Graph theory glossary

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Topics

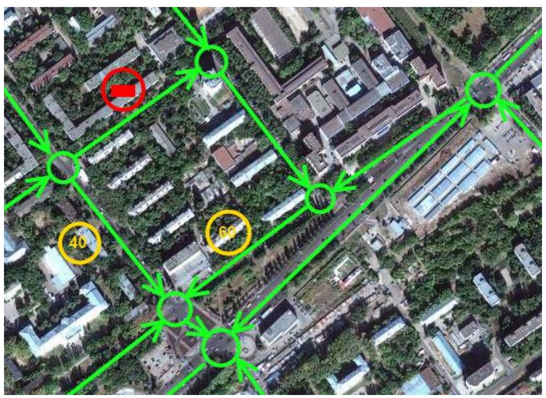
- Basics
- Connectivity
- Paths

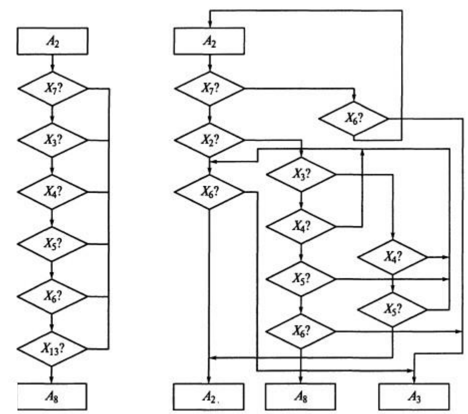
1. Basics

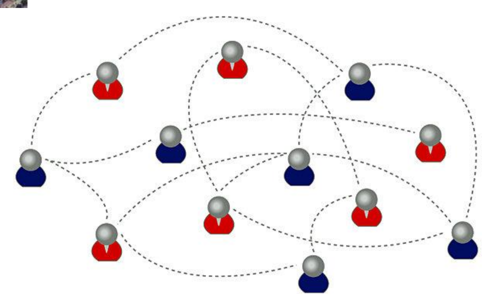
- Graphs and related objects
- Adjacency and incidence
- Isomorphism
- Types of graphs
- Subgraphs

1.1. Graphs and related objects

Graphs are mathematical structures used to model pairwise relations between objects.







Undirected graph (simple graph)

A **simple graph *G*(*V*,*E*)** is a pair of sets:

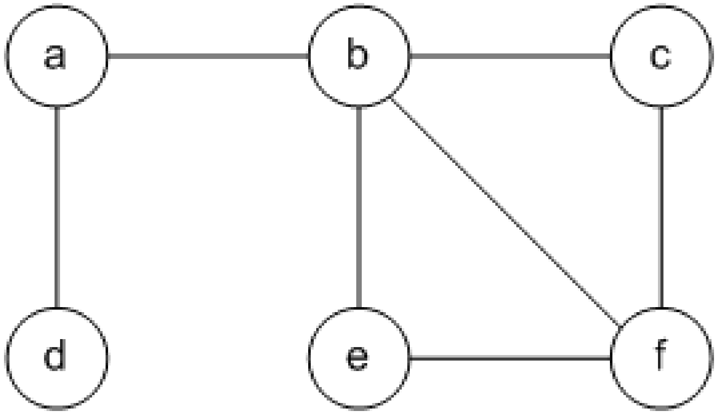
- *V* – the set of **“vertices"** or **"nodes“**;
- *E* – the set of **“edges”** or **“arcs”** that connect pairs of nodes.

An **edge** (an **undirected edge**) is an *unordered* pair of *different* vertices.

An edge *e*=(*a*,*b*) **joins** vertices *a* and *b*. The vertices *a* and *b* are the **end vertices** or the **ends** of the edge *e*.

