8.1. The Transportation Problem

Having decided in Lecture 7 how many desks and tables we must make in order to maximize our profit from making furniture, we now turn to selling our desks through a distribution network of furniture outlets. The stores are at varying distances from the furniture maker's two plants—we have been doing well and have expanded our operations!—and each store has

its own demand, based on its own marketing analyses. Thus, we have the logistical problem of deciding how to allocate the desks among the stores This class of OR problems is called the *transportation problem.*

Three furniture stores have ordered desks: Mary's Furniture Emporium wants 30, Lori's Custom Furniture wants 50, and Jenn's Furniture Bazaar wants 45.We have made 70 desks at Plant 1 and another 80 at Plant 2. The distances between the two plants and the three stores are given in Table 8.1 and the shipping cost is $1.50 per mile per desk. We want to minimize the shipping costs of filling the three orders. Since the cost of shipping a desk is easily calculated, we have to calculate how many desks go from a specified plant (of two) to a particular store (of three).

Table 8.1 The distances (in miles) between Plants 1 and 2, where the desks are made, to Mary's Furniture Emporium, Lori's Custom Furniture, and Jenn's Furniture Bazaar, where the desks will be sold.

	Product		
Plant	(1) Mary's	(2) Lori's	(3) Jenn's
1	10	5	30
2	7	20	5

Is an optimal solution available easily? Doesn't it seem reasonable to ship first along the shortest (and cheapest) routes? Here we would send 50 desks from Plant 1 to Lori's, 45 desks from Plant 2 to Jenn's, and so on; and this approach might yield the optimal solution in this case. However, this is a simple problem that has only two plants and three stores, and thus only six plant-store combinations to consider. In addition, the supply of desks is distributed so that the demand for all three stores can be met with only the shortest routes (see Problem 9.14). Thus, this simple problem does not need the kind of trade-off among alternatives that was needed to maximize profit for making furniture. However, it is a useful template for more complex problems, so let us solve it in a formal way.

This transportation problem of shipping desks from two plants to three stores can be represented as an elementary *network problem*, as depicted in Figure 9.8. The circles represent *nodes* at which desks are either supplied (the plants) or consumed (the stores). In more elaborate problems the nodes may be points to which material is supplied and from which material is distributed. The directed line segments (the arrows) are *links*

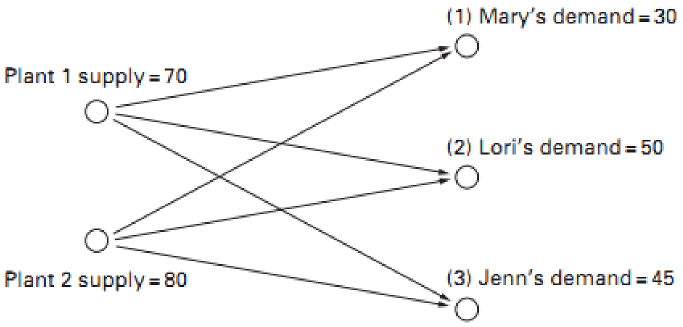

Figure 8.1 A *network representation* of the elementary transportation problem of shipping desks from two plants to three stores. The circles represent *nodes* at which desks are either supplied (the plants) or consumed (the stores). The directed line segments (the arrows) are *links* that represent the routes along which the desks could be shipped.

that represent the routes along which the desks could be shipped, and in more elaborate problems, these directed line segments may thus signify two-way or bi-directional links.

One possible—although *sub-optimal*—solution to this transportation problem is shown in Figure 8.2.We can calculate the shipping cost for this solution as $1387.50, which is substantially higher than the optimal solution.

The transportation problem can be formulated as an LP problem, for which some additional notation will be useful. Thus, we now identify x_{ij} as the number of desks shipped from Plant i = 1,2toStore j = 1, 2, 3. (Note that the two plants were numbered from the beginning, and the stores were assigned numbers in Table 8.1.)

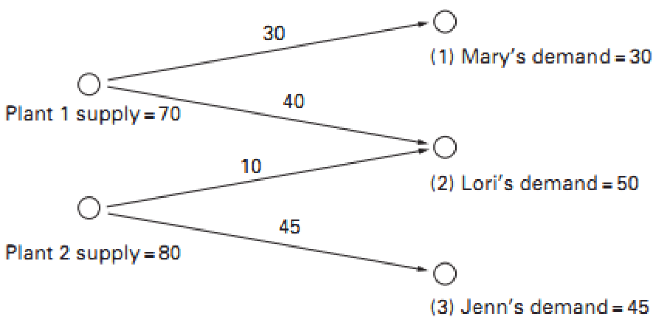

Figure 8.2 One possible solution to the elementary transportation problem of shipping desks from Plants 1 and 2 to Mary's Furniture Emporium, Lori's Custom Furniture, and Jenn's Furniture Bazaar.

