METHODS OF OPTIMIZATION

Lecture 1: Linear Programming

Learning objectives:

The learning objectives of this chapter are

- Introduction to Optimization Techniques.
- Explaining some optimization techniques.
- Introduction to Linear Programming.
- How to solve Linear-programming problems by graphical methods, problems with n – variable and how to overcome that.

Introduction

What is optimization?

- "To optimize is to make as perfect, effective, or functional as possible" … Merriam-Webster dictionary.
- "In engineering domain, optimization is a collection of methods and techniques to design and make use of engineering systems as perfectly as possible with respect to specific parameters"
- "In industrial engineering, one typical optimization problem is in inventory control. For this problem, we want to reduce the costs associated with item stocking and handling in a warehouse"

Optimization problem:

- Maximizing or minimizing some function relative to some set; often representing a range of choices available in a certain situation. The function allows comparison of the different choices for determining which might be "best."
- Optimization requires the representation of the problem in a mathematical model where the decision variables are the parameters of the problem.

Common applications: Minimal cost, maximal profit, minimal error, optimal design, optimal management, and variation principles.

Mathematical Optimization:

- A mathematical optimization problem is one in which some function is either maximized or minimized relative to a given set of alternatives. The function to be minimized or maximized is called the *objective function* and the set of alternatives is called the *feasible region*

Linear programming:

- Linear programming is the name of a branch of applied mathematics that deals with solving optimization problems of a particular form.
- Linear programming problems consist of a linear cost function (consisting of a certain number of variables), which is to be minimized or maximized subject to a certain number of constraints.

History:

- Linear programming is a relatively young mathematical discipline dating from the invention of the simplex method by G. B. Dantzig in 1947.

- Historically, development in linear programming is driven by its applications in economics and management.

- Dantzig initially developed the simplex method to solve U.S. Air Force planning problems, and planning and scheduling problems still dominate the applications of linear programming

Terminology:

- In general case linear programming problem can be formulated as following:

Minimize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$

$$a_{11}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$

....

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$

- n optimization (decision) variables $x_1, x_2, \ldots \ldots, x_n$
- m linear inequality constraints.

Every linear programming problem falls into one of three categories:

1. ***Infeasibl**e*. A linear programming problem is *infeasible* if a feasible solution to the problem does not exist; that is, there is no vector x for which all the constraints of the problem are satisfied.

2. ***Unbounded**.* A linear programming problem is *unbounded* if the constraints do not sufficiently restrain the cost function so that for any given feasible solution, another feasible solution can be found that makes a further improvement to the cost function.

3. ***Has an optimal solution*****.** Linear programming problems that are not infeasible or unbounded have an optimal solution; that is, the cost function has a unique minimum (or maximum) cost function value. This does not mean that the values of the variables that yield that optimal solution are unique, however.

Application Examples of Linear Programming :

1. A farmer has 240 acres of land to plant. He needs to decide how many acres of corn to plant & how many of oats. He can make 40$ per acres profit for corn & 30$ per acres for oats. However corn takes 2 hours of labor per acre to harvest while oats takes 1 hour per acre. He has only 320 labors he can invest. To maximize his profit how many acres of corn & oats should he plant.

Solution: Let x & y be number of acres of corn & oats respectively. Our objective in this problem is to maximize the profit P.

Therefore, $P = 40x + 30y \rightarrow$ max

Constraints: $x \geq 0$; $y \geq 0$; ……… as land can't be negative

Another constraint we have how much land we can allot; we have 240 acres of total land. So, $x + y \leq 240$;

Now, we have constraints on labor also. We have 320 hours of labor to spend. So it can be represented as $2x + y \leq 320$;

Hence, the linear equation of problem can be describes as below:

$$\text{Maximize:} \quad P = 40x + 30y$$

$$\text{Subject to:} \quad x + y \leq 240;$$

$$2x + y \leq 320;$$

Where with respect to problem $x \geq 0$; $y \geq 0$;

2.There are two kinds of feeds: feed I & feed II, containing 3 different kinds of vital nutrients N1, N2 & N3. The table includes the number of nutritional units per kilo for each feed & a required minimum of nutrients

Nutrients	Requirements min. of Nutrients	No. Of nutritional units per kilo for each feed	
		I	II
N_1	9	3	1
N_2	8	1	2
N_3	12	1	6

1 Kilo of the 1st nutrition cost 4 rubles & 1 kilo of the 2nd nutrition costs 6. It is necessary to combine daily food allowances having the smallest possible cost & satisfy these nutritional requirements

Solution:

Let x & y be the quantities of feed I & II. The objective function is to minimize the cost while satisfying the nutritional requirements.

Therefore, Z (X)= 4x+ 6y → min

Constraints: From given table, daily food allowance consists of (3x+y) units of nutrients of N1, (x+2y) units of nutrients of N2 & (x+6y) units of nutrients of N3. The nutrients N1, N2 & N3 must not be smaller than 9,8 & 12 correspondingly. Hence there is subject to constraints: $3x+y \geq 9$, $x+y \geq 8$ & $x+6y \geq 12$.

Hence, the linear equation of problem can be describes as below:

Minimize $Z(X) = 4x + 6y \rightarrow \min$

Subject to:

$3x+y \geq 9$

$x+y \geq 8$

$x+6y \geq 12$

Where with respect to problem: $x \geq 0$, $y \geq 0$.

Standard form of Linear Programming problem:

We say that a linear program is in standard form if the following are all true:

- 1. Non-negativity constraints for all variables.
- 2. All remaining constraints are expressed as equality constraints.
- 3. The right hand side vector, b, is non-negative.

The given LP not in standard form:

Maximize: $Z = 3x_1 + 2x_2 - x_3 + x_4$

Subject to: $x_1 + 2x_2 + x_3 - x_4 \leq 5$ ………not equality

$-2x_1 - 4x_2 + x_3 + x_4 \leq -1$… not equality and negative RHS

$x_1 \geq 0$, $x_2 \leq 0$ ………………… x_2 is required to be non-positive.

x_3 and x_4 may be positive or negative.

Why do we need to know how to convert a linear program to standard form? What's so special about standard form?

- The main reason that we care about standard form is that this form is the starting point for the simplex method, which is the primary method for solving linear programs.
- In addition, it is good practice for students to think about transformations, which is one of the key techniques used in mathematical modeling.

Steps to follow to convert Linear-programming problem to standard form:

Step 1:

If the objective function is minimization, just “negate” the objective function.

For example. “minimize –a +b subject to a-b $\leq$ 2, a, b $\geq$ 0” is equivalent to “maximize a-b subject to a-b $\leq$ 2, a, b $\geq$ 0”.

Step 2:

Converting "≤" constraint into standard form. Consider a simple inequality constraint is $x_1 + 2x_2 + x_3 - x_4 \le 5$;

To convert a "≤" constraint to equality, add a slack variable. In this case, the inequality constraint becomes the equality constraint:

$x_1 + 2x_2 + x_3 - x_4 + s_1 = 5$.

We also require that the slack variable is non-negative. That is $s_1 \ge 0$.

Note: s1 is called a slack variable, which measures the amount of "unused resource."

Step 3:

Converting a "≥" constraint into standard form, and converting inequalities with a negative RHS. Consider the constraint

$-2x_1 - 4x_2 + x_3 + x_4 \leq -1$

First we have to multiply the inequality by -1 in order to obtain a positive RHS. Then we get

$2x_1 + 4x_2 - x_3 - x_4 \geq 1$

Then we add a surplus variable and get

$2x_1 + 4x_2 - x_3 - x_4 - s_2 = 1$

s_2 is called a surplus variable, which measures the amount by which the LHS exceeds the RHS

Step 4:

Getting Rid of Negative Variables. We have to transform the constraint: $x_2 \leq 0$ into standard form.

Let $y_2 = -x_2$. Then $y_2 \geq 0$. And we substitute $-y_2$ for x_2 wherever x_2 appears in the LP.

Step 5:

Getting Rid of Variables that are Unconstrained in Sign. That is, it can be positive or negative. Such variables are also called as free Variables. We replace a free variable by the difference of two non-negative variables. For example, we replace x_3 by $y_3 - w_3$, and require y_3 and w_3 to be non-negative. And $y_3 \geq 0$; $w_3 \geq 0$.

Examples:

1.Convert the Linear-programming problem to the standard form

Maximize: $Z = 3x_1 + 2x_2 - x_3 + x_4$

Subject to: $x_1 + 2x_2 + x_3 - x_4 \leq 5;$

$-2x_1 - 4x_2 + x_3 + x_4 \leq -1;$

$x_1 \geq 0; x_2 \leq 0;$

Solution:

Maximize: $Z = 3x_1 - 2y_2 - y_3 + w_3 + y_4 - w_4$

Subject to: $x_1 - 2y_2 + y_3 - w_3 - y_4 + w_4 + s_1 = 5.$

$2x_1 - 4y_2 - y_3 + w_3 - y_4 + w_4 - s_2 = 1.$

$x_1 \geq 0; y_2 \geq 0; y_3 \geq 0; y_4 \geq 0; w_3 \geq 0; w_4 \geq 0; s_1 \geq 0; s_2 \geq 0$

2.Convert the Linear-programming problem to the symmetrical form

$$\textit{Maximize:} \quad Z = 4x_1 - 5x_2 + x_3 + 2x_4$$

$$\textit{Subject to:} \quad 3x_1 - 2x_2 + x_3 + 4x_4 = 6;$$

$$-7x_1 + 10x_2 + 3x_3 - 4x_4 = 2;$$

$$x_j \geq 0, \quad j = 1,2,3,4.$$

Solution:

Let us transform the constraint system by Jordan-Gauss Method & exclude pivot variables from the objective function.

Note: Don't choose the coefficient of the objective function as pivot.

x_1	x_2	x_3	x_4	b
3	-2	1	4	6
-7	10	3	-4	2
4	-5	1	2	0
3	-2	1	4	6
-16	16	0	-16	-16
1	-3	0	-2	-6
3	-2	1	4	6
-1	1	0	-1	-1
1	-3	0	-2	-6
1	0	1	2	4
-1	1	0	-1	-1
-2	0	0	-5	-9

After that the problem can be written as the following

$$Z = -2x_1 - 5x_4 + 9 \rightarrow \text{Max}$$

Subject to the constraints: $x_1 + x_3 + 2x_4 = 4;$

$$-x_1 + x_2 - x_4 = -1;$$

$$x_j \geq 0, \quad j = 1,2,3,4.$$

As x_2 & x_3 variables are non-negative they can be eliminated and the problem can be formulated in symmetrical form:

$$Z = -2x_1 - 5x_4 + 9 \rightarrow \text{Max}$$

subject to the constraints: $x_1 + 2x_4 \leq 4;$

$$-x_1 - x_4 \leq -1;$$

$$x_j \geq 0, \quad j = 1,4.$$

Geometric Method

MANY PRACTICAL PROBLEMS involve maximizing or minimizing a function subject to certain constraints.