Undirected graph (simple graph)

Example



$G(V,E)$

$V=\{a,b,c,d,e,f\}$

$E=\{(a,b),(a,d),(b,e),(b,c),$
$(b,f),(c,f),(e,f)\}$

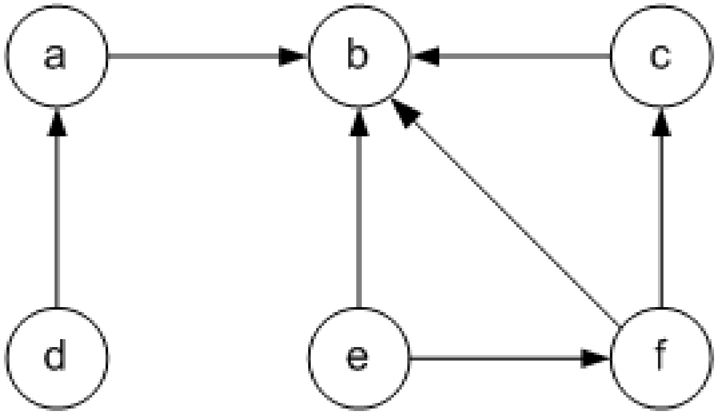
- It is possible to write *ab* instead of (a,b);
- $ab=ba$.

Directed graph (digraph)

If edges are ordered pairs of different nodes, then edges are called **directed edges** and a graph is called **directed graph** or **digraph**.

For an edge *e*=(*a*,*b*) the vertex *a* is its **head** and the vertex *b* is its **tail**.

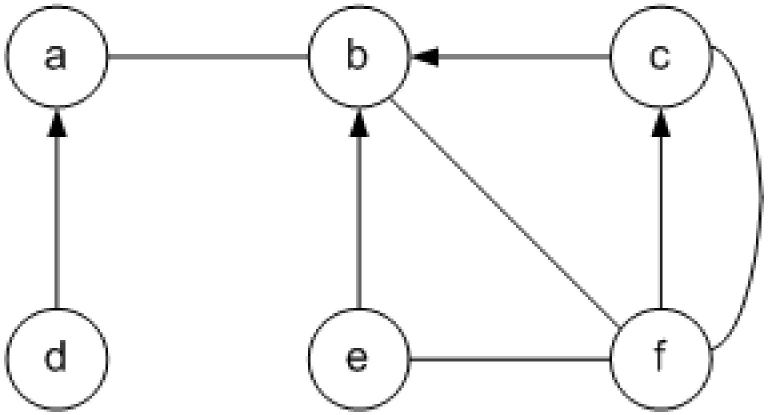
Example



Mixed graph

If both undirected and directed edges are allowed, then a graph is called **mixed graph**.

Example

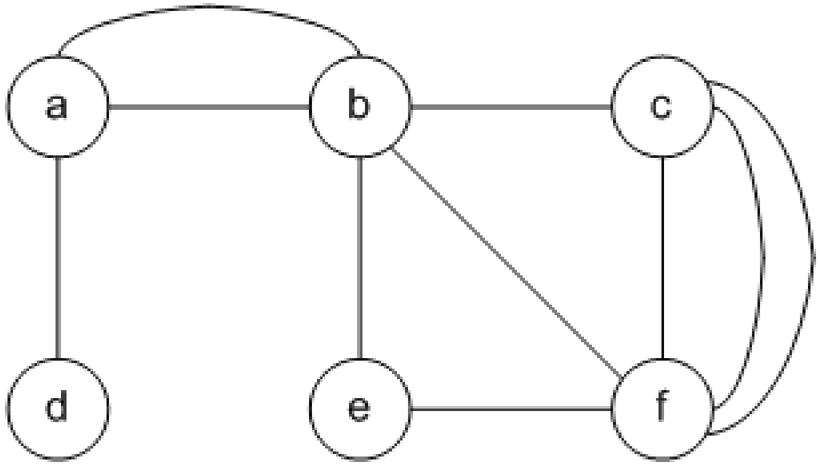


Multigraph

Edges with the same ends (or with the same head and tail) are **multiple edges**.

If multiple edges are allowed, then a graph is called **multigraph**.

