CHAPTER 6. AXONOMETRIC PROJECTIONS

6.1 The Method of Axonometric Projection Coefficient of Distortion

A complex drawing is rather simple and easily measured, although it is hard sometimes to imagine an object in space by means of it. It is often necessary to have in addition to it a drawing of pictorial view, which may be obtained by projecting an object and its co-ordinate axes onto one plane. Then one projection will provide a visual and metrically distinguished image of the object. Such kinds of an object representation are called the axonometric projections.

The method of axonometric projection consists in the following: a given figure and the axes of rectangular co-ordinates to which the figure is related in space are projected on a plane referred to as a plane of projections (it also called a picture plane).

Depending on the distance between the centre of projection and the picture plane all axonometric projections are classified as: the *central projections* - the centre is located at a finite distance from the plane; and the *parallel projections* - the centre is at infinity.

Only parallel axonometric projections are considered in this chapter.

The word “axonometry” is derived from the Greek words “*axon*” which means “axis” and “*metro*” meaning “I measure’, so it can be translated as “the measurement by the axes”. That is, an axonometric representation provides the opportunity to measure an object both by the co-ordinate axes *x*, *y*, *z* and by the directions parallel to them.

Let us construct an axonometric projection of the point *A* related to three mutually perpendicular projection planes (Fig.6.1).

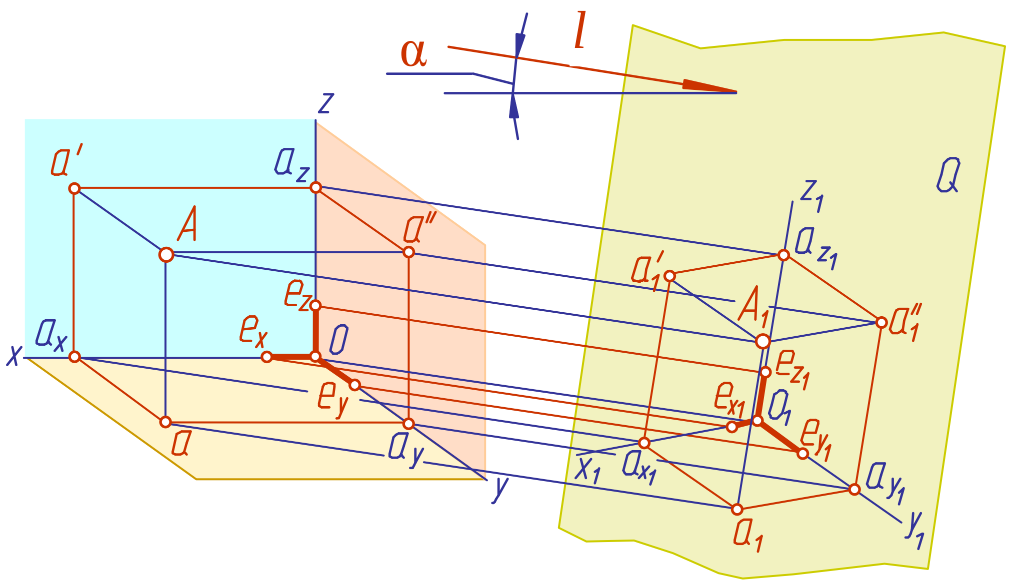


Fig.6.1

The x, y, z co-ordinate axes are called the natural axes. Take a scaled line-segment e (natural scale), lay it off on the axes and designate it as e_x, e_y, e_z ($e = e_x = e_y = e_z$). Now project (by parallel beams) the point A, the projections a, a', a'', the co-ordinate axes and the scaled segments e_x, e_y, e_z onto the picture plane Q.

Let us introduce some designations:

Q - axonometric projection plane (picture plane);
S - direction of projecting;
α - angle of inclination of the direction S to the plane Q;
x_1, y_1, z_1 - axonometric co-ordinate axes or axonometric axes;
A_1 - axonometric projection of the point A;
a_1, a_1', a_1''- secondary projections of the point A;
e_x, e_y, e_z - scaled line-segments;
e_{x1}, e_{y1}, e_{z1}-axonometric (secondary) projections of the scaled line-segments;

Depending on the position of the planes H, W, V, the axonometric projection plane Q and the direction of projecting S in the space, the point co-ordinates will be projected with different distortions.

The ratio of the length of the axonometric projection segment to its true size is referred to as the coefficient of distortion on an axis.

Let us designate the above coefficients:

on the axis x $$m = \frac{e_{x1}}{e_x}$$

on the axis y $$n = \frac{e_{y1}}{e_y}$$

on the axis z $$k = \frac{e_{z1}}{e_z}$$

Depending on the ratio of the coefficient of distortion on the axes, the following axonometric projections are distinguished:

1. Isometric projections when $m = n = k$;
2. Dimetric projections when $m = k \neq n$ *or* $m = n \neq k$;
3. Trimetric projections when $m \neq n \neq k$.