For example, we may wish to maximize a profit function subject to certain limitations on the amount of material and labor available. Maximization or minimization problems that can be formulated in terms of a linear objective function and constraints in the form of linear inequalities are called linear programming problems. In this chapter, we look at linear programming problems involving two variables. These problems are amenable to geometric analysis, and the method of solution introduced here will shed much light on the basic nature of a linear programming problem

Problem with two Variables:

Steps for Solving a Linear Programming Problem

1. Translate the problem into mathematical terms
2. Graph the feasible set described by the constraint inequalities and finds the coordinates of all the corner points. If the region is unbounded, determine whether it's possible for the objective function to obtain the desired extreme value. If not, write "no solution". Otherwise go to step 3.
3. Evaluate the objective function at each of the corner points.
4. Find the corner point that makes the objective function a maximum (minimum). If there's only one such corner point then the value of the objective function at that point is the maximum (minimum). If there are two adjacent corner points that maximize (minimize) the objective function then the maximum (minimum) value of the objective function occurs at any point on the line segment joining the two corner points.

Examples:

1. Mike's Famous Toy Trucks manufactures two kinds of toy trucks—a standard model and a deluxe model. In the manufacturing process each standard model requires 2 hours of grinding and 2 hours of finishing, and each deluxe model needs 2 hours of grinding and 4 hours of finishing. The company has two grinders and three finishers, each of whom works at most 40 hours per week. Each standard model toy truck brings a profit of $3 and each deluxe model a profit of $4. Assuming that every truck made will be sold, how many of each should be made to maximize profits?

Solution:

First, we name the variables:

X = Number of standard models made

Y = Number of deluxe models made

The quantity to be maximized is the profit, which we denote by P:

$$P = \$3x + \$4y$$

This is the objective function. To manufacture one standard model requires 2 grinding hours and to make one deluxe model requires 2 grinding hours. The number of grinding hours needed to manufacture x standard and y deluxe models is $2x + 2y$

But the total amount of grinding time available is only 80 hours per week. This means we have the constraint

$$2x + 2y \leq 80;$$

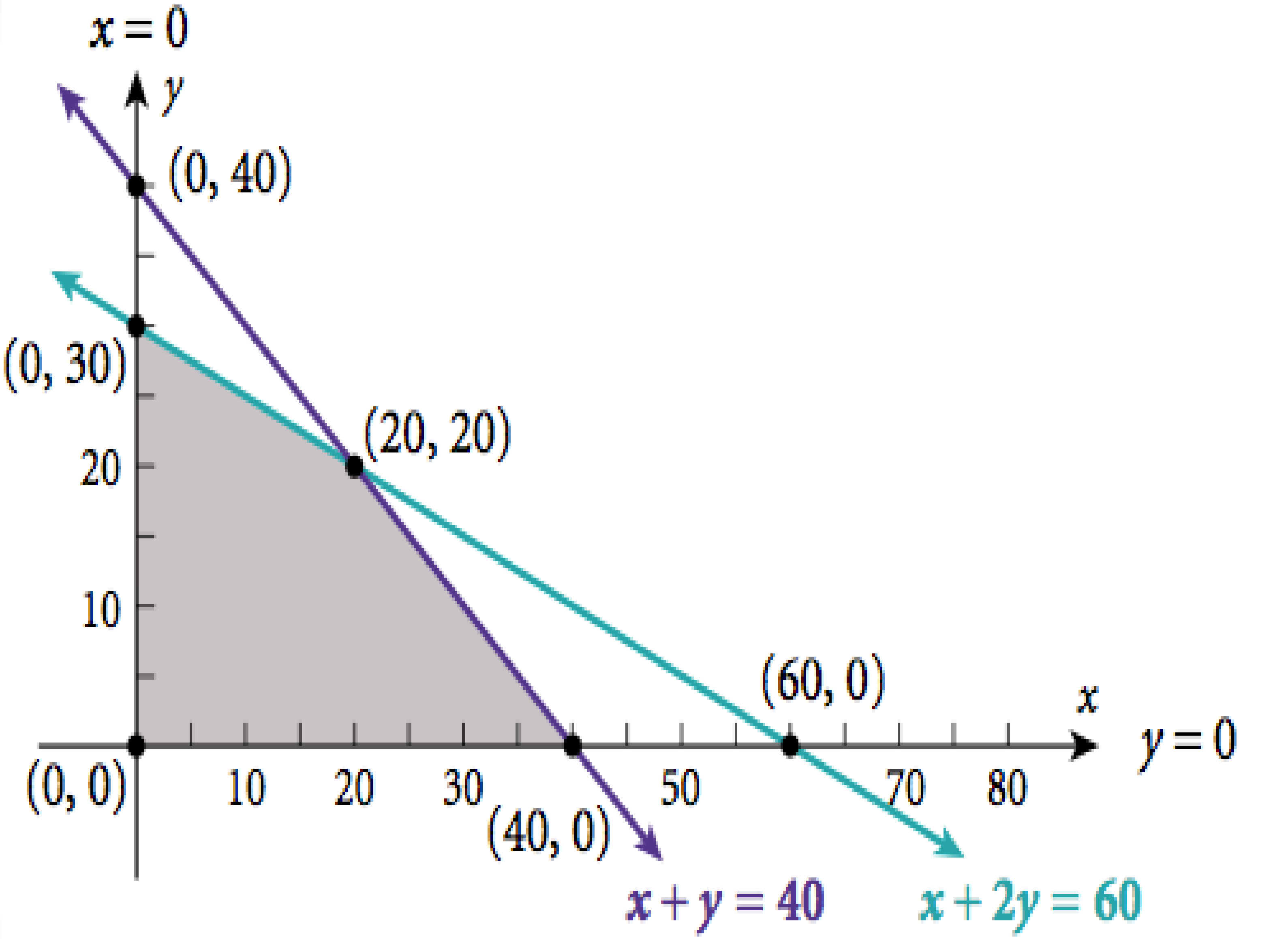
Similarly, for the finishing time we have the constraint

$$2x + 4y \leq 120$$

By simplifying each of these constraints and adding the non-negativity constraints $x \geq 0$ and $y \geq 0$.

we may list all the constraints for this problem:

$x + y \leq 40$
$x + 2y \leq 60$
$x \geq 0$
$y \geq 0$
Figure illustrates the set of feasible points, which is bounded.



x = 0
y
(0, 40)
(0, 30)
20
10
(20, 20)
(60, 0)
x
y = 0
(0, 0)
10
20
30
50
70
80
(40, 0)
x + y = 40
x + 2y = 60

The corner points of the set of feasible points are (0, 0) (0, 30) (40, 0) (20, 20)

Table lists the corresponding values of the objective equation

Corner Points	Value of Objective Function P = $3x + $4y
(0, 0)	P=0
(0, 30)	P=$120
(40, 0)	P=$120
(20, 20)	*P*= 3(20) + 4(20) = $140

A maximum profit is obtained if 20 standard trucks and 20 deluxe trucks are manufactured. The maximum profit is $140.

