We devote this last chapter on fundamentals to discussions of elementary mathematical approximation techniques and of model testing and validation. Approximations are used to simplify both models (as we will see in Chapter 7 where the nonlinear model of the pendulum is simplified to obtain a linear estimate of the pendulum's behavior) and the numerical calculations made with the models. Such approximations and their numerical implementations introduce error, but the magnitudes of these errors can be estimated and limited. We will also discuss means of model validation: checking dimensions and units, testing qualitative behavior and limits, and applying basic statistics.

Taylor's Formula

Engineering and scientific calculations abound with mathematical approximations, in some measure because linear problems are easier to solve, but in larger measure because many of our linearmodels are validated and justified by experiment and by experience. Distinctions such as those between a linearized model and its full nonlinear counterpart also involve mathematical approximations such as those described in this section. How do we approximate a function to properly estimate the behavior it describes?

Many analytical approximations are derived from Taylor's formulas. Advanced numerical techniques such as the finite element method also use Taylor's formulas to approximate functions as polynomials with unknown coefficients that are determined numerically. Thus, we now review some basic results about Taylor's formula and series, including Taylor formulas of trigonometric functions and binomial expansions.

3.1. Taylor's Formula and Series

Any function that is continuous and has derivatives can, in general, be expanded into and approximated by a Taylor's formula. For values of the independent variable, *x*, in a region near *x =a*, a function *f (x)* can be approximated by the polynomial

$$f(x) \cong f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \,. \qquad (3.1)$$

where $f'(a)$ represents the first derivative of f (x), $f''(a)$ the second derivative, and $f^{(n)}(a)$ the nth derivative of f (x) evaluated at the point $x = a$.The series given in eq. (3.1) is called *theTaylor formula of f(x) in the neighborhood of the point x = a*. The point $x = a$ must be such that all derivatives of f (x) exist there and are finite. In addition, and most important for this discussion, if the difference *(x−a)* is very small, then we need only a few terms of the series (3.1) to render a good approximation of *f (x)* in the neighborhood of x = a. The corresponding *Taylor's series* that renders the approximate equality in eq. (4.1) an exact equality is the limit of eq. (3.1) as n →∞:

$$f(x) = \lim_{n\to\infty}\left[f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n\right]. \qquad (3.2)$$

If we want to approximate the function *f (x*) at another point, say *x* = b, we evaluate eq. (3.1) at that point to find Taylor's formula for *f (b):*

$$f(b) \cong f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n \,. \qquad (3.3)$$

If we use only the first termof eq. (3.3), we are approximating *f (b)* as being equal to *f (a)*, as shown in Figure 4.1(a). If we use the first two terms of eq. (3.3), our approximation is improved by incorporating the effect of the slope change $f'(a)$, as shown in Figure 3.1(b). This value is closer to the true value than our simple one-term approximation. Our approximation is still further improved when three terms of the expansion (3.3) are used to approximate *f (b)*, as shown in Figure 3.1(c).