Example

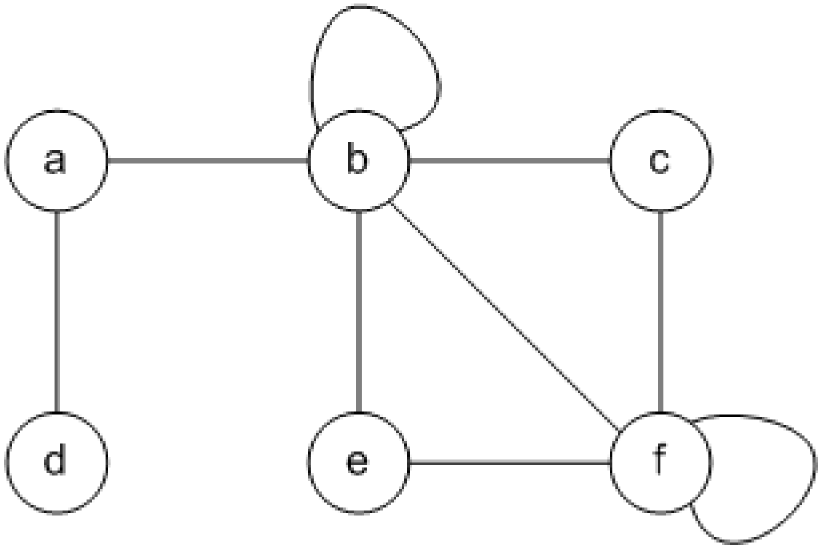


Pseudograph

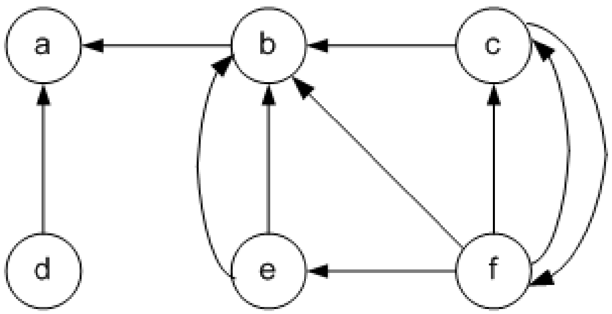
A **loop** is an edge whose endpoints are the same vertex (*e*=*vv*).

If loops are allowed, then a graph is called **pseudograph**.

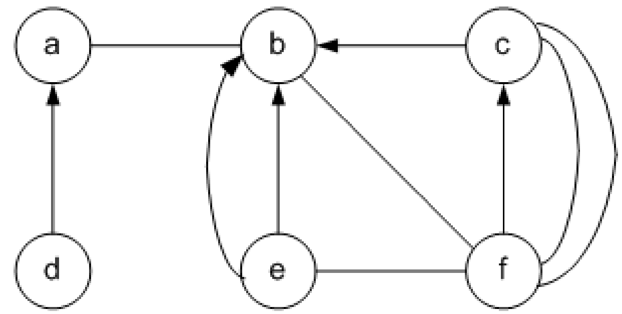
Example



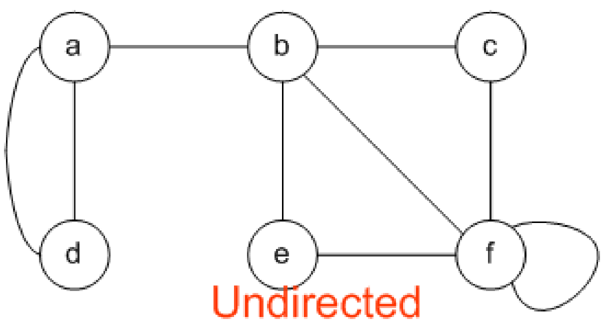
Combinations



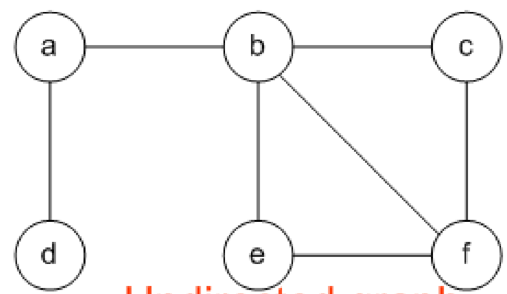
Directed multigraph



Mixed multigraph



Undirected multipseudograph



Undirected graph (graph)

Graph invariants

A **graph invariant** is a property of graphs that depends only on the abstract structure, not on graph representations such as particular labellings or drawings of the graph.

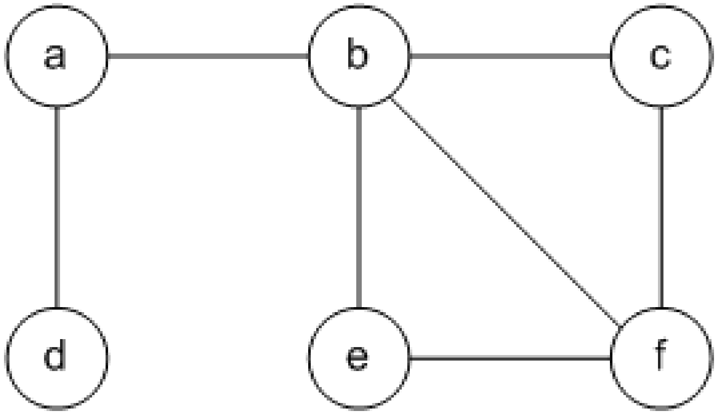
The number of vertices of a graph is its **order** (notation $p=|V|$).

The number of edges of a graph is its **size** (notation $q=|E|$).

Example

$p=6$

$q=7$



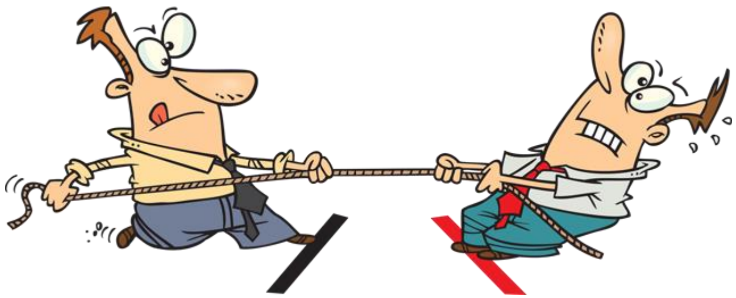
1.2. Adjacency and incidence

Consider an edge *e*=*ab* of a graph (directed or undirected).

The vertices *a* and *b* are **incident** with the edge *e*. The edge e is **incident** with the vertices *a* and *b.*

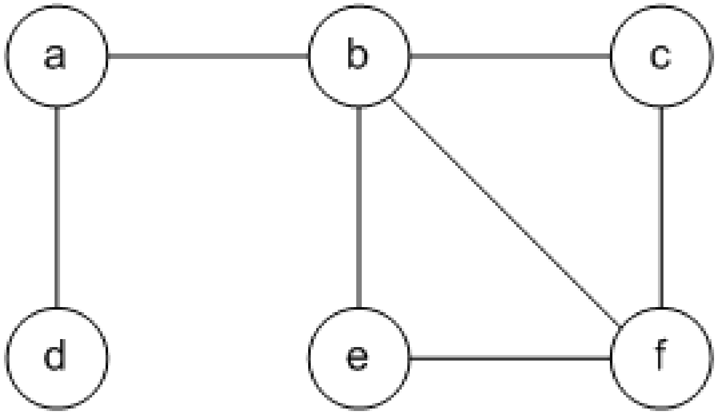
The vertices *a* and *b* are **adjacent**.

Edges incident with the same vertex are **adjacent**.



Adjacency and incidence

Examples



The vertices *a* and *b* are *adjacent*.

The vertices *a* and *e* are *not adjacent*.