2. Determine the graphical solution set for the following system of linear inequalities:

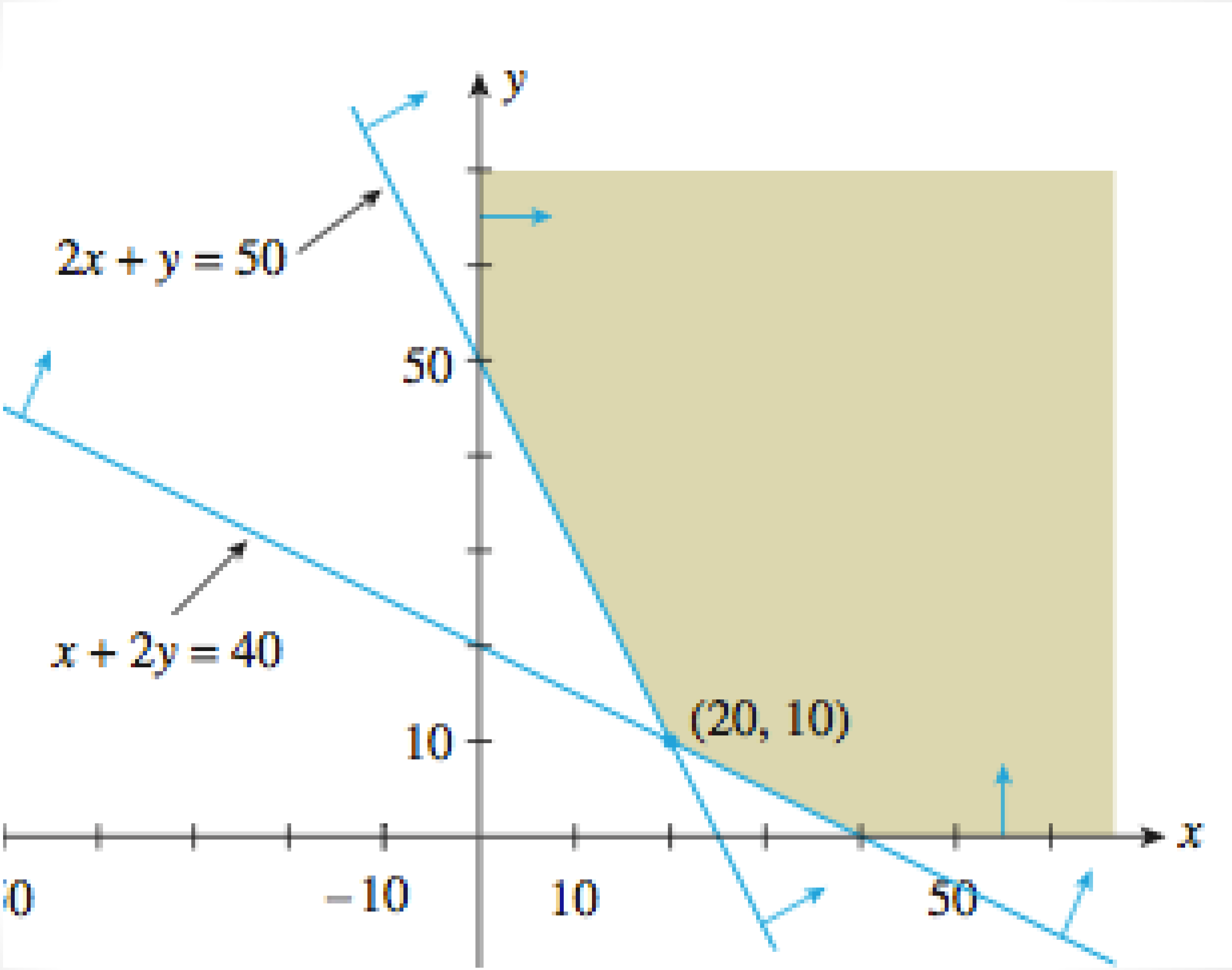
$2x + y=50;$

$x +2y = 40$

$x \geq 0$

$y \geq 0$

Solution: The required solution set is unbounded region shown in Figure



y
2x + y = 50
50
x + 2y = 40
(20, 10)
10
x
–10
10
50

3. Determine graphically the solution set for the following system of inequalities:

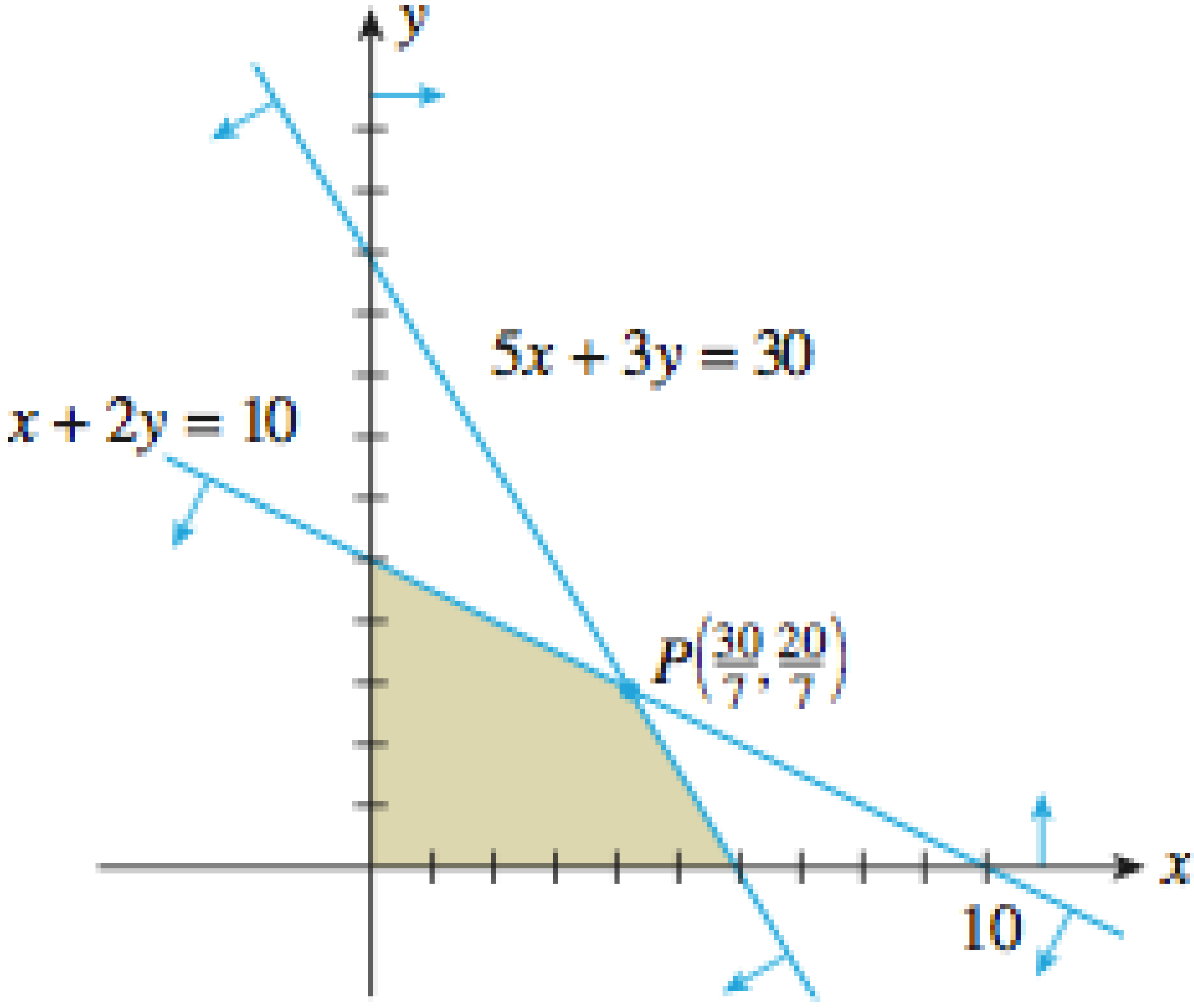
$$x + 2y \leq 10$$

$$5x + 3y \leq 30$$

$$x \geq 0$$

$$y \geq 0$$

Solution: The required solution set is shown in the following figure:



y
5x + 3y = 30
x + 2y = 10
P(30/7, 20/7)
x
10

The point P is found by solving the system of equations

$x + 2y = 10$ ……(1)

$5x + 3y = 30$ ……(2)

Solving the first equation for x in terms of y gives

$x = 10 - 2y$

Substituting this value of x into the second equation of the system gives

$5(10-2y)+3y=30$

$50-10y+3y= 30$

$-7y =-20$

So $y=\frac{20}{7}$ Substituting this value of y into the expression

for x found earlier, we obtain

$x = 10 - 2\left(\frac{20}{7}\right) = \frac{30}{7}$

Giving the point of intersection as $\left(\frac{30}{7},\frac{20}{7}\right)$

Problem of n- Variables:

The graphical method is confined to two variables. However, if the problem has standard form and satisfies the condition $n-r \leq 2$, where n is number of unknowns, & r is rank of constraint vector system it can be solved. If equations of a system are linear independent, then rank r equals the number of equations m.

Example:

Solve the problem by graphical method.

Minimize $Z = -x_1 - x_2 + x_3 + 3x_4 + 7x_5$

Subject to the constraints:

$$-x_1 + x_2 + x_3 + 2x_4 - 3x_5 = 4$$

$$x_1 + x_2 + 4x_3 + x_4 - 8x_5 = 3$$

$$x_2 + x_3 - 4x_5 = -4$$

$$x_j \geq 0, \; j=1,2,3,4,5.$$

Solution:

Let's check up whether the graphical method can be applied to solve the problem. For what we find n-r=5-3=2. Hence, method will be applied.

We can use Jordan-Gauss method to solve this problem & pivot unknowns are excluded from the objective function.

Using the last part of table. We rewrite the linear programming problem as

$$Z = -x_4 + 4x_5 + 22 \rightarrow \text{Min},$$

Subject to the constraints:

$$x_2 - x_4 - 3x_5 = -9$$

$$x_3 + x_4 - x_5 = 5$$

$$x_1 - 2x_4 - x_5 = -8$$

$$x_j \geq 0, \text{ j=1,2,3,4,5.}$$

As pivot unknowns are non-negative we can drop them and change the signs ‘=’ by ‘≤’. And we will get the auxiliary problem with two variables

$$Z = -x_4 + 4x_5 + 22 \rightarrow \text{Min},$$

Subject to the constraints:

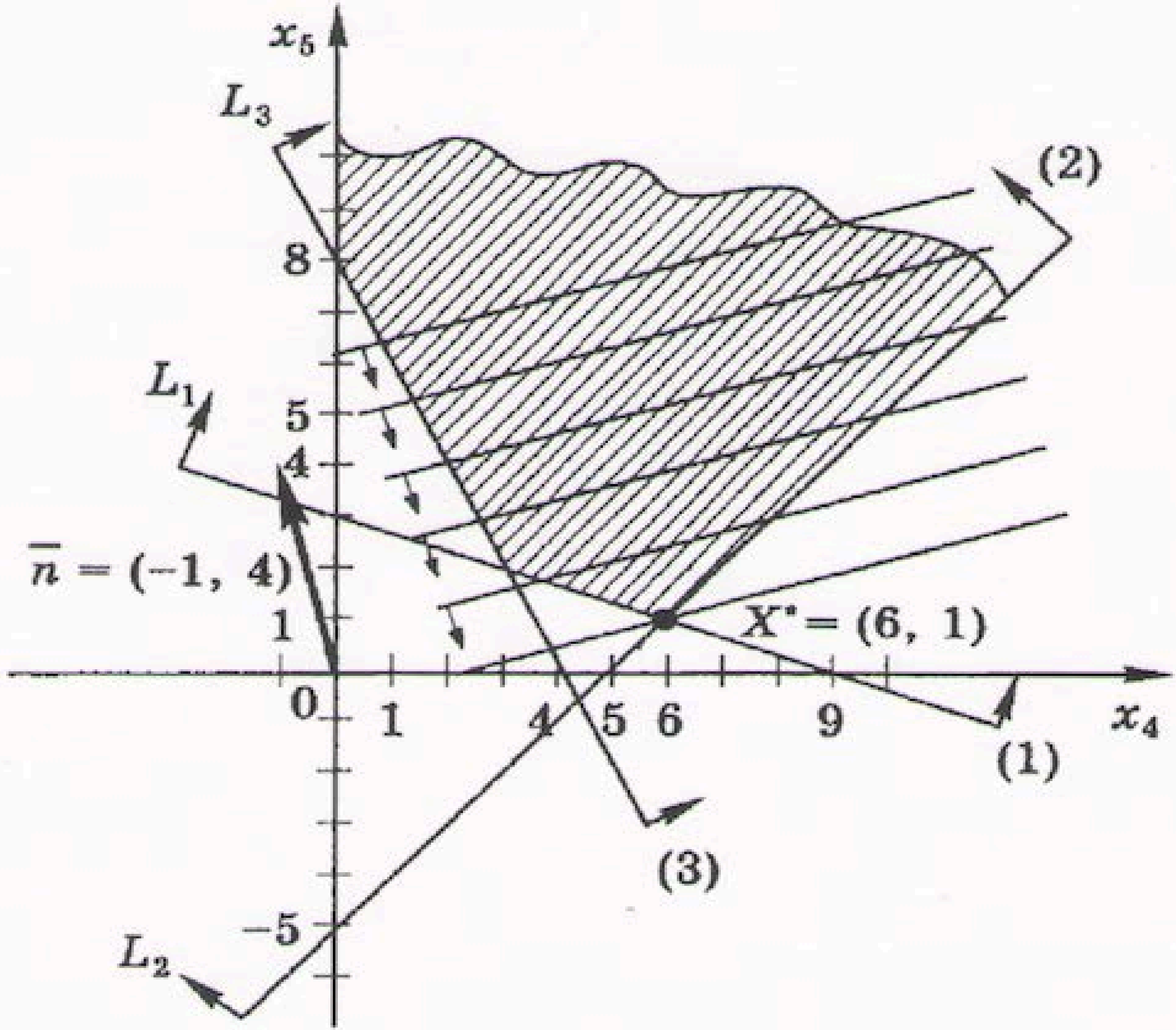
$$-x_4 - 3x_5 \leq -9$$

$$x_4 - x_5 \leq 5$$

$$-2x_4 - x_5 \leq -8$$

$$x_j \geq 0, \text{ j=4,5.}$$

It can be solved by graphical method:



x_5
L_3
(2)
8
L_1
5
4
$\bar{n} = (-1,\ 4)$
1
$X^* = (6,\ 1)$
0
1
4
5
6
9
x_4
(1)
(3)
−5
L_2

From fig, it's seems object function does not influence the finding of optimal solution. It must be taken into account for evaluating of the objective function value. The optimal solution of the auxiliary problem X'= $\square_1 \cap \square_2$: is

$$-\square_4 - 3\square_5 = -9$$

$$+ \quad \square_4 - \square_5 = 5$$

$$-4\square_5 = -4$$

$\square_5 = 1$ & $\square_4 = 6$, X = (6,1)

The optimal solution of the initial problem is evaluated from the system of constraints in solved form

$$\square_2 - \square_4 - 3\square_5 = -9$$

$$\square_3 + \square_4 - \square_5 = 5$$

$$\square_1 - 2\square_4 - \square_5 = -8$$

From which x_2, x_3 & x_1:

x_2= -9+ x_4 + 3x_5 = -9+6+3=0

x_3 = 5- x_4+x_5 = 5-6+1=0

x_1 = -8 + 2x_4 + x_5= -8+12+1=5

Thus X=(5,0,0,6,1).

Therefore, Z=20 at X=(5,0,0,6,1).

Lecture 2: Simplex Method for Maximization

Learning objectives:

The learning objectives of this chapter are

- The simplex Method Algorithm.
- The simplex Method - Basic & non-basic variables.
- Application of simplex method to maximization & how to deal with minimization problems.

Introduction:

The geometric method of solving linear programming problems presented before. The graphical method is useful only for problems involving two decision variables and relatively few problem constraints.

What happens when we need more decision variables and more problem constraints?

We use an algebraic method called the simplex method, which was developed by George B. DANTZIG (1914-2005) in 1947 while on assignment with the U.S. Department of the air force.

Converting a linear program to Standard Form

Before the simplex algorithm can be applied, the linear program must be converted into standard form where all the constraints are written as equations (no inequalities) and all variables are nonnegative (no unrestricted variables). This process of converting a linear program to its standard form requires the addition of slack variable s_i which represents the amount of the resource not used in the *ith* ≤constraint. Similarly, ≥constraints can be converted into standard form by subtracting excess variable e_i.

The standard form of any linear program can then be represented by the following linear system with *n* variables (including decision, slack and excess variables) and *m* constraints.

$$
\begin{array}{llllll}
\max z = & & & & & \\
(or \min) & c_1x_1 + & c_2x_2 + & \ldots + & c_nx_n & \\
s.t. & a_{11}x_1 + & a_{12}x_2 + & \ldots + & a_{1n}x_n & = b_1 \\
 & a_{21}x_1 + & a_{22}x_2 + & \ldots + & a_{2n}x_n & = b_2 \\
 & \ldots & \ldots & \ldots + & \ldots & \ldots \\
 & a_{m1}x_1 + & a_{m2}x_2 + & \ldots + & a_{mn}x_n & = b_m
\end{array}
$$

$$
x_i \geq 0 \quad (i = 1, 2, \ldots, n)
$$

Algorithm for the Simplex Method

The simplex algorithm, instead of evaluating all basic feasible solutions (which can be prohibitive even for moderate-size problems), starts with a basic feasible solution and moves through other basic feasible solutions that successively improve the value of the objective function. The algorithm terminates once the optimal value is reached. Below we present a step-wise description of the simplex algorithm.

1. Convert the linear program into standard form.
2. Obtain a basic feasible solution from the standard form.
3. Determine if the basic feasible solution is optimal.
4. If the current basic feasible solution is not optimal, select a non-basic variable that should become a basic variable and basic variable which should become a non-basic variable to determine a new basic feasible solution with an improved objective function value.
5. Use elementary row operations to solve for the new basic feasible solution. Return to Step 3

Steps 1 and 2 of the algorithm have been previously discussed. Steps 3, 4 and 5 of the algorithm are best executed with the help of a tableau which is simply a table with a particular format that shows a summary of the key information regarding the linear program.

Basic and Non-basic Variables, and Basic Feasible Solutions :

If we define,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \ \mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \text{ and } \mathrm{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix},$$

the constraints of the standard form of a linear program can be simply represented by a system of simultaneous equations

$A\mathrm{x} = \mathrm{b}$.

Basic variables are selected arbitrarily with the restriction that there be as many basic variables as there are equations. The remaining variables are non-basic variables.

This system has two equations, we can select any two of the four variables as basic variables. The remaining two variables are then non-basic variables. A solution found by setting the two non-basic variables equal to 0 and solving for the two basic variables is a basic solution. If a basic solution has no negative values, it is a basic feasible solution.

Examples:

The Cannon Hill furniture Company produces tables and chairs. Each table takes four hours of labor from the carpentry department and two hours of labor from the finishing department. Each chair requires three hours of carpentry and one hour of finishing. During the current week, 240 hours of carpentry time are available and 100 hours of finishing time. Each table produced gives a profit of $70 and each chair a profit of $50. How many chairs and tables should be made?

Resource	Table s ( )	Chairs ( )	Constraints
Carpentry (hrs.)	4	3	240
Finishing (hrs.)	2	1	100
Unit Profit	$70	$50	

Objective Function: $P = 70x_1 + 50x_2$

Carpentry Constraint: $4x_1 + 3x_2 \leq 240$

Finishing Constraint: $2x_1 + 1x_2 \leq 100$

Non-negativity conditions: $x_1, x_2 \geq 0$

The first step of the simplex method, requires that each inequality be converted into an equality.

The standardized form

$$4x_1 + 3x_2 + s_1 + 0s_2 = 240$$

$$2x_1 + x_2 + 0s_1 + s_2 = 100$$

$$P - 70x_1 - 50x_2 - 0s_1 - 0s_2 = 0$$

Notes: All the variables are nonnegative Such a solution is called feasible.

x_1	x_2	S_1	S_2	P	RHS
4	3	1	0	0	240
2	1	0	1	0	100
-70	-50	0	0	1	0

The table represents the initial solution;

$$x_1 = 0, \quad x_2 = 0, \quad s_1 = 240, \quad s_2 = 100, \quad P = 0$$

The slack variables S_1 and S_2 form the initial solution mix. The initial solution assumes that all avaliable hours are unused. i.e. The slack variables takes the largest possible values.

Select the pivot column (determine which variable to enter into the solution mix). Choose the column with the “most negative” element in the objective function row.

x_1	x_2	S_1	S_2	P	RHS
4	3	1	0	0	240
2	1	0	1	0	100
-70	-50	0	0	1	0

Select the pivot row (determine which variable to replace in the solution mix). Divide the last element in each row by the corresponding element in the pivot column. The pivot row is the row with the smallest non-negative result

x_1	x_2	S_1	S_2	P	RHS
4	3	1	0	0	240
2	1	0	1	0	100
-70	-50	0	0	1	0

$$240/4 = 60$$

$$100/2 = 50$$

After Solving,

x_1	x_2	S_1	S_2	P	RHS
4	3	1	0	0	240
1	1/2	0	1/2	0	50
-70	-50	0	0	1	0

$$\frac{R_2}{2}$$

x_1	x_2	S_1	S_2	P	RHS
0	1	1	-2	0	40
1	1/2	0	1/2	0	50
0	-15	0	35	1	3500

$-4.R_2 + R_1$

$70.R_2 + R_3$

Now repeat the steps, till we will have not negative element in the last row.

x_1	x_2	S_1	S_2	P	RHS
0	1	1	-2	0	40
1	1/2	0	1/2	0	50
0	-15	0	35	1	3500

$$40/1 = 40$$

$$50/0,5 = 100$$

x_1	x_2	S_1	S_2	P	RHS
0	1	1	-2	0	40
1	0	-1/2	3/2	0	30
0	0	15	5	1	4100

$$-\frac{1}{2}.R_1 + R_2$$

$$15.R_1 + R_3$$

As the last row contains no negative numbers, this solution gives the maximum value of P.

Result:

This simplex table represents the optimal solution to the LP problem and is interpreted as:

$$x_1 = 30, \quad x_2 = 40, \quad s_1 = 0, \quad s_2 = 0$$

and profit P=$4100.

Practice Examples:

1. Maximize: $P = 3x + 4y$

   subject to: $x + y \le 4$

   $2x + y \le 5$

   $x \ge 0, y \ge 0$

2. Maximize: $P = 3x1 + 5x2$

   Subject to: $x1 \le 4$

   $2x2 \le 12$

   $3x1 + 2x2 \le 18$

   $x1 \ge 0, x2 \ge 0$

Notes:

1. When you solve a simplex problem & find that slack variable takes on a positive value. Basically, it because of when one of the slack variables takes on a positive value it means that in maximizing our objective function we stayed ”below” one of our constraints.

For example, if u is the slack variable corresponding to a constraint on labor hours used and the value of u is 12 in our optimal solution, it means we have 12 remaining labor hours available.

