“Graph theory”
Course for the master degree program “Geographic Information Systems”

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4. Location problems

- Distances in a weighted graph
- Centre
- Median
- Extencions
- Absolute P-centre
- P-median

4.1. Distances in a weighted graph

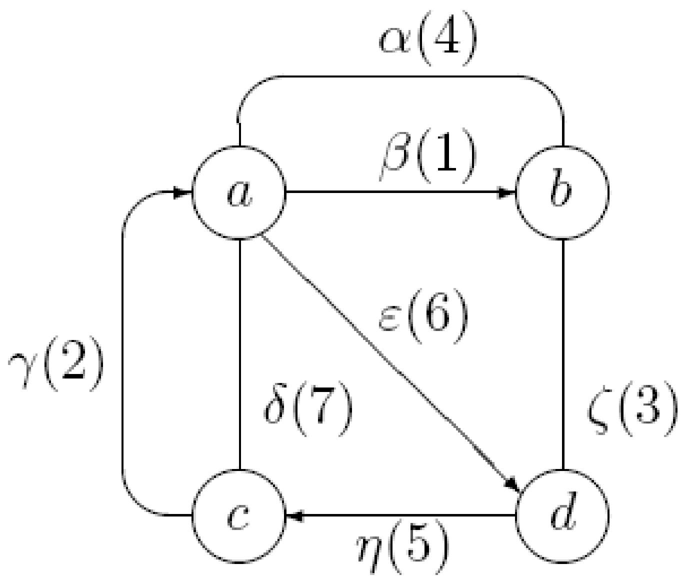
- Vertex-vertex distance
- Point-vertex distance
- Vertex-point distance
- Vertex-edge distance

Vertex-vertex distance

The **vertex-vertex distance** between vertices *i* and *j* (notation *d*(*i*,*j*)) is the weight of the shortest path <*i*,*j*>.

It can be found by the Floyd–Warshall algorithm.

Example.

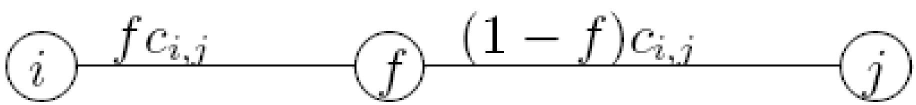


	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

F-point

Consider an edge $e=(i,j)$ with the weight $c_{ij}>0$ and a parameter f : $0 \le f \le 1$.

The point at the edge which divide the edge in proportion f : $(1-f)$ is called the ***f*-point** (notation $f_{(i,j)}$).



The weight of the edge part if is equal to fc_{ij}, the weight of the part fj is equal to $(1-f)c_{ij}$.

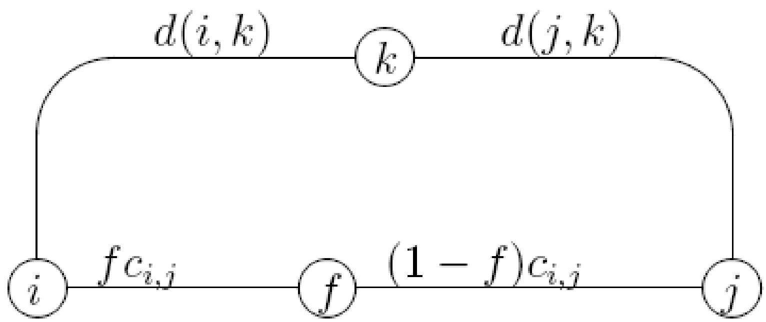
The vertex i is 0-point, the vertex j is 1-point.

The other points are **interior**.

Point-vertex distance

The **point-vertex distance** between a point $f_{(i,j)}$ and a vertex k (notation $d(f_{(i,j)},k)$) is the weight of the minimal path $< f_{(i,j)},k>$.

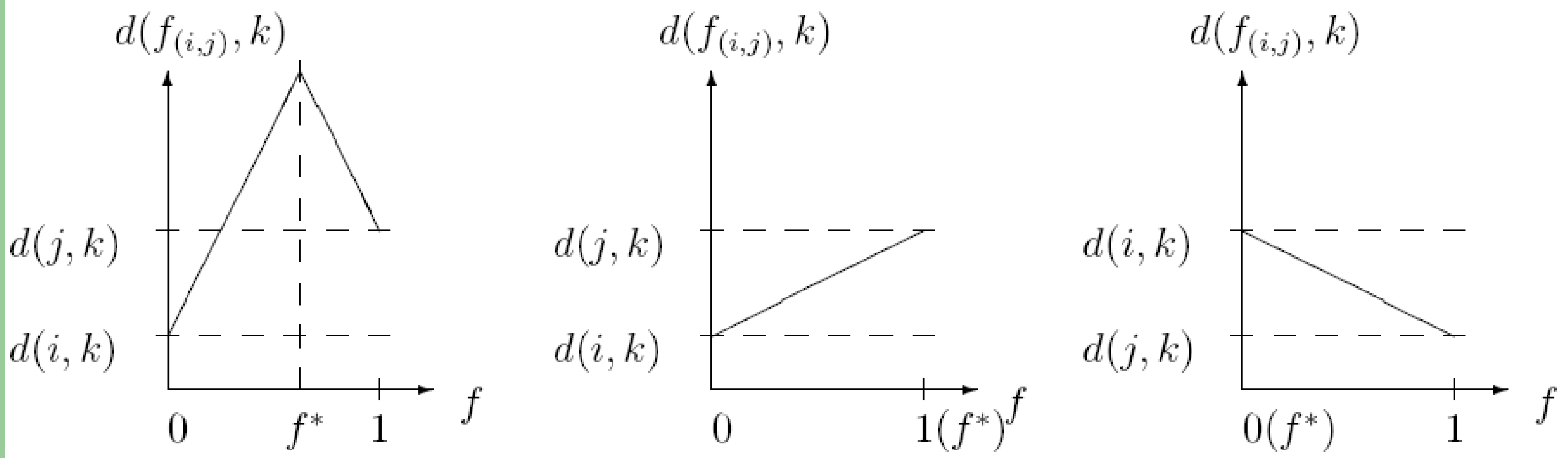
For an undirected edge (i,j):



$$d(f_{(i,j)},k) = \min\{fc_{i,j} + d(i,k), (1-f)c_{i,j} + d(j,k).\}$$

Point-vertex distance

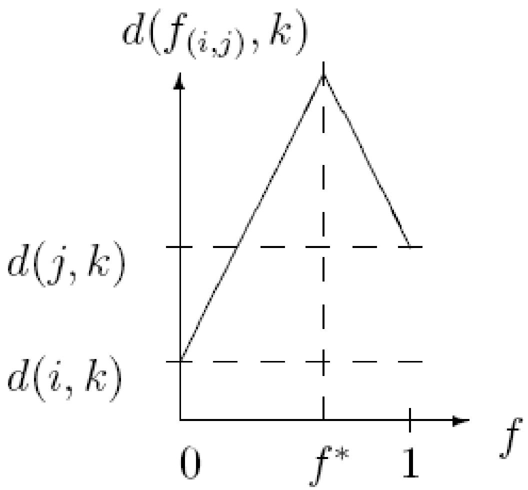
The dependence $d(f_{(i,j)},k)$) of f can be one of three types.



Point-vertex distance

The maximum point f^* is the point of the lines intersection:

$$f c_{i,j} + d(i,k) = (1-f) c_{i,j} + d(j,k)$$



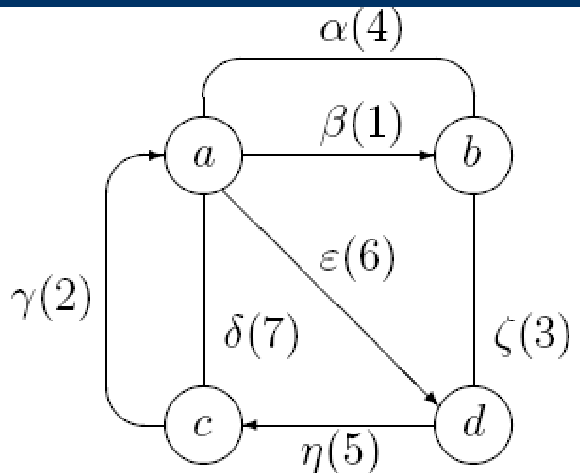
$$f^* = \frac{d(j,k) - d(i,k) + c_{i,j}}{2c_{i,j}}$$

Since
$$d(i,k) \leq c_{i,j} + d(j,k);$$
$$d(j,k) \leq c_{i,j} + d(i,k),$$

so $f^* \in [0,1]$.

Point-vertex distance

Example

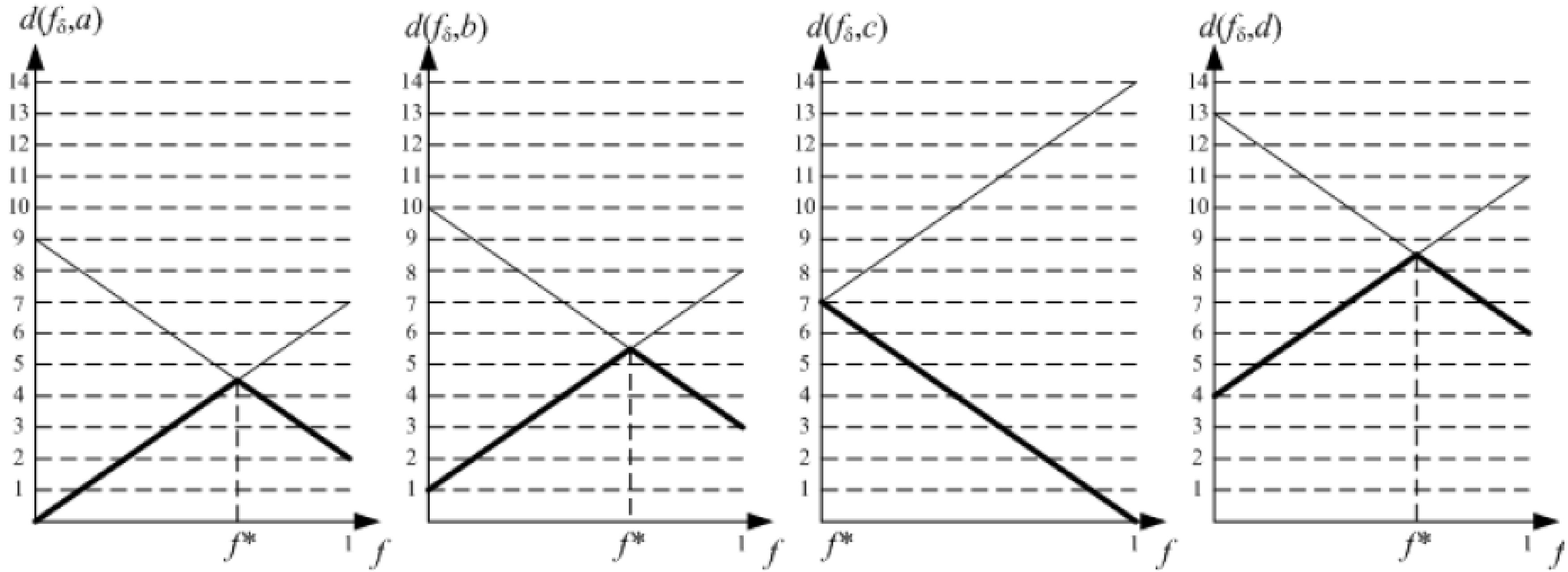


	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$d(f_\delta, a) = \min\{fc_\delta + d(a,a), (1-f)c_\delta + d(c,a)\} = \min\{7f+0, 7(1-f)+2\};$$
$$d(f_\delta, b) = \min\{fc_\delta + d(a,b), (1-f)c_\delta + d(c,b)\} = \min\{7f+1, 7(1-f)+3\};$$
$$d(f_\delta, c) = \min\{fc_\delta + d(a,c), (1-f)c_\delta + d(c,c)\} = \min\{7f+7, 7(1-f)+0\};$$
$$d(f_\delta, d) = \min\{fc_\delta + d(a,d), (1-f)c_\delta + d(c,d)\} = \min\{7f+4, 7(1-f)+6\}.$$

Point-vertex distance

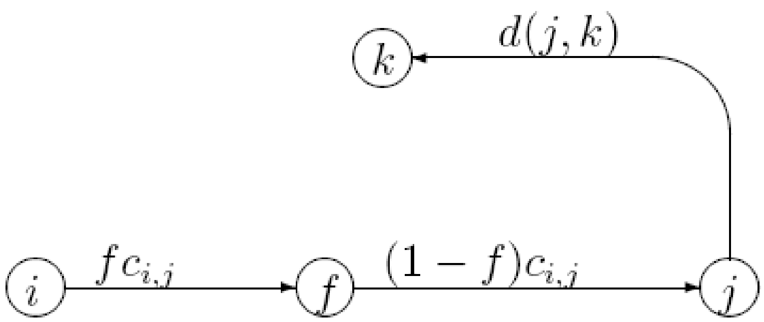
Example:

