8. The solution of ordinary differential equations

A Differential equation is

A Differential equation of n-th order is a relation of the form

$$H\left(t, x, x', x'', \dots, x^{(n)}\right) = 0 \qquad (8.1)$$

Solution of differential equations is the function x (t), which turns the equation into an identity.

The system of differential equations of n-th order is a system of the form:

$$\begin{cases} x_1' = f_1(t, x_1, x_2, \dots, x_n) \\ x_2' = f_2(t, x_1, x_2, \dots, x_n) \\ \dots \\ x_n' = f_n(t, x_1, x_2, \dots, x_n) \end{cases}. \qquad (8.2)$$

Solution of the system is a vector that turns the equations of system into identities:

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{pmatrix}. \qquad (8.3)$$

Differential equations and systems have an infinite number of solutions that differ by constants. For an unambiguous definition of the solution need to set additional initial or boundary conditions. The number of such conditions should coincide with the order of the differential equation or system.

Depending on the type of additional conditions in differential equations are distinguished: the Cauchy problem -all additional conditions specified in a (usually starting) point of the interval; boundary problem -additional conditions are indicated on the boundaries of the interval.

A large number of equations can be solved exactly. However, there are equations, especially the system of equations for which an exact solution can not be recorded. Such equations and solved with the help of numerical methods.

Numerical methods are also used in the case if equations with known analytical solution it is required to find a numerical value for certain initial data.

To solve differential equations and systems provided in Scilab command:

[y,w,iw]=**ode**([type],y0,t0,t [,rtol [,atol]],f [, jac] [,w,iw])

> where mandatory input parameters are y0 – a real vector or matrix, the initial conditions; t0 – a real scalar, the initial time; t – a real vector, the times at which the solution is computed; f – a function, external, string or list, the right hand side of the differential equation (8.2); y – a real vector or matrix. (OUTPUT) (8.3).

Thus, to solve the ordinary differential equation of the form

$$\frac{dy}{dt} = f(t, y), y(t_0) = y_0$$

Solve nonstiff differential equations:

[T,Y] = solver(odefun,tspan,y0)

[T,Y] = solver(odefun,tspan,y0,options)

[T,Y,TE,YE,IE] = solver(odefun,tspan,y0,options) sol = solver(odefun,[t0 tf],y0...)

This page contains an overview of the solver functions: **ode23, ode45, ode113, ode15s, ode23s, ode23t, and ode23tb**. You can call any of these solvers by substituting the placeholder, solver, with any of the function names.

The following table describes the input arguments to the solvers.

odefun	A function handle that evaluates the right side of the differential equations. All solvers solve systems of equations in the form $y' = f(t,y)$ or problems that involve a mass matrix, $M(t,y)y' = f(t,y)$. The ode23s solver can solve only equations with constant mass matrices. ode15s and ode23t can solve problems with a mass matrix that is singular, i.e., differential-algebraic equations (DAEs).
tspan	A vector specifying the interval of integration, [t0,tf]. The solver imposes the initial conditions at tspan(1), and integrates from tspan(1) to tspan(end). To obtain solutions at specific times (all increasing or all decreasing), use tspan = [t0,t1,...,tf]. For tspan vectors with two elements [t0 tf], the solver returns the solution evaluated at every integration step. For tspan vectors with more than two elements, the solver returns solutions evaluated at the given time points. The time values must be in order, either increasing or decreasing.
	Specifying tspan with more than two elements does not affect the internal time steps that the solver uses to traverse the interval from tspan(1) to tspan(end). All solvers in the ODE suite obtain output values by means of continuous extensions of the basic formulas. Although a solver does not necessarily step precisely to a time point specified in tspan, the solutions produced at the specified time points are of the same order of accuracy as the solutions computed at the internal time points. Specifying tspan with more than two elements has little effect on the efficiency of computation, but for large systems, affects memory management.
y0	A vector of initial conditions.
options	Structure of optional parameters that change the default integration properties. This is the fourth input argument. [t,y] = *solver*(odefun,tspan,y0,options) You can create options using the odeset function. See odeset for details.

The following table lists the output arguments for the solvers.

T	Column vector of time points.
Y	Solution array. Each row in Y corresponds to the solution at a time returned in the corresponding row of T.
TE	The time at which an event occurs.
YE	The solution at the time of the event.
IE	The index i of the event function that vanishes.
sol	Structure to evaluate the solution.

Solver	**Problem Type**	**Order of Accuracy**	**When to Use**
ode45	Nonstiff	Medium	Most of the time. This should be the first solver you try.
ode23	Nonstiff	Low	For problems with crude error tolerances or for solving moderately stiff problems.
ode113	Nonstiff	Low to high	For problems with stringent error tolerances or for solving computationally intensive problems.

ode15s	Stiff	Low to medium	If ode45 is slow because the problem is stiff.
ode23s	Stiff	Low	If using crude error tolerances to solve stiff systems and the mass matrix is constant.
ode23t	Moderately Stiff	Low	For moderately stiff problems if you need a solution without numerical damping.
ode23tb	Stiff	Low	If using crude error tolerances to solve stiff systems.