2. If the last row to the left of the vertical line in my simplex table contains all zeros, then there are infinitely many solutions to the optimization problem.

Minimization Problems:

- There are several ways to solve minimization problems. We will see those in next chapter

Lecture 3: Minimization

Problems & Artificial Variable Techniques.

Learning objectives:

The learning objectives of this chapter are

- The simplex Method Algorithm for minimization problems
- Overview of Artificial variable techniques for simplex Method.
- Application of simplex method to maximization & minimization by using

    The Big M Method or Method of Penalties.

    The Two-phase Simplex Method.

Duality Theorem Concept:

Linear programming problems exist in pairs. That is in linear programming problem, every maximization problem is associated with a minimization problem. Conversely, associated with every minimization problem is a maximization problem. Once we have a problem with its objective function as maximization, we can write by using duality relationship of linear programming problems, its minimization version. The original linear programming problem is known as **primal problem**, and the derived problem is known as **dual problem**

Thus, the dual problem uses exactly the same parameters as the primal problem, but in different locations. To highlight the comparison, now look at these same two problems in matrix notation

Primal Problem			Dual Problem	
Minimize	Z=cx		Maximize	W=yb
Subject to	Ax≥b		Subject to	yA≤c
and	x≥0		And	y≥0

Primal problem

$$A=\left|\begin{array}{cc:c} a_{11} & a_{12} & a_{13} \\ b_{11} & b_{12} & b_{13} \\ \hdashline c_{11} & c_{12} & c_{13} \end{array}\right|$$

Dual problem

$$A^{T}=\left|\begin{array}{cc:c} a_{11} & b_{11} & c_{11} \\ a_{12} & b_{12} & c_{12} \\ \hline a_{13} & b_{13} & c_{13} \end{array}\right|$$

Minimization Problems

In Previous Examples, we applied the simplex method only to linear programming problems in standard form where the objective function was to be *maximized*. In this section, we extend this procedure to linear programming problems in which the objective function is to be *minimized*.

A minimization problem is in **standard form** if the objective function is:

$$
\begin{array}{lllllll}
\max z = & & & & & \\
(or \min) & c_1x_1 + & c_2x_2 + & \ldots + & c_nx_n & \\
s.t. & a_{11}x_1 + & a_{12}x_2 + & \ldots + & a_{1n}x_n & = b_1 \\
 & a_{21}x_1 + & a_{22}x_2 + & \ldots + & a_{2n}x_n & = b_2 \\
 & \ldots & \ldots & \ldots + & \ldots & \ldots \\
 & a_{m1}x_1 + & a_{m2}x_2 + & \ldots + & a_{mn}x_n & = b_m
\end{array}
$$

$$x_i \geq 0 \quad (i = 1,2,\ldots,n)$$

Von Neumann Duality Principle

"The objective value w of a minimization problem in standard form has a minimum value if and only if the objective value z of the dual maximization problem has a maximum value. Moreover, the minimum value of w is equal to the maximum value of z. "

Solving a Minimization Problem

To solve this problem we use the following steps

Step 1. Use the coefficients and constants in the problem constraints and the objective function to form a matrix *A* with the coefficients of the objec-tive function in the last row.

Step 2. Interchange the rows and columns of matrix A to form the matrix A^T, the transpose of A.

Step 3. Use the rows of A^T to form a maximization problem with ≤ problem constraints.

Example:

1. Minimize $C = 16 x_1 + 9x_2 + 21x_3$

   subject to

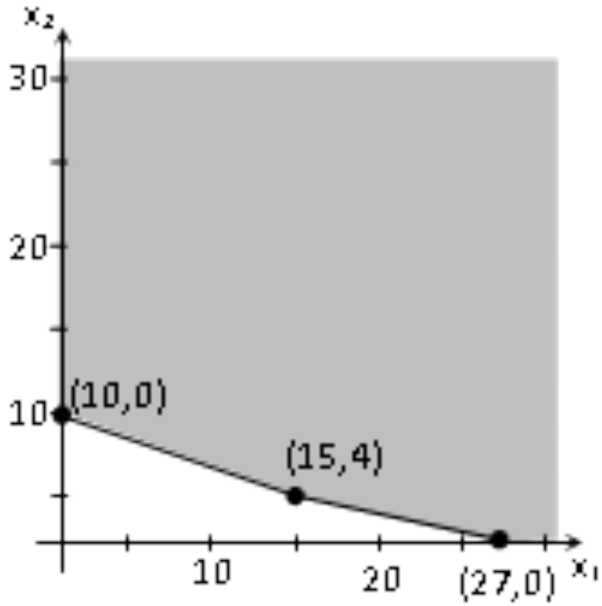
   $x_1 + x_2 + 3x_3 \geq 12$

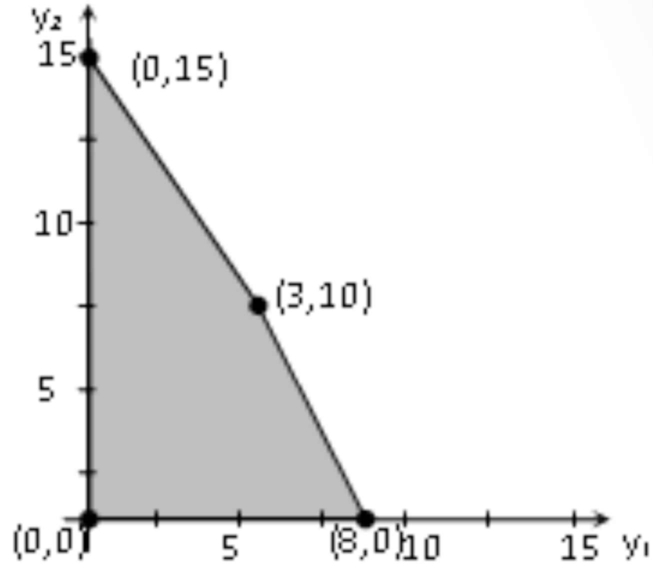
   $2x_1 + x_2 + x_3 \geq 16$

   $x_1, x_2, x_3 \geq 0$

Solution:

ORIGINAL PROBLEM	DUAL PROBLEM
Minimize $C = 16x_1 + 45x_2$	Maximize $P = 50y_1 + 27y_2$
subject to $2x_1 + 5x_2 \geq 50$ $x_1 + 3x_2 \geq 27$ $x_1, x_2 \geq 0$	subject to $2y_1 + y_2 \leq 16$ $5y_1 + 3y_2 \leq 45$ $y_1, y_2 \geq 0$





Corner Point		Corner Point	
(x_1, x_2)	$C = 16x_1 + 45\ x_2$	(y_1, y_2)	$P = 50y_1 + 27\ y_2$
(0,10)	450	(0,0)	0
(15,4)	420	(0,15)	405
(27,0)	432	(3,10)	420
		(8,0)	400
Min C=420 at (15,4)		Max P=420 at (3,10)	

For reasons that will become clear later, we will use the variables x_1 and x_2 from the original problem as the slack variables in the dual problem:

$2y_1 + y_2 + s_1 = 16$ (initial system for the dual problem)

$5y_1 + 3y_2 + s_2 = 45$

$-50y_1 - 27y_2 + P = 0$

2	1	1	0	0	16
5	3	0	1	0	45
-50	-27	0	0	1	0

1	0.5	0.5	0	0	8
5	3	0	1	0	45
-50	-27	0	0	1	0

1	0.5	0.5	0	0	8
0	0.5	-2.5	1	0	5
0	-2	25	0	1	400

1	0	3	-1	0	3
0	1	-5	2	0	10
0	0	15	4	1	420

Since all numbers in the bottom row are nonnegative, the solution to the dual problem is

$$y_1 = 3, \quad y_2 = 10, \quad s_1 = 0, \quad s_2 = 0, \quad P = 420$$

Furthermore, examining the bot-tom row of the final simplex tableau, we see the same optimal solution to the mini-mization problem that we obtained directly by the geometric method:

$$\text{Min } C = 420 \quad \text{at} \quad s_1 = 15, \quad s_2 = 4$$

This is not achieved with mistake.

An optimal solution to a minimization problem always can be obtained from the bottom row of the final simplex tableau for the dual problem.

Note: In dual problem, we will choose slack Variable as solution.

Example 2: (For Practice)

Find the minimum value of

$$w = 3x_1 + 2x_2$$

subject to the constraints

$$2x_1 + x_2 \geq 6$$
$$x_1 + x_2 \geq 4$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Example 3:

A small petroleum company owns two refineries. Refinery 1 costs \$20,000 per day to operate, and it can produce 400 barrels of high-grade oil, 300 barrels of medium-grade oil, and 200 barrels of low-grade oil each day. Refinery 2 is newer and more modern. It costs \$25,000 per day to operate, and it can produce 300 barrels of high-grade oil, 400 barrels of medium-grade oil, and 500 barrels of low-grade oil each day.

The company has orders totaling 25,000 barrels of high-grade oil, 27,000 barrels of medium-grade oil, and 30,000 barrels of low-grade oil. How many days should it run each refinery to minimize its costs and still refine enough oil to meet its orders?

Artificial Variables Techniques

In order to use the simplex method, a BFS is needed. To remedy the predicament, artificial variables are created. These variables are fictitious and cannot have any physical meaning. The artificial variable technique is a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. To solve such LPP there are two methods.

- The Big *M* Method or Method of Penalties.
- The Two-phase Simplex Method.

Big M Method

1. Modify the constraints so that the RHS of each constraint is nonnegative. Identify each constraint that is now an = or ≥ constraint.
2. Convert each inequality constraint to standard form (add a slack variable for ≤ constraints, add an excess variable for ≥ constraints).
3. For each ≥ or = constraint, add artificial variables. Add sign restriction $a_i \geq 0$.
4. Let M denote a very large positive number. Add (for each artificial variable) Ma_i to min problem objective functions or $-Ma_i$ to max problem objective functions.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Remembering M represents a very large number, solve the transformed problem by the simplex.

NOTE: If all artificial variables in the optimal solution equal zero, the solution is optimal. If any artificial variables are positive in the optimal solution, the problem is infeasible.

NOTE :

Why we have to subtract or add M from Objective function?