The names of the projections are derived from Greek: *"isos"* – equal (the isometric projection is a projection of the equal coefficients of distortion on all axes); *"di"* – double (dimetric projection is a projection of the equal coefficients of distortion on two axes); *"treis"* – three (trimetric projection is a projection of different coefficients of distortion on all axes).

Depending on the direction of projecting relative to the axonometric projection plane Q, axonometric projections are classified as rectangular (the projection angle $\alpha = 90°$) and oblique ($\alpha \neq 90°$).

The relationship between the coefficients of distortion, depending on the direction of projecting, may be expressed in the following equations:

for oblique axonometry $m^2 + n^2 + k^2 = 2 + \cot 2\alpha$;

for rectangular axonometry $m^2 + n^2 + k^2 = 2$.

Depending on the location of the co-ordinate axes, the axonometric projection planes and on the directions of projecting, the vast majority of axonometric projections may be different in the direction of the axonometric axes and in their scales.

The principal theorem of axonometry was declared by the German mathematician K.Pohlke in 1853: "Any three line-segments passing from one point on a plane can be referred to as the parallel projections of three equal and mutually perpendicular line-segments in the space". The first generalisation and proof of this theorem was provided by another German mathematician G.Schwarz in 1864. Since then, the theorem is called the Polke – Schwarz theorem.

The definition of axonometry may be derived from it:

Axonometry is a representation of an object on a plane related to a certain coordinate system and completed to a certain scale subject to the coefficients of distortion.

6.2 Rectangular Parallel Isometry

Rectangular parallel isometry is widely used in the practice of technical drawing. In rectangular isometric projection (Fig.6.2) the axonometric axes x, y, z are at 120° to each other and the coefficients of distortion are similar in all three axes ($m=n=k$), and are equal to 0.82 ($m^2+n^2+k^2=2$; $m=n=k=\sqrt{2/3}=0.82$). However, to simplify an isometric projection, a reduction of the coefficients of distortion is usually applied ($m=n=k=1$). In this case, the representation obtained is enlarged by 1.22.

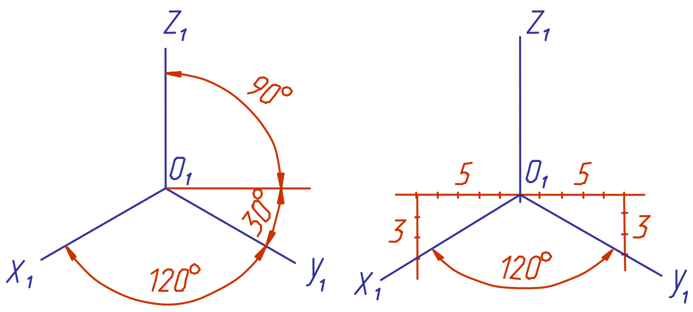


Fig. 6.2

The axis z is positioned vertically, the axes x_1 and y_1 - at 30° to the horizontal direction.

Example: To construct an isometric projection of the point A given its orthogonal projections (Fig.6.3), draw the axonom etric axes at 120° to each other (Fig.6.4). Then from the origin of co-ordinates O_1 on the axis x_1 lay off the line-segment o_1a_{x1} equal to the co-ordinate x_A of the point A. The co-ordinate x_A is taken from the complex drawing (Fig.6.3).

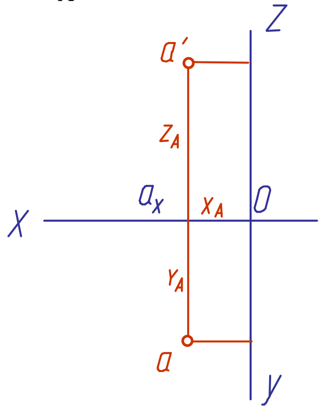


Fig.6.3

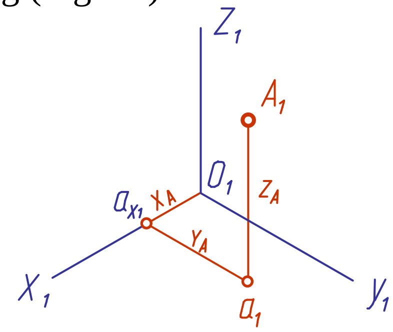


Fig.6.4

From the point a_{x1} pass a straight line parallel to the axis y_1 and on it lay off a segment equal to the co-ordinate y_A of the point A to obtain the point a_1. From this point (a_1) draw a line-segment parallel to the axis z_1 and equal to the co-ordinate z_A of the point A. The point A_1 thus obtained is an isometric projection of the point A.

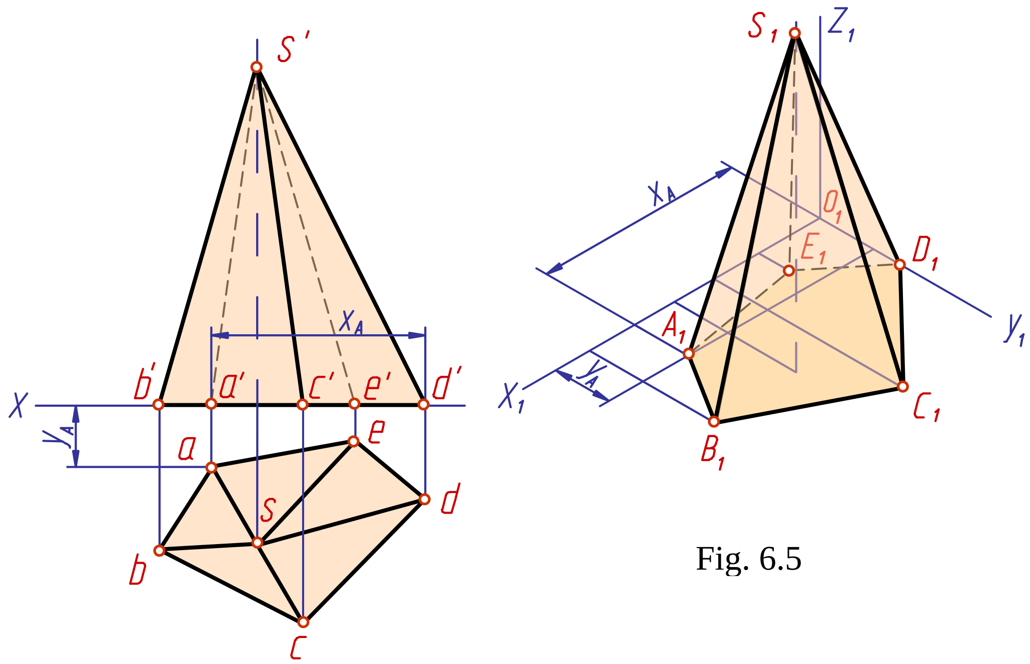


Fig. 6.5

Constructing an isometry of a pentagonal pyramid by its drawing is shown in Fig.6.5. First, determine the co-ordinates of all points of the pyramid base. Then according to x and y co-ordinates construct an isometry of five points - the vertices of the pyramid base. For example, to construct an isometric projection of the point *A* from the origin O_1 on the axis x_1 lay off a segment equal to the co-ordinate $x_A = a'd.'$ From the end of the segment pass a line parallel to the axis y_1, on which lay off a segment equal to the second co-ordinate of the point $y_A = a'a$. Now construct the pyramid height and find its vertex S_1. Joining the point S_1 with the points A_1, B_1, C_1, D_1, E_1 obtain the pyramid isometry.

Fig.6.6 shows the construction an isometry of a hexagonal prism.

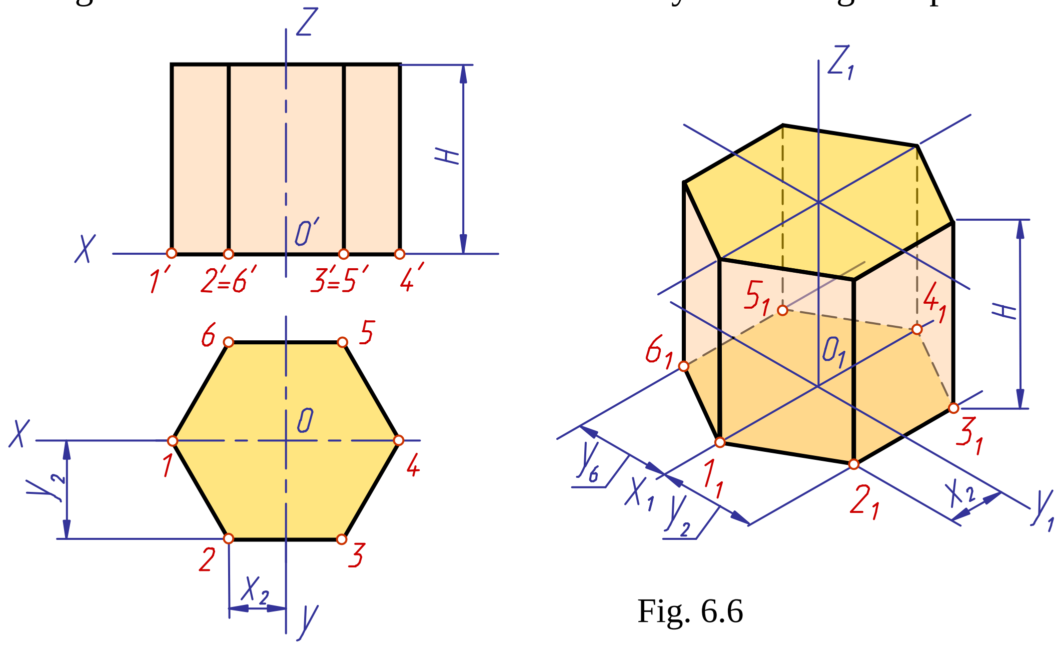


Fig. 6.6

6.3 Rectangular Parallel Dimetry

In the rectangular dimetry the axis z_1 is vertical, the axis x_1 is at 7°10′ and the axis y_1 is at 41°25′ to the horizontal line (Fig.6.7). The coefficients of distortion on the axes x_1 and z_1 are assumed to be equal ($m=k$), those on the axis y_1 - twice less ($n=1/2m$). Then:

$$m^2+k^2+n^2=m^2+m^2+(1/2m)^2=2; \qquad m=\sqrt{8/9}=0.94; \qquad n=0.47$$

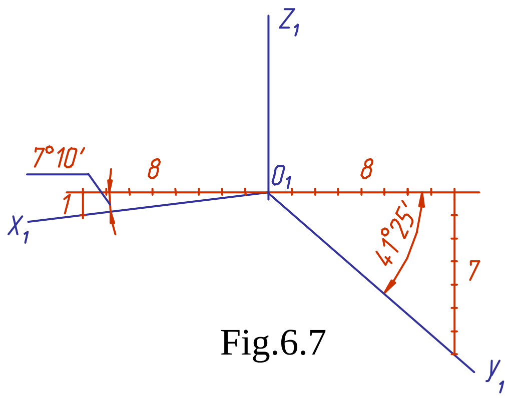


Fig.6.7

In practice the reduction of dimetry is usually used with the coefficients of distortion $m=k=1$, $n=0.5$. In this case the representation is enlarged by *1.06*.

To construct a dimetric projection of the point *A*, given its orthogonal projection (Fig.6.8), pass the axonometric axes at a given angle (Fig.6.9).

On the axis x_1 from the origin of coordinates lay off the line-segment o_1a_{x1}

equal to the coordinate x_A of the point A. From the point a_{x1} draw a straight line parallel to the axis y_1, and on it lay off a segment equal to the half length (as the coefficient of distortion on the axis y_1 is 0.5) of the coordinate y_1 of the point A. From the point a_1 pass the segment a_1A_1 equal to the coordinate z_A. The point A_1 thus obtained is the desired dimetric projection of the point A.

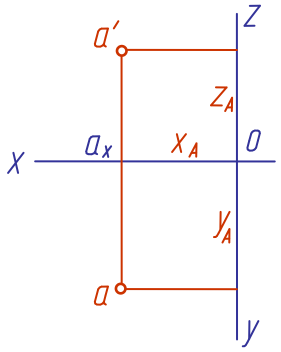


Fig.6.8

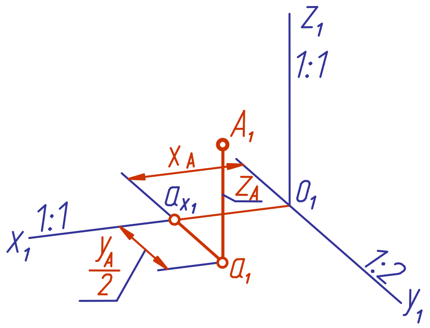


Fig.6.9

Construction of a dimetry of a prism with a prismatic hole (Fig.6.10) is shown in Fig.6.11.

To uncover the inside form of the detail, the axonometric projection is completed with a notch ¼ (the angle contained by the cutting planes is drawn open for a viewer). As the detail is a symmetric one, introduce the origin of the co-ordinates (the point O) in the prism centre and construct the axes x, y, z (Fig.6.10).

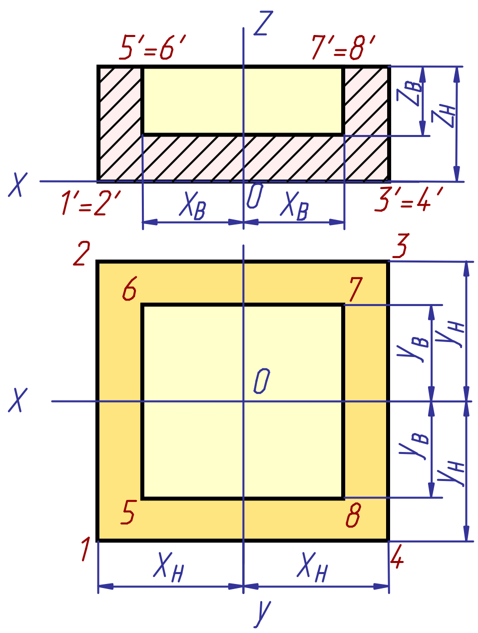


Рис. 6.10

To draw the axonometric projection proceed as follows:

Construct the axonometric axes and the plane figures obtained by cutting the detail with the planes xOz and yOz (Fig.6.11-1)

Designate the vertices of the lower base (the points *1*, *2*, *3*, *4*) and construct the axonometric projections of the points *2*, *3*, *4*. Now pass from the obtained points the segments parallel to the axis z_1 and lay off on them the prism height z_H to construct the upper base of the prism (Fig.6.11-2).

Designate the vertices of the prismatic opening in the upper base (points *5*,*6*,*7*,*8*) and construct the axonometric projections of the points *6*, *7*, *8*. From the obtained points pass the lines parallel to the axis z_1 and lay off on them the depth of the opening z_B. Join the points thus obtained with the thin lines (Fig.6.11-3).

