point out the answer where the unit of voltage on subcircuit, the unit of current density, the unit of resistivity, the unit of emf, the conduction unit and the unit of specific conductivity are in the following order:

1. V; A/m^2; S/m; V; $\Omega\cdot m$; S.
2. V; A; $\Omega\cdot m$; V; O_M; Ω/m.
3. V; A; Ω; V; $\Omega\cdot m$; S.
4. V; A/m^2; $\Omega\cdot m$; V; Ω; $\Omega^{-1}m^{-1}$.
5. V; A/m^2; Ω; V; Ω; Ω/m.

3. What determines the numerical value of the magnetic moment of the contour with current?

1. Product of current multiplied by the contour length.
2. Product of current multiplied by the contour square.
3. Product of magnetic induction of the field by the contour square.
4. Mechanic moment acting on the contour with current in magnetic field.
5. The work performed when the contour with current turns in magnetic field.

4. Point out the expression defining the circulation of the induction vector of magnetic field along the closed contour L:

1. $\oint_L B\,dl\cos\left(\widehat{\vec{B}\,\vec{dl}}\right)$; 3. $\int_S B_n dS$;

2. $\oint_L \left(\vec{B}\,\vec{dl}\right)$; 4. $\int_S B\,dS_n$.

5. Which of the following values determine the emf of induction, which appears in the closed contour?

1. The value of magnetic flux through the surface limited by the given contour.
2. The velocity of magnetic flux change through the surface limited by the contour.
3. The contour resistance.
4. The induction value of the external magnetic field.
5. The velocity of induction change of the external magnetic field.

IV. SPECIAL THEORY OF RELATIVITY

1. Which of the formulas determines the velocity of light in the given medium?

1. $\upsilon = \dfrac{c}{\sqrt{\mu\varepsilon}}$; 2. $\upsilon = \dfrac{1}{\sqrt{\varepsilon_0\varepsilon\,\mu_0\mu}}$; 3. $\upsilon = \dfrac{2\pi R}{T}$;
4. $\upsilon = \sqrt{\dfrac{\gamma RT}{\mu}}$; 5. $\upsilon = \sqrt{k/\rho}$.

2. Which of the following formulas are included in Lorentz transform?

1. $u = \dfrac{u'+\upsilon}{1+u'\upsilon/c^2}$; 2. $x = \dfrac{x'-\upsilon t}{\sqrt{1-\upsilon^2/c^2}}$; 3. $t' = \dfrac{t - x\upsilon/c^2}{\sqrt{1-\upsilon^2/c^2}}$;
4. $l' = l\sqrt{1-\upsilon^2/c^2}$; 5. $\nu' = \nu\dfrac{c+\upsilon}{c-\upsilon}$.

3. Which of the following facts is postulated by the principles of relativity by Einstein?

1. The velocity of light in vacuum is constant.
2. The velocity of any motion in a given medium is less than the velocity of light in this medium.
3. The laws of mechanics are invariant in inertial systems.
4. The laws of mechanics are invariant in any systems.
5. All the physical values are relative.

4. What form does the formula $t' = \dfrac{t + x\upsilon/c^2}{\sqrt{1-\upsilon^2/c^2}}$ take for non-relativistic motions?

1. $t' = t$; 2. $t' = t - x/\upsilon$; 3. $t' = t + x/\upsilon$;
4. $l' = l\sqrt{1-\beta^2}$; 5. $l' = \sqrt{l}$.

5. What form does the formula $l' = l\sqrt{1-\upsilon^2/c^2}$ take for non-relativistic motions?

1. $l' = l + \upsilon t$; 2. $l' = l$; 3. $l' = l - \upsilon t$;
4. $l' = l\sqrt{1-\beta^2}$; 5. $l' = \sqrt{l}$.

6. A spaceship moves to the star, which is at the distance of 4.3 light years, and goes back at a speed 1000 km/h. How many days (twenty-four hours) will the ship clock be slow in comparison with the clocks on the Earth? The answer should be rounded up to a whole number and be put in the computer.

7. How many years will pass on the Earth if it passes 10 years in the ship, which moves at a speed 0,99s with respect to the Earth?

1. 20 years; 2. 99 years; 3. 10,99 years; 4. 71 years; 5. 99 years.

8. The electrons flying out of a cyclotron have kinetic energy 0.67 MeV. How many percents of velocity of light is the speed of the electrons?

1. 50; 2. 96; 3. 90; 4. 99; 5. 47.

9. Which of the following formulas express the relativistic law of composition of speeds?

1. $\nu' = \nu \dfrac{c+\upsilon}{c-\upsilon}$; 2. $\upsilon = \dfrac{\upsilon' + V}{1+\upsilon V/c^2}$;

3. $\upsilon = \upsilon' + V$; 4. $\nu' = \nu \dfrac{1+V/c}{\sqrt{1-V^2/c^2}}$.

APPENDIX

A.1. Basic elements of the theory of field

A.1.1. Formulas of the vector analysis

First derivatives

1. $\vec{\nabla} = \vec{e}_1 \frac{\partial}{\partial x} + \vec{e}_2 \frac{\partial}{\partial y} + \vec{e}_3 \frac{\partial}{\partial z}$.

2. $grad\, f \equiv \vec{\nabla} f = \vec{e}_1 \frac{\partial f}{\partial x} + \vec{e}_2 \frac{\partial f}{\partial y} + \vec{e}_3 \frac{\partial f}{\partial z}$.

3. $\left(\vec{A} \cdot grad\right) f \equiv \left(\vec{A} \cdot \vec{\nabla}\right) f = A_x \frac{\partial f}{\partial x} + A_y \frac{\partial f}{\partial y} + A_z \frac{\partial f}{\partial z}$.

4. $div\, \vec{A} = \left(\vec{\nabla} \cdot \vec{A}\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$.

5. $rot\, \vec{A} = \left[\vec{\nabla} \vec{A}\right] = \vec{e}_1 \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \vec{e}_2 \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \vec{e}_3 \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$.