To prevent an artificial variable from becoming part of an optimal solution to the original problem, a very large "penalty" is introduced into the objective function. This penalty is created by choosing a positive constant *M* so large that the artificial variable is forced to be 0 in any final optimal solution of the original problem.

Example:

Maximize

$$P = 2x_1 + x_2$$

subject to

$$x_1 + x_2 \leq 10$$

$$-x_1 + x_2 \geq 2$$

$$x_1, x_2 > 0$$

We now express the linear programming problem as a system of equations:

$$x_1 + x_2 + s_1 = 10$$

$$-x_1 + x_2 - s_2 = 2$$

$$-2x_1 - x_2 + P = 0$$

$$x_1, x_2, s_1, s_2 \geq 0$$

It can be shown that a basic solution of a system is not feasible if any of the variables (excluding P) are negative. Thus a surplus variable is required to satisfy the nonnegative constraint.

An initial basic solution is found by setting the nonbasic variables x_1 and x_2 equal to 0. That is, $x_1 = 0$, x_2,= 0,, s_1= 10, s_2 = -2, P = 0.

This solution is not feasible because the surplus variable s_2 is negative.

we introduce an artificial variable a_1 into the equation involving the surplus variable s_2:

$$x_1 + x_2 - s_2 + a_1 = 2$$

To prevent an artificial variable from becoming part of an optimal solution to the original problem, a very large "penalty" is introduced into the objective function. This penalty is created by choosing a positive constant *M* so large that the artificial variable is forced to be 0 in any final optimal solution of the original problem.

We then add the term $-Ma_1$ to the objective function:

$$P = 2x_1 + x_2 - Ma_1$$

We now have a new problem, called the **modified problem:**

Maximize

$$P = 2x_1 + x_2 - Ma_1$$

subject to

$$x_1 + x_2 + s_1 \qquad = 10$$

$$x_1 + x_2 - s_2 + a_1 = 2$$

$$x_1, x_2, s_1, s_2, a_1 \geq 0$$

The initial system for the modified problem is

$$x_1 + x_2 + s_1 \qquad = 10$$

$$-x_1 + x_2 - s_2 + a_1 = 2$$

$$-2x_1 - x_2 + Ma_1 + P = 0$$

$$x_1, x_2, s_1, s_2, a_1 \geq 0$$

We next write the augmented coefficient matrix for this system, which we call the **preliminary simplex tableau** for the modified problem.

$$
\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & a_1 & P & \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 10 \\
-1 & 1 & 0 & -1 & 1 & 0 & 2 \\
\hline
-2 & -1 & 0 & 0 & M & 1 & 0
\end{array}
$$

To start the simplex process we require an **initial simplex tableau**, described on the next slide. The preliminary simplex tableau should either meet these requirements, or it needs to be transformed into one that does.

For a system tableau to be considered an **initial simplex tableau**, it must satisfy the following two requirements:

1. The requisite number of basic variables must be selectable. Each basic variable must correspond to a column in the tableau with exactly one nonzero element. Different basic variables must have the nonzero entries in different rows. The remaining variables are then selected as non-basic variables.
2. The basic solution found by setting the non-basic variables equal to zero is feasible.

The preliminary simplex tableau from our example satisfies the first requirement, since s_1, s_2, and P can be selected as basic variables according to the criterion stated.

However, it does not satisfy the second requirement since the basic solution is not feasible ($s_2 = -2$.)

To use the simplex method, we must first use row operations to transform the tableau into an equivalent matrix that satisfies all initial simplex tableau requirements. **This transformation is not a pivot operation.**

If you inspect the preliminary tableau, you realize that the problem is that s_2 has a negative coefficient in its column. We need to replace s_2 as a basic variable by something else with a positive coefficient. We choose a_1.

We want to use a_1 as a basic variable instead of s_2. We proceed to eliminate M from the a_1 column using row operations:

$$(-M)R_2 + R_3 \rightarrow R_3 \qquad \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & a_1 & P \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ -1 & 1 & 0 & -1 & 1 & 0 & 2 \\ \hline -2 & -1 & 0 & 0 & M & 1 & 0 \end{array}\right] \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ -1 & 1 & 0 & -1 & 1 & 0 & 2 \\ \hline M-2 & -M-1 & 0 & M & 0 & 1 & -2M \end{array}\right]$$

We now continue with the usual simplex process, using pivot operations. When selecting the pivot columns, keep in mind that M is unspecified, but we know it is a very large positive number.

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 10 \\ -1 & 1 & 0 & -1 & 1 & 0 & 2 \\ \hline M-2 & -M-1 & 0 & M & 0 & 1 & -2M \end{array}\right]$$

In this example, $M - 2$ and M are positive, and $-M - 1$ is negative. The first pivot column is column 2.

If we pivot on the second row, second column, and then on the first row, first column, we obtain:

$$
\begin{array}{c} \\ x_1 \\ x_2 \\ \\ P \end{array}
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & a_1 & P \end{array} \\
\left[\begin{array}{cccccc|c}
1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & 0 & 4 \\
0 & 1 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & 0 & 6 \\
\hline
0 & 0 & \frac{3}{2} & \frac{1}{2} & M-\frac{1}{2} & 1 & 14
\end{array}\right]
\end{array}
$$

Since all the indicators in the last row are nonnegative, we have the optimal solution:

$$\text{Max } P = 14 \text{ at } x_1 = 4,\ x_2 = 6,\ s_1 = 0,\ s_2 = 0,\ a_1 = 0.$$

Example:

Use Big M method to solve the Turkey Feed Problem as given in Table:

Turkey meal's Data

Ingredient	Compostitoin of each Pound Feed(Oz.)		Minimum Monthly Requirement Per Turkey(Oz.)
	Brand Feed 1	Brand Feed 2	
A	5	10	90
B	4	3	48
C	1/2	0	3/2
Cost per pound	Rs.2	Rs.3	

From Table, We can Formulate our data like this:

$$\text{Minimize } z = 2x_1 + 3x_2$$

subject to these constraints:

$5x_1 + 10x_2 \geq 90$ ounces (ingredient A constraint)

$4x_1 + 3x_2 \geq 48$ ounces (ingredient B constraint)

$\frac{1}{2}x_1 \geq \frac{3}{2}$ ounces (ingredient C constraint)

$x_1 \geq 0, x_2 \geq 0$

After Converting into Standard Form:

$$\text{Minimize } z = 2x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to these constraints:

$$5x_1 + 10x_2 - s_1 = 90$$

$$4x_1 + 3x_2 - s_2 = 48$$

$$x_1 - s_3 = 3$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Find initial basic feasible solution

Putting $x_1 = x_2 = 0$, we get $s_1 = -90$, $s_2 = -48$ and $s_3 = -3$ as the first basic solution but it is not feasible as s_1, s_2 and s_3 have negative values that do not satisfy the non-negativity restrictions. Therefore, we introduce artificial variables A_1, A_2, A_3 in the constraints, which take the form

$$\text{Minimize } z = 2x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to these constraints:

$$5x_1 + 10x_2 - s_1 + A_1 = 90$$

$$4x_1 + 3x_2 - s_2 + A_2 = 48$$

$$x_1 - s_3 + A_3 = 3$$

$$x_1, x_2, s_1, s_2, s_3, A_1, A_2, A_3 \geq 0$$

Now artificial variables with values greater than zero violate the equality in constraints established in step 1. Therefore, A_1,A_2,and A_3 should not appear in the final solution. To achieve this, they are assigned a large unit penalty (a large postive value, +M) in the objective function, which can be written as

$$\text{Minimize } z = 2x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3 + MA_1 + MA_2 + MA_3$$

subject to these constraints:

$$5x_1 + 10x_2 - s_1 + A_1 = 90$$

$$4x_1 + 3x_2 - s_2 + A_2 = 48$$

$$x_1 - s_3 + A_3 = 3$$

$$x_1, x_2, s_1, s_2, s_3, A_1, A_2, A_3 \geq 0$$

Eliminating A_1, A_2, A_3 from the first, second and third equations modified objective function can be written as:

$$\text{Minimize } z = -(10M-2)x_1 - (13M-3)x_2 + Ms_1 + Ms_2 + Ms_3 + 141M$$

$$z + (10M-2)x_1 + (13M-3)x_2 - Ms_1 - Ms_2 - Ms_3 = 141M$$

Problem, now, has eight variables and three constraints. five of the variables have to be zeroised to get initial basic feasible solution to the 'artificial system'. Putting,

$x_1 = x_2 = s_1 = s_2 = s_3 = 0$, and $A_1 = 90, A_2 = 48, A_3 = 3$ we get

The starting feasible solution is $A_1 = 90, A_2 = 48, A_3 = 3$ and $z = 141M$.

x_1	x_2	s_1	s_2	s_3	A_1	A_2	A_3	b	**B.V.**	**Ratio**
5	10	-1	0	0	**1**	**0**	**0**	90	s_1	90/5=19
4	3	0	-1	0	**0**	**1**	**0**	48	s_2	48/4=12
1	0	0	0	-1	**0**	**0**	**1**	3	s_3	3/1=3
10M-2	13M-3	-M	-M	-M	**0**	**0**	**0**	141M	z	

x_1	x_2	s_1	s_2	s_3	A_1	A_2	A_3	b	**B.V.**	**Ratio**
5	10	-1	0	0	**1**	**0**	**0**	90	s_1	90/10=9 *Leaving variable*
4	3	0	-1	0	**0**	**1**	**0**	48	s_2	48/3=16
1	0	0	0	-1	**0**	**0**	**1**	3	s_3	--
498	647	-50	-50	-50	**0**	**0**	**0**	7050	z	

Entering (647)

Current z-value (7050)

x_1	x_2	s_1	s_2	s_3	A_1	A_2	A_3	b	**B.V.**	**Ratio**
1/2	**1**	-1/10	0	0	1/10	**0**	**0**	9	s_1	9/0.5=18
5/2	**0**	3/10	-1	0	-3/10	**1**	**0**	21	s_2	42/5=8.4
1	**0**	0	0	-1	0	**0**	**1**	3	s_3	3/1=3 *Leaving variable*
174.5	**0**	14.7	-50	-50	-64.7	**0**	**0**	1227	z	

Entering (174.5, x_1 column) *Current z-value* (1227)

x_1	x_2	s_1	s_2	s_3	A_1	A_2	A_3	b	**B.V.**	**Ratio**
0	**1**	-1/10	0	1/2	1/10	**0**	-1/2	7.5	x_2	7.5/0.5=15
0	**0**	3/10	-1	5/2	-3/10	**1**	-5/2	13.5	A_1	27/5=5.4 *Leaving variable*
1	**0**	0	0	-1	0	**0**	1	3	x_1	
0	**0**	14.7	-50	124	-64.7	**0**	-174.5	703.5	z	

Entering (124, s_3 column) *Current z-value* (703.5)

x_1	x_2	s_1	s_2	s_3	A_1	A_2	A_3	b	B.V.
0	1	-0.16	0.20	0	0.16	-0.20	0	4.8	x_2
0	0	0.12	-0.40	1	-0.12	0.40	-1	5.4	s_3
1	0	0.12	-0.40	0	-0.12	0.40	0	8.4	x_1
0	0	-0.24	-0.20	0	-49.76	-49.8	-50	31.20	z

Since all z coefficients are negative it becomes optimal solution with minimum cost 31.20.
Hence, the minimum cost solution is to purchase 8.4 pounds of brand 1 feed and 4.8 pounds of brand 2 feed per turkey per month.