Problem 8.1.

Solve the Cauchy problem

$$\frac{dx}{dt} + x = \sin(xt), \quad x(0) = 1.5$$

Rewrite the equation

$$\frac{dx}{dt} = -x + \sin(xt), \quad x(0) = 1.5$$

To simulate this equation, create a function rigid containing the equations:

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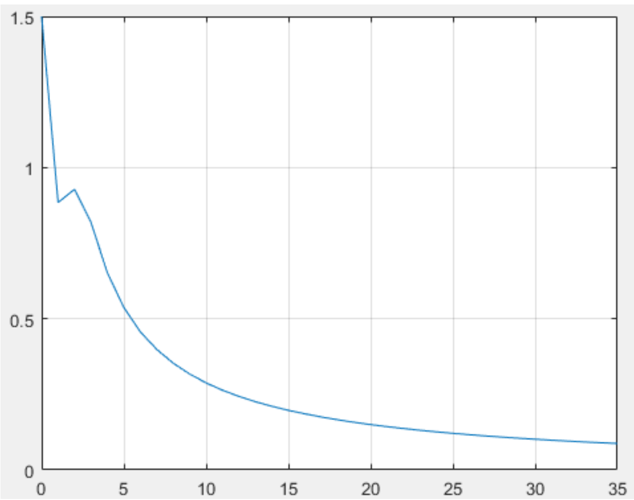
```
function dy = rigid(t,x)
dy=-x+sin(t*x);
```

Command Window

```
>> x0=1.5;t0=0;t=0:1:35;
>> [T,X]=ode45(@rigid,t,x0);
>> plot(T,X);
>> grid on;
```

Result:



Problem 8.2

Solve the Cauchy problem

$$\begin{cases} x' = \cos(xy), \\ y' = \sin(x + ty), \\ \mathrm{x}(0) = 0, \mathrm{y}(0) = 0. \end{cases}$$

in the interval [a,b]

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systdyn.m | syst1.m

```
function dy=syst1(t,y)
dy=zeros(2,1); %a column vector
dy(1)=cos(y(1)*y(2));
dy(2)=sin(y(1)+t*y(2));
```

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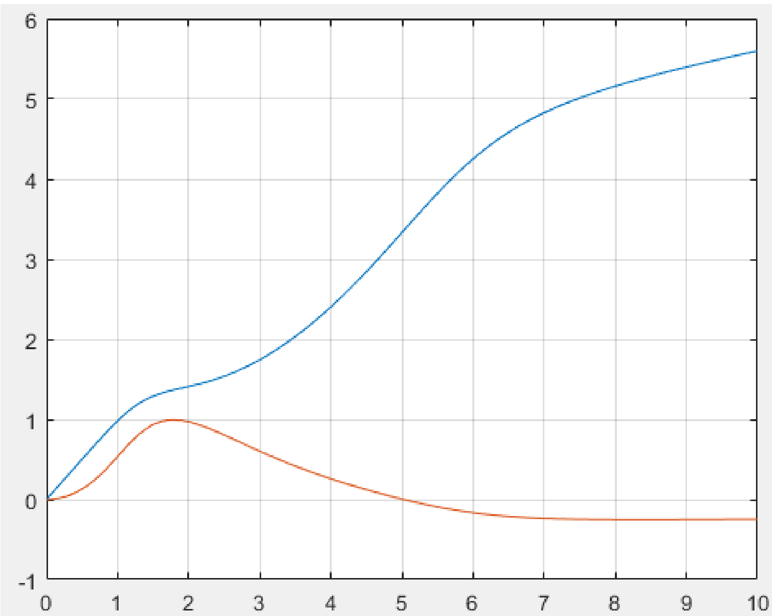
systdyn.m | syst1.m

```
x0=[0,0]; t0=0; t=t0:0.1:10;
[t,y]=ode45(@syst1,t,x0);
plot(t,y);
grid on;
```

Result:



Problem 8.3.