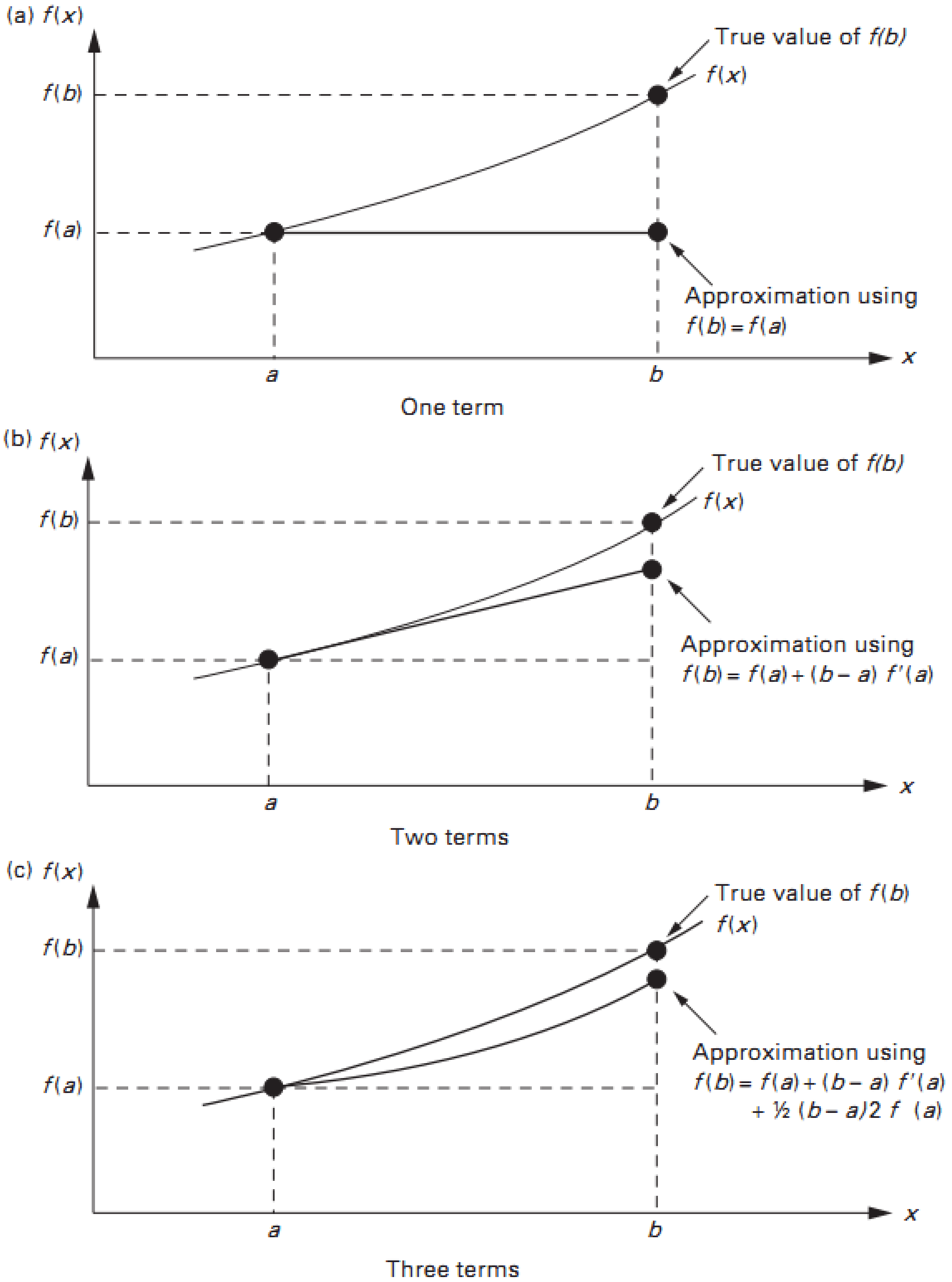

Figure 3.1. Improving the approximations obtained with a Taylor expansion by retaining more terms: (a) a one-term series approximation; (b) a two-term estimate; and (c) a three-term

approximation. Note that the higher-order approximations depend on derivatives of *f(x)* at the reference point of the Taylor series, $x=a$.

The accuracy of an approximation for any function *f (x)* improves with the number of terms used in the expansion. Similarly, the approximation in eq. (3.1) can be turned into an exact formula like eq. (3.2) by adding a *remainder term* R_{n+1} to eq. (3.1):

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}\,. \qquad (3.4)$$

where the remainder term (which can be cast in several forms) is here shown as:

$$R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}. \qquad (3.5)$$

The derivative in eq. (3.5) is calculated at a "suitably chosen" point ξ some where in the interval between a and x. Even though the precise location of ξ is not known, the remainder formula can be used to estimate the error made if a Taylor formula to order n is applied. How many terms do we have to keep in a Taylor formula to ensure that the error is negligible, or at least acceptable? As we will see below, it depends on what we're trying to do, on the specifics of the model we're trying to build.

3.2. Taylor Series of Trigonometric and Hyperbolic Functions

The Taylor series expansions of the trigonometric functions for $a = 0$ are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \qquad (3.6a)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \qquad (3.6b)$$

where *x* is expressed in (dimensionless) radians to ensure dimensional homogeneity. The corresponding Taylor expansions for the hyperbolic functions are:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots, \qquad (3.7a)$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots. \qquad (3.7b)$$

We will now use a Taylor formula for the hyperbolic cosine (eq. (3.7b)) to estimate the sag of a tightly stretched string or cable that is weighted down only by its own weight. Such a cable is called a *catenary* after the

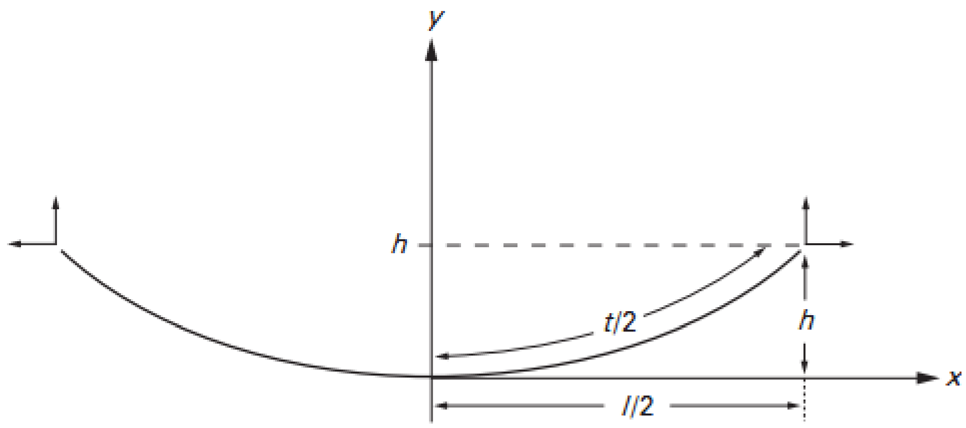

Figure 3.2. A long measurement tape stretched between two fixed points, A and B, for which the sag, h, is exaggerated. The mathematical model of the stretched tape is a hyperbolic cosine that

can be approximated to varying degrees, dependence signifies the fact that actual tape readings, t, must be corrected to property measure the distance, l, on the ground.

Latin word for chain. Estimating the sag of a catenary may not sound all that interesting, but it does have a practical side that had been, until recently, a real engineering application. Until theodolites were introduced to measure large distances in construction projects, surveyors and engineers relied on tape measures. A surveyor's tape acts as a catenary because its only vertical load while measuring is its self-weight. We show such a tape in Figure 3.2, stretched between two supports at the same elevation that are separated by the length, l , with the cable's sag, h, exaggerated. Since cosh (0) = 1, the equation of the catenary is

$$y(x) = c\left(\cosh\frac{x}{c} - 1\right), \qquad (3.8)$$

where c is the *catenary parameter* and the coordinates of the vertex or low point of the cable are (x = 0, y = c). The catenary parameter is a function of T_0, the (constant) horizontal component of the tension in the stretched cable, and of γ , the string's weight per unit length. We see from Figure 3.2 that the sag is given by

$$h = y\left(\frac{l}{2}\right) = c\left(\cosh\frac{x}{2c} - 1\right). \qquad (3.9)$$

Now we substitute the Taylor series (3.7b) of the hyperbolic cosine to find the sag:

$$\begin{aligned} h = c\cosh\frac{l}{2c} - c &= c\left(1 + \frac{1}{2!}\frac{l^2}{4c^2} + \frac{1}{4!}\frac{l^4}{16c^4} + \frac{1}{6!}\frac{l^6}{64c^6} + \cdots - 1\right) \\ &= c\left(\frac{1}{2!}\frac{l^2}{4c^2} + \frac{1}{4!}\frac{l^4}{16c^4} + \frac{1}{6!}\frac{l^6}{64c^6} + \cdots\right). \qquad (3.10) \end{aligned}$$

Note that this Taylor series for the sag has the correct physical dimensions since both c and *h* are measures of length and the ratio *l/c* is dimensionless, as it should be as the argument of the hyperbolic function. Further, for a tightly stretched string, the sag, *h,* is very small compared to the length, l , that is, *h/l* << 1. This suggests that the ratio *l/2c* is also quite small compared to 1 because a one-term approximation of eq. (3.9) is found by retaining only the first term in the last of eq. (3.10):

$$h \cong c\left(\frac{1}{2!}\frac{l^2}{4c^2}\right) = \frac{l^2}{8c}. \qquad (3.11)$$

Equation (3.11) confirms the suggestion that large values of the dimensionless catenary parameter, *2c/l* , correspond to small values of the dimensionless sag, *h/l* , because this result can be arranged as:

$$\frac{2c}{l} = \frac{l}{4h} \gg 1. \qquad (3.12)$$

Further, had we approximated the hyperbolic cosine for small values of *l*/2c independently of eqs. (3.9) and (3.10), we would have calculated that

$$c\ cosh\frac{l}{2c} \cong c\left(1 + \frac{1}{2!}\frac{l^2}{4c^2} + \frac{1}{4!}\frac{l^4}{16c^4} + \frac{1}{6!}\frac{l^6}{64c^6}\right) \cong c, \qquad (3.13)$$

and we would then have found, quite mistakenly, that the sag was identically zero because we had used an inadequate approximation! How do these results affect the measurements of long distances with a tape? The answer is found by calculating the length of tape, *t* , needed to measure the horizontal distance, *l* , as shown in Figure 3.2. An element of arc length along the tape, *ds*, is given by