The invariant determinations of divergence and rotor:

6. $div\, \vec{A} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint A_n dS$.

7. $rot\, \vec{A} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint A_l dl$.

Surface derivatives:

8. $Div\, \vec{A} = A_{n_2} - A_{n_1}$.

9. $Rot\, \vec{A} = [\vec{n}(\vec{A}_2 - \vec{A}_1)]$.

Derivatives of product:

10. $grad(f\, g) = \vec{\nabla}(f\, g) = g\vec{\nabla} f + f\vec{\nabla} g = g\, grad\, f + f\, grad\, g$.

11. $div\left(f\, \vec{A}\right) = (\vec{\nabla} \cdot \left(f\, \vec{A}\right)) = f\left(\vec{\nabla} \cdot \vec{A}\right) + \vec{A} \cdot (\vec{\nabla} f) = f\, div\, \vec{A} + \vec{A}\, grad\, f$.

12. $rot\left(f\, \vec{A}\right) = \left[\vec{\nabla}\left(f\, \vec{A}\right)\right] = f\left[\vec{\nabla} \vec{A}\right] + \left[\left(\vec{\nabla} f\right) \vec{A}\right] = f\, rot\, \vec{A} - \left[\vec{A} \cdot grad\, f\right]$.

13. $div[\vec{A}\vec{B}] = \vec{\nabla} \cdot [\vec{A} \cdot \vec{B}] = \vec{B} \cdot [\vec{\nabla} \vec{A}] - \vec{A} \cdot [\vec{\nabla} \vec{B}] = \vec{B} \cdot rot\, \vec{A} - \vec{A} \cdot rot\, \vec{B}$.

14. $rot[\vec{A}\vec{B}] = [\vec{\nabla}[\vec{A}\vec{B}]] = \left(\vec{B} \cdot \vec{\nabla}\right)\vec{A} - \left(\vec{A} \cdot \vec{\nabla}\right)\vec{B} + \vec{A}\left(\vec{\nabla} \cdot \vec{B}\right) - \vec{B}\left(\vec{\nabla} \cdot \vec{A}\right) = \left(\vec{B}\, grad\right)\vec{A} -$
$- \left(\vec{A}\, grad\right)\vec{B} + \vec{A}\, div\, \vec{B} - \vec{B}\, div \vec{A}$.

The second derivatives:

15. $div\, grad\, f = \nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

16. $rot\, grad\, f \equiv 0$.

17. $div\, rot\, \vec{A} \equiv 0$.

18. $rot\, rot\, \vec{A} = [\vec{\nabla}[\vec{\nabla}\vec{A}]] = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \vec{\nabla}^2\vec{A} = grad\, div\, \vec{A} - \Delta\vec{A}$.

19. $\Delta\vec{A} = \frac{\partial^2\vec{A}}{\partial x^2} + \frac{\partial^2\vec{A}}{\partial y^2} + \frac{\partial^2\vec{A}}{\partial z^2}$.

In above brought formulas f, g, $\vec{A}$, $\vec{B}$: are a functions of coordinates: $f = f(x, y, z)$ and etc.

Double vector product:

20. $[\vec{A}[\vec{B}\vec{C}]] = \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$.

A.1.2. Basic orthogonal system (u_1, u_2, u_3)

System	u_1	u_2	u_3	h_1	h_2	h_3
Cartesian	x	y	z	1	1	1
Cylindrical	r	θ	z	1	r	1
Spherical	r	θ	φ	1	r	$r\sin\theta$

$$d\vec{s} \equiv h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3,$$

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2,$$

$$dV = h_1 h_2 h_3 du_1 du_2 du_3,$$

$$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\hat{u}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\hat{u}_3,$$

$$\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right],$$

$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\times\left\{h_1\hat{u}_1\left[\frac{\partial}{\partial u_2}(h_3A_3) - \frac{\partial}{\partial u_3}(h_2A_2)\right] + h_2\hat{u}_2\left[\frac{\partial}{\partial u_3}(h_1A_1) - \frac{\partial}{\partial u_1}(h_3A_3)\right] + \right.$$

$$\left. + h_3\hat{u}_3\left[\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\partial}{\partial u_2}(h_1A_1)\right]\right\},$$

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right].$$

Cylindrical system:

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2,$$

$$dV = r\, dr\, d\theta\, dz.$$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{r} + \frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{\partial\psi}{\partial z}\hat{z},$$

$$\nabla\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(r A_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{1}{r}\frac{\partial A_z}{\partial z},$$

$$\nabla\times\vec{A} = \hat{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right) + \hat{\theta}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \frac{1}{r}\hat{z}\left(\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right),$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}.$$

Spherical system

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2,$$

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi,$$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\hat{\varphi},$$

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi},$$

$$\nabla\times\vec{A} = \frac{1}{r\sin\theta}\hat{r}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\varphi\right) - \frac{\partial A_\theta}{\partial\varphi}\right] + \hat{\theta}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}\left(r A_\varphi\right)\right] + \frac{1}{r}\hat{\varphi}\left[\frac{\partial}{\partial r}\left(r A_\theta - \frac{\partial A_r}{\partial\theta}\right)\right],$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}.$$

The conditions on the boundary of two mediums

$$\left(\vec{D}_2 - \vec{D}_1\right)\cdot\hat{n} = \sigma$$

$$\left(\vec{B}_2 - \vec{B}_1\right)\cdot\hat{n} = 0$$

$$\left(\vec{E}_2 - \vec{E}_1\right)\times\hat{n} = 0$$

$$\left(\vec{H}_2 - \vec{H}_1\right)\times\hat{n} = -\vec{J}_\sigma$$

Connection between quantities β, cp, T, E, γ and their derivatives

	β	cp	T	E	γ
$\beta =$	β	$\left[(E_0/cp)^2+1\right]^{-1/2} = $ $= cp/E$	$\sqrt{1-\left(1+\frac{T}{E_0}\right)^{-2}}$	$\sqrt{1-\left(\frac{E_0}{E}\right)^2} =$ $= cp/E$	$\sqrt{1-\gamma^{-2}}$
$cp=$	$E_0/\sqrt{\beta^{-2}-1} =$ $= E\beta$	cp	$\left[T(2E_0+T)\right]^{1/2} =$ $= T\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}$	$\sqrt{E^2-E_0^2} =$ $= E\beta$	$E_0\sqrt{\gamma^2 -}$
$E_0 =$	$cp/\beta\gamma =$ $= E\left(1-\beta^2\right)^{1/2}$	$cp\left(\gamma^2-1\right)^{-1/2}$	$T/(\gamma-1)$	$\sqrt{E^2-c^2p^2}$	E/γ
$T=$	$\left[\frac{1}{\sqrt{1-\beta^2}}-1\right]E_0$	$\sqrt{E_0^2+c^2p^2-E_0} =$ $= cp\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}$	T	$E-E_0$	$E_0(\gamma-1)$

$\gamma =$	$\left(1-\beta^2\right)^{-1/2}$	$cp/E_0\beta =$ $= \left[1-\left(\frac{cp}{E_0}\right)^2\right]^{1/2}$	$1+T/E_0$	E/E_0	γ

First derivatives

	$d\beta$	$d(cp)$	$d\gamma = dE/E_0 = dT/E_0$
$d\beta =$	$d\beta$	$\left[1+(cp/E_0)^2\right]^{-3/2} d(cp)/E_0 =$ $= \gamma^{-3} d(cp)/E_0$	$\gamma^{-2}\left(\gamma^2-1\right)^{-1/2} d\gamma =$ $= \beta^{-1}\gamma^{-3} d\gamma$
$d(cp) =$	$E_0\left(1-\beta^2\right)^{-3/2} d\beta =$ $= E_0\gamma^3 d\beta$	$d(cp)$	$E_0\gamma\left(\gamma^2-1\right)^{-1/2} d\gamma =$ $= E_0\beta^{-1} d\gamma$
$d\gamma = dE/E_0 =$ $= dT/E_0 =$	$\beta\left(1-\beta^2\right)^{-3/2} d\beta =$ $= \beta\,\gamma^3 d\beta$	$\left[1+(E_0/cp)^2\right]^{-1/2} d(cp)/E_0 =$ $= \beta\, d(cp)/E_0$	$d\gamma$

Logarithmic first derivatives

	$d\beta/\beta$	dp/p	dT/T	$dE/E = d\gamma/\gamma$
$d\beta/\beta =$	$d\beta/\beta$	$\gamma^{-2}\, dp/p =$ $= dp/p - d\gamma/\gamma$	$[\gamma(\gamma+1)]^{-1}\, dT/T$	$\left(\gamma^2-1\right)^{-1} d\gamma/\gamma =$ $= (\beta\gamma)^{-2}\, d\gamma/\gamma$
$dp\ /\ p\ =$	$\gamma^2\, d\beta/\beta$	dp/p	$[\gamma/(\gamma+1)]dT/T$	$\beta^{-2}\, d\gamma/\gamma$
$dT/T =$	$\gamma(\gamma+1)d\beta/\beta$	$\left(1+\gamma^{-1}\right)dp/p$	dT/T	$\gamma(\gamma-1)^{-1} d\gamma/\gamma$
$dE/E =$ $= d\gamma/\gamma =$	$(\beta\gamma)^2\, d\beta/\beta =$ $= \left(\gamma^2-1\right)d\beta/\beta$	$\beta^2 dp/p =$ $= dp/p - d\beta/\beta$	$\left(1-\gamma^{-1}\right)dT/T$	$d\gamma/\gamma$

A.2. Special theory of relativity

A.2.1. Basic properties of space and time in the classical physics

1. There are inertial systems concerning which the free particle (a material point) moves uniformly and rectilinearly. The rest condition is a special case of such movement.
2. In any inertial coordinate system the free space from a matter is homogeneous and is isotropic, and time is homogeneous.
3. Any mechanical phenomena at identical entry conditions proceed equally in all inertial systems (a relativity principle).
4. Interactions between material bodies and signals with which help the information is transferred, can instantly extend (with infinite speed).
5. Coordinates and time in two inertial systems are connected by Galilean transformations.

Let system K' moves concerning system K with the velocity $\vec{V} = const$ along axis x and at the moment of $t = 0$ the beginnings of co-ordinates O and O' coincided, and corresponding axes were parallel. Then co-ordinates x', y', z' and time t' in system K' are connected with x, y, z, t transformations Galilee

$$x' = x - Vt, \quad y' = y, \quad z' = z, \quad t' = t. \qquad \text{(A.2.1)}$$

The time in systems K and K one and too, that is has absolute character. From

Galilean transformations follow the law of addition of speeds. The velocity of material point $\vec{\upsilon}'$ in system K' is connected with the velocity of this point $\vec{v}$ in system K parity

$$v'_x = v_x - V, \quad v'_y = v_y, \quad v'_z = v_z. \qquad \text{(A.2.2)}$$

A.2.2. Basic properties of space time and movement in the special theory of relativity (STR)

1. In the STR postulate 1 remains. This is postulate of the classical mechanics about existence of inertial systems.
2. Limits of applicability of the statement about homogenity and isotropy free space and homogenity of time extend on all physical phenomena (electromagnetic, thermal, processes with participation of elementary particles, etc.).
3. The relativity principle extends on all physical phenomena.
4. Interactions between bodies and the signals transferring the information cannot extend with infinite speed. There is a speed limit coinciding with a velocity of light in vacuum, and this speed is identical in all inertial systems of readout and does not depend on movement of a source or light receiver. Set of postulates 3 and 4 is called as a principle of a relativity of Einstein.

Lorentz's transformations

From Einstein's postulate 4 follows, that if at the moment of $t = t' = 0$ coordinate systems K and K coincided, let out from the beginning of coordinates observers in K and K will see light in the form of the front of a spherical light wave described by the equation

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \qquad \text{(A.2.3)}$$

and the equation

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0. \qquad \text{(A.2.4)}$$

The assumption about the spherical form of a wave in both systems seems inconsistent, but this contradiction is eliminated if to accept, that events, simultaneous in one inertial system, are not necessarily simultaneous in another. Time is not the absolute size independent of spatial variables and relative movement with the velocity $\vec{V}$. Having accepted it, with necessity we come to Lorenz's transformations

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \left(V/c^2\right)x}{\sqrt{1 - \frac{V^2}{c^2}}}. \qquad \text{(A.2.5)}$$

If speed of movement of inertial system K rather K has any direction $\vec{V}$ it is obvious, that the previous parities (A.2.3) concern components of radius-vector $\vec{r}$, perpendicular and parallel $\vec{V}$:

$$\vec{r}_\perp' = \vec{r}_\perp, \quad \vec{r}_\parallel{}' = \frac{\vec{r}_\parallel - \vec{V}t}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad t' = \frac{t - \vec{V}\vec{r}/c^2}{\sqrt{1 - \frac{V^2}{c^2}}}. \qquad \text{(A.2.6)}$$

A.2.3. Relativistic transformation of velocity

Let at the moment of t in system K in point (x, y, z) there is a body of the small sizes, its position is defined by radius-vector $\vec{r}$. In system K the same body is characterized by radius-vector $\vec{r}'(t') = (x'(t'), y'(t'), z'(t'))$. By definition, the Cartesian components of the velocity of a body in the frame K equals:

$$\upsilon_x = dx/dt, \quad \upsilon_y = dy/dt, \quad \upsilon_z = dz/dt, \qquad \text{(A.2.7)}$$

and in the frame *K*:

$$\upsilon_x' = dx'/dt', \quad \upsilon_y' = dy'/dt', \quad \upsilon_z' = dz'/dt'. \qquad \text{(П.2.8)}$$

Let's differentiate (5) and divide first three parities on dt'. We will receive the relativistic law of addition of velocities

$$\upsilon_x' = \frac{\upsilon_x - V}{1 - \upsilon_x V/c^2}, \quad \upsilon_y' = \frac{\upsilon_y\sqrt{1 - V^2/c^2}}{1 - \upsilon_x V/c^2}, \quad \upsilon_z' = \frac{\upsilon_z\sqrt{1 - V^2/c^2}}{1 - \upsilon_x V/c^2}. \qquad \text{(A.2.9)}$$

In limit $V/c << 1$ the formulas (A.2.7) pass in (A.2.8) under the law of addition of velocities in the classical mechanics.

From transformations (A.2.5) reduction of the sizes of a body (core) in a longitudinal direction follows

$$l_0 = x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1 - V^2/c^2}} = \frac{l}{\sqrt{1 - V^2/c^2}}. \qquad \text{(A.2.10)}$$

And time delay in moving system. If one hours are based in system *K*, and others in K' it agree (A.2.5) while in system K' there will pass time interval $\Delta\tau = t_2' - t_1'$, in system K there will pass more time:

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - V^2/c^2}}. \tag{A.2.11}$$

For differentials it is received

$$d\tau = dt\sqrt{1 - V^2/c^2}. \tag{A.2.12}$$

That is in moving system time flows more slowly. Time counted on hours, motionless concerning some object, is called as own time of this object. Own time can be defined and for non-uniformly moving object, keeping for $d\tau$ definition (A.2.12) if under V to understand the velocity of the inertial system instantly accompanying given object and which in this connection, coincides with the velocity of the object.

A.2.4. Four-dimensional formulation of the special theory of relativity

Relativistic electrodynamics

Coordinates in four-dimensional space is set of sizes $\{x_\mu\}$ where $x_1 = x$, $x_2 = y$, $x_3 = z$ – spatial coordinates and the fourth coordinate are connected in due course: $x_4 = ict$. Lorentz's transformation registers by means of Lorentz's $L_{\mu\nu}$ matrix:

$$x'_\mu = L_{\mu\nu} x_\nu. \tag{A.2.13}$$

Also translates components x_μ set in system K, in components x'_μ of system K'.

It is supposed, that on repeating indexes ν there is summation $(\mu, \nu = 1,2,3,4)$. The matrix of transformation of Lorentz looks like

$$\begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \tag{A.2.14}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = \frac{V}{c}$ and V – the velocity of movement of inertial system K' rather K along axis x, measured in system K. Lorenz's return transformations are reduced to replacement $\beta \to -\beta$. Set of four sizes $\{A_\mu\}$ which will be transformed as co-ordinate, is called as 4-vector components. Examples of 4-vectors are lower resulted.

4-vector of a particle velocity:

$$u_\mu = \frac{dx_\mu}{d\tau} = \left(\frac{\vec{v}}{\sqrt{1 - v^2/c^2}}, \frac{ic}{\sqrt{1 - v^2/c^2}} \right), \tag{A.2.15}$$

where $\vec{v} = d\vec{r}/dt$ – the velocity of a particle, $d\tau = dt\sqrt{1-\beta^2}$ – own time of a particle which are invariant of transformation of Lorenz.

4-vector of the energy-impulse, or the 4-momentum:

$$p_\mu = (\vec{p}, i\varepsilon/c), \tag{A.2.16}$$

where

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} \tag{A2.17}$$

– an impulse of a relativistic particle;

$$\varepsilon = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \sqrt{c^2 \vec{p}^2 + m^2 c^4} \tag{A.2.18}$$

– the full energy connected with kinetic energy T and energy of rest $\varepsilon_0 = mc^2$ by parity

$$T = \varepsilon - mc^2. \tag{A.2.19}$$

Wave 4-vector:

$$k_\mu = \left(\vec{k},\ i\omega/c\right), \tag{A.2.20}$$

where $\vec{k}$ – a wave vector and ω – frequency in expression for a flat electromagnetic wave.

4-vector of density of a current:

$$j_\mu = \left(\vec{j}, ic\rho\right), \tag{A.2.21}$$

where $\vec{j}$ – a vector of density of a current, ρ – density of charges.

4-dimensional potential:

$$A_\mu = \left(\vec{A},\ i\varphi\right), \tag{A.2.22}$$

where $\vec{A}$ – vector, and φ – scalar potentials of an electromagnetic field.

The 4-dimensional operator of differentiation:

$$\partial/\partial x_\mu = \left(\vec{\nabla},\ \frac{\partial}{\partial x_4}\right). \qquad \text{(A.2.23)}$$

If A_μ and B_μ are two 4-vectors,

$$(AB) = A_\mu B_\mu = (A'_\mu B'_\mu) = (A'B') = inv\,. \qquad \text{(A.2.24)}$$

Scalar product of two 4-vectors remains invariant at Lorentz's transformation. In particular, length of a 4-vector-invariant concerning Lorentz's transformations. The size is invariant

$$S_{12}^2 = \left[(\vec{r}_1 - \vec{r}_2)^2 - c^2(t_1 - t_2)^2\right]. \qquad \text{(A.2.25)}$$

Named an interval between events with co-ordinates $(\vec{r}_1, t_1)$ and $(\vec{r}_2, t_2)$. Time counted on hours, moving together with the given object, is called as own time of the given object. An interval of own time $d\tau$ выражается through time interval dt системы K the formula:

$$d\tau = dt\sqrt{1 - V^2/c^2}\,. \qquad \text{(A.2.26)}$$

If any core in the system of rest has length l_0 at movement with a speed V along the axis, it has in system K length

$$l = l_0\sqrt{1 - V^2/c^2}\,. \qquad \text{(A.2.27)}$$

Any 4-vector A_μ преобразуется under the formula

$$A'_\mu = L_{\mu\nu} A_\nu\,,$$

where $L_{\mu\nu}$ – Lorent's matrix.

Components of usual velocity of a particle will be transformed under the formulas:

$$v'_x = \frac{v_x - V}{1 - v_x V/c^2},\quad \upsilon'_y = \frac{v_y\sqrt{1 - V^2/c^2}}{1 - v_x V/c^2},\quad v'_z = \frac{v_z\sqrt{1 - V^2/c^2}}{1 - v_x V/c^2}\,. \qquad \text{(A.2.28)}$$

If a particle velocity makes with axis x corners θ and θ' in systems K and K', accordingly, that

$$tg\,\theta' = \frac{v\sin\theta\sqrt{1 - V^2/c^2}}{v\cos\theta - V}\,, \qquad \text{(A.2.29)}$$

where

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}\,. \qquad \text{(A.2.30)}$$

As four-dimensional acceleration is called the 4-vector

$$w_\mu = \frac{du_\mu}{d\tau} = \frac{d^2 x_\mu}{d\tau^2}. \qquad \text{(A.2.31)}$$

As wave 4-vector $k_\mu = \left(\vec{k}, i\,\omega/c\right)$, phase $\varphi = k_\mu x_\mu = inv$

$$\vec{k}'\cdot\vec{x}' - \omega' t' = \vec{k}\cdot\vec{x} - \omega t, \qquad \text{(A.2.32)}$$

hence, with the account of transformations

$$x' = \gamma(x - Vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{Vx}{c^2}\right), \qquad \text{(A.2.33)}$$

follows

$$k'_x = \gamma\left[k_x - \left(V/c^2\right)\omega\right], \quad k'_y = k_y, \quad k'_z = k_z, \quad \omega' = \frac{\omega - V k_x}{\sqrt{1 - V^2/c^2}}. \qquad \text{(A.2.34)}$$

With the account of that for light waves $\left|\vec{k}\right| = \omega/c$, $\left|\vec{k}'\right| = \omega'/c$, it is possible to write down the received parities in the form of frequency transformation

$$\omega' = \frac{\omega\left(1 - V/c\cos\theta\right)}{\sqrt{1 - V^2/c^2}} \qquad \text{(A.2.35)}$$

(Doppler effect-displacement with the relativistic correction).

$$tg\theta' = \frac{\sin\theta\sqrt{1 - V^2/c^2}}{\cos\theta - V/c}, \qquad \text{(A.2.36)}$$

where θ and θ' – angles which form vectors $\vec{k}$ and $\vec{k}'$ with the velocity $\vec{v}$. At $\theta = \pi/2$ it is received

$$\omega' = \frac{\omega}{\sqrt{1 - V^2/c^2}} \qquad \text{(A.2.37)}$$

(The transversal Doppler displacement).

The electromagnetic field tensor

4-tensor the second rank is the set of quantities $A_{\mu\nu}$, which at transformations of four-dimensional coordinates,

$$x'_\mu = L_{\mu\nu} x_\nu \qquad \text{(A.23.8a)}$$

will be transformed as follows

$$A'_{\mu\nu} = L_{\mu\alpha} L_{\nu\beta} A_{\alpha\beta}. \qquad \text{(A.2.38b)}$$

The example of 4-dimensional tensor is antisymmetric tensor of the second-rank of an electromagnetic field in vacuum which it is possible to present through components of fields $\vec{E}$ and $\vec{B}$:

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}, \tag{A2.39}$$

where

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{A.2.40}$$

and first index μ numbers a line, and the second ν - a column.

Maxwell Equations:

$$div\, E = 4\pi\,\rho\,,$$
$$rot\, B - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}\,, \tag{A.2.41}$$

in the covariant form look like

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} j_\mu\,. \tag{A.2.42}$$

Equations of Maxwell

$$div\vec{B} = 0\,,$$
$$rot\,\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0\,, \tag{A.2.43}$$

are reduced to the four-dimensional equations

$$\frac{\partial F_{\mu\nu}}{\partial x_\alpha} + \frac{\partial F_{\nu\alpha}}{\partial x_\mu} + \frac{\partial F_{\alpha\mu}}{\partial x_\nu} = 0\,, \tag{A.2.44}$$

where μ,ν,α – any three of numbers 1, 2, 3, 4.

At transition from system K to system K' field components will be transformed as follows

$$\begin{aligned} E'_x &= E_x\,, \quad & E'_y &= \gamma\left(E_y - \beta H_z\right), \quad & E'_z &= \gamma\left(E_z + \beta H_y\right), \\ H'_x &= H_x\,, \quad & H'_y &= \gamma\left(H_y + \beta E_z\right), \quad & H'_z &= \gamma\left(H_z - \beta E_y\right). \end{aligned} \tag{A2.45}$$

Thus quantities $\vec{H}^2 - \vec{E}^2$ and $\vec{E}\cdot\vec{H}$ are invariants of the Lorentz's transformations:

$$\vec{H}^2 - \vec{E}^2 = inv,$$
$$\vec{E}\cdot\vec{H} = inv.$$

At Lorentz's general transformation from system K to system K', moving with a velocity $\vec{\upsilon}$ rather K, fields will be transformed as follows:

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \qquad \vec{B}'_{\parallel} = \vec{B}_{\parallel},$$
$$\vec{E}'_{\perp} = \gamma\left(\vec{E}_{\perp} + \frac{1}{c}\vec{V}\times\vec{B}\right), \qquad \vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} + \frac{1}{c}\vec{V}\times\vec{E}\right). \tag{A.2.46}$$

Indexes || and $\perp$ specify in parallelism and perpendicularity of fields by the velocity $\vec{V}$.

Density of Lorentz force

$$\vec{f} = \rho\vec{E} + \frac{1}{c}\vec{j}\times\vec{B}, \tag{A.2.47}$$

it is possible to present in the form of a 4-vector

$$f_{\mu} = \frac{1}{c}F_{\mu\nu j\nu}, \tag{A.2.48}$$

where

$$f_0 = \vec{E}\vec{J}, \tag{A.2.49}$$

the work made by a field over charges in individual volume in unit of time. Spatial part f_k defines speed of change of quantity of movement in volume units.

Using non-uniform equations Максвелла, it is possible to write down f_{μ} in such form

$$f_{\mu} = \frac{1}{4\pi}F_{\mu\nu}\frac{\partial F_{\nu\alpha}}{\partial x_{\alpha}}, \tag{A.2.50}$$

and by means of electromagnetic тензора energy-impulse

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_{\alpha\nu} + \frac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta}\right), \tag{A.2.51}$$

the equations (A.2.51) will take the form

$$f_{\mu} = \frac{\partial T_{\mu\nu}}{\partial x_{\nu}}. \tag{A.2.52}$$

Components тензора $T_{\mu\nu}$ can be connected with fields $\vec{B}$ and $\vec{E}$. Тензор $T_{\mu\nu}$ looks like

$$T_{\mu\nu} = \begin{pmatrix} T_{11} & T_{12} & T_{13} & -icq_1 \\ T_{21} & T_{22} & T_{23} & -icq_2 \\ T_{31} & T_{32} & T_{33} & -icq_3 \\ -icq_1 & -icq_2 & -icq_3 & \omega \end{pmatrix}, \tag{A.2.53}$$

where T_{ik} – symmetric Maxwell stress tensor;

$$T_{ij} = \frac{1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}\left(E^2 + B^2\right)\right], \qquad (A.2.54)$$

and

$$\vec{q} = \frac{1}{4\pi c}\vec{E}\times\vec{B} = \frac{1}{c^2}\vec{S} \qquad (A.2.55)$$

– density of an impulse of the electromagnetic field, connected with the vector Pointing $\vec{S}$:

$$\omega = \frac{1}{8\pi}\left(E^2 + B^2\right) \qquad (A.2.56)$$

– density of energy of a field.

Maxwell's equations in the environment can be written down in the covariant form

$$\partial H_{\mu\nu} / \partial x_\nu = \frac{4\pi}{c} j_\mu ,$$
$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0 , \qquad (A.2.57)$$

where

$$H_{\mu\nu} = \begin{pmatrix} 0 & H_3 & -H_2 & -iD_1 \\ -H_3 & 0 & H_1 & -iD_2 \\ H_2 & -H_1 & 0 & -iD_3 \\ iD_1 & iD_2 & iD_3 & 0 \end{pmatrix} \qquad (A.2.58)$$

and тензор $F_{\mu\nu}$ the same, as for electrodynamics in vacuum.

A.3. The symmetry of electrostatic and magnetostatic equations

Electrostatics	**Magnetostatics**
$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$,	$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$,
$\vec{\nabla}\times\vec{E} = 0$,	$\vec{\nabla}\cdot\vec{B} = 0$,
$\vec{E} = -\vec{\nabla}\varphi$,	$\vec{B} = \vec{\nabla}\times\vec{A}$,
$\vec{\nabla}\cdot\vec{\nabla}\varphi = -4\pi\rho$,	$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \frac{4\pi}{c}\vec{j}$,
$\varphi(\vec{r}) = \int \rho(\vec{r}\,')\frac{dV'}{R}$,	$\vec{A}(\vec{r}) = \int \vec{j}(\vec{r}\,')\frac{dV'}{R}$,
$\vec{E}(\vec{r}) = \int \rho(\vec{r}\,')\frac{dV'}{R^3}$,	$\vec{B}(\vec{r}) = \int \vec{j}(\vec{r}\,')\frac{dV'}{R^3}$,

$$\varphi_{\partial un} = \frac{\vec{p}\vec{r}}{r^3}, \qquad \vec{A}_{\partial un} = \frac{\vec{m}\times\vec{r}}{r^3},$$

$$\vec{E}_{\partial un} = \frac{3\vec{r}(\vec{r}\cdot\vec{p}) - r^2\vec{p}}{r^5}, \qquad \vec{B}_{\partial un} = \frac{3\vec{r}(\vec{r}\cdot\vec{m}) - r^2\vec{m}}{r^5},$$

$$\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')dV', \qquad \vec{m} = \frac{1}{2c}\int \vec{r}\,'\times\vec{j}(\vec{r}\,')dV',$$

$$W = \frac{1}{2}\int \rho\varphi\, dV, \qquad W = \frac{1}{2}\int \vec{j}\cdot\vec{A}\, dV,$$

$$W = \frac{1}{2}\sum_{i,j} C_{ij}V_iV_j, \qquad W = \sum_{i,j} L_{ij}I_iI_j,$$

$$\omega(\vec{r}) = \frac{1}{8\pi}|E(\vec{r})|^2. \qquad \omega(\vec{r}) = \frac{1}{8\pi}|B(\vec{r})|^2.$$

