7. Optimization: What Is the Best…?

Why? How?

This final chapter is about achieving the *best result*, obtaining the *maximum gain*, finding the *optimal outcome*. Thus, this chapter is about *optimization*—an especially interesting subject because finding an optimum result may be difficult, and at times even impossible. Our experience with finding maxima and minima in calculus suggests that we can often find a point where the derivative of a function vanishes and an extreme value exists. But in engineering design and in life generally, we often have to "satisfice," that is, in the word of Herbert A. Simon, be satisfied with an acceptable outcome, rather than an optimal one. Here, however, we will focus on modeling the ways we seek optimal solutions. In so doing, we will see that the formulation of an optimization problem depends strongly on how we express the *objective function* whose extreme values we want and the *constraints* that limit the values that our variables may assume.

Much of the work on finding optimal results derives from an interest in making good decisions. Many of the ideas about formulating optimization problems emerged during and after World War II, when a compelling need to make the very best use of scarce military and economic resources translated in turn into a need to be able to formulate andmake the *best decisions* about using those resources. Thus, with improved decision making as the theme, we will also present (in Section 9.4) a method of choosing the best of an available set of alternatives that can be used in a variety of settings.

We will close with a miscellany of interesting, "practical" optimization problems.

7.1. Continuous Optimization Modeling

Find? How?

We start with a basic minimization problem whose solution is found using elementary calculus. Suppose that we want to find the minimum values of the *objective function*

$$U(x) = \frac{x^2}{2} - x, \qquad (7.1)$$

which we have drawn in Figure 7.1. That picture of the objective function *U(x)*—so called because we set our objective as finding its extreme value— is a parabolic function of x, as the algebraic form of eq. (7.1) confirms.

Thus, it has only a single minimum value, called the *global* minimum. The value of x at which this global minimum is found is determined by setting the first derivative of U*(x)* to zero:

$$\frac{dU(x)}{dx} = x - 1 = 0, \qquad (7.2)$$

from which it follows that the minimum value of *U(x)* occurs when $x_{\min} = 1$ and is

$$U_{min} = U(x_{min}) = -\frac{1}{2}. \qquad (7.3)$$

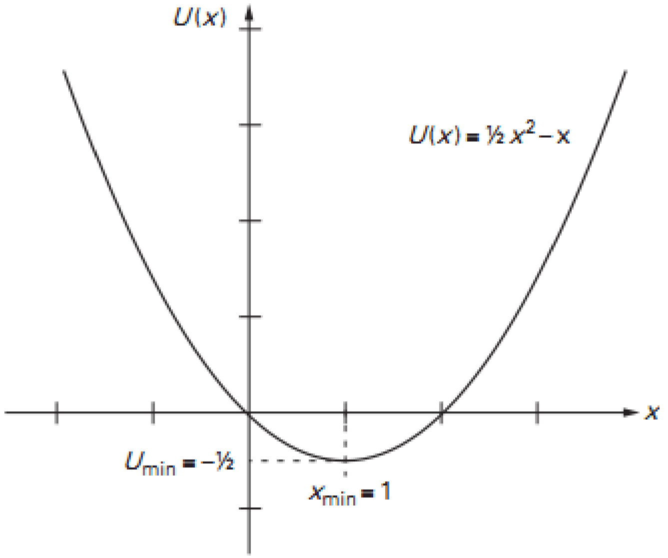

Figure 7.1 The objective function *U(x)* = $x^2/2 - x$ plotted over the unrestricted range of $-\infty \leq x \leq +\infty$. The minimum value of the objective function, $U_{min} = -1/2$, occurs at $x_{min} = 1$.

We also note from eq. (7.2) that the slope of *U(x)* increases monotonically as x goes from$-\infty$to$+\infty$, which means that *U(x)* itself can have only one flat spot.We can confirmthis by calculating the rate of change or derivative of the slope,

$$\frac{d^2U(x)}{dx^2} = 1, \qquad (7.4)$$

which is always positive. Thus, there is only one minimum, and it is a global minimum. In fact, we can go a step further and identify the minimum value of eq. (7.3) as an *unconstrained minimum* because we did not constrain or limit the values that the variable x could assume.