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(y'(x)\right)^2}. \qquad (3.14)$$

If we substitute the catenary shape (3.8) into eq. (3.14) and apply a standard identity, we find that

$$ds = \cosh\frac{x}{c}\,dx. \qquad (3.15)$$

Equation (3.15) can be straightforwardly integrated, that is,

$$\int_0^{t/2} ds = \int_0^{l/2} \cosh\frac{x}{c}\,dx,$$

to yield:

$$t = 2c\sinh\frac{l}{2c}. \qquad (3.16)$$

We can expand eq. (3.16) in a Taylor formula, again based on the assumption that *l/2c* is quite small, but for reasons that will soon become evident, we will retain the first two terms in the series, that is:

$$t \cong 2c\left(\frac{l}{2c} + \frac{1}{3!}\frac{l^3}{8c^3}\right). \qquad (3.17)$$

With the aid of either eq. (3.11) or eq. (3.12), eq. (3.17) can be written as a quadratic equation in the distance *l* :

$$l^2 - lt + \frac{8}{3}h^2 = 0. \qquad (3.18)$$

The quadratic equation (3.18) can be solved for its roots:

$$2l = t\left(1 \pm \sqrt{1 - \frac{32}{3}\left(\frac{h}{t}\right)^2}\right). \qquad (3.19)$$

Only the positive root is physically viable here. In the next section, we will see that the radicand in eq. (3.19) is an ideal candidate to be written as a *binomial expansion*, which is a special form of Taylor's formula. For small values of *h/l* and to two term accuracy,

$$2l \cong t\left(1 + \left(1 - \frac{32}{6}\left(\frac{h}{t}\right)^2\right)\right) = t\left(2 - \frac{32}{6}\left(\frac{h}{t}\right)^2\right). \qquad (3.20)$$

Thus, the actual length, *l* , that is measured by a tape reading of t is given by

$$l \cong t\left(1 - \frac{8}{3}\left(\frac{h}{t}\right)^2\right). \qquad (3.21)$$

Obviously, the larger the sag, *h*, the larger the correction that must be applied to the tape reading, *t* , to ensure an accurate measurement of the distance, *l* .

Lastly on the expansion (3.10), we point out that it is an approximation in the spirit of the *small angle* approximation that appears frequently in engineering and scientific models. For example, from eq. (3.6b) we know that the second-order Taylor formula for the elementary cosine can be written as

$$\cos x \cong 1 - \frac{x^2}{2!}, \qquad (3.22)$$

where *x* is measured in radians. To approximate the cosine function for very small angles in the neighborhood of x = 0, we can safely ignore the second-order term in eq. (3.22) and take $\cos x \cong 1$. However, as we will see in the formal development of the pendulum model in Lecture 5, we often have reason to approximate a slightly different function, (1 −cos *x*). If we neglected or ignored the second-order term here, the resulting approximation would be $(1 - \cos x) \cong 0$, which is a bad approximation that results from throwing out the dependence on x. Thus, as in so many other aspects of modeling, it is important to know where we're going when truncating Taylor formulas or series.

There is another approach to approximating trigonometric functions that is worth mentioning. Suppose we wanted to replace sin x by x in a model or a calculation. We could look at the numerical values of both functions to see where the substitution would be acceptable. For example, if we are willing to accept an error of 5%, we could replace sin x by x for $x \leq \pi/6$. For an error of only 2%, the substitution would be acceptable for $x \leq \pi/12$. (And while it is important that all angles in these arguments be either rendered as dimensionless ratios of variables or expressed as angles measured in radians, it is worth noting that the two examples just given correspond to small angles of, respectively, 30◦ and 15◦.) Thus, by exploring the numerical ranges of interest and the associated errors, we can often justify replacing a trigonometric function by an algebraic approximation.

3.3. Binomial Expansions

Algebraic Approximations

Numerical Approximations: Significant Figures

Problems

4.35. Estimate the error made in approximating y(x)= sin x with a Taylor's formula to n = 4 by evaluating the remainder R5.

4.36. Do the statements that sin x 1 and tan x 1 produce similar approximations? Confirm and explain your answer.

4.37. The readings of an old-fashioned analog voltmeter—it has dials, not digital readouts!—are subject to some systematic error where all of its readings are too large. The magnitude of the error has been found to vary linearly from 1 V at a dial reading of 5 V to 4 V at a dial reading of 80 V.

(a) What are the correct voltages for dial readings of 80, 100, 50, 1, 35, and 10V?