Then we have

x_{11} = number of desks from Plant 1 to Store 1 (Mary's),

x_{12} = number of desks from Plant 1 to Store 2 (Lori's),

x_{13} = number of desks from Plant 1 to Store 3 (Jenn's).

x_{21} = number of desks from Plant 2 to Store 1 (Mary's),

x_{22} = number of desks from Plant 2 to Store 2 (Lori's), and

x_{23} = number of desks from Plant 2 to Store 3 (Jenn's).

Since the unit cost of shipping is a constant ($1.50 per desk per mile), we can use the data in Table 1 to establish an objective function that is equal to the shipping cost:

$$\text{shipping cost} = (\$1.50)(10x_{11} + 5x_{12} + 30x_{13} + 7x_{21} + 20x_{22} + 5x_{23}). \quad (8.1)$$

The constraints for this problem arise from the supply of desks produced by the two plants and the demand for the desks by the three stores. The two plants cannot exceed their capacities for producing desks:

$$x_{11} + x_{12} + x_{13} \leq 70, \qquad (8.2)$$

$$x_{21} + x_{22} + x_{23} \leq 80.$$

The stores, in turn, must have enough desks shipped to them to meet their demands:

$$x_{11} + x_{21} \geq 30,$$

$$x_{12} + x_{22} \geq 50, \qquad (8.3)$$

$$x_{13} + x_{23} \geq 45.$$

Finally, the numbers of desks must satisfy a non-negativity constraint because the desks are real, so that:

$$x_{ij} \geq 0. \qquad (8.4)$$

Thus, to sum up the formulation of our shipping problem as an LP problem, we want to find the

minimum of $(\$1.50)(10x_{11} + 5x_{12} + 30x_{13} + 7x_{21} + 20x_{22} + 5x_{23})$,

$$\text{subject to} \quad \begin{cases} x_{11} + x_{12} + x_{13} \leq 70, \\ x_{21} + x_{22} + x_{23} \leq 80, \\ x_{11} + x_{21} \geq 30, \\ x_{12} + x_{22} \geq 50, \\ x_{13} + x_{23} \geq 45, \\ x_{ij} \geq 0. \end{cases} \qquad (8.5)$$

The shipping problem posed thus far has supply exceeding demand. Amore restricted version, the *classical transportation problem*, sets the total supply equal to the total demand. The five inequality constraints (8.2) and (8.3) become simple equality constraints, which reduces the number of independent constraints by one. Suppose that Plant 1 produces only 45 desks (instead of 70). This reduces the (total) supply to a level that equals the demand level of 125 desks. Thus, the supply constraints (8.2) become

$$x_{11} + x_{12} + x_{13} = 45, \qquad (8.6)$$

$$x_{21} + x_{22} + x_{23} = 80.$$

whose sum, a constraint on the total supply, can then be found to be:

$$x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} = 125. \qquad (8.7)$$

Similarly, the demand constraints (8.3) become

$$x_{11} + x_{21} = 30,$$

$$x_{12} + x_{22} = 50, \qquad (8.8)$$

$$x_{13} + x_{23} = 45.$$

and their sum, a constraint on the total demand, adds up to the same result as for the total supply [eq. (8.7)]. Since the total demand and the total supply equations are the same, the set of constraints (8.6) and (8.8) represent only four—not five—independent equations.

This reduction in the number of independent constraints produces some real computational benefits in solving classical transportation problems. One of the benefits is that all of the variables turn out to be integers when the constraints are expressed as integers. Further, the "extra" constraint produced by equating supply to demand results in comparatively straightforward and efficient computations of the optimum.

If the demand exceeds the supply, the LP model cannot even get started because it is impossible to get into the feasible region—from which the solutions derive. This is clear from summing the supply constraint inequalities (8.2),

$$\sum_{i,j} x_{ij} \leq \text{supply}, \qquad (8.9)$$

and comparing it to the sum of the demand in equalities (8.3),

$$\sum_{i,j} x_{ij} \geq \text{demand}. \qquad (8.10)$$

Of course, if supply exceeds demand, so that there is a net, positive surplus, an LP solution can proceed in a straightforward fashion.

8.2. Choosing the Best Alternative

People choose among alternatives all of the time: voters rank candidates; designers rank objectives; and students rank colleges. In each of these circumstances, the *voter* or *decision maker* is charged with choosing among the alternatives.

8.2.1. Rankings and Pairwise Comparisons

In recent years, questions have been raised about *how* voters establish rankings of alternatives. Further, since people seem to compare objects in a list on a pairwise basis before rank ordering the entire list, there is a special focus on how pairwise comparisons are performed as a means of assembling information for doing rank orderings. In *pairwise comparisons*, the elements in a set (i.e., the candidates, design objectives, or colleges) are ranked two at a time, on a pair-by-pair basis, until all of the permutations have been exhausted. Points are awarded to the winner of each comparison. Then the points awarded to each element in the set are summed, and the rankings are obtained by ordering the elements according to points accumulated. However, it is worth noting that as both described here and practiced, the number of points awarded in such pairwise comparisons is often non-uniform and arbitrarily weighted. But, as we will note below, it is quite important that the points awarded be measured in fixed increments.

The pairwise comparison methodology has been criticized particularly because it violates the famous *Arrow impossibility theorem* for which Kenneth J. Arrow was awarded the 1972 Nobel Prize in Economics. In that theorem, Arrow proved that a perfect or fair voting procedure cannot be developed whenever there are more than two candidates or alternatives that are to be chosen. He started by analyzing the properties that would typify a *fair* election system, and stated (mathematically) that a voting procedure can be characterized as fair *only* if four axioms are obeyed:

1. *Unrestricted:* All conceivable rankings registered by individual voters are actually possible.

2. *No Dictator :* The system does not allow one voter to impose his/her ranking as the group's aggregate ranking.

3. *Pareto Condition*: If every individual ranks A over B, the societal ranking has A ranked above B.

4. *Independence of Irrelevant Alternatives (IIA):* If the aggregate ranking would choose A over B when C is not considered, then it will not choose B over A when C is considered.

Arrow proved that at *least one of these properties must be violated* for problems of reasonable size (at least three voters expressing only ordinal preferences among more than two alternatives). It is worth noting that a consistent social choice (voting) procedure can be achieved by violating any one of the four conditions. Further, some *voting procedures* based on pairwise comparisons are faulty in that they can produce ranking results that offend our intuitive sense of a reasonable outcome—and quite often a desired final ranking can be arrived at by specifying a voting procedure.

Among pairwise comparison procedures, the *Borda count* (which we describe below, in Section 9.4.2) most "respects the data" in that it avoids the counter-intuitive results that can arise with other methods. As D. G. Saari notes, the Borda count "*never elects the candidate which loses all pairwise elections ... always ranks a candidate who wins all pairwise comparisons above the candidate who loses all such comparisons*."

The Borda count does violate Arrow's final axiom, the *independence of irrelevant alternatives* (IIA). What does it mean that IIA is violated? And, is that important? The meaning depends to some extent on the domain and whether or not there are meaningful alternatives or options that are being excluded. In an election with a finite number of candidates, the IIA axiom is likely not an issue. In conceptual design, where the possible space of design choice is large or even infinite, IIA could be a problem. However, rational designers must find a way to limit their set of design alternatives to a finite, relatively small set of options. Thus, options that don't meet some criteria or are otherwise seen as poor designs may be eliminated. It is unlikely that IIA matters much if it is violated for one of these two reasons—and there is some evidence to support this—unless it is shown that promising designs were wrongly removed early in the process.

The violation of IIA leads to the possibility of *rank reversals*, that is, changes in order among n alternatives that may occur when one alternative is dropped from a once-ranked set before a second ranking of the remaining n–1 alternatives (as we will soon see below). The elimination of designs or candidates can change the tabulated rankings of those designs or candidates that remain under consideration. The determination of which design is "best" or which candidate is "preferred most" may well be sensitive to the set of designs considered.

Rank reversals occur when there are *Condorcet cycles* in the voting patterns: [A $\succ$ B $\succ$ C , B $\succ$ C $\succ$ A, C $\succ$ A $\succ$ B]. When aggregated over all voters and alternatives, these cycles cancel each other out because each option has the same Borda count. When one of the alternatives is removed, this cycle no longer cancels. Thus, removing C from the above cycle unbalances the Borda count between A and B, resulting in a unit gain for A that is propagated to the final ranking results. Thus, the rank reversals symbolize a loss of information that occurs when an alternative is dropped or removed from the once-ranked set.

We now describe a way to use pairwise comparisons in a structured approach that parallels the role of the Borda count in voting procedures and, in fact, produces results that are identical to the accepted vote-counting standard, the Borda count. The method is a structured extension of pairwise comparisons to a *pairwise comparison chart* (PCC) or matrix. The PCC produces consistent results quickly and efficiently, and these results are identical with results produced by a Borda count.

8.2.2. Borda Counts and Pairwise Comparisons

We begin with an example that highlights some of the problems of (non-Borda count) pairwise comparison procedures. It also suggests the equivalence of the Borda count with a structured pairwise comparison chart (PCC).

Twelve (12) voters are asked to rank order three candidates: A, B, and C. In doing so, the twelve voters have, collectively, produced the following sets of orderings:

$$\begin{array}{ll} 1 \text{ preferred } A \succ B \succ C, & 4 \text{ preferred } B \succ C \succ A, \\ 4 \text{ preferred } A \succ C \succ B, & 3 \text{ preferred } C \succ B \succ A. \end{array} \quad (8.11)$$

Pairwise comparisons other than the Borda count can lead to inconsistent results for this case. For example, in a widely used plurality voting process called *the best of the bes*t, A gets 5 first-place votes, while B and C each get 4 and 3, respectively. Thus, A is a clear winner. On the other hand, in an "antiplurality" procedure characterized as *avoid the worst of the* worst, C gets only 1 last-place vote, while A and B get 7 and 4, respectively. Thus, under these rules, C could be regarded as the winner. In an iterative process based on *the best of the best*, if C were eliminated for coming in last, then a comparison of the remaining pair A and B quickly shows that B is the winner:

$$\begin{array}{ll} 1 \text{ preferred } A \succ B, & 4 \text{ preferred } B \succ A, \\ 4 \text{ preferred } A \succ B, & 3 \text{ preferred} B \succ A. \end{array} \quad (8.12)$$

On the other hand, a Borda count produces a clear result. The Borda count procedure assigns numerical ratings separated by a common constant to each element in the list. Thus, sets such as (3, 2, 1), (2, 1, 0) and (10, 5, 0) could be used to rank a three-element list. If we use (2, 1, 0) for the rankings presented in eq. (9.30), we find total vote counts of (A:2+8+0+0 = 10), (B:1+0+8+3

= 12) and (C:0+4+4+6 = 14), which clearly shows that C is the winner. Furthermore, if A is eliminated and C is compared only to B in a second Borda count,

$$\begin{array}{ll} 1 \text{ preferred } B \succ C, & 4 \text{ preferred } B \succ C, \\ 4 \text{ preferred } C \succ B, & 3 \text{ preferred } C \succ B. \end{array} \quad (8.13)$$

C remains the winner, as it also would here by a simple vote count. It must be remarked that this consistency cannot be guaranteed, as the Borda count violates the IIA axiom.

We now make the same comparisons in a PCC matrix, as illustrated in Table 8.2. As noted above, a point is awarded to the winner in each pairwise comparison, and then the points earned by each alternative are summed. In the PCC of Table 8.2, points are awarded row-by-row, proceeding along each row while comparing the row element to each column alternative in an individual pairwise comparison.

Table 8.2 A pairwise comparison chart (PCC) for the ballots cast by twelve (12) voters choosing among the candidates A, B and C (see eq. (8.11)).

Win/Lose	*A*	*B*	*C*	Sum/Win
A	– –	1+4+0+0	1+4+0+0	10
B	0+0+4+3	– –	1+0+4+0	12
C	0+0+4+3	0+4+0+3	– –	14
Sum/Lose	14	12	10	– –

This PCC result shows that the rank ordering of preferred candidates is entirely consistent with the Borda results just obtained:

$$C \succ B \succ A. \quad (8.14)$$

Note that the PCC matrix exhibits a special kind of symmetry, as does the ordering in the “Win” column (largest number of points) and the “Lose” row (smallest number of points): the sum of corresponding off-diagonal elements, $X_{ij} + X_{ji}$, is a constant equal to the number of comparison sets.

We have noted that a principal complaint about some pairwise comparisons is that they lead to rank reversals when the field of candidate elements is reduced by removing the lowest-ranked element between orderings. (Strictly speaking, rank reversal can occur when any alternative is removed. In fact, and as we note further in Section 8.2.3, examples can be constructed to achieve a specific rank reversal outcome. Such examples usually include a dominated option that is not the worst. Also, rank reversals are possible if new alternatives are *added*.) Practical experience suggests that the PCC generally preserves the original rankings if one alternative is dropped. If element A is removed above and a two-element runoff is conducted for B and C, we find the results given in Table 9.3.Hence, once again we find

$$C \succ B. \quad (8.15)$$

The results in inequality (8.15) clearly preserve the ordering of inequality (8.14), that is, no rank reversal is obtained as a result of applying the PCC approach. In those instances where some rank reversal does occur, it is often among lower-ranked elements where the information is strongly influenced by the removed element (see Section 8.2.3).

Table 8.3 A reduced pairwise comparison chart (PCC) for the problem in Table 8.2 wherein the “loser” A in the first ranking is removed from consideration.

Win/Lose	*B*	*C*	Sum/Win
B	− −	1+0+4+0	5
C	0+4+0+3	− −	7
Sum/Lose	7	5	− −

8.2.3. Pairwise Comparisons and Rank Reversals

8.2.4. Pay Attention to All of the Data

8.2.5. On Pairwise Comparisons and Making Decisions