A.4. The experimental base for Maxwell's equations

The differential form of the Gaussian law in electrostatics

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho. \tag{A.4.1}$$

For electrostatics it takes place

$$\vec{\nabla}\times\vec{E} = 0. \tag{A.4.2}$$

The experimentally observed absence of the magnetic monopoles means that the magnetic field satisfies the equation

$$\vec{\nabla}\cdot\vec{B} = 0. \tag{A.4.3}$$

Ampere's law is fulfilled for magnetostatics:

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}. \tag{A.4.4}$$

The equations (1) are (3) are always valid but the equation (2) must be modified in the presence of the magnetic field changing in time (the account of the Faraday's law of electromagnetic induction) as well as the differential form of Ampere's law (4) must be changed to satisfy the equation of continuity:

$$d\rho/dt + \vec{\nabla}\cdot\vec{j} = 0. \tag{A.4.5}$$

It is achieved by the replacement of equation (2) to

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \tag{A.4.6}$$

and equation (4) to

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\left(\vec{j} + \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}\right), \tag{A.4.7}$$

where $\frac{1}{4\pi}\frac{\partial \vec{E}}{\partial t}$ presents the displacement current suggested by Maxwell.

Thus, the equations describing the dynamics of the electromagnetic field are based on 4 laws experimentally proved

$$\vec{\nabla} \cdot \vec{E} = 4\pi\,\rho \qquad \text{(A.4.8)}$$

is Gauss' law;

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} \qquad \text{(A.4.9)}$$

is the modified Ampere's law;

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{(A.4.10)}$$

is the absence of monopoles;

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \qquad \text{(A.4.11)}$$

is Faraday's law.

A.5. Таблица перевода выражений и формул из гауссовой системы в систему СИ и обратно

Наименование	Гауссова система	Система СИ
Скорость света	c	$1/\sqrt{\varepsilon_0\mu_0}$
Напряжённость электрического поля, потенциал	$\vec{E}$, φ	$\sqrt{4\pi\varepsilon_0}\left(\vec{E},\varphi\right)$
Электрическая индукция	$\vec{D}$	$\sqrt{4\pi/\varepsilon_0}\ \vec{D}$
Заряд, плотность заряда, ток, плотность тока, поляризация	q, ρ, I, $\vec{j}$, $\vec{P}$	$\frac{1}{\sqrt{4\pi\varepsilon_0}}\left(q,\rho,I,\vec{j},\vec{P}\right)$
Магнитная индукция, магнитный поток	$\vec{B}$, Φ	$\sqrt{4\pi/\mu_0}\left(\vec{B},\Phi\right)$
Напряжённость магнитного поля	$\vec{H}$	$\sqrt{4\pi\mu_0}\ \vec{H}$
Магнитный момент, намагниченность	$\vec{m}$, $\vec{M}$	$\sqrt{\mu_0/4\pi}\left(\vec{m},\vec{M}\right)$
Электрическая проницаемость, магнитная проницаемость (относительные)	ε, μ	ε, μ
Электрическая поляризуемость, восприимчивость, магнитная восприимчивость	α, χ, k	$\frac{1}{4\pi}\left(\alpha,\chi,k\right)$
Удельная проводимость	σ	$\frac{\sigma}{4\pi\varepsilon_0}$

Сопротивление	R	$4\pi\varepsilon_0 R$
Ёмкость	C	$\frac{1}{4\pi\varepsilon_0}C$
Индуктивность	L	$\frac{4\pi}{\mu_0}L$

A.6. The list of transition of numerical values of physical notions from SI system in Gauss system

Designation	Sign	SI system	Gauss system
Length	ℓ	1 m (meter)	10^2 cm
Mass	m	1 kg (kilogram)	10^3 g
Time	t	1 s (second)	1 s
Force	$\vec{F}$	1 N (Newton)	10^5 dyne
Work, energy	A, ε	1 J (joule)	10^7 erg
Power	P	1 W (watt)	10^7 erg/s
Pressure	p	1 Pa (Pascal)	10 dyne/cm^2
Electric current	I	1 A (ampere)	$3{\cdot}10^9$
Electric charge	q	1 C (coulomb)	$3{\cdot}10^9$
Electric filed	$\vec{E}$	1 V/m (volt per meter)	$\frac{1}{3}\cdot 10^{-4}$
Electric potential	φ	1 V (volt)	$\frac{1}{3}\cdot 10^{-2}$
Polarisation	$\vec{P}$	1 C/m^2 (coulomb per square meter)	$3{\cdot}10^5$
Displacement	$\vec{D}$	1 C/m^2 (coulomb per square meter)	$12\pi{\cdot}10^5$
Capaciitance	C	1 F (farad)	$9{\cdot}10^{11}$ cm
Electric resistance	R	1 Ω (ohm)	$\frac{1}{9}\cdot 10^{-11}$ s·cm^{-1}
Resistivity	ρ	1 Ω·m (ohm-meter)	$\frac{1}{9}\cdot 10^{-9}$ s
Conductivity	$\Lambda = 1/R$	1 S (siemens)	$9{\cdot}10^{11}$ cm·s^{-1}
Specific conductivity	σ	1 S/m (siemens per meter)	$9{\cdot}10^9$ s^{-1}
Magnetic flux	Φ	1 Wb (Weber)	10^8 Mx (Maxwell)
Magnetic induction	$\vec{B}$	1 T (Tesla)	10^4 Gs
Magnetic field	$\vec{H}$	1 A/m (ampere per meter)	$4\pi{\cdot}10^{-3}$ Oe
Magnetisation	$\vec{M}$	1 A/m (ampere per meter)	$\frac{1}{4\pi}\cdot 10^4$ Gs
Inductance	L	1 H (Henry)	10^9 cm

A.7. Some physical constants

Velocity of light in vacuum	$2{,}997924562\cdot10^{8}$ m/s = $2{,}997924562\cdot10^{10}$ cm/s
Charge of an electron	$1{,}602\cdot10^{-19}$ C = $4{,}803\cdot10^{-10}$ cm$^{3/2}$ g$^{1/2}$/ s
Planck's constant	h=$6{,}626\cdot10^{-34}$ J·s = $6{,}626\cdot10^{-27}$ erg·s
Avogadro constant	$6{,}022\cdot10^{23}$ mol^{-1}
Boltzmann constant	$1{,}381\cdot10^{-23}$ J/K = $1{,}381\cdot10^{-16}$ erg/K
Gas constant	8,314· J/(mol·K) = $8{,}314\cdot10^{7}$ erg/(mol·K)
Faraday constant	$0{,}9648\cdot10^{5}$ C/mol = 9, $648\cdot10^{3}$ cm$^{3/2}$g$^{1/2}$/s
Rest mass of electron	$9{,}11\cdot10^{-31}$ kg = $9{,}11\cdot10^{-28}$ g=0,511 MeV
Rest mass of proton	$1{,}6727\cdot10^{-27}$ kg = $1{,}6727\cdot10^{-24}$ g
Rest mass of neuron	$1{,}6750\cdot10^{-27}$ kg = $1{,}6750\cdot10^{-24}$ g

1 eV=$1{,}6\cdot10^{-12}$ erg=$1{,}6\cdot10^{-19}$ J

A.8. The list of some constants characterizing properties of matter

A.8.1. Dielectric permittivity

Dielectric	ε	Dielectric	ε
Water	81	Polyethylene	2,3
Air	1,00058	Mica	7,5
Wax	7,8	Spirit	26
Kerosene	2,0	Glass	6,0
Paraffin	2,0	Chine	6,0
Acrylic plastic	3,5	Ebonite	2,7

A.8.2. Resistivity of conductors and non-conductors

Conductors	Resistivity (at 20 °C) ρ, nΩ·m	Temperature coefficient α, kK^{-1}	Non-conductors	Resistivity ρ, nΩ·m
Aluminium	25	4,5	Paper	10^{10}
Tungsten	50	4,8	Paraffin	10^{15}
Iron	90	6,5	Mica	10^{13}
Aurum	20	4,0	China	10^{13}
Copper	16	4,3	Shellac	10^{14}
Plumbum	190	4,2	Ebonite	10^{14}
Argentum	15	4,1	Amber	10^{17}

A.8.3. Magnetic susceptibilities of para- and diamagnetics

Paramagnetic	K = μ – 1, 10^{-6}	Diamagnetic	K = μ – 1, 10^{-6}
Nitrogen	0,013	Hydrogen	–0,063
Air	0,38	Benzol	–7,5
Oxygen	1,9	Water	–9,0
Ebonite	14	Copper	–10,3

Aluminium	23	Glass	–12,6
Tungsten	176	Mine salt	–12,6
Platinum	360	Quartz	–15,1
Liquid oxygen	3400	Bismuth	–176

A.8.4. Index of refraction n

Gases	n	Liquids	n	Solids	n
Nitrogen	1,00030	Benzol	1,50	Diamond	2,42
Air	1,00029	Water	1,33	Fused	1,46
Oxygen	1,00027	Glycerin	1,47	quartz	
		Carburet of sulfur	1,63	Glass (common)	1,50
Note. Index of refraction also depend on the length of the light wave, that is why the given values should be regarded as conventional.					

A.8.5. The edge of the absorption band

Z	Element	λ_K, пм	Z	Element	λ_K, пм
23	V	226,8	47	Ag	48,60
26	Fe	174,1	50	Sn	42,39
27	Co	160,4	74	W	17,85
28	Ni	148,6	78	Pt	15,85
29	Cu	138,0	79	Au	15,35
30	Zn	128,4	82	Pb	14,05
42	Mo	61,9	92	U	10,75

A.9. The scale of electromagnetic waves

Type of radiation	Range of		Energy of photons, eV
	Wave lengths	frequencies, sec^{-1}	
γ-radiation	$< 10^{-11} m = 0{,}1 A$	$> 3\cdot10^{19}$	$> 10^{5}$
X-ray	$10^{-11} \div 10^{-8} m = 0{,}1 \div 100 A$	$3\cdot10^{19} \div 3\cdot10^{16}$	$10^{5} \div 10^{2}$
Ultra-violet	$10^{-8} \div 4\cdot10^{-7} m = 100 \div 4000 A$	$3\cdot10^{16} \div 0{,}75\cdot10^{14}$	$10^{2} \div 3$
Visible	$4\cdot10^{-7} \div 7{,}5\cdot10^{-7} m = 4000 \div 7500 A$	$7{,}5\cdot10^{14} \div 4\cdot10^{14}$	$3 \div 1{,}6$
Infra-red	$7{,}5\cdot10^{-7} \div 10^{-3} m$	$4\cdot10^{14} \div 3\cdot10^{11}$	$1{,}6 \div 10^{-3}$
Microwave	$10^{-3} \div 10^{-1} m = 0{,}1 \div 10 cm$	$3\cdot10^{11} \div 3\cdot10^{9} = (3\cdot10^{5} \div 3\cdot10^{3}) MHz$	$10^{-3} \div 10^{-5}$
Radiowave	$> 10^{-1} m = 10 cm$	$< 3\cdot10^{9}$ $(3\cdot10^{3} MHz)$	$< 10^{-5}$

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