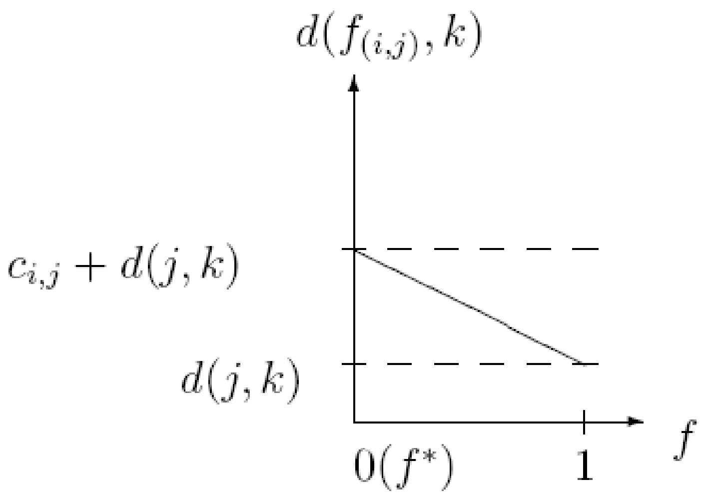


Point-vertex distance

For a directed edge (*i*,*j*):

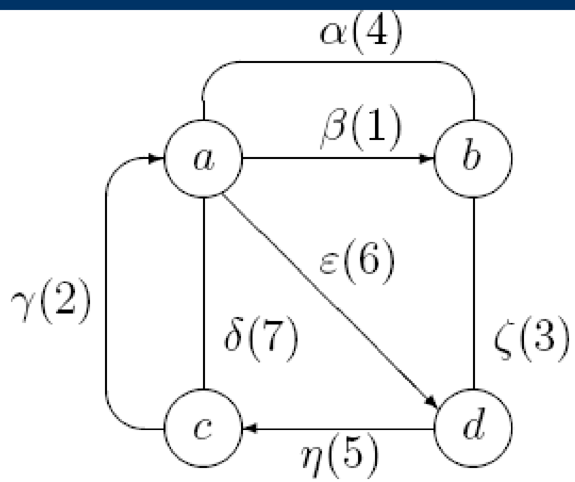
$$d(f_{(i,j)}, k) = (1 - f)c_{i,j} + d(j,k).$$





Point-vertex distance

Example

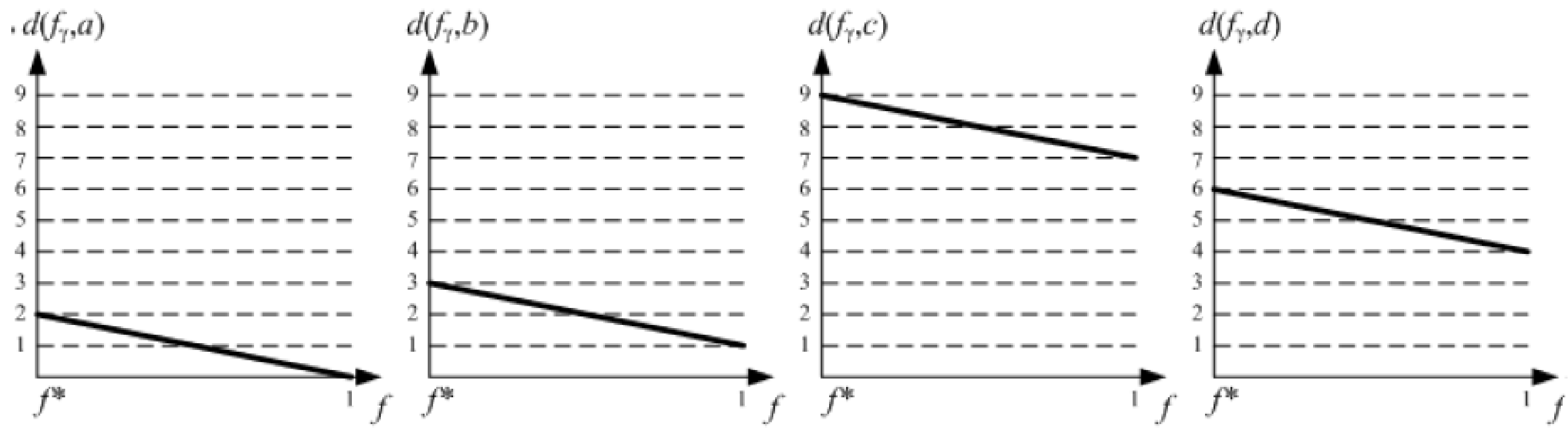


	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$d(f_\gamma, a) = (1-f)c_\gamma + d(a,a) = 2(1-f) + 0 = 2 - 2f;$$
$$d(f_\gamma, b) = (1-f)c_\gamma + d(a,b) = 2(1-f) + 1 = 3 - 2f;$$
$$d(f_\gamma, c) = (1-f)c_\gamma + d(a,c) = 2(1-f) + 7 = 9 - 2f;$$
$$d(f_\gamma, d) = (1-f)c_\gamma + d(a,d) = 2(1-f) + 4 = 6 - 2f;$$

Point-vertex distance

Example:



Vertex-point distance

The **vertex-point distance** between a vertex k and a point $f_{(i,j)}$ (notation $d(k, f_{(i,j)})$) is the weight of the minimal path $<k, f_{(i,j)}>$.

For an undirected edge ij:

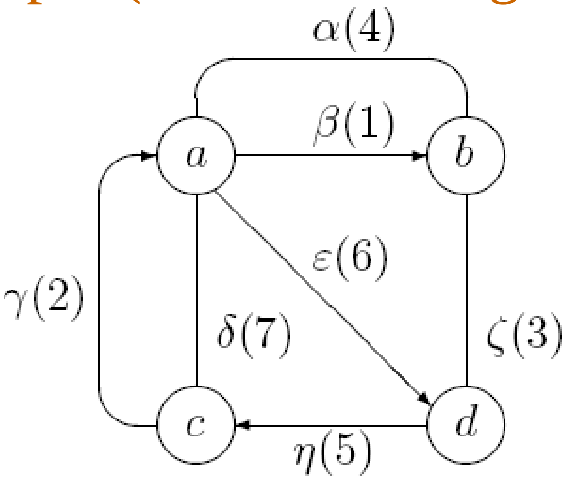
$$d(k, f_{(i,j)}) = \min\{d(k,i) + fc_{i,j}, d(k,j) + (1-f)c_{i,j}.\}$$

For a directed edge ij:

$$d(k, f_{(i,j)}) = d(k,i) + fc_{i,j}.$$

Vertex-point distance

Example (undirected edges):

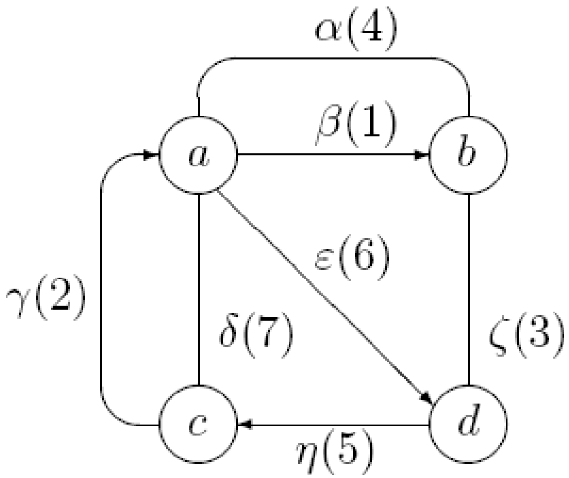


	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$\alpha = (a,b): \quad d(a, f_\alpha) = \min\{d(a,a) + f c_\alpha, d(a,b) + (1-f)c_\alpha\} = \min\{0 + 4f, 1 + 4(1-f)\};$$
$$\delta = (a,c): \quad d(a, f_\delta) = \min\{d(a,a) + f c_\delta, d(a,c) + (1-f)c_\delta\} = \min\{0 + 7f, 7 + 7(1-f)\};$$
$$\zeta = (b,d): \quad d(a, f_\zeta) = \min\{d(a,b) + f c_\zeta, d(a,d) + (1-f)c_\zeta\} = \min\{1 + 3f, 1 + 3(1-f)\}.$$

Vertex-point distance

Example (directed edges):



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$\beta = (a,b): \quad d(a, f_\beta) = d(a,a) + f c_\beta = 0 + f;$$
$$\gamma = (c,a): \quad d(a, f_\gamma) = d(a,c) + f c_\gamma = 7 + 2f;$$
$$\varepsilon = (a,d): \quad d(a, f_\varepsilon) = d(a,a) + f c_\varepsilon = 0 + 6f;$$
$$\eta = (d,c): \quad d(a, f_\eta) = d(a,d) + f c_\eta = 4 + 5f;$$

Vertex-edge distance

The **vertex-edge distance** between a vertex k and an edge ij (notation $d(k,(i,j))$) is the maximum vertex-point distance $d(k, f_{(i,j)})$:

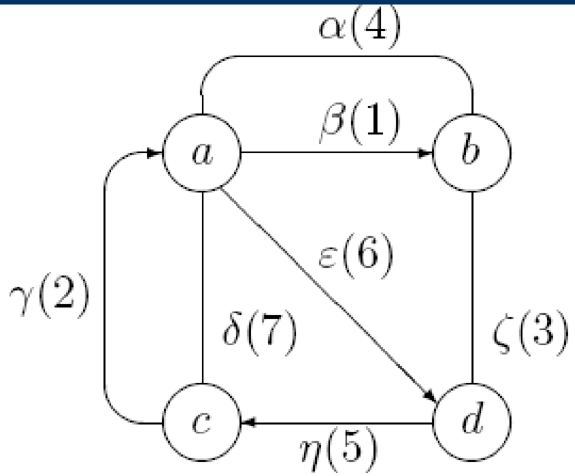
$$d(k,(i,j)) = \max_{f \in [0,1]} d(k, f_{(i,j)}).$$

For a directed edge (i,j) the maximum point $f^*=1$ and the vertex-edge distance

$$d(k,(i,j)) = d(k,i) + c_{i,j}.$$

Vertex-edge distance

Example (directed edges):

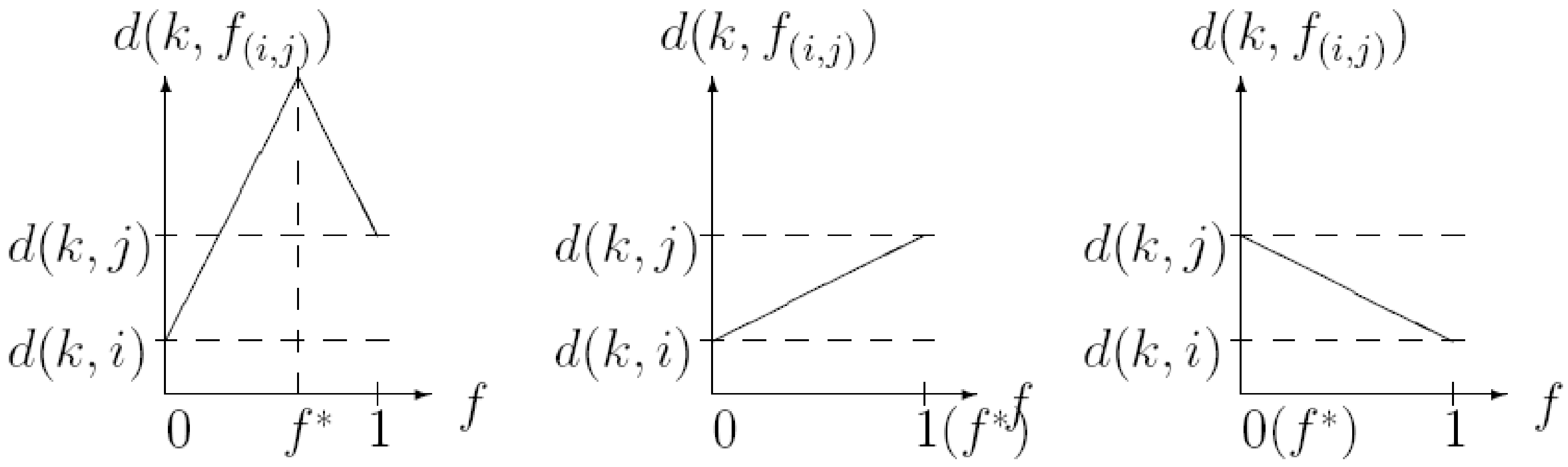


	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$\begin{aligned}
\beta = (a,b):&\quad d(a,\beta) = d(a,a) + c_\beta = 0 + 1 = 1;\\
\gamma = (c,a):&\quad d(a,\gamma) = d(a,c) + c_\gamma = 7 + 2 = 9;\\
\varepsilon = (a,d):&\quad d(a,\varepsilon) = d(a,a) + c_\varepsilon = 0 + 6 = 6;\\
\eta = (d,c):&\quad d(a,\eta) = d(a,d) + c_\eta = 4 + 5 = 9.
\end{aligned}$$

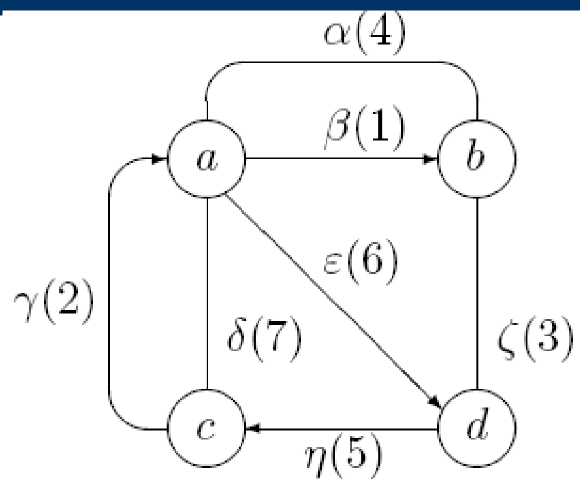
Vertex-edge distance

For an undirected edge (i,j) the dependence $d(k,f_{(i,j)})$ of f can be one of three types.



Vertex-edge distance

Example (undirected edges):



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$\alpha = (a, b): \quad d(a, \alpha) = \frac{d(a,a) + d(a,b) + c_\alpha}{2} = \frac{0+1+4}{2} = 2,5;$$

$$\delta = (a, c): \quad d(a, \delta) = \frac{d(a,a) + d(a,c) + c_\delta}{2} = \frac{0+7+7}{2} = 7;$$

$$\zeta = (b, d): \quad d(a, \zeta) = \frac{d(a,b) + d(a,d) + c_\zeta}{2} = \frac{1+4+3}{2} = 4.$$

Point-edge distance

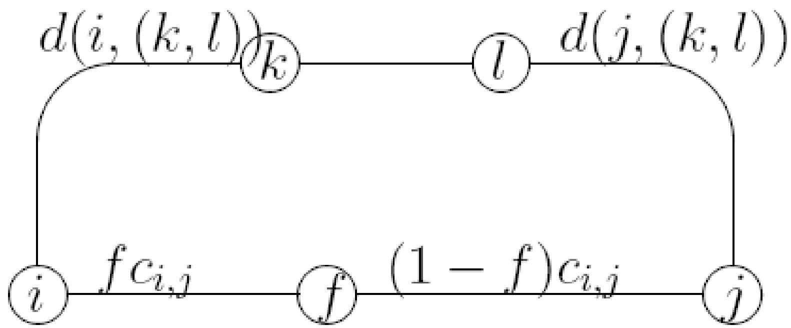
The **point-point distance** between a point $f_{(i,j)}$ and a point $g_{(k,l)}$ (notation $d(f_{(i,j)},g_{(k,l)})$) is the weight of the minimal path $<f_{(i,j)},g_{(k,l)}>$.

The **point-edge distance** between a point $f_{(i,j)}$ and an edge (k,l) (notation $d(f_{(i,j)},(k,l))$) is the maximum point-point distance $d(f_{(i,j)},g_{(k,l)})$:

$$d(f_{(i,j)},(k,l)) = \max_{g\in[0,1]} d(f_{(i,j)},g_{(k,l)}).$$

Point-edge distance

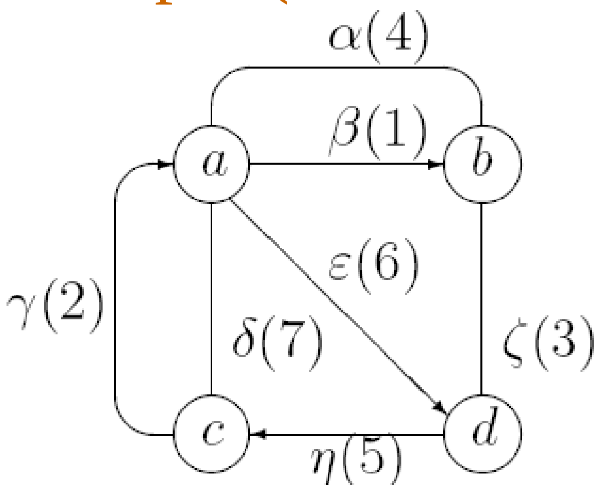
For an undirected edge $(i,j) \neq (k,l)$ the minimal path can pass through the vertex i or the vertex j:



$$d(f_{(i,j)},(k,l)) = \min\{fc_{i,j} + d(i,(k,l)), (1-f)c_{i,j} + d(j,(k,l))\}.$$

Point-edge distance

Example (undirected edge):



	α	β	γ	δ	ε	ζ	η
a	2, 5	1	9	7	6	4	9
b	4	5	10	9, 5	10	3	8
c	4, 5	3	2	4, 5	8	6	11
d	7	8	7	9, 5	13	3	5

$$\begin{aligned} d(f_\delta, \eta) &= \min\{f c_\delta + d(a,\eta), (1-f)c_\delta + d(c,\eta)\} = \\ &= \min\{7f + 9, 7(1-f) + 11\}\,. \end{aligned}$$

$$7f + 9 = 18 - 7f, \qquad f^* = 9/14.$$

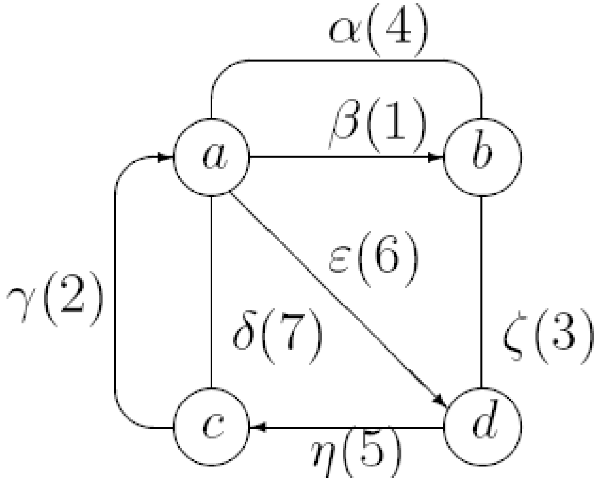
Point-edge distance

For a directed edge (i,j)$\neq$(k,l) the minimal path can pass only through the vertex j:

$$d(f_{(i,j)},(k,l)) = (1-f)c_{i,j} + d(j,(k,l)).$$

Point-edge distance

Example (directed edge):

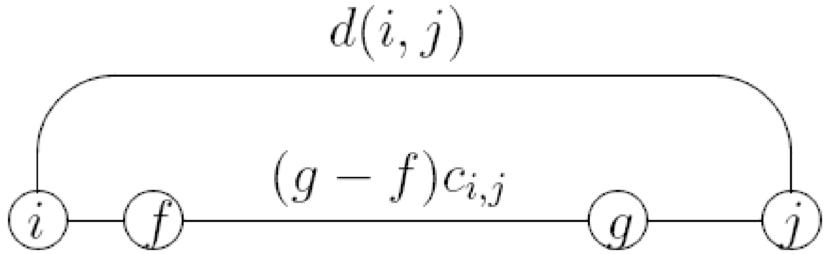


	α	β	γ	δ	ε	ζ	η
a	2, 5	1	9	7	6	4	9
b	4	5	10	9, 5	10	3	8
c	4, 5	3	2	4, 5	8	6	11
d	7	8	7	9, 5	13	3	5

$$d(f_\gamma, \eta) = (1 - f)c_\gamma + d(a, \eta) = 2(1 - f) + 9 = 11 - 2f.$$

Point-edge distance

For an undirected edge $(i,j)=(k,l)$ and $f<1/2$ the most distant points g are close to the vertex j. If $d(i,j)<c_{i,j}$ then the minimal path $\langle f_{(i,j)},g_{(i,j)}\rangle$ can pass through the vertex i:



$$d(f_{(i,j)}, g_{(k,l)}) = \min\{(g-f)c_{i,j}, d(i,j) + (1-g+f)c_{i,j}\}$$

Point-edge distance

The maximum point g^* is the point of the lines intersection:

$$(g - f)c_{i,j} = d(j,i) + (1 - g + f)c_{i,j}.$$

Hence $$\max_g d(f_{(i,j)}, g_{(i,j)}) = \frac{d(i,j) + c_{i,j}}{2}$$

Point-edge distance

If the minimal path $<f_{(i,j)},g_{(i,j)}>$ passes only through the edge (i,j) then:

$$d(f_{(i,j)}, g_{(i,j)}) = (g - f)c_{i,j}.$$

The maximum point $g^*=1$.

$$\max_g d(f_{(i,j)}, g_{(i,j)}) = (1 - f)c_{i,j}.$$

Point-edge distance

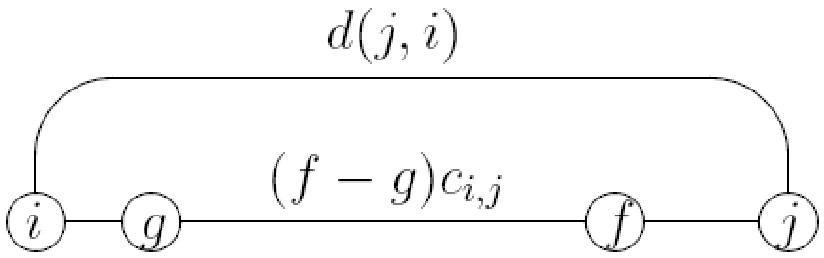
Hence the point-edge distance for $f<1/2$

$$d(f_{(i,j)}, (i,j)) = \min\left\{(1-f)c_{i,j}, \frac{d(i,j)+c_{i,j}}{2}\right\}$$

This distance is maximum for $f=0$ and minimum for $f=1/2$. The minimum distance is equal to $c_{i,j}/2$.

Point-edge distance

For an undirected edge $(i,j)=(k,l)$ and $f>1/2$ the most distant points g are close to the vertex i. If $d(j,i)<c_{j,i}$ then the minimal path $<f_{(i,j)},g_{(i,j)}>$ can pass through the vertex j:



$$d(f_{(i,j)}, g_{(k,l)}) = \min\{(f-g)c_{i,j}, d(j,i) + (1-f+g)c_{i,j}\}$$

Point-edge distance

The maximum point g^* is the point of the lines intersection:

$$(f - g)c_{i,j} = d(j, i) + (1 - g + f)c_{i,j}.$$

Hence

$$\max_g d(f_{(i,j)}, g_{(i,j)}) = \frac{d(j, i) + c_{i,j}}{2}$$

Point-edge distance

If the minimal path <$f_{(i,j)}$,$g_{(i,j)}$> passes only through the edge (i,j) then:

$$d(f_{(i,j)}, g_{(i,j)}) = (f - g)c_{i,j}.$$

The maximum point $g^*=0$.

$$\max_{g} d(f_{(i,j)}, g_{(i,j)}) = f c_{i,j}.$$

Point-edge distance

Hence the point-edge distance for $f>1/2$

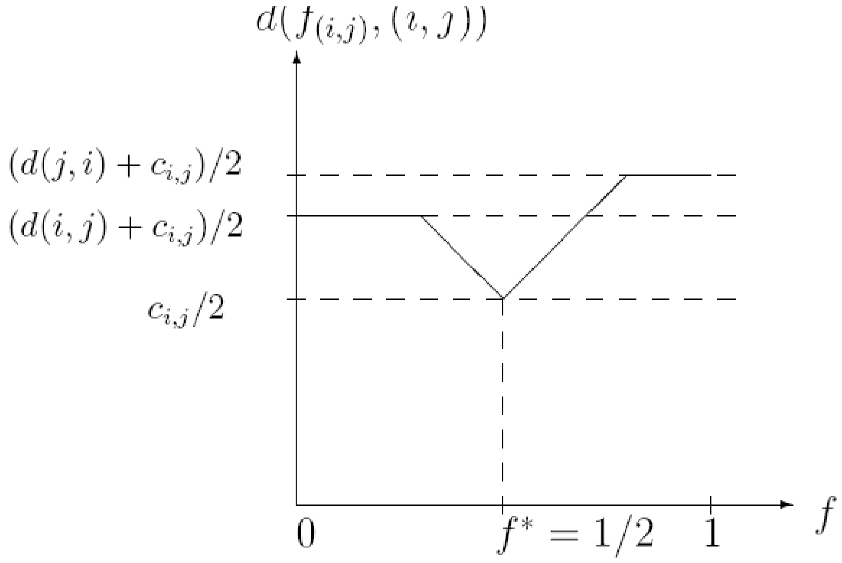
$$d(f_{(i,j)},(i,j)) = \min\left\{ f c_{i,j}, \frac{d(j,i)+c_{i,j}}{2} \right\}$$

This distance is maximum for $f=1$ and minimum for $f=1/2$. The minimum distance is equal to $c_{i,j}/2$.

Point-edge distance

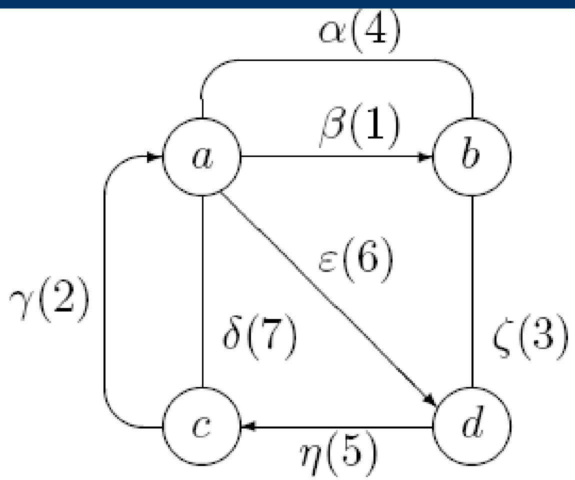
Finally, the point-edge distance is

$$\max\left\{\min\left\{(1-f)c_{i,j}, \frac{d(i,j)+c_{i,j}}{2}\right\}, \min\left\{f c_{i,j}, \frac{d(j,i)+c_{i,j}}{2}\right\}\right\}.$$



Point-edge distance

Example (undirected edges):



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$d(f_\delta, \delta) =$$

$$= \max\left\{\min\left\{(1-f)c_\delta, \frac{d(a,c)+c_\delta}{2}\right\}, \min\left\{f c_\delta, \frac{d(c,a)+c_\delta}{2}\right\}\right\} =$$

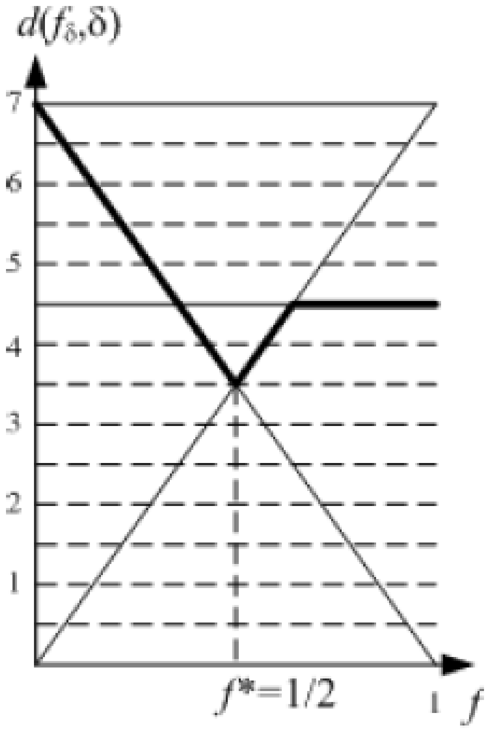
$$= \max\left\{\min\left\{7(1-f), \frac{7+7}{2}\right\}, \min\left\{7f, \frac{2+7}{2}\right\}\right\} =$$

$$= \max\left\{\min\left\{7-7f, 7\right\}, \min\left\{7f, 4,5\right\}\right\}.$$

Point-edge distance

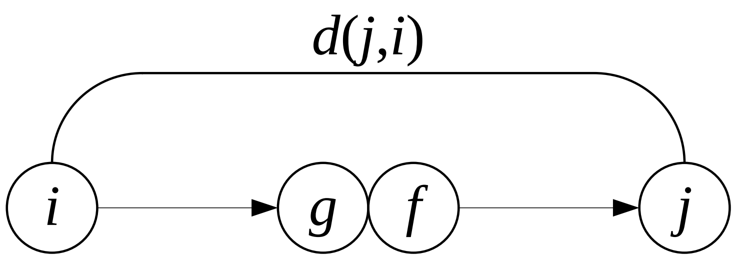
Example (undirected edges):

$$\max\{\min\{7-7f,7\},\min\{7f,4,5\}\}$$



Point-edge distance

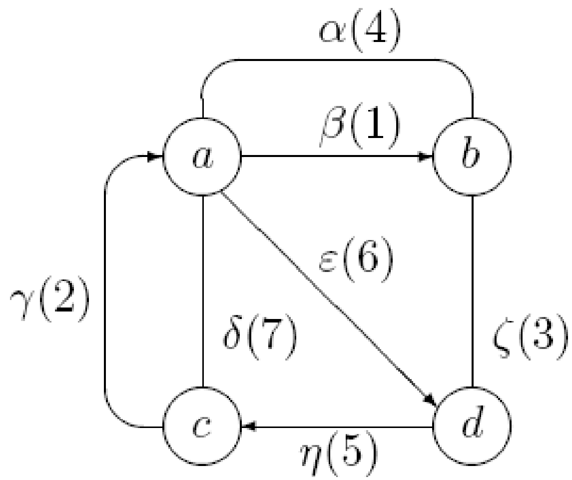
For a directed edge $(i,j)=(k,l)$ the most distant points g are situated between the vertex i and the point f close to the point f .



$$d(f_{(i,j)},(i,j)) = d(j,i) + c_{i,j}.$$

Point-edge distance

Example (directed edges):



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$d(f_\gamma, \gamma) = d(a, c) + c_\gamma = 7 + 2 = 9.$$

Maximum distances

Maximum vertex-vertex: $\mathrm{MVV}(i) = \max_{j}\{d(i, j)\}.$

Maximum point-vertex: $\mathrm{MPV}(f_{(i,j)}) = \max_{k}\{d(f_{(i,j)}, k)\}.$

Maximum vertex-edge: $\mathrm{MVE}(i) = \max_{(k,l)}\{d(i, (k, l))\}.$

Maximum point-edge: $\mathrm{MPE}(f_{(i,j)}) = \max_{(k,l)}\{d(f_{(i,j)}, (k, l))\}.$

Total distances

Total vertex-vertex: $$\mathrm{TVV}(i) = \sum_j \{d(i,j)\}.$$

Total point-vertex: $$\mathrm{TPV}(f_{(i,j)}) = \sum_k \{d(f_{(i,j)}, k)\}.$$

Total vertex-edge: $$\mathrm{TVE}(i) = \sum_{(k,l)} \{d(i,(k,l))\}.$$

Total point-edge: $$\mathrm{TPE}(f_{(i,j)}) = \sum_{(k,l)} \{d(f_{(i,j)},(k,l))\}.$$

4.2. Centers of a graph

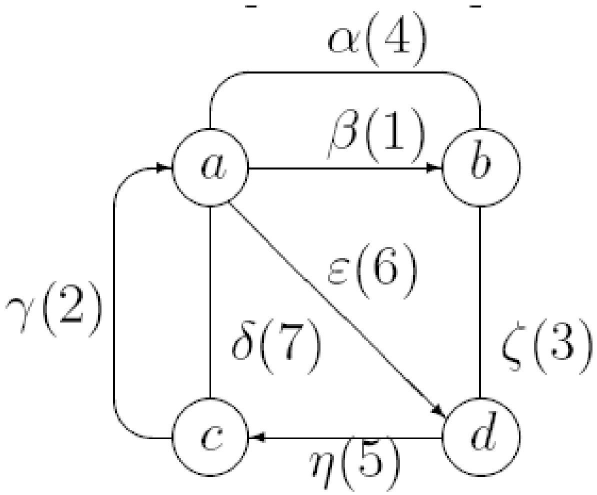
- Center
- General center
- Absolute center
- General absolute center

Center

A **center** of graph G is any vertex v of graph G such that

$$\mathrm{MVV}(v) = \min_j \mathrm{MVV}(j).$$

Example. Vertex *c* is the center.



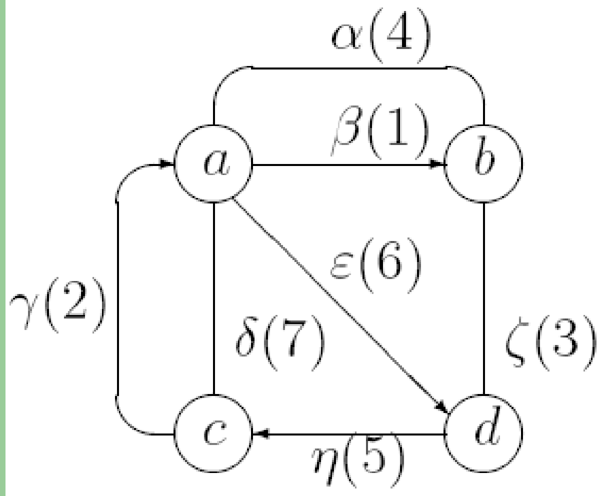
	a	b	c	d	$\mathrm{MVV}(v)$	
a	0	1	7	4	7	
b	4	0	8	3	8	
c	2	3	0	6	6	min
d	7	3	5	0	7	

General center

A **general center** of graph G is any vertex v of graph G such that

$$\mathrm{MVE}(v) = \min_j \mathrm{MVE}(j).$$

Example. Vertex *a* is the general center.



	α	β	γ	δ	ε	ζ	η	MVE(v)	
a	2, 5	1	9	7	6	4	9	9	min
b	4	5	10	9, 5	10	3	8	10	
c	4, 5	3	2	4, 5	8	6	11	11	
d	7	8	7	9, 5	13	3	5	13	

Absolute center

An **absolute center** of graph *G* is any point *g* of graph *G* such that

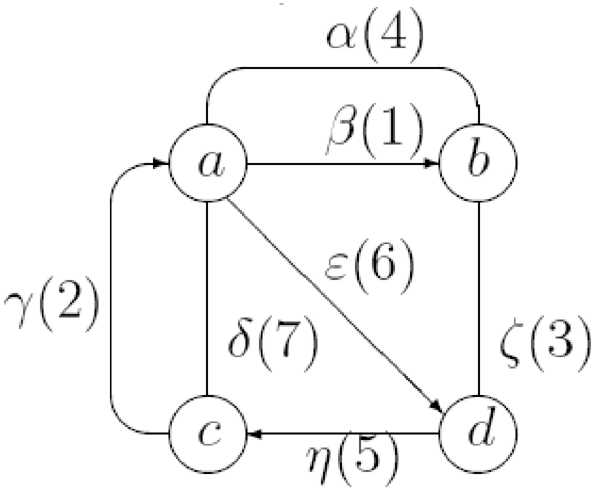
$$\mathrm{MPV}(g_{(v,u)}) = \min_{f_{(i,j)}} \mathrm{MPV}(f_{(i,j)}).$$

Theorem. No interior point of a directed edge can be an absolute center.

Point *f** of an undirected edge can be a candidate for absolute center if it is gives the minimal value of the upper portion of the point-vertex distance from point *f** to all the vertices.

