5. Modeling Free Vibration

We now turn to modeling vibration, the behavior of something moving back and forth, to and fro, usually in an evident rhythmic pattern. Vibration not only occurs all around us, but within us as well, as noted in 1965 by a well-known British mechanical engineer, R. E. D. Bishop:

After all, our hearts beat, our lungs oscillate, we shiver when we are cold, we sometimes snore, we can hear and speak because our eardrums and our larynges vibrate. The light waves which permit us to see entail vibration. We move by oscillating our legs. We cannot even say 'vibration' properly without the tip of the tongue oscillating. And the matter does not end there—far from it. Even the atoms of which we are constituted vibrate.

Why?

Other vibratory phenomena that come to mind are pendulums, clocks, conveyor belts, machines and engines, buildings subjected to a broad array of moving forces (e.g., pedestrians, air conditioners, elevators, wind, earthquakes), as well as tides and seasons. Clearly, we could go on. But the more interesting questions for us are: Do these diverse instances of vibration have anything in common? If so, what? How do we model their common features?

We devote most of this chapter to modeling a well-known "golden oldie," the swinging or vibrating pendulum. It provides a familiar platform upon which we can lay out a number of modeling strategies. Then we will provide a few examples of freely vibrating phenomena. We will also illustrate how the mathematics of free vibration can be used to model stability phenomena. In Lecture 6 we will provide some more examples and then go on to model forced vibration.

5.1. The Freely-Vibrating Pendulum–I: Formulating a Model

Given?

We will now model the free vibration of a pendulum, starting with some experimental results and using dimensional analysis, some basic physics, and some basic mathematics (e.g., linearity, second-order differential equations) to model that motion.

5.1.2. Some Experimental Results

How?

We started by building some very simple pendulums in the laboratory, each consisting of a lead-filled wooden ball suspended from a stand by an ordinary piece of string. A basic schematic of the laboratory set-up is shown in Figure 5.1. The balls were initially held at rest at some angle, θ_0, and then they were let go to swing back and forth until they all stopped moving. As each pendulum swung, we measured its period of free vibration, the time T_0 it takes to swing through two complete arcs (from $\theta = \theta_0$ to $\theta = -\theta_0$ and back again). The periods of vibration were measured with photoelectric cells that were placed at the lowest point on the pendulum arc ($\theta = 0$) and were in turn connected to digital counters operating with a gated pulse. The counters were turned on by the first passing of the pendulum and then off again at the second passing, thus providing a direct read of one-half of the period T_0.

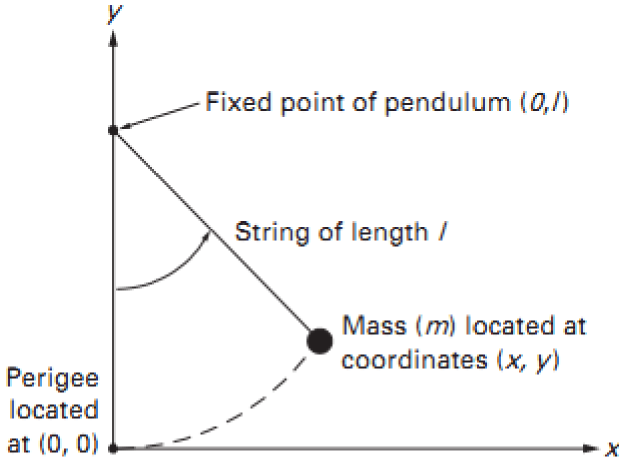

Figure 5.1. The geometry of a planar pendulum. Note that the origin of the coordinate system is located at the pendulum's *perigee*, the lowest point of its arc.

Table 5.1 The dependence of the period, T_0, of a freely-vibrating pendulum on its initial amplitude of vibration, θ_0. The mass is 390 gm and the string length is 276 cm.

θ_0 (deg)	θ_0 (rad)	T_0 measured (sec)	(T_0 measured)/(3.372)
8.34	0.1456	3.368	1.00
13.18	0.2300	3.368	1.00
18.17	0.3171	3.372	1.00
23.31	0.4068	3.372	1.00
28.71	0.5011	3.390	1.01
33.92	0.5920	3.400	1.01
39.99	0.6980	3.434	1.02
46.62	0.8137	3.462	1.03

The experiments were done with two different masses (237 gm and 390 gm), each of which was hung from strings of two different lengths (276 cm and 226 cm). The experimental data thus obtained are shown in Tables 5.1 and 5.2; note that each data point shown represents the average of five measured values. Thus, the data presented result from a consistent, repeatable experiment. The data in Table 5.1, for the larger mass (390 gm) and the shorter string (276 cm), show how the period, T_0, varies with different starting values of θ_0. We see that the period varies with the initial starting angle, θ_0, but the dependence is very weak and exceeds 1% only when $\theta_0 \geq 40^\circ$.

Table 5.2. The dependence of the period, T_0, of a freely-vibrating pendulum on its length and on its mass. The data show a marked change with length, but virtually no change with mass.

	$m=237$ gm	$m=390$ gm
$l=226$ cm	3.044 sec	3.058 sec
$l=276$ cm	3.350 sec	3.372 sec

The data in Table 5.2 summarize the periods across the four possible combinations of mass and length that were available for the pendulums used in this experiment. This data suggest that the period varies very little, if at all, with mass: increasing the mass by some 65%from237 gmto 390 gm changes the period by a fraction of 1%. On the other hand, increasing the length by 22% from 226 cm to 276 cm increases the period by approximately 10%. Thus, the data suggest that the free motion of a vibrating pendulum is periodic, and that the period of vibration does not depend on the pendulum's mass, but that it does depend on the pendulum's length.

5.1.2. Dimensional Analysis

We will now apply some dimensional analysis results to formalize the results we obtained in the laboratory. In Lecture 2 we used the Buckingham Pi theorem to determine that the period of vibration, T_0, of a pendulum was related to its length, l , and the gravitational acceleration, g [see the first of eq. (2.30)]:

$$T_0 = \Pi_1 \sqrt{\frac{l}{g}}. \qquad (5.1)$$

Valid?

Note that the pendulum's period does not depend on mass, a result supported by the data in Table 5.2, and that the constant, Π_1 is dimensionless. We can determine the value of Π_1 from the data given in Table 5.2. For the pendulum of length l = 276 cm, one measured value of the period is T_0 = 3.372 sec, so that with g = 980 cm/sec/sec,

$$\Pi_1 = \frac{3.372}{\sqrt{276/980}} \cong 6.35\,. \qquad (5.2)$$

Is the number "6.35" in eq. (5.2) some new universal constant? Actually, no. Rather, it is an approximation of another well-known constant: $2\pi \cong 6.28$. Thus, substituting this judgment call about the constant into eq. (5.2) yields the final result,

$$T_0 = 2\pi \sqrt{\frac{l}{g}}. \qquad (5.3)$$

Table 5.3 Calculated values of the period, T_0,of a freely-vibrating pendulum that provide support for the experimental data presented in Table 5.2.

l(cm)	T_0(sec)
226	3.02
276	3.33

Predict? Verified?

We can use eq. (5.3) to predict values of the period to match the remaining values displayed in Table 5.2, as shown in Table 5.3. The calculated pre dictions and the experimental data agree to within less than 1.5%. Thus it seems that we have a pretty good model—determined from dimensional analysis and use of some experimental data—that works quite well and predicts the remaining experimental data, including both the period' dependence *on* length and its independence *of* mass. We will confirm the model (5.3) again before we're done with the pendulum.

5.1.3. Equations of Motion

How?

We formulate the problem by writing the mathematical expression of a balance or conservation principle (see Lecture 1) from physics. The principle is Newton's second law: *The time rate of change of momentum is equal to the net force producing it; that momentum change is in the same direction as the net force.* Newton's second law is both a balance principle and a conservation principle: it reflects a balance of the forces acting on a particle or system, and it also reflects the conservation of momentum. Written as a balance principle, Newton's second law in a plane is:

$$\sum F_x = m\frac{d^2x}{dt^2}, \qquad (5.4a)$$

$$\sum F_y = m\frac{d^2y}{dt^2}, \qquad (5.4b)$$

where *x(t)* and *y(t)* are the time-dependent coordinates of amass, *m,* acted on by net forces $\sum F_x$ and $\sum F_y$, respectively.

We want to apply Newton's second law, commonly referred to as *equations of equilibri*um, to the pendulum depicted in Figure 5.1. The pendulum is simply a mass, m, attached to the end of a string of length, l .Its wings in a plane from an attachment point with coordinates *(0, l)* so that the origin of the coordinates coincides with the *perigee* or low point of the pendulum's arc. The coordinates (*x, y)* of the pendulum mass can be written in terms of the string length and the angle θ between the string and the *y*-axis:

$$x(t) = l\sin\theta\,(t), \qquad (5.5a)$$

$$y(t) = l\big(1 - \cos\theta\,(t)\big), \qquad (5.5b)$$

In Figure 5.2 we show a *free-body diagram* (FBD) of the two forces that act on the mass: the tension in the string, T, and the weight, mg , which acts due to the pull of gravity. Then we can identify the net forces along the coordinates from the FBD, so that eqs. (5.4) can then be written as *equations of motion*:

$$m\frac{d^2x}{dt^2} = \sum F_x = -T\sin\theta, \qquad (5.6a)$$

$$m\frac{d^2y}{dt^2} = \sum F_y = T\cos\theta - mg. \qquad (5.6b)$$

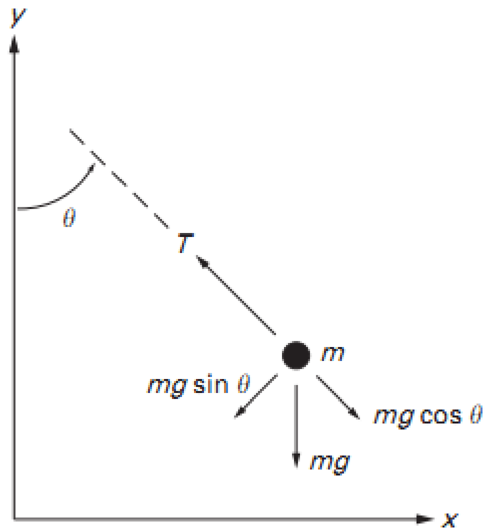

Figure 5.2 A *free-body diagram* (FBD) of the oscillating planar pendulum. It shows the two forces acting on the pendulum's mass, *m*, the string tension, *T* , and the weight, *mg*, and their components in the radial and tangential directions.

Improve?

In principle, all we need to do now is integrate eqs. (5.6a–b) to find how the pendulum's coordinates vary with time, from which we can then find out whatever else we might want to know about the pendulum. However life's not that easy, for a number of reasons. First, we don't know the tension in the string, T, so that the right-hand sides of both of eqs. (5.6a–b) are unknown. Second, since we have two equations with *three* unknowns— $x(t)$, $y(t)$, T—we are prompted to wonder how Newton's second law would look if rewritten in *radial* (along the string) and *tangential* (to the pendulum's arc) coordinates. In fact, those equations are

$$\sum F_{\text{radial}} = ml\left(\frac{d\theta}{dt}\right)^2, \qquad (5.7a)$$

$$\sum F_{\text{tangential}} = ml\frac{d^2\theta}{dt^2}. \qquad (5.7b)$$

Equation (5.7a) clearly displays the familiar centripetal acceleration. If we sum the forces in the FBD of Figure 5.2 in the radial and tangential directions, we would find that

$$T = ml\left(\frac{d\theta}{dt}\right)^2 + mg\cos\theta, \qquad (5.8a)$$

$$ml\frac{d^2\theta}{dt^2} + mg\sin\theta = 0. \qquad (5.8b)$$

Equations (5.8a–b) show two equations for two dependent variables, the tension, T, and the angle, θ. Equation (5.8b) is a single equation with a single unknown, θ, so it can in principle be solved on its own, which thus determines the location of the mass [see also eqs. (5.5a–b)]. Then the tension, T, can be obtained directly by substituting the newly-found θ into eq. (5.8a). We also note that eqs. (5.8a–b) are equivalent to eqs. (5.6a–b): both are representations of Newton's second law, eqs. (5.8a–b) written in radial and tangential coordinates (l , θ), eqs. (5.6a–b) in rectangular coordinates (x, y).

We further note that eqs. (5.8a–b) are decidedly nonlinear because the dependent variable *θ(t)* or its derivatives have an exponent different than 1. This is most obvious in eq. (5.8a) because of the centripetal acceleration, but it is equally true of eq. (5.8b) because

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots. \qquad (5.9)$$

As we noted in Section 1.3.4, the presence of such nonlinear terms means that superposition, one of the most powerful weapons in the arsenal of mathematics, is no longer available. We will return to this point in greater detail in Section 5.3.

5.1.4. More Dimensional Analysis

5.1.5. Conserving Energy as the Pendulum Moves

5.1.6. Dissipating Energy as the Pendulum Moves

5.2. The Freely-Vibrating Pendulum–II: The Linear Model

5.2.1. Linearizing the Nonlinear Model

5.2.2. The Differential Equation $md^2x/dt^2 + kx = 0$

5.2.3. The Linear Model

5.3. The Spring-Mass Oscillator–I: Physical Interpretations

5.4. Stability of a Two-Mass Pendulum

5.5. The Freely-Vibrating Pendulum–III: The Nonlinear Model

Problems:

5.1. Assume a hypothetical relationship, $T_0 = am^b$, for the dependence of the period of a pendulum on its mass. Determine the unknown parameters, *a* and *b*, using the data in Table 5.2. (Hint : Logarithms may be useful here.)

5.2. Assume a hypothetical relationship, $T_0 = cl^d$, for the dependence of the period of a pendulum on its length. Determine the unknown parameters c and d using the data in Table 5.2. (Hint : Logarithms may be useful here.)

5.3. Why do eqs. (5.4a–b) represent Newton's second law as a balance principle?

5.4. How would eqs. (5.4a–b) be written as a conservation principle?

5.5. Identify and explain *all* of the nonlinearities in eq. (5.8a).