The edges *ab* and *ad* are *not adjacent*.

The edges *be* and *ad* are *not adjacent*.

The vertex *a* and the edge *ab* are *incident*.

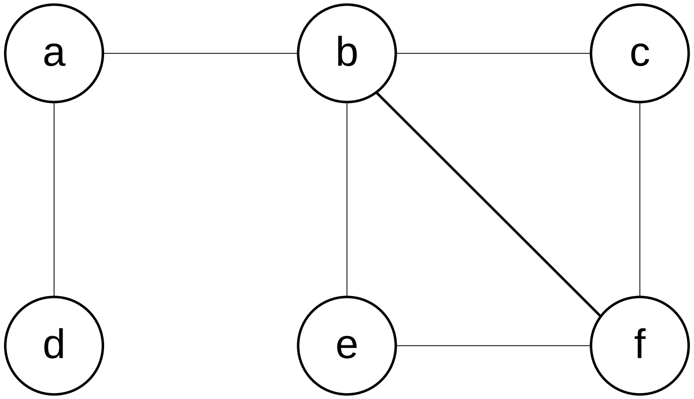
The vertex *c* and the edge *ab* are *not incident*.

Neighbours

Consider an undirected graph $G(V,E)$ and a vertex $a \in V$.

The set $N(a)=\{b: ab \in E\}$ is the set of **neighbors** of the vertex a.

Example



$N(a)=\{b,d\}$

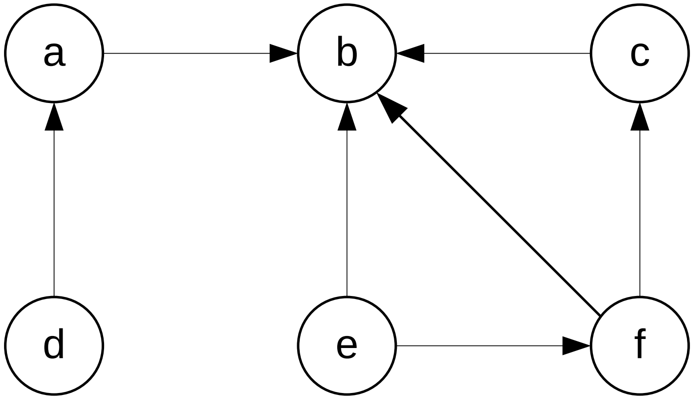
Neighbours

Consider a directed graph $G(V,E)$ and a vertex $a \in V$.

The set $N^+(a)=\{b: ab \in E\}$ is the set of **out-neighbors** of the vertex a.

The set $N^-(a)=\{b: ba \in E\}$ is the set of **in-neighbors** of the vertex a.