Outline the visible lines of the drawing and take away the auxiliary constructions. Draw the cross-hatching lines (Fig.6.11-4)

The cross-hatching lines in axonometric projections are drawn parallel to one of the diagonals of the squares lying in the corresponding co-ordinate planes, the sides of which are parallel to the axonometric axes (Fig.6.12-for isometry, Fig.6.13 - for dimetry).

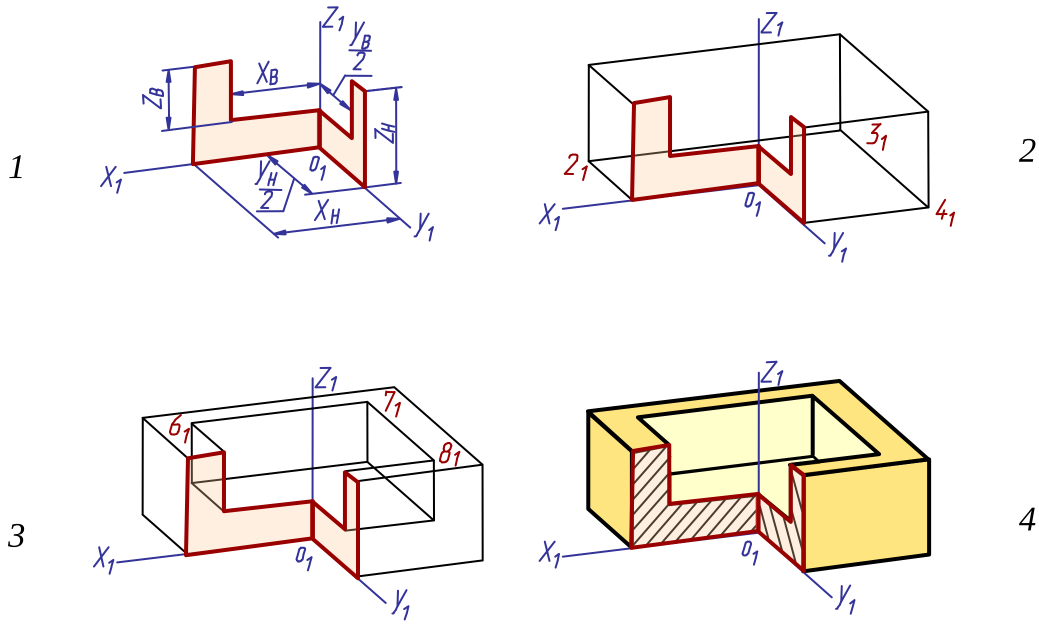


Fig.6.11

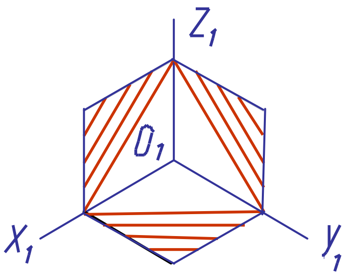


Fig.6.12

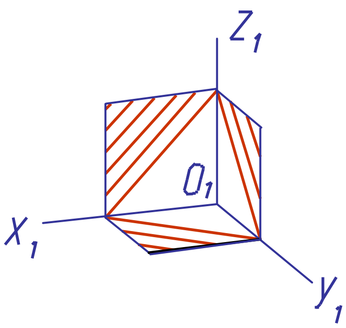


Fig.6.13

6.4 Representation of a Circle and a Sphere in Axonometry

A circle in axonometry is generally projected in an ellipse. When constructing an ellipse, it is necessary to know the direction of its axes and their dimensions. Note: the minor axis of an ellipse is always perpendicular to the major one.

When a circle projection is constructed (the circle lies in one of the coordinate planes), the minor axis of the ellipse is directed parallel to the axonometric axis which does not participate in the formation of the plane the drawing is in.

Isometric Projection of a Circle

When an accurate axonometry of a circle is constructed, the length of the ellipse major axis is equal to the diameter of the above circle. When a reduced axonometry is drawn, the dimensions are enlarged by *1.22*, so the ellipse major axis' length makes *1.22D*, the minor one's - *0.71D*. Fig.6.14 presents a graphical method of determination of the ellipse axes' dimensions. Draw a circle of the diameter *D*, the chord *AB=0.71D* (the length of the ellipse minor axis). Assuming the points *A* and *B* as the centre, with the radius equal to *AB* draw the arcs to meet each other in *E* and *F*. Join the obtained points with a straight line. *EF=1.22D* (the length of the ellipse major axis).