(b) What is the percentage error for each of the six (6) readings in part (a)?

4.38. (a) Is it possible to have a set of measurements that are precise but not accurate? Explain.

(b) Is it possible to have a set ofmeasurements that are accurate but not precise? Explain.

4.39. (a) Write the Taylor series expansion for ex about x = 0.

(b) Calculate e0.5 to five significant figures using the first four terms of the series found it part (a).

4.40. (a) What percentage error was incurred in the calculation of part

(b) of Problem 4.39 if the "true value" of e0.5 is 1.6487?

(b) Use the Taylor remainder (eq. (4.5)) to calculate the error in e0.5 after only four terms. Is the error calculated in part (a) of this problem acceptable? Explain.

4.41. Evaluate the following function by hand (no calculators or computers, please) for x = 4:

$$\left(1+\frac{2}{x}\right)^{1/4}$$

4.42. How does an observer know when enough is enough, that enough measurements have been taken?

4.43. Make a list of five new (i.e., not found in the text) examples of systematic errors.

4.44. Make a list of five new (i.e., not found in the text) examples of random errors.

4.45. The resistance of a resistor, R, ismade by passing several currents, I , through it and measuring the corresponding voltage drops, V, and currents with imprecise, analog meters. The resulting data are:

xi = V(V) 10 20 30 40 50 60 70 80

yi = I (A) 0.8 1.1 2.5 4.2 4.3 4.7 5.8 6.4

(a) What kinds error will be found in the data?

(b) Assuming that V = IR, plot the data (by hand!) and "eyeball" in the best-fit line for that data.

4.46. Use the method of least squares to plot a V versus I curve for the data of Problem4.45. How does it compare with the "eyeball" result of Problem 4.45?

4.47. The data presented belowcomprise 100 readings of noise levels taken 6 mi away from an airport, taken late in an evening at 15 s intervals.

Find the mean, median, and standard deviation of these data.

Observed Decibel Values (dB), $n = 100$

50	50	53	48	45	51	57*	75*	85*	82*
75*	71*	65*	61*	60*	60*	55*	55*	51	50
49	49	48	51	49	54	48	48	47	49
49	49	49	49	48	47	50	49	48	49
47	48	48	50	50	54	48	47	47	48
48	49	48	47	50	49	48	48	48	48
48	48	52	50	53	49	49	48	49	47
49	55	51	50	49	48	49	45	48	50
50	51	49	50	47	47	47	47	47	47
48	50	49	49	49	49	49	49	56	49

4.48. The starred numbers in the data of Problem 4.47 are readings taken while an aircraft was flying directly overhead. If these data are deleted, what are the mean, median, and standard deviation of the remaining 88 data points?

4.49. Draw (a) a histogram of all of the data of Problem 4.47 and (b) a continuous curve of the number of readings as a function of the measured noise level.

4.50. Determine a *far-field approximation* of the function *f (r)* given below as a binomial expansion for values of *r* >> *a*.

$$f(r) = \sqrt{a^2 + r^2}$$

4.51. The electric potential, Ve , at a distance, r, along the axis of revolution of a disk of radius a is given by

$$V_e = \frac{q}{2\pi a^2 \varepsilon_0}\left(\sqrt{a^2 + r^2} - r\right)$$

where q is the total charge that is distributed uniformly over the surface of the disk and ε0 is the permittivity constant. Using the results of Problem4.50, find a far-field approximation for the electric potential for values of r >> a.

4.52. Compare the minimum number of terms kept in the binomial expansions of the solutions to Problems 4.50 and 4.51. Are those numbers the same, or not?Why are those numbers the same, or not?

4.53. Suppose we need to calculate the radial extension or deflection w of a very thin, spherical balloon, meaning that the sphere’s radius extends from R to R + w as the balloon is pressurized. It is made of an elasticmaterial. Acolleague finds a textbook that shows a formula for the pressure, p, that looks reasonable:

$$\frac{w}{R} = \frac{pR}{Eh}$$

where h is the balloon’s wall thickness, and E is the modulus of the material of which the sphere ismade. Is this equation dimensionally consistent?

4.54. Analyze the limit behavior of the equation presented in Problem4.53 as the pressure, modulus, radius, and thickness both go to zero and become infinitely large. Does this limit behavior conformwith your intuitive estimate of what should happen?

4.55. Use the equation in Problem 4.53 to derive an estimate of the magnitude of the pressure, p, as a fraction of the modulus, E. Estimate the pressure fraction for a thin-walled sphere, for which h/R << 1.