Absolute center

Example.



	a	b	c	d	$\mathrm{MVV}(v)$
a	0	1	7	4	7
b	4	0	8	3	8
c	2	3	0	6	6
d	7	3	5	0	7

Absolute center

Example. Edge δ=(a,c).

$$\begin{aligned} d(f_\delta, a) &= \min\{f c_\delta + d(a,a), (1-f)c_\delta + d(c,a)\} = \\ &= \min\{7f + 0, 7(1-f) + 2\}\,; \\ d(f_\delta, b) &= \min\{f c_\delta + d(a,b), (1-f)c_\delta + d(c,b)\} = \\ &= \min\{7f + 1, 7(1-f) + 3\}\,; \\ d(f_\delta, c) &= \min\{f c_\delta + d(a,c), (1-f)c_\delta + d(c,c)\} = \\ &= \min\{7f + 7, 7(1-f) + 0\}\,; \\ d(f_\delta, d) &= \min\{f c_\delta + d(a,d), (1-f)c_\delta + d(c,d)\} = \\ &= \min\{7f + 4, 7(1-f) + 6\}\,. \end{aligned}$$

Absolute center

Example. Edge α=(a,b).

$$
\begin{aligned}
d(f_\alpha, a) &= \min\{f c_\alpha + d(a,a), (1-f)c_\alpha + d(b,a)\} = \\
&= \min\{4f + 0, 4(1-f) + 4\}\,; \\
d(f_\alpha, b) &= \min\{f c_\alpha + d(a,b), (1-f)c_\alpha + d(b,b)\} = \\
&= \min\{4f + 1, 4(1-f) + 0\}\,; \\
d(f_\alpha, c) &= \min\{f c_\alpha + d(a,c), (1-f)c_\alpha + d(b,c)\} = \\
&= \min\{4f + 7, 4(1-f) + 8\}\,; \\
d(f_\alpha, d) &= \min\{f c_\alpha + d(a,d), (1-f)c_\alpha + d(b,d)\} = \\
&= \min\{4f + 4, 4(1-f) + 3\}\,.
\end{aligned}
$$

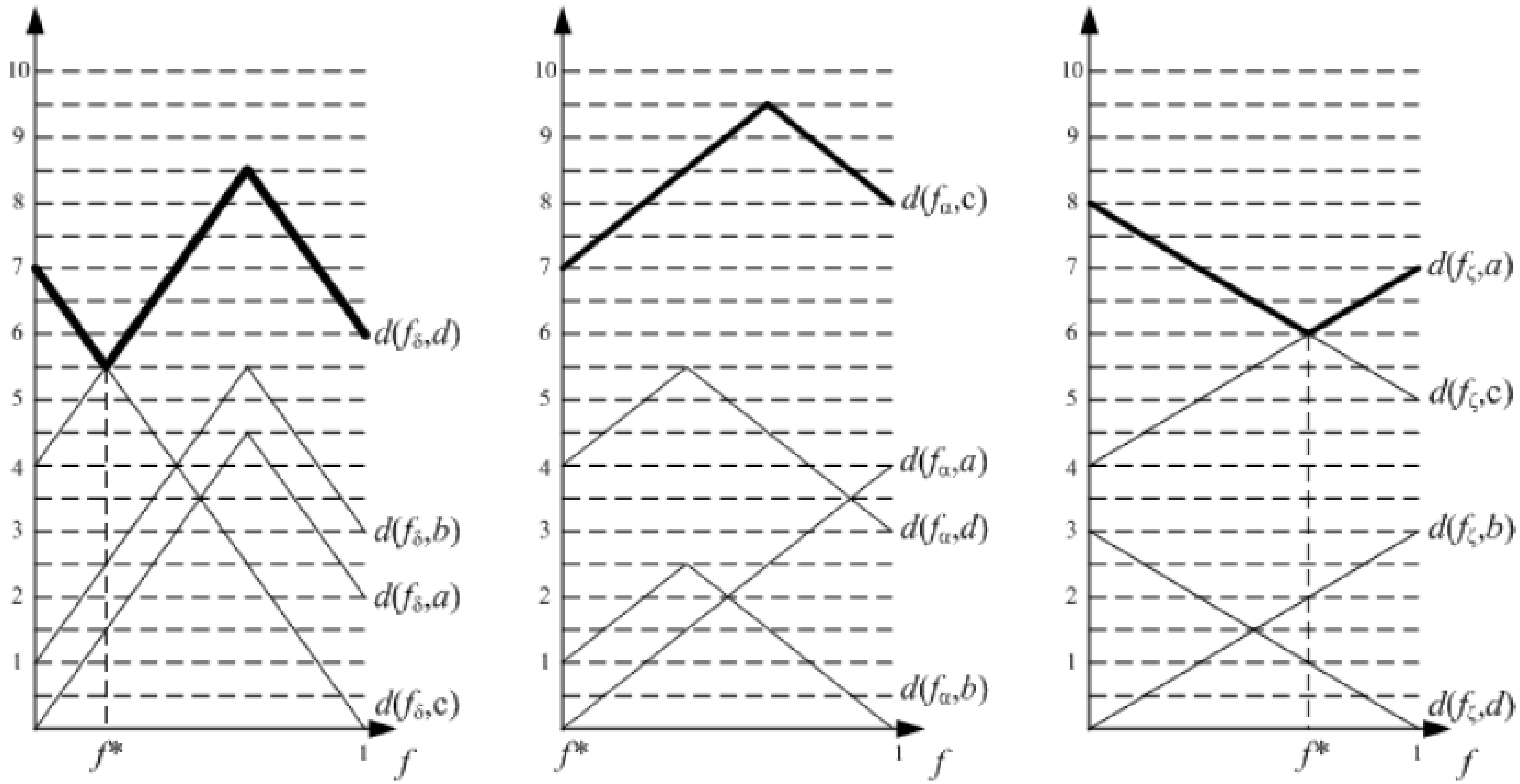
Absolute center

Example. Edge ζ=(b,d).

$$
\begin{aligned}
d(f_\zeta, a) &= \min\left\{f c_\zeta + d(b,a), (1-f)c_\zeta + d(d,a)\right\} = \\
&= \min\left\{3f + 4, 3(1-f) + 7\right\}; \\
d(f_\zeta, b) &= \min\left\{f c_\zeta + d(b,b), (1-f)c_\zeta + d(d,b)\right\} = \\
&= \min\left\{3f + 0, 3(1-f) + 3\right\}; \\
d(f_\zeta, c) &= \min\left\{f c_\zeta + d(b,c), (1-f)c_\zeta + d(d,c)\right\} = \\
&= \min\left\{3f + 8, 3(1-f) + 5\right\}; \\
d(f_\zeta, d) &= \min\left\{f c_\zeta + d(b,d), (1-f)c_\zeta + d(d,d)\right\} = \\
&= \min\left\{3f + 3, 3(1-f) + 0\right\}.
\end{aligned}
$$

Absolute center

Example. Plots of point-vertex distances.



Absolute center

Example.

For edge δ=(a,c):

$$7 - 7f = 4 + 7f,$$

$$f^* = 3/14, \quad \mathrm{MPV}(f^*_\delta) = 7 - 7 \times 3/14 = 5,5.$$

For edge α=(a,b): f*=0 (vertex a).

For edge ζ=(b,d):

$$8 - 3f = 4 + 3f,$$

$$f^* = 2/3, \quad \mathrm{MPV}(f^*_\delta) = 8 - 3 \times 2/3 = 6.$$

Absolute center: point $3/14_{\delta}$, MPV($3/14_{\delta}$)=5,5.

General absolute center

An **general absolute center** of graph *G* is any point *g* of graph *G* such that

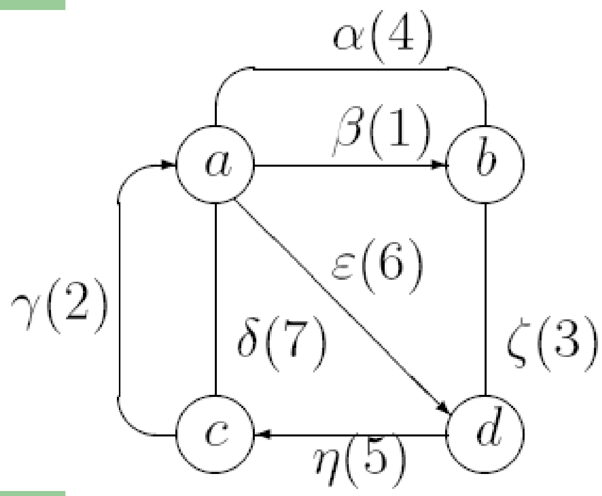
$$\mathrm{MPE}(g_{(v,u)}) = \min_{f_{(i,j)}} \mathrm{MPE}(f_{(i,j)}).$$

Theorem. If an interior point of a directed edge is a general absolute center then its end is also a general absolute center.

Point *f** of an undirected edge can be a candidate for general absolute center if it is gives the minimal value of the upper portion of the point-edge distance from point *f** to all the edges.

General absolute center

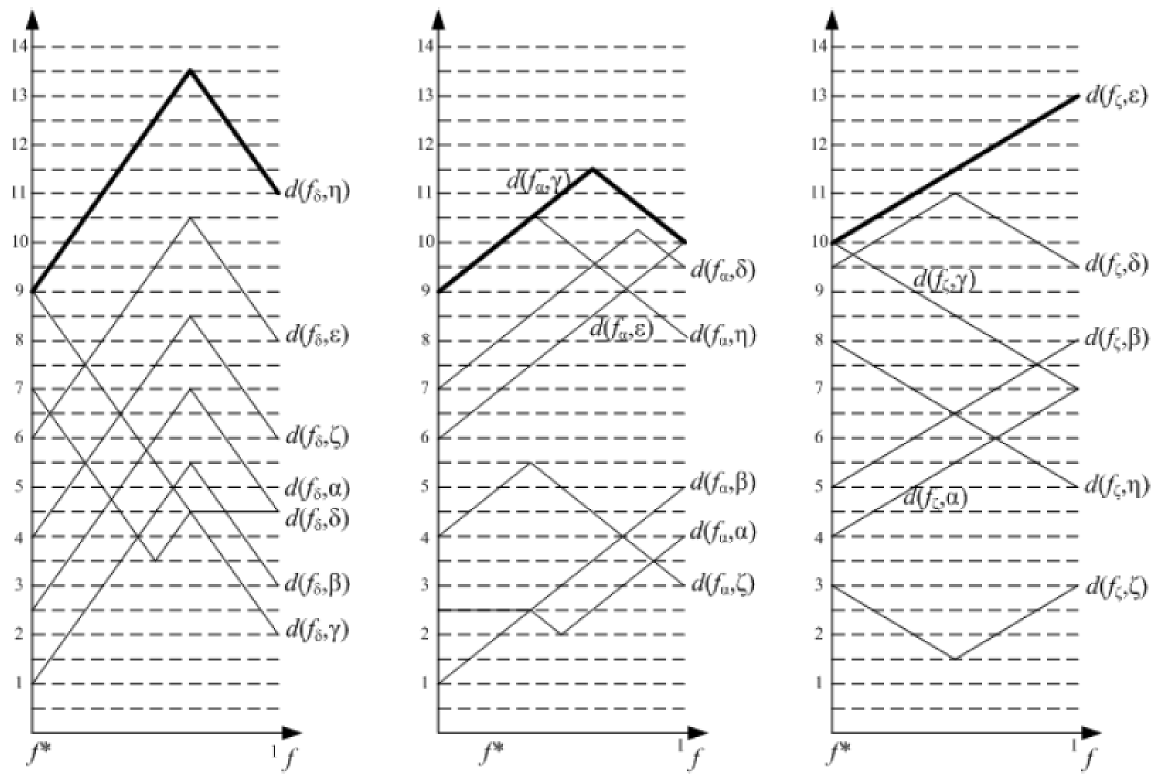
Example.



	α	β	γ	δ	ε	ζ	η	$\mathrm{MVE}(v)$	
a	2, 5	1	9	7	6	4	9	9	min
b	4	5	10	9, 5	10	3	8	10	
c	4, 5	3	2	4, 5	8	6	11	11	
d	7	8	7	9, 5	13	3	5	13	

General absolute center

Example. Plots of point-edge distances. Vertex *a* is the general absolute center.



4.3. Medians of a graph

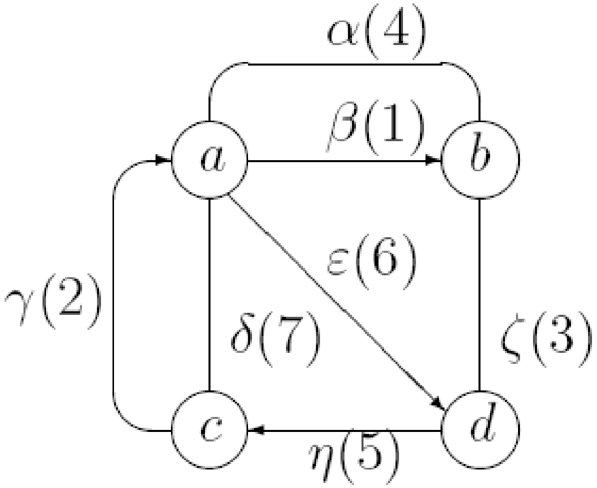
- Median
- General median
- Absolute median
- General absolute median

Median

A **median** of graph G is any vertex v of graph G such that

$$\mathrm{TVV}(v) = \min_j \mathrm{TVV}(j).$$

Example. Vertex *c* is the median.



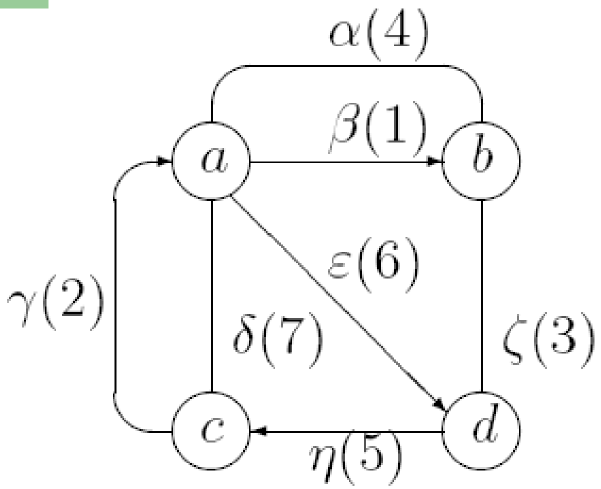
	a	b	c	d	TVV(v)	
a	0	1	7	4	12	
b	4	0	8	3	15	
c	2	3	0	6	11	min
d	7	3	5	0	15	

General median

A **general median** of graph G is any vertex v of graph G such that

$$\mathrm{TVE}(v) = \min_j \mathrm{TVE}(j).$$

Example. Vertex *a* is the general median.