Find the solution of the Cauchy problem for the next rigid system:

$$\frac{dX}{dt} = \begin{pmatrix} 119.46 & 185.38 & 126.88 & 121.03 \\ -10.395 & -10.136 & -3.636 & 8.577 \\ -53.302 & -85.932 & -63.182 & 54.211 \\ -115.58 & -181.75 & -112.8 & -199 \end{pmatrix} X; \qquad X(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

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syst2.m | pr8_syst2.m

```
function dx=syst2(t,x)
B=[119.46 185.38 126.88 121.03;-10.395 -10.136 -3.636 8.577;...
    -53.302 -85.932 -63.182 -54.211;-115.58 -181.75 -112.8 -199];
dx=B*x;
```

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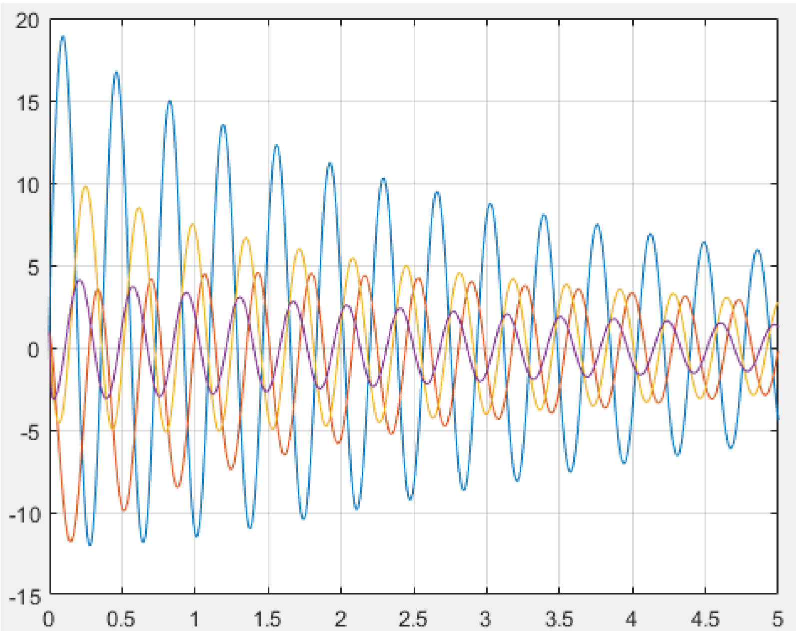
syst2.m | pr8_syst2.m

```
x0=[1;1;1;1]; t0=0; t=t0:0.01:5;
[t,y]=ode23s(@syst2,t,x0);
plot(t,y);
grid on;
```

Result:



Problem 8.4.

Solve the following nonlinear rigid system of differential equations:

$$\left\{ \begin{aligned} \frac{dx_1}{dt} &= -7x_1 + 7x_2 \\ \frac{dx_2}{dt} &= 157x_1 + x_2 - 1.15x_1x_3 \\ \frac{dx_3}{dt} &= 0.96x_1x_2 - 8.36x_3 \end{aligned} \right\}, \qquad X(0) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

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```
function dx=syst3(t,x)
dx=zeros(3,1);
dx(1)=-7*x(1)+7*x(2);
dx(2)=157*x(1)+x(2)-1.15*x(1)*x(3);
dx(3)=0.96*x(1)*x(2)-8.36*x(3);
```

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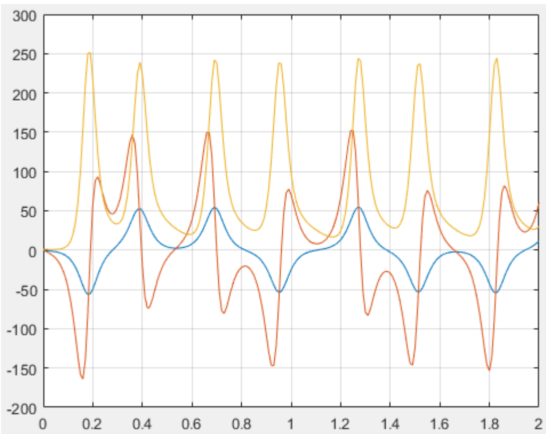
```
x0=[-1;0;1]; t0=0; t=0:0.01:2;
[t,y]=ode45(@syst3,t,x0);
plot(t,y);
grid on;
```

Result: 

Problem 8.5.

Solve the following boundary problem on the interval [0.25; 2]

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13 = e^{\sin(t)}, \quad x(0.25) = -1, \quad x'(0.25) = 1.$$

We transform the equation to the system, making the change

$$y = \frac{dx}{dt}$$

$$\frac{dy}{dt} = -4y - 13x + e^{\sin(t)}, \quad \frac{dx}{dt} = y, \quad y(0.25) = 1, \quad x(0.25) = -1.$$

We form the function calculation system and to solve it:

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FF.m | Pr8_FF.m

```
function F=FF(t,x)
F=[-4*x(1)-13*x(2)+exp(t); x(1)];
```

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FF.m | Pr8_FF.m

```
x0=[1;-1]; t0=0.25; t=0.25:0.05:2;
[t,y]=ode45(@FF, t,x0);
plot(t,y);
grid on;
```

Result: