6. Applying Vibration Models

As we noted in Lecture 5, vibration is omnipresent in our lives, both in people-made and living objects and devices. Vibration is also complex. For example, sound is modeled as a sum of harmonics, of vibrations with different periods or natural frequencies. Certainly buildings and cars and airplanes and dentists' drills vibrate in complex, multi-modal ways as well, with a lot of modes having different frequencies and different amplitudes. Given that life seems so complex, is it worth doing elementary vibration modeling? Yes, it is, as so eloquently said by one of the great pioneers of the field of vibration, Sir John William Strutt, third Baron Rayleigh, known quite widely as Lord Rayleigh:

The material systems, with whose vibrations Acoustics is concerned, are usually of considerable complication, and are susceptible of very various modes of vibration, any or all of which may coexist at any particular moment. Indeed in some of the most important musical instruments, as strings and organ-pipes, the number of independent modes is theoretically infinite, and the consideration of several of them is essential to the most practical questions relating to the nature of the consonant chords. Cases, however, often present themselves, in which one mode is of paramount importance; and even if this were not so, it would still be proper to commence the consideration of the general problem with the simplest case—that of one degree of freedom. It need not be supposed that the mode treated of is the only one possible, because so long as vibrations of other modes do not occur their possibility under other circumstances is of no moment.

Why?

Guided by Lord Rayleigh's insight, we will continue to limit our discussion of models of vibratory behavior to those having but a single degree of freedom. We will focus on two important elements. First, we develop the *mechanical-electrical analogy*, wherein we make more explicit the several commonalities of vibration behavior that we had identified in Lecture 5. In our second focus, we note a dividing line that is extraordinarily powerful for modeling vibration: some phenomena seem to go on indefinitely, quite on their own, while others appear as responses to repetitive stimulation. Thus far, our models have been in the first category, called *free* or *unforced* vibration, referring to phenomena that continue after some initial jolt gets themgoing. It includes the vibration of struck piano strings and the tides of the seas. The second category that we take up in this chapter, *forced* vibration, occurs when there is a persistent, repetitive input, such as the kind an air conditioning system imparts to the building it cools or an engine imparts to the vehicle it powers.

6.1. The Spring–Mass Oscillator–II: Extensions and Analogies

How?

In Lecture 5.3 we noted that the pendulum could be modeled as a *spring-mass oscillator*, a model we now develop by applying once again the force balance embodied in Newton's second law. We show such a spring-mass system in Figure 6.1. Newton's law states that (see Lecture 5) the motion of the oscillator's mass, *m*, is governed by

$$\text{net force} = m\frac{d^2x(t)}{dt^2}. \qquad (6.1)$$

Given?

Two forces are shown acting on the mass: a specified *applied force*, *F(t)*, and a force exerted by the spring. The spring is an ideal elastic spring that has no mass and dissipates no energy. Its attachment points at each end are called *nodes.*

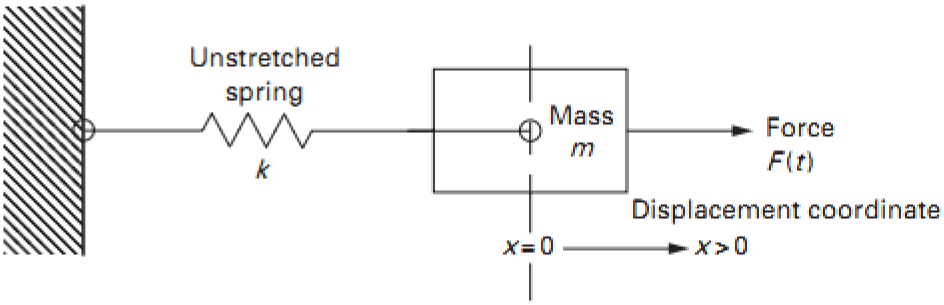

Figure 6.1 An elementary *spring-mass system* the shows an ideal spring exerting a restoring force on a mass, *m*, as does a specified applied force, *F(t).* The spring's stiffness is *k*, and the displacement or movement of the mass to which the spring's right end is attached is *x(t).*

The left node of the spring in Figure 6.1 is attached to a fixed point, say on a wall, while the right node is attached to a mass whose movement, *x(t)*, is the system's single degree of freedom. Moreover, the spring always exerts a *restoring force* on the node or mass that returns the spring to its original, unextended position. Thus, if moved a positive distance to the right, x(t), the spring pulls the node back to the left; if the spring is compressed a distance to the left, −x(t), it pushes the node back to the right. The magnitude of the spring force is given by

$$F_{\text{spring}} = kx(t). \qquad (6.2)$$

The net force on the mass is the difference between the applied and the spring forces,

$$\text{net force} = F(t) - F_{\text{spring}}. \qquad (6.3)$$

so that the equation of motion is found by combining eqs. (6.1), (6.2), and (6.3):

$$m\frac{d^2x(t)}{dt^2} + kx(t) = F(t). \qquad (6.4)$$

Equation (6.4) was already introduced as an analog of the pendulum in Lecture 5, where we made the argument that the gravitational pull on the pendulum mass exerted a spring-like force on the pendulum. For free, unforced vibration, there is no applied force, and the governing equation is

$$m\frac{d^2x(t)}{dt^2} + kx(t) = 0. \qquad (6.5)$$

If we introduce a scaling factor, ω_0, to make the time dimensionless, as we did in eq. (5.10), the oscillator equation (6.5) becomes

$$m\omega_0^2\frac{d^2x(\tau)}{d\tau^2} + kx(\tau) = 0. \qquad (6.6)$$

which suggests that the scaling factor for the spring-mass system is

$$\omega_0 = \sqrt{\frac{k}{m}}. \qquad (6.7)$$

Equation (6.7) can be confirmed to be dimensionally correct and, as for the pendulum, ω_0 can be identified as the *circular frequency* of the spring-mass oscillator. The circular frequency can be related to the frequency and the period:

$$f_0 = \frac{1}{T_0} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \qquad (6.8)$$

Again, both f_0 and ω_0 have the physical dimensions of (time)$^{-1}$, but the units of f_0 are number of cycles per unit time, while those of ω_0 are radians per unit time.

Use? Predict?

Equation (6.7) is actually far more important than its simple appearance suggests. It provides a fundamental paradigm for thinking about the vibration of systems: The natural frequency of the oscillator is proportional to the square root of the *stiffness-to-mass* ratio. Thus, natural frequencies increase (and periods decrease) with increasing stiffness, *k*, while natural frequencies decrease (and periods increase) with increasing mass, *m*. We will refer back to this paradigm often, and we will also see that it captures a very useful design objective.

Why? How?

We now extend the spring-mass model to incorporate non-ideal, dissipative behavior. We do this by attaching to the mass a *damping* or *dissipative element,* sometimes called a *dashpot* or *damper,* which exerts a restoring force proportional to the speed at which the element is extended or compressed:

$$F_{\text{damper}} = c\dot{x}(t). \qquad (6.9)$$

The damper acts *in parallel* with the spring, as shown in Figure 6.2, so that the net force exerted on the mass is

$$\text{net force} = F(t) - F_{\text{spring}} - F_{\text{damper}}, \qquad (6.10)$$

and the corresponding equation of motion for a *spring-mass-damp system* is

$$m\frac{d^2x(t)}{dt^2} + c\dot{x}(t) + kx(t) = F(t). \qquad (6.11)$$

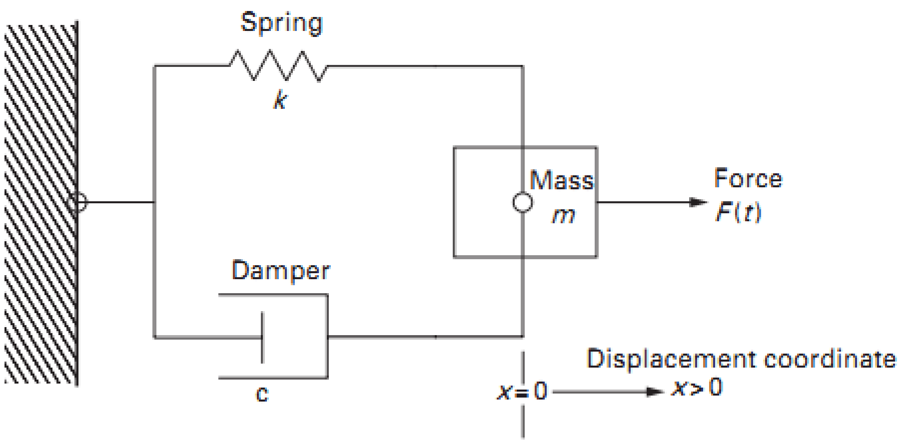

Figure 6.2 An elementary *spring-mass-damper system* that shows the ideal spring (of stiffness, *k*) exerting a force on a mass, *m*, the specified applied force, *F (t)*, and a viscous damping element that exerts a restoring force that is proportional to the speed, $\dot{x}(t)$, at which the mass moves.

This result is very similar to the corresponding result for the damped pendulum, eq. (5.27), save for the facts that the present result includes a forcing function, *F(t)*, and its spring term is (already) linear.

6.1.1. Restoring and Dissipative Forces and Elements

Equation (6.11) offers the prospect of generalizing the energy ideas of Lecture 5 in rather broad terms. The spring-mass-damper system is itself a paradigm for a very broad range of vibration models— physical, biological, chemical, and so on. Thus, we will not only be able to identify a system's mass, but we will also be able to identify a spring-like element with a stiffness, such as the gravitational pull of the pendulum, and a dissipative element with a damping constant, much like the shock absorber of an auto suspension (see Section 6.3). There is one salient feature common to each of these elements that will be true no matter what physical, biological, chemical or other model we are analyzing: Each element either *stores* energy or *dissipate*s energy. Two elements store energy in the spring-mass-damper: the mass, which stores kinetic energy,

$$KE = \frac{1}{2}m(\dot{x}(t))^2, \qquad (6.12)$$

and the spring, which stores potential energy,

$$PE = \frac{1}{2}k(x(t))^2. \qquad (6.13)$$

In an ideal system, where there is no damping, the spring and the mass exchange energy from potential to kinetic to potential, and so on indefinitely. Thus, the two storage elements exchange their forms of energy repetitively as the ideal spring-mass system vibrates.

The damping element dissipates energy according to (see eq. (5.29))

$$\frac{dE(t)}{dt} = -\frac{1}{2}c(\dot{x}(t))^2. \qquad (6.14)$$

As a spring-mass-damper vibrates or oscillates, energy is no longer simply passed back and forth between the spring and the mass. Rather, the damping element draws energy out of the system and dissipates it as wasted power or energy, typically through the heat transfer we associate with frictional devices.

Again, these characterizations turn out to be useful for helping us analyze systems or phenomena as we try to build models of their behavior.

6.1.2. Electric Circuits and the Electrical-Mechanical Analogy

Electric circuits and their elements offer a parallel paradigm for analyzing oscillatory behavior. Consider the elementary, *parallel* RLCcircuit shown in Figure 6.3. It has three ideal elements connected in parallel that are driven by a *current source* that produces a current $i_{source}(t)$. The three elements are idealized in the same way that themass of a spring-mass systemis perfectly rigid and that its spring is mass-less. The first element we introduce is the ideal *capacitor* that, when discharged, transmits a voltage drop, *V(t)*, that is proportional to the electric charge, *q(t)*, stored on two plates separated

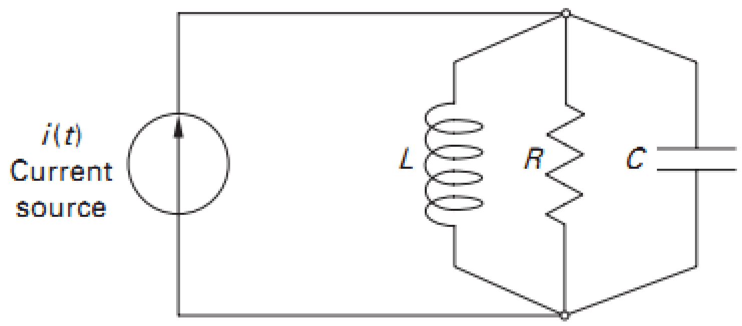

Figure 6.3 A *parallel RLC circuit* that has a current source as its driver. The elements are the *capacitor* of capacitance, C, the *inductor* with inductance, L, and the resistor with resistance, R. The current source provides a current of magnitude, $i_{source}(t)$.

by an insulator:

$$V(t) = \frac{q(t)}{C}. \tag{6.15}$$

The constant, C, is the *capacitance* of the capacitor and its units are farads, named after the British chemist and physicist Michael Faraday (1791–1867). The capacitor stores energy in an amount proportional to the square of the voltage across it:

$$E_C = \frac{1}{2} C (V(t))^2. \tag{6.16}$$

Notwithstanding the elegant simplicity of eqs. (6.15) and (6.16), electrical circuit models are generally cast in terms of the time rate of change of charge, called the *current*, because it is hard to measure charge:

$$i(t) = \frac{dq(t)}{dt}. \tag{6.17}$$

This form of the capacitor model is an element that carries a current, $i_C(t)$, that is directly proportional to the time rate of change of the voltage drop, $V(t)$, across the capacitor:

$$i_C = C \frac{dV(t)}{dt}. \tag{6.18}$$

The second element we introduce is the *inductor*, which is a coil that builds up a magnetic field rate when a current flows through it. The magnetic field causes a voltage drop across the inductor that is proportional to the time rate of change of the current flowing through it:

$$\frac{di_L}{dt} = \frac{V(t)}{L}. \tag{6.19}$$

The constant, L, is the *inductance*, which is measured in henrys, named after the American physicist Joseph Henry (1797–1878). Now we integrate eq. (6.19) with respect to time,

$$i_L = \frac{1}{L} \int_{-\infty}^{t} V(t')dt', \tag{6.20}$$

where t' is a dummy variable of integration in the integral in eq. (6.20). The inductor stores energy in an amount proportional to the square of the current flowing through it:

$$E_L = \frac{1}{2} L (i_L(t))^2 = \frac{1}{2L} \left(\int_{-\infty}^{t} V(t')dt' \right)^2. \tag{6.21}$$

The third element is the resistor. It impedes (or resists) the flow of charge in proportion to the time rate of change of charge, or the current. The resulting voltage drop across the resistor is directly proportional to the current flowing through it:

$$i_R = \frac{V(t)}{R}, \qquad (6.22)$$

where the constant, R, is the *resistance*, which ismeasured in ohms, named after the German physicist Georg Simon Ohm (1787–1854). The resistor, like itsmechanical counterpart, the dashpot, dissipates energy by throwing it off as waste heat or power. Thus, in the context of Section 8.1.1, we can regard the resistor and the dashpot as similar dissipative elements, and the capacitor (like themass) and the inductor (like the spring) as elements that store energy.

Can we draw an analogy between the electrical elements just introduced and the spring-mass-damper system described earlier in this section? Yes. In fact, there are two well-known electrical-mechanical analogies. The choice of analogy is to some extent a matter of taste, and we describe here the one we prefer; this book's first edition presented the other.

We first invoke Gustav Robert *Kirchhoff 's* (1824–1887) *current law* (KCL) to derive the governing equations for the parallel RLC circuit in Figure 6.3. The KCL states that the time rate of change of the electrical charge flowing into or out of a node or connection in a circuit must be zero. In other words, a node cannot accumulate charge. Expressed mathematically, the KCL states that

$$\frac{dq_{\text{node}}(t)}{dt} = \sum_{n=1}^{N} i_n(t) = 0, \qquad (6.23)$$

where the $i_n(t)$ are the currents taken as positive flowing into the node through the N elements connected at that node. Thus, looking at the indicated currents going into and out of either of the two nodes in Figure 6.3, we see that

$$\sum_{n=1}^{N} i_n(t) = i_{\text{source}}(t) - i_C - i_L - i_R = 0, \qquad (6.24)$$

where, again, $i_{\text{source}}(t)$ is the current provided by the *current source* in the circuit, and the remaining terms are the currents flowing through the capacitor, the inductor, and the resistor, respectively. Note that eq. (6.24) looks remarkably like a force balance equation [e.g., eqs. (6.3) and (6.10)]! We now replace the currents in the elements by their respective *constitutive equations* (6.18), (6.20), and (6.22), that describe how the current flows through each relates to the voltage across each. Then eq. (6.24) becomes:

$$C\frac{dV(t)}{dt} + \frac{V(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} V(t')dt' = i_{\text{source}}(t). \qquad (6.25)$$

If we differentiate eq. (6.25) once with respect to time, we find:

$$C\frac{d^2V(t)}{dt^2} + \frac{1}{R}\frac{dV(t)}{dt} + \frac{1}{L}V(t) = \frac{di_{\text{source}}(t)}{dt}. \qquad (6.26)$$

Use? Predict?

Equation (6.26) is a second-order, linear differential equationwith constant coefficients. Its dimensions can be shown to be consistent and correct (see Problem 8.4). When solved, it yields the common voltage across the three parallel elements, from which both the currents through each and the energy stored by the capacitor and inductor can be calculated [using eqs. (6.18), (6.20), and (6.22)].

What is most noteworthy about eq. (6.26) is its uncanny resemblance to eq. (6.11), the equilibrium equation for the spring-mass-damper. It is most tempting to conclude that voltage is analogous to displacement, and that

$$C \sim m, \quad \frac{1}{R} \sim c, \quad \frac{1}{L} \sim k. \qquad (6.27)$$

Some further expressions of this *electrical-mechanical analogy* are shown in Table 6.1. The analogy is interesting and useful. Consider, for example, the fact that we described the RLC circuit in Figure 6.3 as a parallel circuit. In the spring-mass-damper of Figure 6.2, we specifically inserted the dashpot as an element in parallel with the spring. The mass can also be said to be in parallel with the spring and the dashpot since it shares their common endpoint displacement. Further, the analogy extends into the context of system characterization: A system can be said to be very stiff if k is large or its inductance, *L*, is small, or as having a large effective mass or inertia if either its mass, *m*, or its capacitance, *C*, is large.

Now, to complete this introduction to the electrical-mechanical analogy, we repeat the thought that the choice of analogies is a matter of taste. The analogy presented here allows us to draw distinctions between behaviors that go *through* elements (force and current), and those measured *across* elements (displacement and voltage). The analogy also enables us to identify Newton's second law and Kirchhoff 's current law as similar expressions of balance (force or current) or conservation (momentum or charge). The other analogy identifies force with voltage and displacement with charge. It, therefore, does offer some more immediately recognizable appeal because the resemblance of basic equations is even more obvious.

Table 6.1 Elements of one electrical-mechanical analogy.

Mechanical	Electrical
Momentum (~ Speed): *mv*(*t*)	Charge: *q*(*t*)
Force (~ *d*(Momentum)/*dt*):	Current(~ *d*: (Charge)/*dt*):
$F = m\frac{dv(t)}{dt}$	$i(t) = \frac{dq(t)}{dt}$
Displacement: *x*(*t*)	Voltage: *V*(*t*)
	Kirchhoff 's Current Law:

Newton's 2nd @Massless Node:

$$\sum_{n=1}^{N} F_n(t) = \frac{d\big(mv_{\text{node}}(t)\big)}{dt} = 0$$

$$F_{\text{spring}} = k\int_{-\infty}^{t} v(t')dt' = kx(t)$$

$$F_{\text{damper}} = cv(t) = c\dot{x}(t)$$

$$F_{\text{net}} = m\dot{v}(t) = m\ddot{x}(t)$$

$$PE = \frac{1}{2}k\big(x(t)\big)^2$$

$$KE = \frac{1}{2}m(\dot{x}(t))^2$$

$$\frac{dq_{\text{node}}(t)}{dt} = \sum_{n=1}^{N} i_n(t) = 0$$

$$i_L = \frac{1}{L}\int_{-\infty}^{t} V(t')dt'$$

$$i_R = \frac{1}{R}V(t)$$

$$i_C = C\dot{V}(t)$$

$$E_C = \frac{1}{2}C(V(t))^2$$

$$E_L = \frac{1}{2}L(i_L(t))^2 = \frac{1}{2L}(\dot{q}(t))^2$$

However, the preferred analogy described above is more consistent with physical principles and conforms better to our intuition of how such systems behave.

6.2. The Fundamental Period of a Tall, Slender Building

Problems

6.1. We experience the pull of gravity as constant and not dependent on position. How does it come to be interpreted as exerting a spring force that is linearly proportional to position? (*Hint*: Think about the equation of motion in which the relevant term appears.)

6.2. Identify the fundamental physical dimensions of the spring stiffness, *k*, and the mass, *m*, and use them to determine the physical dimensions of ω_0 for a spring-mass oscillator.

6.3. Taking as fundamental the dimensions of current, I, as charge per unit time and voltage (or *electromotive force*), V, as (force × distance) per unit charge, determine the fundamental physical dimensions of the capacitance, C, the inductance, L and the resistance, R.

6.4. Using the fundamental dimensions identified in Problem 6.3, confirm that eq. (6.26) is dimensionally consistent and correct.

6.5. Using the fundamental dimensions identified in Problem 6.3, determine whether the energy expressions for E_C and E_L given inTable 6.1 are dimensionally correct.

6.6. Determine the governing equation for the free oscillation of the voltage in a parallel LC circuit with ideal elements.

6.7. Determine the natural frequency of free vibration and the period of the ideal parallel LC circuit of Problem 6.6.