Assume? How?

Suppose we did impose a constraint, say of the form $x \leq x_0$, which requires the independent variable, x, to always be less than or equal to a given constant, x0. This means that search for the minimum of *U(x)* is limited to the *admissible values* of *x:* $x \leq x_0$.We can visualize a procedure for implementing this constraint as putting a line on the same graph as the curve, *U(x)*, and then "moving" this line to different values of x_0, as shown in Figure 7.2. The constraint then shows as the set of lines, $x_{01} < x_{02} < x_{03}$ so we can now briefly consider the three problems of determining the minimum values of *U(x)* with $x \leq x_{0i}$, *i* = 1, 2, 3. In the first case, *i* = 1, the admissible range of *x* is so restricted that the constrained minimum value

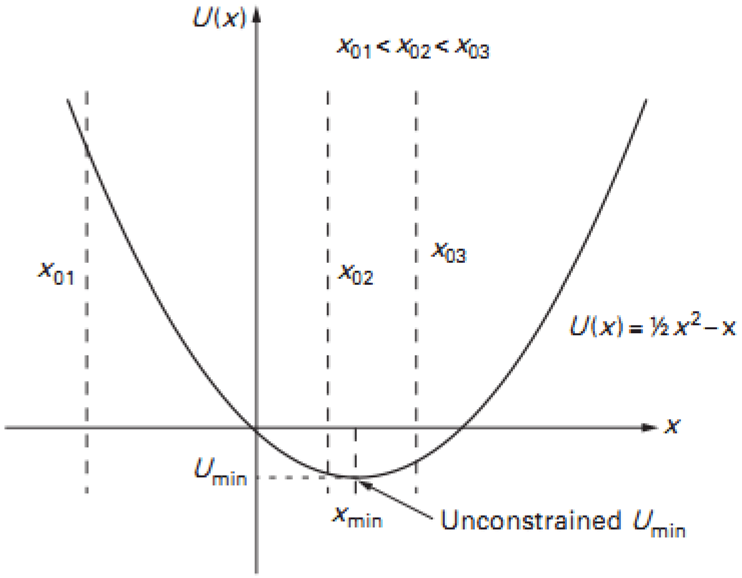

Figure 7.2. The objective function $U(x) = x^2/2 - x$ plotted together with three constraints that restrict the range of admissible values: the set of lines, $x_{01} < x_{02} < x_{03}$. These lines allow us to consider the three problems of determining the minimum values of $U(x)$ with $x \le x_{0i}$, $i = 1, 2, 3$.

of $U(x)$ is apparently significantly greater than the unconstrained minimum of eq. (7.3). For example, if $x_{01} = -3$, the corresponding constrained minimum is $U(-3) = 7.5$. As the constraint "moves" further to the right (i = 2, 3), we approach and then go through the unconstrained minimum. Thus, the range of *feasible solutions* for the minimum of $U(x)$ may include the unconstrained minimum, U_{min}—or it may not—depending on just where the constraint boundaries happen to be.

The constraints so far imposed are *inequality constraints*, $x \le x_0$, that bound the range of feasible values at the upper end by the equality, $x = x_0$, and include the interior region, $x < x_0$.Wemight have posed only a simple equality constraint, $x = x_0$, in which case we would have found a (highly) constrained minimum $U(x_0)$.

If our objective function were only slightly more complicated, the search for extreme points would become significantly more complicated. Consider the objective function

$$U(x) = \sin x, \qquad (7.5)$$

This elementary function could have, depending on the limits placed on the range of admissible values of x, an infinite number of maxima and of minima, or a constrained extremum somewhere between the two. The point of this seemingly trivial example is simple. Characterizing and finding the extrema can be complicated even when the objective function is well known and its properties well understood.

Why?

The objective functions (7.1) and (7.5) have only a single variable. However, *multi-dimensional optimization problems* are almost always the norm in engineering practice because engineered devices and processes rarely, if ever, depend only on a single variable. One simple example can be found at the local post office, where postal regulations typically stipulate that the rectangular package shown in Figure 7.3 can be mailed only if the sum of its girth ($2x$ +$2y$) and length (z) do not exceed 84 in (2.14 m). What is the largest volume that such a rectangular package can enclose?