Examples for Practice:

1. Minimize $z = 12x_1 + 20x_2$

   Subject to

   $6x_1 + 8x_2 \geq 100$

   $7x_1 + 12x_2 \geq 120$

   $x_1, x_2 \geq 0$

[Ans: $x_1 = 15$, $x_2 = 4$ and Min $z = 205$]

2. Maximize $z = 2x_1 + x_2 + 3x_3$,

   Subject to

$$x_1 + x_2 + 2x_3 \le 5$$

$$2x_1 + 3x_2 + 4x_3 = 12$$

$$x_1,\ x_2,\ \ x_3 \ge 0$$

[**Ans:** $x_1 = 3$, $x_2 = 2$, $x_3, = 0$ and Max $z = 8$]

Two Phase Method

The two-phase method is another method to handle these artificial variables. Here the L.P. problem is solved in two phases.

PHASE I

In this phase we find an initial basic feasible solution to the original problem. For this all artificial variables are to be driven to zero. To do this an artificial (Auxiliary) objective function (r) is created which is the sum of all the artificial variables. This new objective function is then minimized, subject to the constraints of the given (original) problem, using the simplex method. At the end of phase I, two cases arise:

TWO PHASE METHOD : NO FEASIBLE SOLUTION

If the minimum value of $r > 0$, and at least one artificial variable appears in the basis at a positive level, then the given problem has no feasible solution and the procedure terminates.

TWO PHASE METHOD: OPTIMALITY

If the minimum value of $r = 0$, and no artificial variable appears in the basis, then a basic feasible solution to the given problem is obtained. The artificial column (s) are deleted for phase II computations.

PHASE II

Use the optimum basic feasible solution of phase I as a starting solution for the original LPP. Assign the actual costs to the variable in the objective function and a zero cost to every artificial variable in the basis at zero level. Delete the artificial variable column from the table which is eliminated from the basis in phase I. Apply simplex method to the modified simplex table obtained at the end of phase I till an optimum basic feasible is obtained or till there is an indication of unbounded solution.

Example:

$$\text{Minimize} \quad z = 12x_1 + 18x_2 + 15x_3$$

Subject to

$$4x_1 + 8x_2 + 6x_3 \geq 64$$

$$3x_1 + 6x_2 + 12x_3 \geq 96$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

The canonical form of the given problem is shown below:

Minimize $z = 12x_1 + 18x_2 + 15x_3$

Subject to

$$4x_1 + 8x_2 + 6x_3 - s_1 + A_1 = 64$$

$$3x_1 + 6x_2 + 12x_3 - s_2 + A_2 = 96$$

$$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$$

Phase 1

Auxilary objective function:

$$\text{Minimize} \quad r = A_1 + A_2$$
$$= 160 - 7x_1 - 14x_2 - 18x_3 + s_1 + s_2$$

subject to

$$4x_1 + 8x_2 + 6x_3 \quad - s_1 + A_1 = 64$$
$$3x_1 + 6x_2 + 12x_3 - s_2 + A_2 = 96$$
$$x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$$

x_1	x_2	x_3	s_1	s_2	A_1	A_2	b	**B.V.**
4	8	6	-1	0	**1**	**0**	64	A_1
3	6	12	0	-1	**0**	**1**	96	A_2
7	14	18	-1	-1	**0**	**0**	160	r

x_1	x_2	x_3	s_1	s_2	A_1	A_2	b	**B.V.**
4	8	6	-1	0	**1**	**0**	64	A_1
3	6	12	0	-1	**0**	**1**	96	x_3 *Leaving variable*
7	14	18	-1	-1	**0**	**0**	160	r

Entering (↑ x_3)

Current r-value (↑ 160)

x_1	x_2	x_3	s_1	s_2	A_1	A_2	b	**B.V.**
2.5	5	0	-1	0.5	**1**	0.5	64	A_1
0.25	0.5	1	0	-0.093	**0**	-0.093	96	A_2
2.5	5	0	-1	0.5	**0**	1.5	16	r

Leaving variable (A_1 row)

Entering (x_2 column)

Current r-value (16)

x_1	x_2	x_3	s_1	s_2	A_1	A_2	b	**B.V.**
0.5	**1**	**0**	-0.2	0.1	-0.1	-0.1	3.2	x_2
0	**0**	**1**	0.1	-0.13	-0.093	0.13	6.4	x_3
0	**0**	**0**	0	0	0	0	0	r

The set of basic variables in the optimal table of phase 1 does not contain artificial variables. So, the given problem has a feasible solution.

Phase 2:

The optimal results are presented by x_1 =0, x_2 =3.2=6/5, x_3 = 6.4 and min z=153.6

x_1	x_2	x_3	s_1	s_2	b	**B.V.**
0.5	**1**	**0**	-0.2	0.1	3.2	x_2
0	**0**	**1**	0.1	-0.13	6.4	x_3
-12	**-15**	**-18**	0	2.1	0	z

Initial Table of Phase 2

x_1	x_2	x_3	s_1	s_2	b	**B.V.**
0.5	**1**	**0**	-0.2	0.1	3.2	x_2
0	**0**	**1**	0.1	-0.13	6.4	x_3
3	**0**	**0**	0	2.1	153.6	z

The optimal results are presented by x_1 =0, x_2 =3.2=6/5, x_3 = 6.4 and min z=153.6

Lecture 4: Transportation Models & Optimization

Learning objectives:

The learning objectives of this chapter are

- Introduction to Transportation model & applications.
- North west Corner rule to solve transportation problem.
- Least cost Method & application.
- Vogel's approximation method.

Introduction:

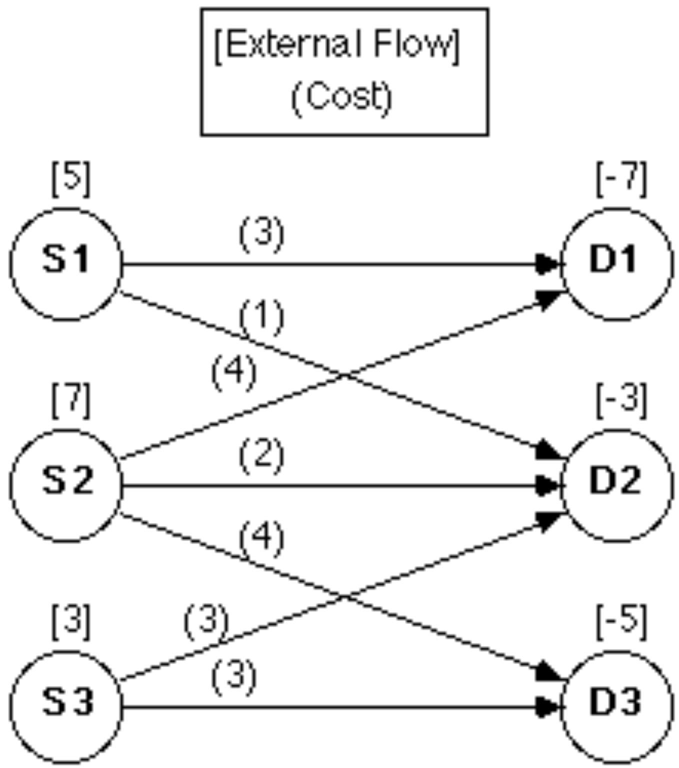
A typical transportation problem is shown in Fig. It deals with sources where a supply of some commodity is available and destinations where the commodity is demanded. The classic statement of the transportation problem uses a matrix with the rows representing sources and columns representing destinations. The algorithms for solving the problem are based on this matrix representation. The costs of shipping from sources to destinations are indicated by the entries in the matrix. If shipment is impossible between a given source and destination, a large cost of M is entered. This discourages the solution from using such cells. Supplies and demands are shown along the margins of the matrix.

As in the example, the classic transportation problem has total supply equal to total demand.

	D1	D2	D3	Supply
S1	3	1	M	5
S2	4	2	4	7
S3	M	3	3	3
Demand	7	3	5	

Matrix model of a transportation problem

The network model of the transportation problem is shown in Fig. Sources are identified as the nodes on the left and destinations on the right. Allowable shipping links are shown as arcs, while disallowed links are not included.



North West Corner Rule

The **North West corner rule** is a method for computing a basic feasible solution of a **transportation problem**, where the basic variables are selected from the North – West corner (i.e., top left corner).

Steps in North West Corner Rule

- Select the upper left-hand corner cell of the transportation table and allocate as many units as possible equal to the minimum between available supply and demand, i.e., min(s1, d1).
- Adjust the supply and demand numbers in the respective rows and columns.
- If the demand for the first cell is satisfied, then move horizontally to the next cell in the second column.
- If the supply for the first row is exhausted, then move down to the first cell in the second row.
- If for any cell, supply equals demand, then the next allocation can be made in cell either in the next row or column.
- Continue the process until all supply and demand values are exhausted.

Application:

Find the initial basic feasible solution of the following transportation problem using North – West Corner Method.