	α	β	γ	δ	ε	ζ	η	TVE(v)	
a	2,5	1	9	7	6	4	9	38,5	min
b	4	5	10	9,5	10	3	8	49,5	
c	4,5	3	2	4,5	8	6	11	39	
d	7	8	7	9,5	13	3	5	52,5	

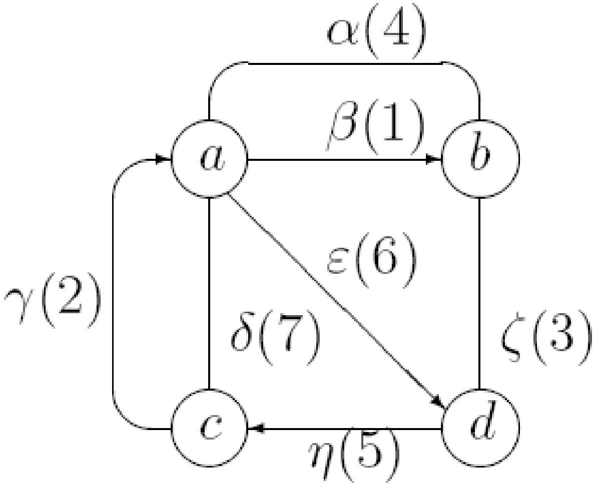
Absolute median

An **absolute median** of graph G is any point g of graph G such that

$$\mathrm{TPV}(g_{(v,u)}) = \min_{f_{(i,j)}} \mathrm{TPV}(f_{(i,j)}).$$

Theorem. There is always a vertex that is an absolute median.

Example. Vertex *c* is the median and the absolute median.



	a	b	c	d	$\mathrm{TVV}(v)$	
a	0	1	7	4	12	
b	4	0	8	3	15	
c	2	3	0	6	11	min
d	7	3	5	0	15	

General absolute median

A **general absolute median** of graph G is any point g of graph G such that

$$\mathrm{TPE}(g_{(v,u)}) = \min_{f_{(i,j)}} \mathrm{TPE}(f_{(i,j)}).$$

Theorem. No interior point of a directed edge can be a general absolute median.

Theorem. There is always a vertex or the middle point of an undirected edge that is a general absolute median.

General absolute median

$$(i, j) \neq (k, l)$$

$$d((1/2)_{(i,j)}, (k, l)) = \frac{1}{2} c_{i,j} + \min\{d(i, (k, l)), d(j, (k, l))\}.$$

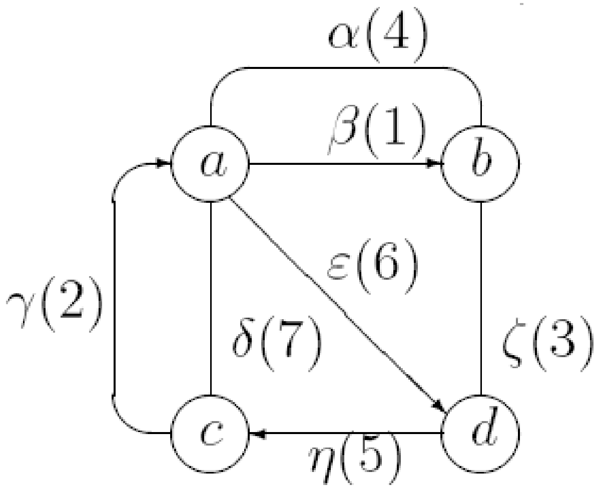
$$(i, j) = (k, l)$$

$$d((1/2)_{(i,j)}, (i, j)) = \frac{1}{2} c_{i,j}.$$

$$\mathrm{TPE}\left(\frac{1}{2}_{(i,j)}\right) = \frac{q}{2} c_{i,j} + \sum_{(k,l) \neq (i,j)} \min\{d(i, (k, l)), d(j, (k, l))\}.$$

General absolute median

Example.



	α	β	γ	δ	ε	ζ	η	TVE(v)	
a	2,5	1	9	7	6	4	9	38,5	min
b	4	5	10	9,5	10	3	8	49,5	
c	4,5	3	2	4,5	8	6	11	39	
d	7	8	7	9,5	13	3	5	52,5	

General absolute median

Example.

$$\mathrm{TPE}\left(\frac{1}{2}_{\alpha}\right) = \frac{7}{2}c_\alpha + \sum_{e\neq\alpha} \min\{d(a,e), d(b,e)\} =$$
$$= \frac{7}{2}4 + \min\{1,5\} + \min\{9,10\} +$$
$$+ \min\{7,\ 9,5\} + \min\{6,10\} + \min\{4,3\} + \min\{9,8\} =$$
$$= 14 + 1 + 9 + 7 + 6 + 3 + 9 = 49;$$
$$\mathrm{TPE}\left(\frac{1}{2}_{\delta}\right) = \frac{7}{2}c_\delta + \sum_{e\neq\delta} \min\{d(a,e), d(c,e)\} =$$
$$= \frac{7}{2}7 + \min\{2,5,4,5\} + \min\{1,3\} +$$
$$+ \min\{9,2\} + \min\{6,8\} + \min\{4,6\} + \min\{9,11\} =$$
$$= 24,5 + 2,5 + 1 + 2 + 6 + 4 + 9 = 49;$$

General absolute median

Example

$$\mathrm{TPE}\left(\frac{1}{2}\zeta\right) = \frac{7}{2}c_\zeta + \sum_{e \neq \zeta} \min\{d(b,e), d(d,e)\} =$$
$$= \frac{7}{2}3 + \min\{4,7\} + \min\{5,8\} +$$
$$+ \min\{10,7\} + \min\{9,5,\ 9,5\} + \min\{10,13\} + \min\{8,5\} =$$
$$= 10,5 + 4 + 5 + 7 + 9,5 + 10 + 5 = 51.$$

Vertex *a* is the general absolute median.

4.4. Extensions

- Weighted location
- Multicentres and multimedians

Weighted location

Suppose that different weights $W(j)$ ($W(i,j)$) are associated with vertex j (edge (i,j)). This weights can be considered as probabilities or frequencies of visiting the vertex or the edge.

Vertex-vertex distance:

$$d^*(i,j) = W(j)d(i,j)$$

Vertex-edge distance:

$$d^*(i,(k,l)) = W(k,l)d(i,(k,l)).$$

Multicentres and multimedians

Let X_r be a subset of points of graph $G(V,E)$ containing r points.

Set-vertex distance $d(X_r,j)$ is the minimum distance between any one of the points in set X_r and vertex j; i.e.

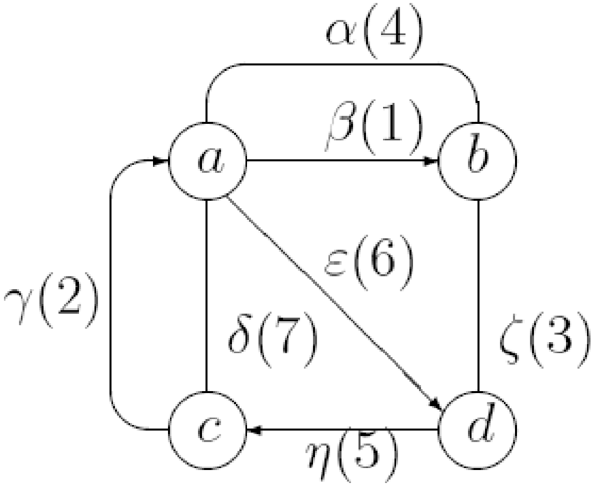
$$d(X_r, j) = \min_{i \in X_r} d(i, j).$$

Set-edge distance $d(X_r,(k,l))$ is the minimum distance between any one of the points in set X_r and edge (k,l), i.e.

$$d(X_r, (k, l)) = \min_{i \in X_r} d(i, (k, l)).$$

Multicentres and multimedians

Example. $X_3 = \{c, (2/7)_\delta, (1/2)_\alpha\}$



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

Multicentres and multimedians

Example.

$$d(c,d) = 6$$

$$\begin{aligned} d\left(\frac{2}{7}\delta, d\right) &= \min\left\{\tfrac{2}{7}c_\delta + d(a,d), \left(1-\tfrac{2}{7}\right)c_\delta + d(c,d)\right\} = \\ &= \min\left\{\frac{2}{7}7+4, \frac{5}{7}7+6\right\} = 6; \\ d\left(\frac{1}{2}\alpha, d\right) &= \min\left\{\tfrac{1}{2}c_\alpha + d(a,d), \left(1-\tfrac{1}{2}\right)c_\alpha + d(b,d)\right\} = \\ &= \min\left\{\frac{1}{2}4+4, \frac{1}{2}4+3\right\} = 5. \end{aligned}$$

$$d(X_3, d) = \min\left\{d(c,d), d\left(\frac{2}{7}\delta, d\right), d\left(\frac{1}{2}\alpha, d\right)\right\} = \min\{6,6,5\} = 5.$$

Multicentres and multimedians

Example.

$$d(c,\eta) = 11$$

$$\begin{aligned} d\left(\frac{2}{7}\delta,\eta\right) &= \min\left\{\tfrac{2}{7}c_\delta + d(a,d\eta), \left(1-\tfrac{2}{7}\right)c_\delta + d(c,\eta)\right\} = \\ &= \min\left\{\frac{2}{7}7+9, \frac{5}{7}7+11\right\} = 11; \\ d\left(\frac{1}{2}\alpha,\eta\right) &= \min\left\{\tfrac{1}{2}c_\alpha + d(a,\eta), \left(1-\tfrac{1}{2}\right)c_\alpha + d(b,\eta)\right\} = \\ &= \min\left\{\frac{1}{2}4+9, \frac{1}{2}4+8\right\} = 10. \end{aligned}$$

$$d(X_3,\eta) = \min\left\{d(c,\eta), d\left(\frac{2}{7}\delta,\eta\right), d\left(\frac{1}{2}\alpha,\eta\right)\right\} = \min\{11, 11, 10\} = 10.$$

Multicentres and multimedians

Multicenter and **multimedian** problems arise when there is a need to locate a number of facilities in the best possible way. The following distances can be minimize:

- Maximum set-vertex distance (MSV)
- Maximum set-edge distance (MSE)
- Total set-vertex distance (TSV)
- Total set-edge distance (TSE)

4.5. Absolute p-centre

Problems:

- (a) Find the optimal location anywhere on the graph of a given number (say p) of centres so that the distance required to reach the most remote vertex from its nearest centre is a minimum.
- (b) For a given "critical" distance, find the smallest number (and location) of centres so that all the vertices of the graph lie within this critical distance from at least one of the centres.

Δ-matrix

Vertex k is **reachable from point $f_{(i,j)}$ within distance Δ** if

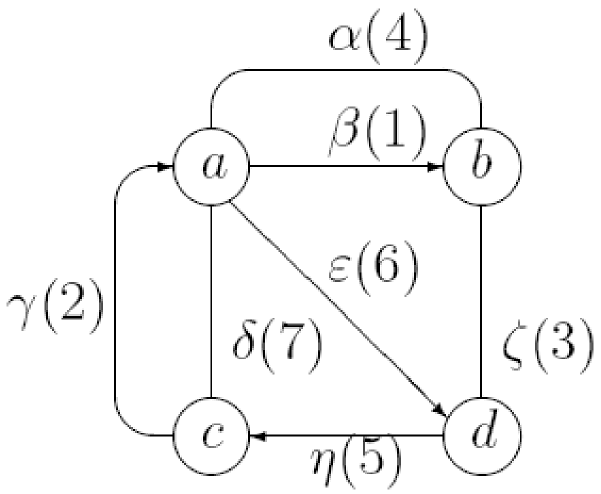
$$d(f_{(i,j)}, k) \leq \Delta.$$

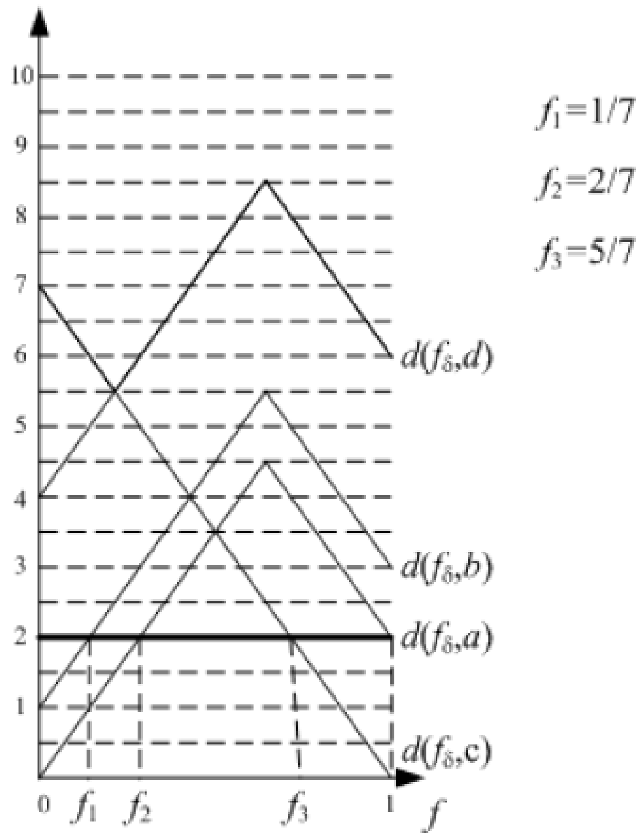
Any set of points of the graph is called a **region**.