The objective function is the package's volume,

$$V(x, y, z) = xyz, \qquad (7.6)$$

Assume?

where *x* and y are the two smaller dimensions whose sum comprises the package's girth, and the length, *z*, is its longest dimension. We assume that these three dimensions are positive real numbers (i.e., $x > 0$, $y > 0$, $z > 0$).

The constraint on the package dimensions stemming from the postal regulations can be written as:

$$\underbrace{2x + 2y}_{\text{girth}} + \underbrace{z}_{\text{length}} \leq 84, \qquad (7.7)$$

Since we seek the largest possible volume, this inequality constraint on the package dimensions can be expressed as an equality constraint:

$$2x + 2y + z = 84. \qquad (7.8)$$

How?

Thus, the volume maximization problem is expressed as the objective function (7.6) to be maximized, subject to the equality constraint (7.8). Although the problem is formulated in three dimensions, we can use the equality constraint to eliminate one variable, say the length, z , so that the objective function becomes:

$$V(x, y) = xy(84 - 2x - 2y) = 84xy - 2x^2y - 2xy^2. \qquad (7.9)$$

Now we want to find the maximum value of *V(x, y)* as a function of x and y. As we recall from calculus, the necessary condition that *V(x, y)* takes on an extreme value is:

$$\frac{\partial V(x, y)}{\partial x} = 84y - 4xy - 2y^2 = 2y(42 - 2x - y) = 0, \qquad (7.10a)$$

$$\frac{\partial V(x, y)}{\partial y} = 84x - 2x^2 - 4xy = 2x(42 - x - 2y) = 0. \qquad (7.10b)$$

Equations (7.10a–b) can be reduced to a pair of linear algebraic equations whose non-trivial solution can be found ($x = y = 14$ in) to determine the corresponding package volume, $V = 5488$ in3. This volume can be confirmed to be a maximum.

The package problem, albeit multi-dimensional, was still relatively simple because its inequality constraint could logically and appropriately be reduced to an equality constraint that could, in turn, be used to reduce the dimensionality of the problem. Then we found the maximum volume of the package by applying standard calculus tools and seemingly without any further reference to constraints. Consider for a moment the problem of finding the minimum of the following objective function:

$$U(x,y) = x^2 + 2(x-y)^2 + 3y^2 - 11y. \qquad (7.11)$$

Find?

We show a three-dimensional rendering of this parabolic surface in Figure 7.3. It has an unconstrained minimum at the point (x =1, y =1.5), where U_{min} =−8.25. What happens if an equality constraint is imposed? That is, in the style and terminology of the field of operations research, suppose that we want to find the

$$\text{minimum of} \quad U(x,y) = x^2 + 2(x-y)^2 + 3y^2 - 11y, \qquad (7.12)$$
$$\text{subject to} \quad x + y = 3.$$

We could again use standard calculus techniques to show that the constrained minimum occurs at the point (x = 31/24, y = 41/24), where U_{min} =−385/48 . Note that the minimum is located on the boundary plane where the constraint intersects $U(x, y)$, that is, at a point such that $x + y$ = 31/24 + 41/24 = 72/24 = 3. If the equality constraint of eq. (7.12) was replaced with the (strict) inequality constraint $x + y < 3$, we would find that the minimum sought lies inside the intersecting boundary plane.

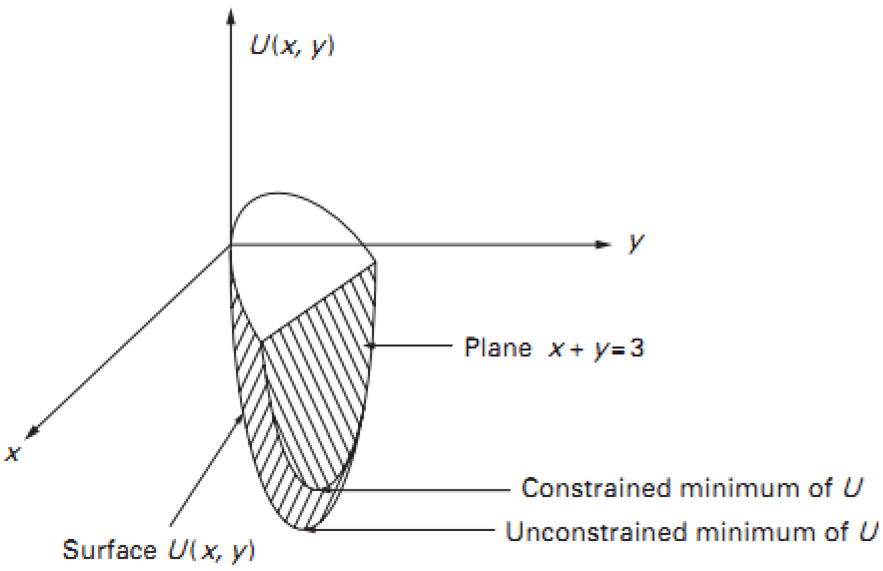

Figure 7.3 The objective function $U(x,y) = x^2 + 2(x - y)^2 + 3y^2 - 11y$ “plotted” in three dimensions, along with the plane $x + y = 3$ that could form the boundary of an equality constraint or of a corresponding inequality constraint.

7.2. Optimization with Linear Programming

The section just completed showed that the search for an optimum or extreme value of a function subject to an inequality constraint requires a search over the interior of the region defined by the constraint boundary. Thus, as shown in Figure 7.2, we must search for all values of $x \leq x_{0i}$. This is true more generally because an objective function may fluctuate in value, perhaps like the sinusoid of eq. (7.5). Consider, for example, the sketch of a generic objective function in Figure 7.4. The good news is that the standard methods of calculus are usually adequate for searches where the objective functions are relatively tractable. The bad news is that, in such cases, we generally need to search the entire domain, $x_{04} \leq x \leq x_{05}$ to find global optima. However, there is a very important class of problems where a search of the interior region is not required because the optimum point must occur on one of the constraint boundaries. This class of problems is made up of objective functions that are linear functions of the independent variables, and their optimization searches are known as *linear programming* (LP).

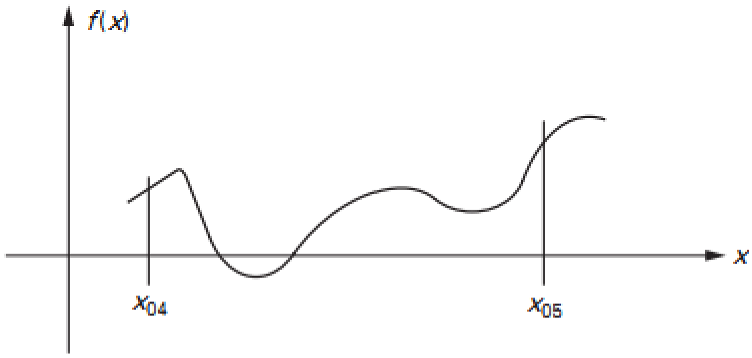

Figure 7.4 A generic sketch of an objective function that shows some variation or fluctuation, with peaks and valleys in the domain of interest. The bad news is that here we do need to search the entire domain, $x_{04} \leq x \leq x_{05}$, to find a global optimum. The good news is that the standard methods of calculus are usually adequate for searches if the objective functions are relatively straightforward.

Suppose we want to find (see Figure 7.5) the

$$\text{minimum of} \quad U(x,y) = mx + b, \qquad (7.13)$$
$$\text{subject to} \quad x_1 \leq x \leq x_2.$$

Now, the minimum of $U(x)$ must lie within the admissible range of values of x, defined by the two inequality constraints just given. Geometry, however, tells us that the optimal values of the linear objective function,