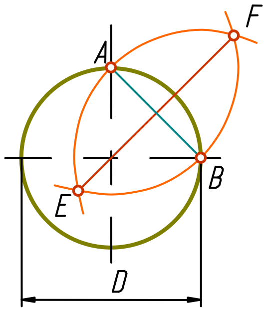


Fig.6.14

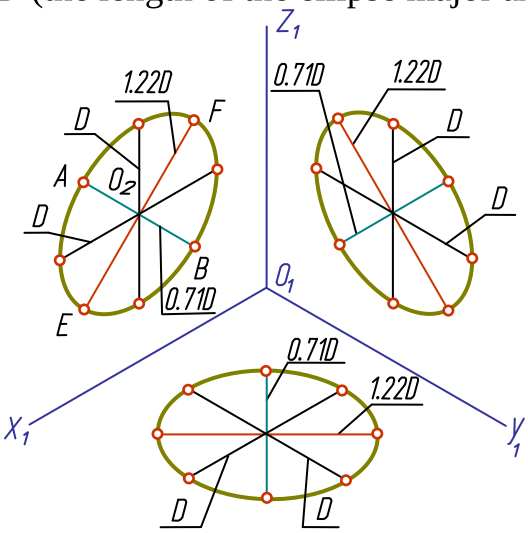


Fig.6.15

Construct the axonometric axes z_1, y_1, z_1 and specify in the plane x_1, o_1, z_1 the directions of the major and minor axes (Fig.6.15). Lay off the segments equal in length to the major *EF* and the minor *AB* axes, to meet in the centre of the ellipse - the point O_2. Through this point pass the lines parallel to the axes x_1 and z_1 generating the given plane. On the lines, lay off the values equal to the diameter *D* of the circle. Join the obtained *8* points to get an ellipse. Another method may also be used for ellipse construction.

Construction of an ellipse in the other planes is similar, only the directions of the axes change.

Dimetric Projection of a Circle

In contradiction to the isometry where the sizes of the ellipse major and minor axes remain equal whatever the plane of the circle is, in dimetry only the length of the major axis is always constant (*1.06D*). The size of the minor axis in the horizontal (*H*) and profile (*W*) planes makes *0.35D*, in the frontal (*V*) plane it makes *0.94D*.

To determine the size of an ellipse axes by means of the graphical method let us construct a right triangle (Fig.6.16) given the legs (100 mm and 35 mm) and the hypotenuse (106 mm). If we lay off the segment *AB* equal to the circle diameter *D* on the longer leg, the segment *DC* will make *0.35D*, i.e. will be equal to the length of the minor ellipse axis on the planes *H* and *W*.

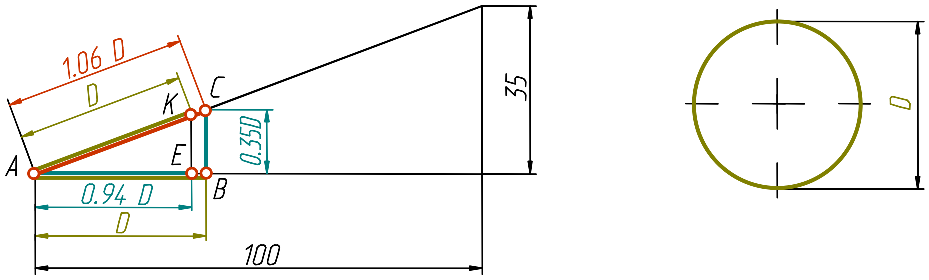


Fig.6.16

The line-segment *AC* is equal to *1.06D*, that is to the length of the major ellipse axis. If we lay off the length of the diameter *D* (the segment *AK*) on the hypotenuse and then drop a perpendicular from the point *K* to the longer leg of the triangle, the segment *AE* will be equal to *0.94D*, i.e. to the length of the ellipse minor axis on the plane *V*.

The representation of a circle in the rectangular dimetric projection is shown in Fig.6.17.

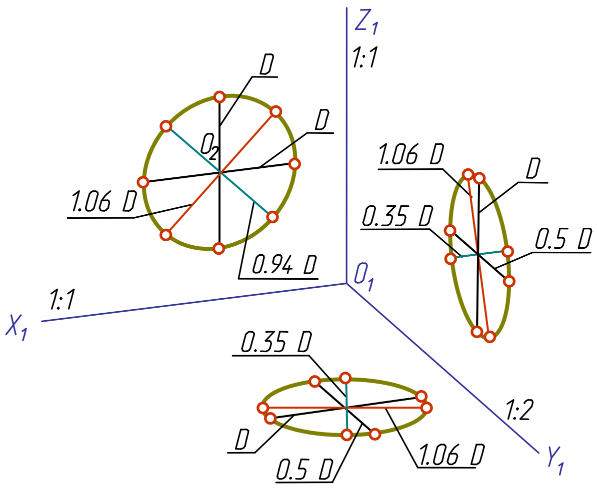


Fig. 6.17

Draw the lines parallel to the axes x_1 and z_1 and lay off on them the segments equal to the circle diameter; then draw a line parallel to the axis y_1 and lay off on it a segment of $0.5D$. Construct the major and minor axes of the ellipse. Join the points thus obtained with a smooth line.

In rectangular parallel axonometry, a sphere is represented as a circle. When a sphere is constructed by the true values of distortion, its axonometric projection is a circle of the diameter equal to the diameter of the sphere. When a sphere is constructed by reduction, the diameter of the circle enlarges in conformity with the reduction coefficient: in isometry it is 1.22; in dimetry - 1.06.

Fig.6.18 shows an isometric projection of a torus produced by means of the auxiliary spheres inscribed in it.

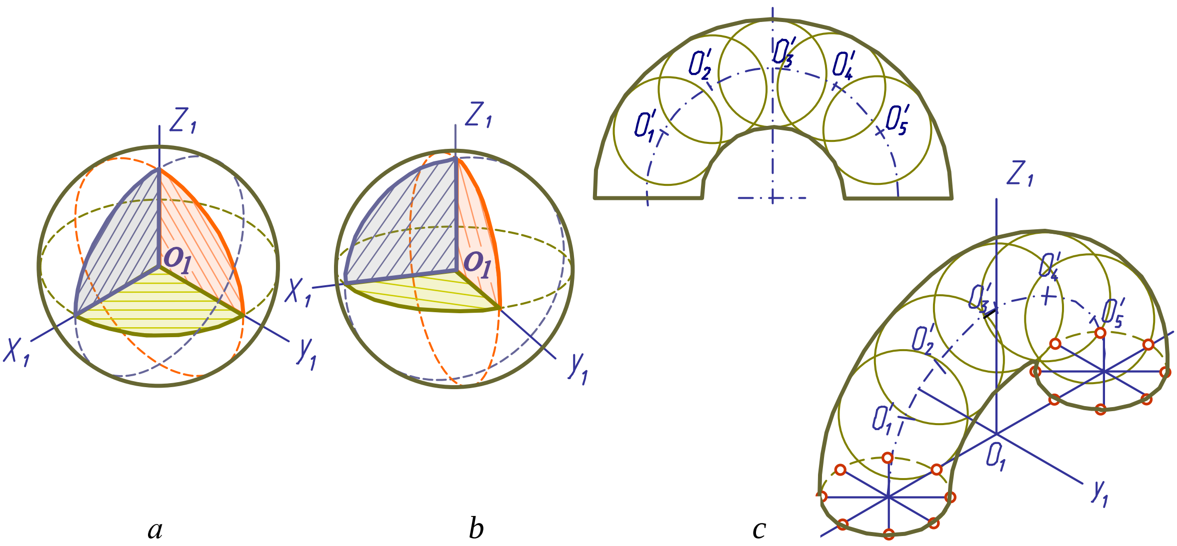


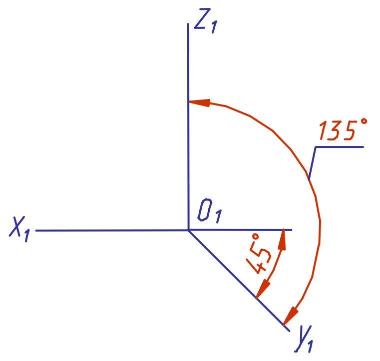
Fig.6.18

6.5 Oblique Axonometry

The Frontal Isometric Projection

The position of the axonometric axes is shown in Fig.6.19.

It is admissible to apply the frontal isometric projections with the angle of inclination of the axis y_1 of 30° and 60°. The frontal isometric projection is completed without distortion on the axes x_1, y_1, z_1.



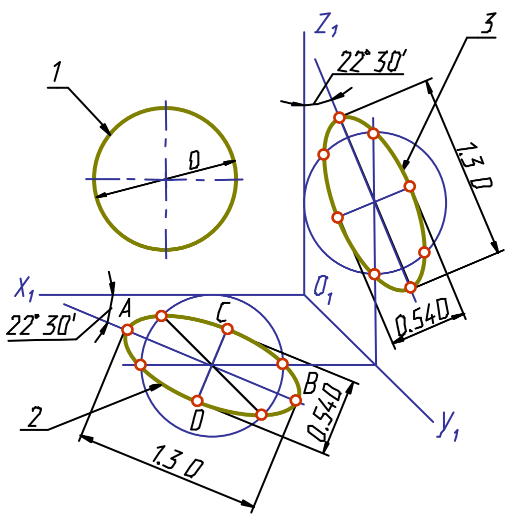


Fig.6.19 Fig.6.20

The circles lying in the planes parallel to the frontal projection plane *V*, are projected on the axonometric plane as circles. The circles lying in the planes parallel to the planes *H* and *W*, are projected as ellipses (Fig.6.20).

The major axis of ellipses *2* and *3* makes *1.3D*, the minor one - *0.54D* (*D* is the diameter of the circle). The major axis of ellipses *2* and *3* is directed along the acute bisectrix between the lines parallel to the axonometric axes and passing through the ellipse centres.

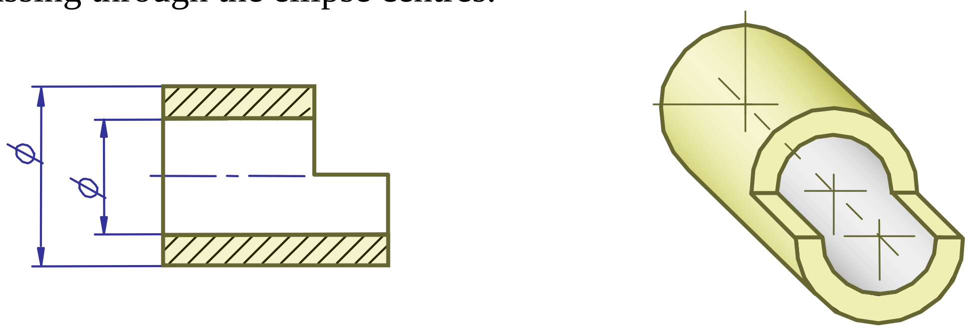


Fig.6.21

A detail in the frontal isometry should be positioned relative to the axes so that the complex plane figures, circles and arcs of the plane curves are located in the planes parallel to the frontal projection plane (Fig.6.21). In this case their representations are distortionless and the drawing work is simpler to do.

The Frontal Dimetric Projection

The location of the axonometric axes is similar to that of the frontal isometric projection (Fig. 6.22).

It is admissible to apply the frontal dimetric projections with the angle of inclination of the axis y_1 of 30° and 60°.

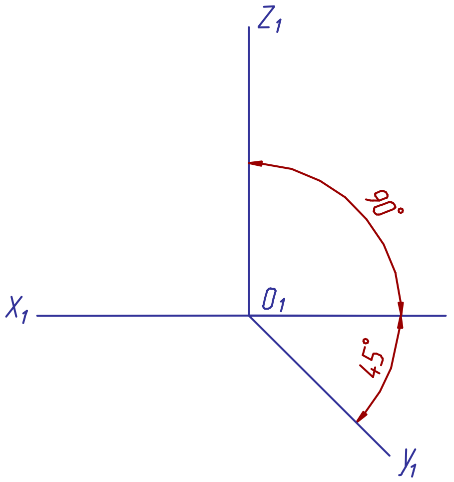


Fig.6.22

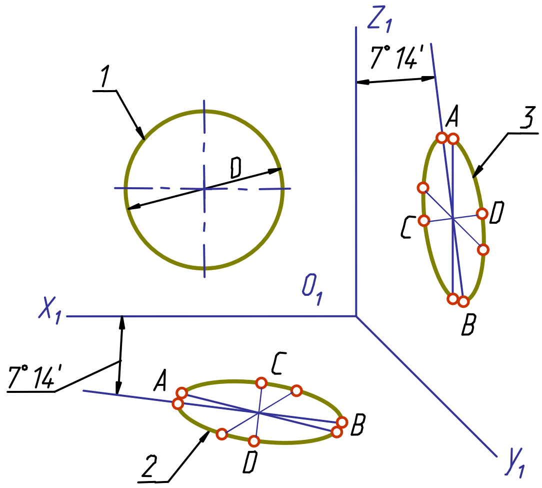


Fig.6.23

The distortion coefficient on the axis y_1 is *0.5*, on the axes x_1 and z_1 it is *1*.

The circles lying in the planes parallel to the frontal projection plane *V* are projected on the axonometric plane as circles, those lying in the planes parallel to *H* and *W* planes - as ellipses (Fig.6.23-6.24).

The major axis of ellipses *2* and *3* is *1.07*, the minor one - *0.33* of the circle diameter. The major axis A_1B_1 of ellipse *2* is inclined to the horizontal axis x_1 at the angle of *7°14′*, the major axis of ellipse *3* - at the same angle to the vertical axis z_1.

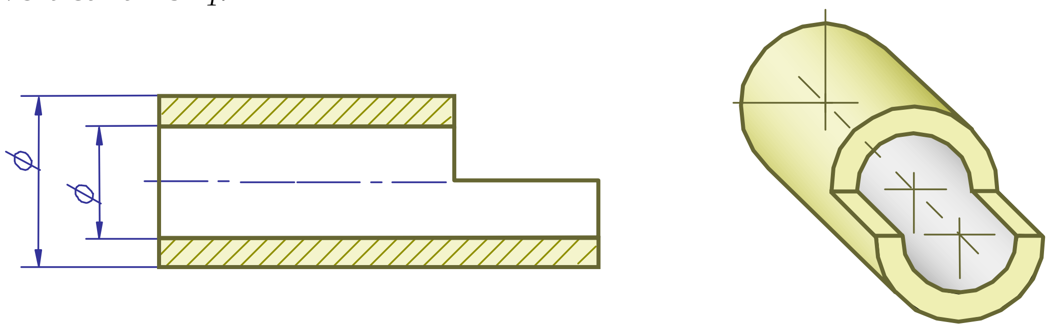


Fig.6.24

Questions to Chapter 6

1. What is the essence of the method of axonometric projection?
2. Formulate the principal theorem of axonometry.
3. What is the coefficient of distortion?
4. How are the coefficients of distortion related to each other?
5. How are the axonometric projections classified according to the direction of projecting and the comparable value of the coefficients of distortion?
6. What is the way of determining the direction of the major and minor axes of an ellipse, if ellipse is the isometric and dimetric projection of a circle?