FACTORY	WAREHOUSE				SUPPLY
	W1	W2	W3	W4	
F1	14	25	45	5	6
F2	65	25	35	55	8
F3	35	3	65	15	16
DEMAND	4	7	6	13	30

Solution:

Factory	Warehouse								Supply
	W1		W2		W3		W4		
F1	4	14	2	25		45		5	6/2/0
F2		65	5	25	3	35		55	8/3/0
F3		35		3	3	65	13	15	16/13/0
Demand	4/0		7/5/0		6/3/0		13/0		30

The initial basic feasible solution for the given problem is:

From	To	Units shipped	Cost per Unit	Total Cost
F1	W1	4	14	56
F1	W2	2	25	50
F2	W2	5	25	125
F2	W3	3	35	105
F3	W3	3	65	195
F3	W4	13	15	195
				726

Practice Example:

Find the initial basic feasible solution of the following transportation problem using North – West Corner Method

W→ F↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Solution: 1015

Least Cost Method:

This method usually provides a better initial basic feasible solution than the North-West Corner method since it takes into account the cost variables in the problem.

Steps in Least Cost Method

Step1: Select the cell having lowest unit cost in the entire table and allocate the minimum of supply or demand values in that cell.

Step2: Then eliminate the row or column in which supply or demand is exhausted. If both the supply and demand values are same, either of the row or column can be eliminated.

In case, the smallest unit cost is not unique, then select the cell where maximum allocation can be made.

Step3: Repeat the process with next lowest unit cost and continue until the entire available supply at various sources and demand at various destinations is satisfied.

Application:

Find the initial basic feasible solution of the following transportation problem using Least Cost Method

FACTORY	DISTRIBUTION CENTERS			SUPPLY
	D1	D2	D3	
F1	2	7	4	5
F2	3	3	1	8
F3	5	4	7	7
F4	1	6	2	14
DEMAND	7	9	18	34

Solution:

Factory	Warehouse						Supply
	D1		D2		D3		
F1		2		7		4	5
F2	*	3	*	3	8	1	8/0
F3		5		4		7	7
F4		1		6		2	14
Demand	7		9		18/10		34

Factory	Warehouse						Supply
	D1		D2		D3		
F1	*	2		7		4	5
F2	*	3	*	3	8	1	8/0
F3	*	5		4		7	7
F4	7	1		6		2	14/7
Demand	7/0		9		18/10		34

Factory	Warehouse						Supply
	D1		D2		D3		
F1	*	2		7		4	5
F2	*	3	*	3	8	1	8/0
F3	*	5		4		7	7
F4	7	1	*	6	7	2	14/7/0
Demand	7/0		9		18/10/3		34

Factory	Warehouse						Supply
	D1		D2		D3		
F1	*	2		7	3	4	5/2
F2	*	3	*	3	8	1	8/0
F3	*	5		4	*	7	7
F4	7	1	*	6	7	2	14/7/0
Demand	7/0		9		18/10/3/0		34

Factory	Warehouse						Supply
	D1		D2		D3		
F1	*	2		7	3	4	5/2
F2	*	3	*	3	8	1	8/0
F3	*	5	7	4	*	7	7/0
F4	7	1	*	6	7	2	14/7/0
Demand	7/0		9/2		18/10/3/0		34

Factory	Warehouse						Supply
	D1		D2		D3		
F1	*	2	2	7	3	4	5/2/0
F2	*	3	*	3	8	1	8/0
F3	*	5	7	4	*	7	7/0
F4	7	1	*	6	7	2	14/7/0
Demand	7/0		9/2/0		18/10/3/0		34

The initial basic feasible solution for the given problem is:

From	To	Units shipped	Cost per Unit	Total Cost
F1	D2	2	7	14
F1	D3	3	4	12
F2	D3	8	1	8
F3	D2	7	4	28
F4	D1	7	1	7
F4	D3	7	2	14
				83

Vogel's Approximation Method

This method also takes costs into account in allocation. The Vogel's approximation method (VAM) usually produces an optimal or near- optimal starting solution. One study found that VAM yields an optimum solution in 80 percent of the sample problems tested.

Steps to solve Vogel's Approximation method:

<u>Step1</u>: Calculate penalty for each row and column by taking the difference between the two smallest unit costs. This penalty or extra cost has to be paid if one fails to allocate the minimum unit transportation cost.

<u>Step2</u>: Select the row or column with the highest penalty and select the minimum unit cost of that row or column. Then, allocate the minimum of supply or demand values in that cell. If there is a tie, then select the cell where maximum allocation could be made.

<u>Step3</u>: Adjust the supply and demand and eliminate the satisfied row or column. If a row and column are satisfied simultaneously, only of them is eliminated and the other one is assigned a ***zero*** value. Any row or column having ***zero*** supply or demand, can not be used in calculating future penalties.

<u>Step4</u>: Repeat the process until all the supply sources and demand destinations are satisfied.

Application:

Find the initial basic feasible solution of the following transportation problem using VAM (Vogel's Approximation Method) .

W→ F ↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Solution:

Factory	Warehouse								Supply	
	W1		W2		W3		W4			
F1		19		30		50		10	7	9
F2		70		30		40		60	9	10
F3		40		8		70	13	20	18	12
Demand	5		8		7		14		34	
	21		22		10		10			

Factory	Warehouse								Supply	
	W1		W2		W3		W4			
F1		19	*	30		50		10	7	9
F2		70	*	30		40		60	9	10
F3		40	8	8		70	13	20	18/10	12
Demand	5		8/0		7		14		34	
	21		22		10		10			

Factory	Warehouse								Supply	
	W1		W2		W3		W4			
F1	5	19	*	30		50		10	7/2	9
F2	*	70	*	30		40		60	9	20
F3	*	40	8	8		70	13	20	18/10	20
Demand	5/0		8/0		7		14		34	
	21				10		10			

Factory	Warehouse								Supply	
	W1		W2		W3		W4			
F1	5	19	*	30		50		10	7/2	40
F2	*	70	*	30		40		60	9	20
F3	*	40	8	8	*	70	10	20	18/10/0	50
Demand	5/0		8/0		7		14/4		34	
						10	10			

Factory	Warehouse								Supply	
	W1		W2		W3		W4			
F1	5	19	*	30	*	50	2	10	7/2/0	40
F2	*	70	*	30		40		60	9	20
F3	*	40	8	8	*	70	10	20	18/10/0	
Demand	5/0		8/0		7		14/4/2		34	

10 50

Factory	Warehouse								Supply
	W1		W2		W3		W4		
F1	5	19	*	30	*	50	2	10	7/2/0
F2	*	70	*	30	7	40		60	9/2
F3	*	40	8	8	*	70	10	20	18/10/0
Demand	5/0		8/0		7/0		14/4/2		34

20

Factory	Warehouse								Supply
	W1		W2		W3		W4		
F1	5	19	*	30	*	50	2	10	7/2/0
F2	*	70	*	30	7	40	2	60	9/2/0
F3	*	40	8	8	*	70	10	20	18/10/0
Demand	5/0		8/0		7/0		14/4/2/0		34

The initial basic feasible solution for the given problem is:

From	To	Units shipped	Cost per Unit	Total Cost
F1	W1	5	19	95
F1	W4	2	10	20
F2	W3	7	40	280
F2	W4	2	60	120
F3	W2	8	8	64
F3	W4	10	20	200
				779

Lecture 5:

Optimization Techniques for Transportation Model

Learning objectives:

The learning objectives of this chapter are

- Introduction to Optimality tests unbalanced problems for transportation problem .
- Stepping Stone Method for finding optimal Solution
- Distributed Modified Problem for prohibited route problem.
- Degeneracy

Optimality Tests.

There are two tests to check whether basic feasible solution fit for optimality test or not:

1. If there are m number of rows & n number of columns then number of allocation must be *((m+n)-1)*

2. All allocations should be in independent positions i.e. there should not be able to make closed loop from allocations.

Which one Optimal Solution?

In last lecture, we learn about Northeast corner method, Least cost method & VAM out of which found VAM generate more optimal solution but it's not always better:

Solutions method to get Optimal solution:

1. Stepping-stone method

2. Modified distributed method (MODI

Stepping Stone Method

Is the following an optimal solution for the transportation problem?
If not, how to modify it?

Plant	Warehouse				Supply
	W_1	W_2	W_3	W_4	
P_1	6	1	(50) 9	(20) 3	70
P_2	(55) 11	5	2	8	55
P_3	(30) 10	(35) 12	4	(25) 7	90
Demand	85	35	50	45	

Solution:

Lecture Handouts. (For better understanding solutions for this chapter provided in lecture Handouts)

Modified Distribution Method:

A company has three factories which supply their products to four warehouses. Monthly capacities of the factories are 120, 200 and 180 units respectively. Monthly requirements of warehouses are 100, 140, 110 and 120 respectively. Unit shipping costs are as follows:

Factory	Warehouses			
	P	Q	R	S
I	15	-	30	20
II	-	24	12	15
III	25	15	-	20

Shipment from I to Q, II to P and III to R is not possible due to certain unavoidable reasons. Find the optimum distribution program to minimize shipping costs.

Solution:

Lecture Handouts. (For better understanding solutions for this chapter provided in lecture Handouts)

Degeneracy

Table B-11
The Minimum Cell Cost Solution

From \ To	A	B	C	Supply
1	[6]	[8] 25	[10] 125	150
2	[7]	[11]	[11] 175	175
3	[4] 200	[5] 75	[12]	275
Demand	200	100	300	600

m rows + n column – 1 = the number of cells with allocations
3 + 3 - 1 = 5

It satisfied.

If failed? …… considering …..

Total demand ≠ total supply

From \ To	A	B	C
1	6	8	10
2	7	11	11
3	4	5	12

Note that, total demand=650, and total supply = 600

How to solve it?

We need to add a dummy row and assign o cost to each cell as such ..

Extra row, since Demand > supply

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Dummy	0	0	0	50
Demand	200	100	350	650

Extra Column, since Demand < supply

Table B-30
An Unbalanced Model
(Supply > Demand)

From \ To	A	B	C	Dummy	Supply
1	6	8	10	0	150
2	7	11	11	0	175
3	4	5	12	0	375
Demand	200	100	300	100	700

Practice Example: Degeneracy

We have three reservoirs with daily supplies of 15, 20 and 25 million litres of water respectively. On each day, we must supply four cities A, B, C and D whose demands are 8, 10, 12 and 15 million litres respectively. The cost of pumping in Rs. per million litre is as given below:

Reservoirs	Cities			
	A	B	C	D
R1	2	3	4	5
R2	3	2	5	2
R3	4	1	2	3

Use the transportation algorithm to determine the cheapest pumping schedule if excess water can be disposed off at no extra cost.

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