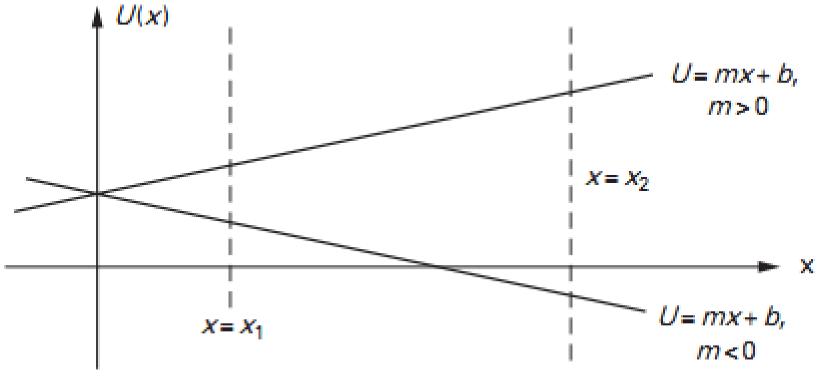

Figure 7.5 A generic linear programming problem which is characterized by an objective function that is a linear function of the variable x. Note that the optimal values, both maxima and minima, for $m > 0$or $m < 0$, occur at points where the objective function intersects the constraint boundaries, that is, on the constraint boundaries themselves.

$U(x) = mx + b$, occur at points where $U(x)$ intersects one of the two constraint boundaries. For $m > 0$, Umin must occur at $x = x_1$ and U_{max} must occur at $x = x_2$. Thus, for this linear programming problem, we can find the optima of $U(x)$ without searching the interior region defined by the constraint boundaries: We know *a priori* that the optima must occur on the constraint boundaries. In fact, it can be shown that the optimum solutions for LP problems are found by searching only at the boundary intersections or *vertices*. The search problem is thus "reduced" to solving for a set of intersection points defined by various linear equations.

Is requiring an objective function to be linear too much of a simplification? Are LP problems useful, or a cute mathematical artifact? In fact, LP is extremely important and useful, and is one of the cornerstones of the field of *operations research*. The field of operations research (OR)—pronounced "oh r"—developed first in Britain and then in the United States during World War II when there was a compelling interest in optimizing scarce military and economic resources. Since that time, OR has been applied to both military and civil problems, including in the latter a wide variety of commercial enterprises, allocating medical resources, managing traffic, and modeling the criminal justice system. The hallmark of LP is the determination of optimal results for *single* objectives: *minimizing* transportation costs, *optimizing* the product mix, *maximizing* hospital bed availability, *minimizing* the number of highway toll attendants when traffic is slack, or *minimizing* drivers' waiting times when traffic is heavy.

7.2.1. Maximizing Profitinthe Furniture Business

Suppose that we are in the furniture business and making desks and tables that are made of oak and maple. Desks and tables consume different amounts of lumber: a desk requires 6 board-feet (bft) each of oak and maple, while a table requires 3 bft of oak and 9 bft of maple. The local lumber mill will supply up to 1200 bft of oak at $6.00/bft and up to 1800 bft of maple at $4.00/bft. The market for desks and tables is such that they can be sold for, respectively, $90.00 and $84.00. How many desks and how many tables should we make to maximize our profit?

We will soon find out (see eq. (7.16)) that under the conditions assigned here, the profits earned by selling a desk are the same as the profits earned by selling a table, namely, $30.00 each.

Suppose that this was not the case and that the profit in selling a table was only $18.00. Then it might seem reasonable to first make only desks to maximize profit—except that we will run out of oak after only 200 desks are made and have an excess, unusable supply of maple left over. It will also turn out that the profit earned in this case, $6000, is not the maximum profit possible. This problem is interesting because the constraints supply *limits* on the available materials, which means in turn that we must make *trade-offs* between desks and tables to maximize our overall profit.

We formulate this profit optimization problem as an LP problem, meaning that we build an objective function—the difference between sales income and cost of manufacture—and the relevant operating constraints. If x_1 is the number of desks made, and x_2 the number of tables, the income derived by selling desks and tables is:

$$\text{income} = (\$/\text{desk})x_1 + (\$/\text{table})x_2 = \$(90x_1 + 84x_2). \qquad (7.14)$$

7.2.2 On Linear Programming and Extensions

7.2.3 On Definingand Assessing Optima