Vertex k is **reachable from a region within distance Δ** if it is reachable from any point of the region within distance Δ.

Δ-matrix

Example. Edge δ=(a,c).





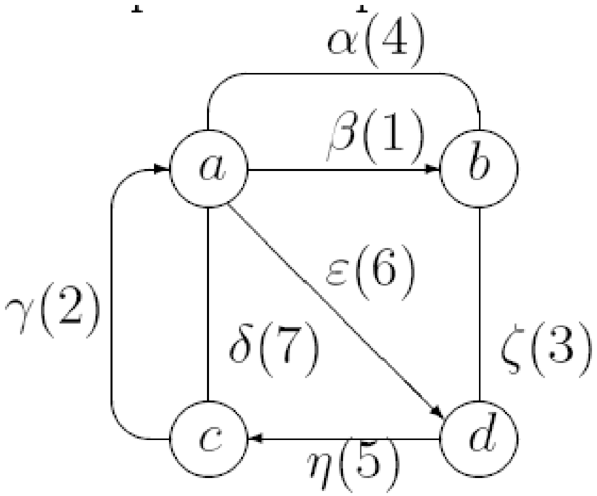
Δ-matrix

Let **Δ-matrix** be a Boolean matrix where rows correspond to regions and columns correspond to vertices. Element Δ_{ij}=1 if vertex *j* is reachable from region *i* within distance Δ.

The shortest cover of Δ-matrix gives us an absolute p-center of the minimum cardinal number *p*.

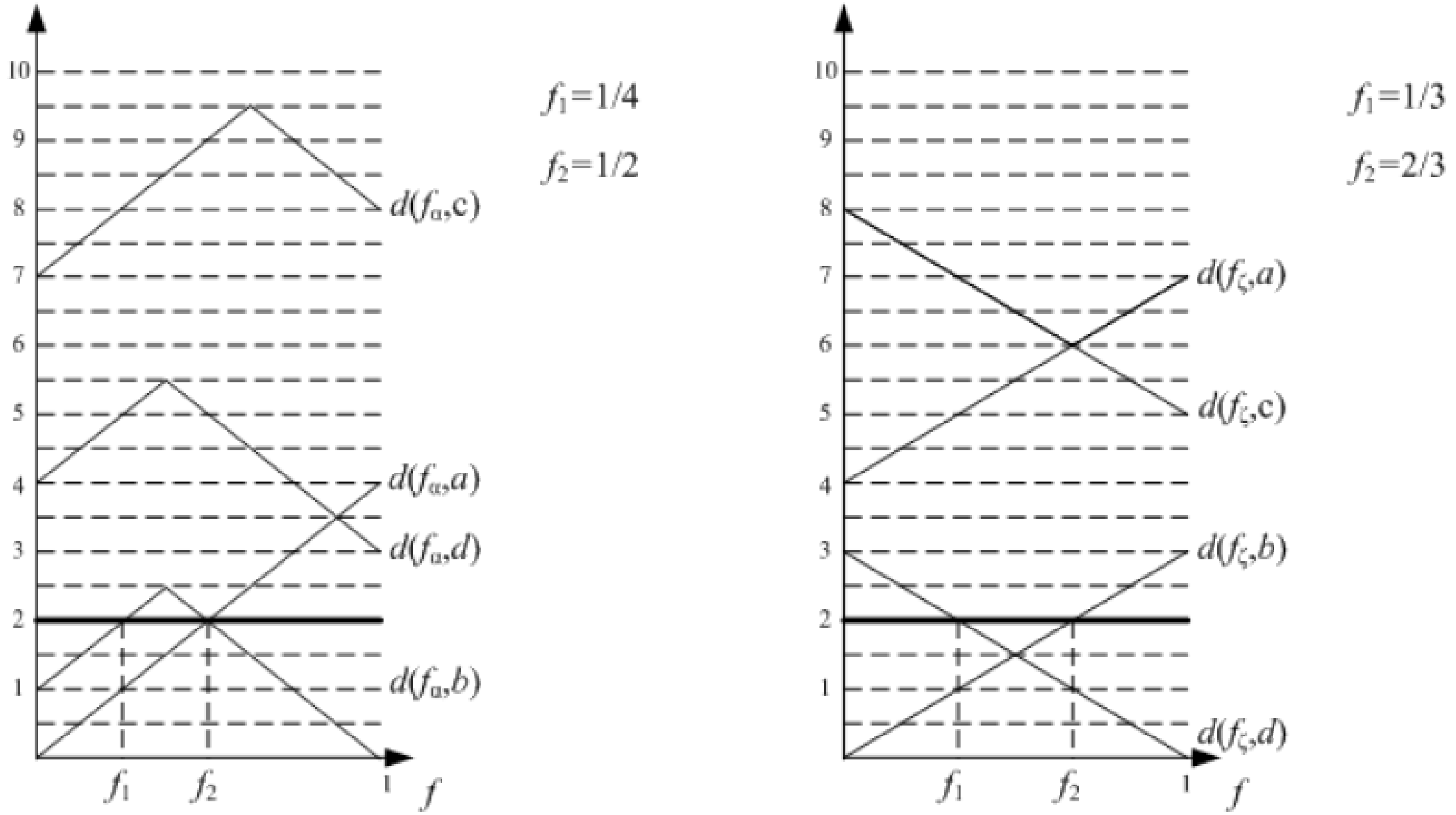
Δ-matrix

Example. Δ=2.



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

Δ-matrix


10
9
8
7
6
5
4
3
2
1
d(fα,c)
d(fα,a)
d(fα,d)
d(fα,b)
f1
f2
1
f
f1=1/4
f2=1/2
10
9
8
7
6
5
4
3
2
1
d(fζ,a)
d(fζ,c)
d(fζ,b)
d(fζ,d)
f1
f2
1
f
f1=1/3
f2=2/3

Δ-matrix

Example.

	a	b	c	d
c	1		1	
$[a, (1/7)_\delta]$	1	1		
$((1/7)_\delta, (2/7)_\delta]$	1			
$[(5/7)_\delta, c)$			1	
$[a, (1/4)_\alpha)$	1	1		
$[(1/4)_\alpha, (1/2)_\alpha)$	1			
$[(1/2)_\alpha, b]$		1		
$[b, (1/3)_\zeta)$		1		
$[(1/3)_\zeta, (2/3)_\zeta]$		1		1
$((2/3)_\zeta, d]$				1

$$X_2 = \{c, f_\zeta\}$$

$$f_\zeta \in [(1/3)_\zeta, (2/3)_\zeta]$$

Shortest cover of a Boolean matrix

Consider Boolean matrix Q.

Row i **covers** column j if $q_{ij}=1$, i.e. if row i contains 1 in column j.

A **cover** of a Boolean matrix is any set of its rows covering all its columns.

The **length** of a cover is the number of rows in the cover.

A **shortest cover** is a cover of the minimal length.

Shortest cover of a Boolean matrix

Example.

Row H covers columns 2, 3, 6, 8.

1	1									A
		1	1							B
				1	1					C
							1	1		D
						1			1	E
				1		1				F
								1	1	G
	1	1			1		1			H
1	2	3	4	5	6	7	8	9	10	

Shortest cover of a Boolean matrix

Example.

Covers: {A,B,C,D,E,F}, {A,B,C,E,G,H}…

Shortest cover: {A,B,C,D,E}

1	1									A
		1	1							B
				1	1					C
							1	1		D
						1			1	E
				1		1				F
								1	1	G
	1	1			1		1			H
1	2	3	4	5	6	7	8	9	10	

Essential row rule

An **essential row** is a row covering a column contained one and only one 1.

Essential row rule. If a Boolean matrix has an essential row hence this row is contained in any cover. An essential row is deleted from the matrix with all the columns covered by the row.

Essential row rule

Example.

Essential rows: A, B.

1	1									A
		1	1							B
				1	1					C
							1	1		D
						1			1	E
				1		1				F
								1	1	G
	1	1			1		1			H
1	2	3	4	5	6	7	8	9	10	

Essential row rule

Example.

After applying the essential row rule.

$$
Q' = \begin{array}{cccccc|c}
\hline
1 & 1 & & & & & C \\
\hline
 & & & 1 & 1 & & D \\
\hline
 & & 1 & & & 1 & E \\
\hline
1 & & 1 & & & & F \\
\hline
 & & & & 1 & 1 & G \\
\hline
 & 1 & & 1 & & & H \\
\hline
5 & 6 & 7 & 8 & 9 & 10 &
\end{array}
$$

Petrick's method

Column cover function is the disjunction of variables corresponding to rows covering the column.

Matrix cover function is the conjunction of all the column cover functions, i.e., conjunctive normal form (CNF).

If the CNF of the matrix cover function is transformed to the disjunctive normal form (DNF) then every conjunction of the DNF gives us a cover of the matrix. The shortest disjunction gives us the shortest cover.

Petrick's method

Example.

$$Q' = \begin{array}{|c|c|c|c|c|c|l} \hline 1 & 1 & & & & & C \\ \hline & & & 1 & 1 & & D \\ \hline & & 1 & & & 1 & E \\ \hline 1 & & 1 & & & & F \\ \hline & & & & 1 & 1 & G \\ \hline & 1 & & 1 & & & H \\ \hline \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & \multicolumn{1}{c}{8} & \multicolumn{1}{c}{9} & \multicolumn{1}{c}{10} & \end{array}$$

$$(C \vee F)(C \vee H)(E \vee F)(D \vee H)(D \vee G)(E \vee G).$$

$$= CDE \vee CEGH \vee CDFG \vee DEFH \vee FGH$$

Predecessor row rule and successor column rule

Boolean vector $\alpha = a_1 a_2 \ldots a_n$ **precedes** Boolean vector $\beta = b_1 b_2 \ldots bn$ if for every $i=1, \ldots, n$:

$$a_i \leq b_i.$$

Here vector α is the **predecessor**, vector β is the **successor**.

Predecessor row rule. If in a Boolean matrix row α precedes row β then predecessor row α is deleted from the matrix. The shortest cover does not lost because in every cover row α can be replaced by row β.

Successor column rule. If in a Boolean matrix column γ precedes column δ then successor column δ is deleted from the matrix. The shortest cover does not lost because every cover of column γ also covers column δ..

Predecessor row rule and successor column rule

Example. $A \preceq C$

$$Q_1'' = \begin{array}{|c|c|c|c|c|c|l}
\hline
 & 1 & & 1 & & & A \\
\hline
1 & 1 & & 1 & & & C \\
\hline
1 & & & & 1 & 1 & D \\
\hline
1 & 1 & & & 1 & & E \\
\hline
 & & 1 & 1 & & 1 & F \\
\hline
 & & 1 & & 1 & 1 & G \\
\hline
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & \multicolumn{1}{c}{8} &
\end{array}
\qquad
Q_2'' = \begin{array}{|c|c|c|c|c|c|l}
\hline
1 & 1 & & 1 & & & C \\
\hline
1 & & & & 1 & 1 & D \\
\hline
1 & 1 & & & 1 & & E \\
\hline
 & & 1 & 1 & & 1 & F \\
\hline
 & & 1 & & 1 & 1 & G \\
\hline
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & \multicolumn{1}{c}{8} &
\end{array}$$

Predecessor row rule and successor column rule

Example. $4 \leq 1$.

$$Q_2'' = \begin{array}{c c}
\left[\begin{array}{cccccc}
1 & 1 &   & 1 &   &   \\
1 &   &   &   & 1 & 1 \\
1 & 1 &   &   & 1 &   \\
  &   & 1 & 1 &   & 1 \\
  &   & 1 &   & 1 & 1
\end{array}\right] & \begin{array}{c} C \\ D \\ E \\ F \\ G \end{array} \\
\begin{array}{cccccc} 1 & 4 & 5 & 6 & 7 & 8 \end{array} &
\end{array}
\qquad
Q_3'' = \begin{array}{c c}
\left[\begin{array}{ccccc}
1 &   & 1 &   &   \\
  &   &   & 1 & 1 \\
1 &   &   & 1 &   \\
  & 1 & 1 &   & 1 \\
  & 1 &   & 1 & 1
\end{array}\right] & \begin{array}{c} C \\ D \\ E \\ F \\ G \end{array} \\
\begin{array}{ccccc} 4 & 5 & 6 & 7 & 8 \end{array} &
\end{array}$$

Algorithm

Step 1. If there is a row covering all columns then add this row to the shortest cover and go to the end.

Step 2. Apply the essential row rule. If there were core rows then go to step 1.

Step 3. Apply the predecessor row rule. If there were predecessor rows then go to step 2.

Step 4. Apply the successor column rule. If there were successor columns then go to step 3.

Step 5. Write the CNF of the matrix cover function and transform it into DNF.

Step 6. Choose the shortest conjunction of the DNF and add core rows to the rows from this conjunction. The shortest cover is obtained.

Algorithm

Example. 8=5.

$$Q''_4 = \begin{array}{|c|c|c|c|c|c} \hline 1 & & 1 & & & C \\ \hline 1 & & & 1 & & E \\ \hline & 1 & 1 & & 1 & F \\ \hline & 1 & & 1 & 1 & G \\ \hline \multicolumn{1}{c}{4} & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & \multicolumn{1}{c}{8} & \end{array} \qquad Q''_5 = \begin{array}{|c|c|c|c|c} \hline 1 & & 1 & & C \\ \hline 1 & & & 1 & E \\ \hline & 1 & 1 & & F \\ \hline & 1 & & 1 & G \\ \hline \multicolumn{1}{c}{4} & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & \end{array}$$

$$(C \vee E)(F \vee G)(C \vee F)(E \vee G) =$$

$$= (C \vee EF)(G \vee EF) = CG \vee EF$$

The shortest cover {B,C,G}.

Direct tree search

Consider Boolean matrix M and row X.

X=1: row X is included into a cover. Delete row X and all columns covered by it, hence obtain matrix M’.

X=0: row X is not included into a cover. Delete row X, hence obtain matrix M’.

Direct tree search

Example. A, B – essential rows.

1	2	3	4	5	6	7	8	9	10	
1	1									A
		1	1							B
				1	1					C
							1	1		D
						1			1	E
				1		1				F
								1	1	G
	1	1			1		1			H

Direct tree search

Example. Choose row C. Matrix N: C=1, matrix P: C=0.

$$
Q = \begin{array}{|c|c|c|c|c|c|l}
\hline
1 & 1 &   &   &   &   & C \\ \hline
  &   &   & 1 & 1 &   & D \\ \hline
  &   & 1 &   &   & 1 & E \\ \hline
1 &   & 1 &   &   &   & F \\ \hline
  &   &   &   & 1 & 1 & G \\ \hline
  & 1 &   & 1 &   &   & H \\ \hline
\multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & \multicolumn{1}{c}{8} & \multicolumn{1}{c}{9} & \multicolumn{1}{c}{10} &
\end{array}
$$

Direct tree search

Example. Choose row C. Matrix N: C=1, matrix P: C=0.

$$N = \begin{array}{|c|c|c|c|l}
\hline
 & 1 & 1 & & D \\ \hline
1 & & & 1 & E \\ \hline
1 & & & & F \\ \hline
 & & 1 & 1 & G \\ \hline
 & 1 & & & H \\ \hline
\multicolumn{1}{c}{7} & \multicolumn{1}{c}{8} & \multicolumn{1}{c}{9} & \multicolumn{1}{c}{10} &
\end{array}$$

$$P = \begin{array}{|c|c|c|c|c|c|l}
\hline
 & & & 1 & 1 & & D \\ \hline
 & & 1 & & & 1 & E \\ \hline
1 & & 1 & & & & F \\ \hline
 & & & & 1 & 1 & G \\ \hline
 & 1 & & 1 & & & H \\ \hline
\multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & \multicolumn{1}{c}{8} & \multicolumn{1}{c}{9} & \multicolumn{1}{c}{10} &
\end{array}$$

Direct tree search

Example. Consider matrix N.

$$N = \begin{array}{c c}
\begin{array}{|c|c|c|c|}
\hline
 & 1 & 1 & \\ \hline
1 & & & 1 \\ \hline
1 & & & \\ \hline
 & & 1 & 1 \\ \hline
 & 1 & & \\ \hline
\end{array} & \begin{array}{c} D \\ E \\ F \\ G \\ H \end{array} \\
\begin{array}{cccc} 7 & 8 & 9 & 10 \end{array} &
\end{array}$$

$$F \preceq E, \; H \preceq D$$

$$N = \begin{array}{c c}
\begin{array}{|c|c|c|c|}
\hline
 & 1 & 1 & \\ \hline
1 & & & 1 \\ \hline
 & & 1 & 1 \\ \hline
\end{array} & \begin{array}{c} D \\ E \\ G \end{array} \\
\begin{array}{cccc} 7 & 8 & 9 & 10 \end{array} &
\end{array}$$

Direct tree search

Example. E – an essential row.

$$N = \begin{array}{|c|c|c|c|l} \cline{1-4} & 1 & 1 & & D \\ \cline{1-4} 1 & & & 1 & E \\ \cline{1-4} & & 1 & 1 & G \\ \cline{1-4} \multicolumn{1}{c}{7} & \multicolumn{1}{c}{8} & \multicolumn{1}{c}{9} & \multicolumn{1}{c}{10} & \end{array} \qquad N = \begin{array}{|c|c|l} \cline{1-2} 1 & 1 & D \\ \cline{1-2} & 1 & G \\ \cline{1-2} \multicolumn{1}{c}{8} & \multicolumn{1}{c}{9} & \end{array}$$

Row D covers all columns. We obtain a cover {A,B,C,D,E}.

Direct tree search

Example. Consider matrix P. F, H – essential rows.

$P =$

	5	6	7	8	9	10
D				1	1	
E			1			1
F	1		1			
G					1	1
H		1		1		

$P =$

	9	10
D	1	
E		1
G	1	1

Row G covers all columns. We obtain a cover {A,B,F,G,H}.

4.6. P-median

Problems:

- (a) Find the optimal location anywhere on the graph of a given number (say p) of medians so that the total distance required to reach all the vertices from its nearest median is a minimum.
- (b) For a given "critical" distance, find the smallest number (and location) of medians so that the total distance required to reach all the vertices from its nearest median lie within this critical distance.

Problem statement

X_p – multimedian (p-median)

$v \in X_p$ – **median vertex**

$v \notin X_p$ – **non-median vertex**

Vertex j is **allocated** to vertex i if vertex i is a median vertex and

$$d(Xp,j)=d(i,j).$$

Any median vertex i is allocated to vertex i itself.

Problem statement

$$\xi_{i,j} = \begin{cases} 1, & \text{if vertex } j \text{ is allocated to vertex } i; \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
&\sum_{i \in V} \sum_{j \in V} d(i,j)\xi_{i,j} \to \min_{\xi_{i,j}}; \\
&\sum_{i \in V} \xi_{i,j} = 1, \quad \forall j \in V; \\
&\sum_{i \in V} \xi_{i,i} = p; \\
&\xi_{i,j} \le \xi_{i,i}, \qquad \forall i \in V,\ j \in V; \\
&\xi_{i,j} \in \{0,1\}, \quad \forall i \in V,\ j \in V.
\end{aligned}$$

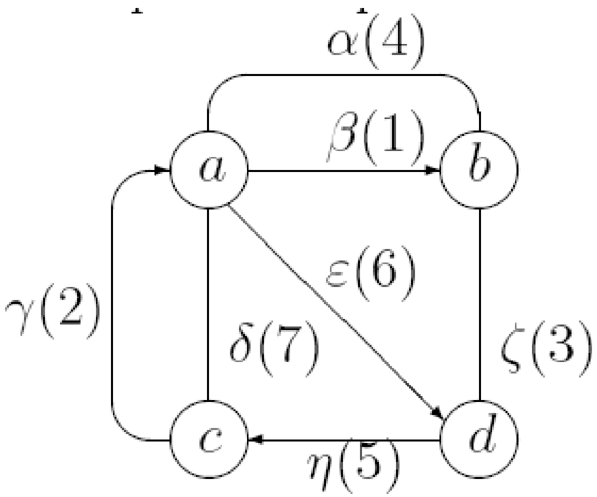
Direct tree search

Set up a matrix M:$n \times n$ the j-th column of which contains all the vertices of the graph G arranged in ascending order of their distance to vertex j. Thus, if m_{ij} = k, then there are i–1 vertices, such that the distance from them to vertex j does not exceed $d(k,j)$ and n–i vertices, such that the distance from them to vertex j is not less than $d(k,j)$.

Obviously, the nearest vertex to vertex j is itself, i.e. m_{1j} =j.

Direct tree search

Example.



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$M = \begin{array}{c|cccc} & a & b & c & d \\ \hline 1 & a & b & c & d \\ 2 & c & a & d & b \\ 3 & b & c & a & a \\ 4 & d & d & b & c \end{array}$$

Direct tree search

For every vertex j we define index k_j as a number of a row of matrix M.

$$x_j = m_{k_j,j}$$

At the subproblem under consideration, vertex x_j is the best variant for vertex j to be allocated to.

A lower bound of the cost of the optimal solution

$$\underline{C} = \sum_{j \in V} d(x_j, j).$$

Direct tree search

For the start problem for every vertex $j \in V$

$$k_j = 1,\ x_j = j,\ d(x_j, j) = 0.$$

An upper bound of the cost of the optimal solution

$$C^* = \min_j \mathrm{TVV}(j).$$

Two new subproblems are generated from the current subproblem choosing variable ξ_{ij} and setting $\xi_{ij} = 1$ and $\xi_{ij} = 0$.

Direct tree search

S^+ – set of median vertices;

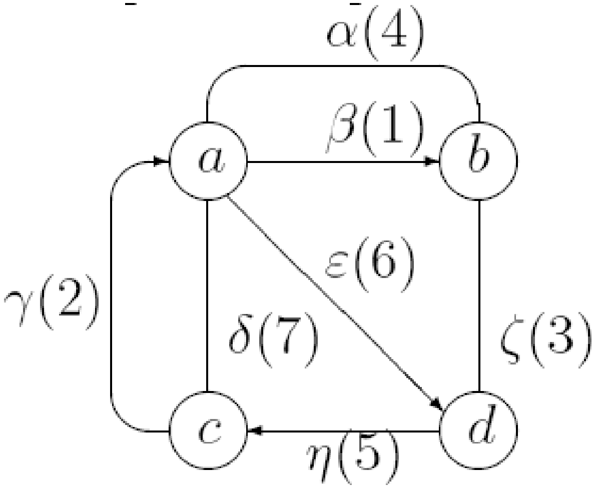
S^- – set of non-median vertices;

F – set of non-allocated vertices.

Every median vertex is allocated to itself, thus, $S^+ \cap F = \varnothing$.

Direct tree search

Example. Start problem A (p=2).



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

Direct tree search

$S+(A)=\varnothing$;

$S-(A)= \varnothing$;

$F(A)=\{a,b,c,d\}$;

$C(A)=0$.

$$M(A) = \begin{array}{c|cccc} & a & b & c & d \\ \hline 1 & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ 2 & c & a & d & b \\ 3 & b & c & a & a \\ 4 & d & d & b & c \end{array}$$

Choose variable ξ_{aa}.

Direct tree search

$$S^+(B) = \{a\}; \quad S^-(B) = \emptyset; \quad F(B) = \{b, c, d\}; \quad \xi_{a,a} = 1.$$

$$M(B) = \begin{array}{c|cccc} & a & b & c & d \\ \hline 1 & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ 2 & c & a & d & b \\ 3 & b & c & a & a \\ 4 & d & d & b & c \end{array} \qquad \underline{C}(B) = 0.$$

$$S^+(C) = \emptyset; \quad S^-(C) = \{a\}; \quad F(C) = \{a, b, c, d\}; \quad \xi_{a,a} = 0.$$

$$M(C) = \begin{array}{c|cccc} & a & b & c & d \\ \hline 1 & a & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ 2 & \mathbf{c} & a & d & b \\ 3 & b & c & a & a \\ 4 & d & d & b & c \end{array} \qquad \underline{C}(C) = 2.$$

Direct tree search