Example

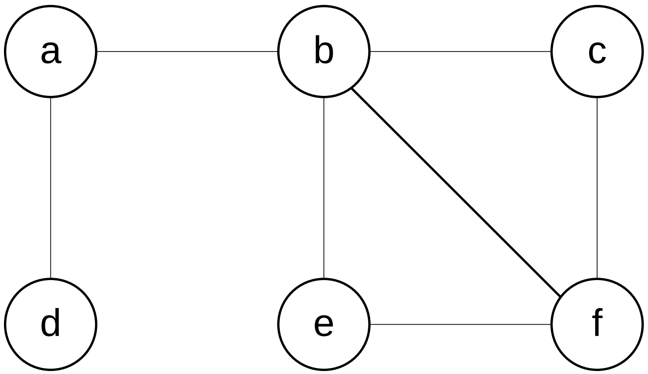


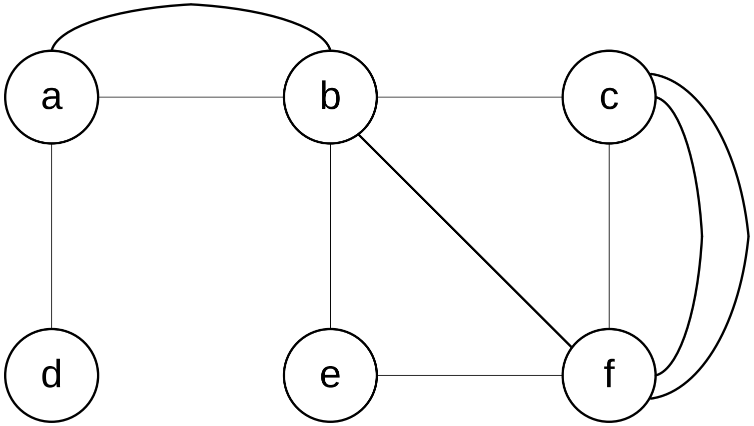
$$N^+(a)=\{b\},\ N^-(a)=\{d\}$$

The degree of a vertex

For an undirected graph **the degree of a vertex** *v* (notation *d*(*v*)) is the number of edges incident with *v*.

Examples





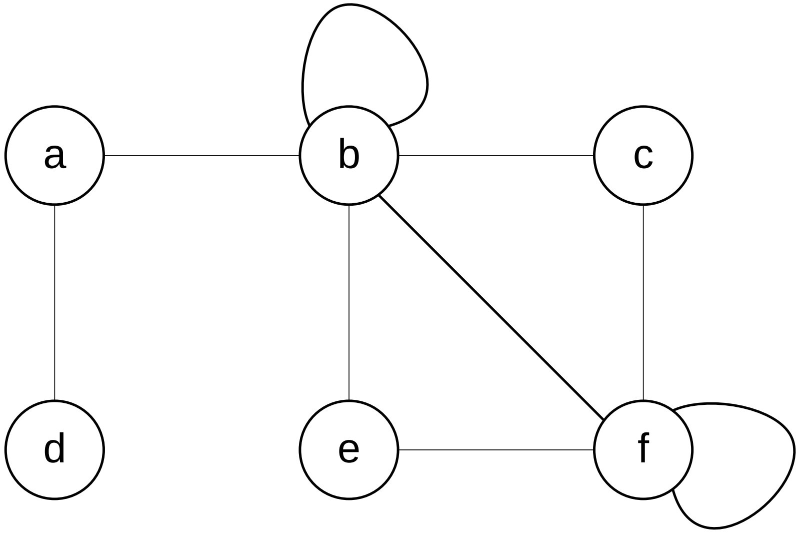
$d(a)=2$, $d(b)=4$, $d(f)=3$

$d(a)=3$, $d(b)=5$, $d(f)=5$

The degree of a vertex

A loop *vv* adds 2 to **the degree of a vertex** *v*.

Example



$d(a)=2$, $d(b)=6$, $d(f)=5$

The degree of a vertex

For a directed graph there are three characteristics:

- the **out-degree of a vertex** *v* (notation $d^+(v)$) is the number of edges with the tail in *v*:

$$d^+(v)=|\{vu: u \in V, vu \in E\}|;$$

- the **in-degree of a vertex** *v* (notation $d^-(v)$) is the number of edges with the head in *v*:

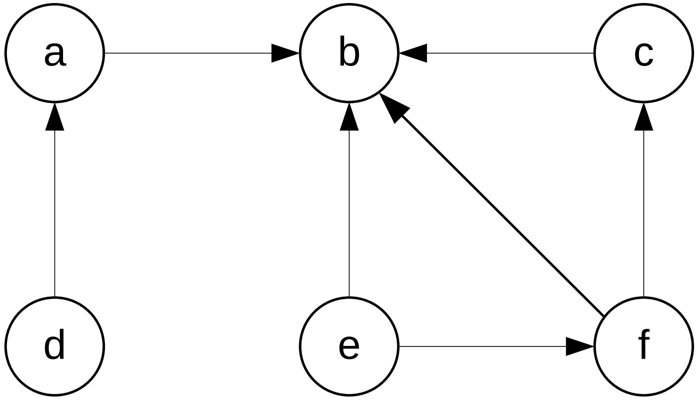
$$d^-(v)=|\{uv: u \in V, uv \in E\}|;$$

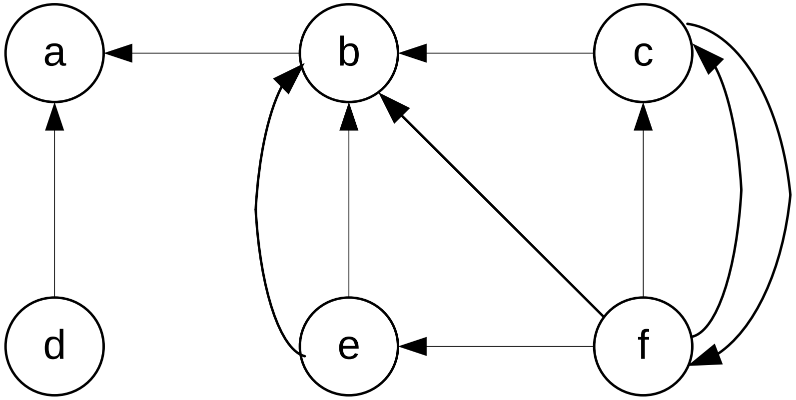
- the **degree of a vertex** *v* (notation $d(v)$) is the sum of the out-degree and the in-degree of *v*:

$$d(v)= d^+(v)+d^-(v).$$

The degree of a vertex

Examples



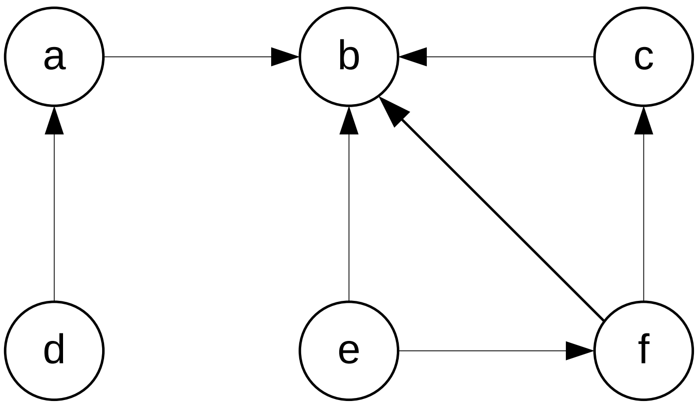


$d^+(a)=1$, $d^-(a)=1$, $d(a)=2$;
$d^+(f)=2$, $d^-(f)=1$, $d(f)=3$

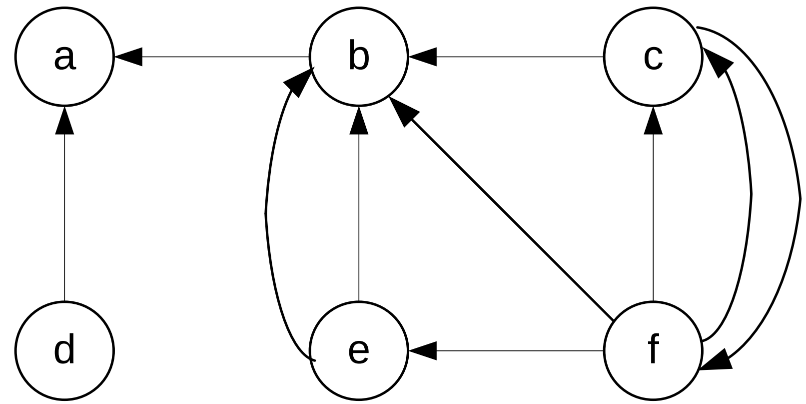
$d^+(a)=1$, $d^-(a)=1$, $d(a)=2$;
$d^+(f)=4$, $d^-(f)=1$, $d(f)=5$

The degree of a vertex

Examples



$d^+(a)=1$, $d^-(a)=1$, $d(a)=2$;
$d^+(f)=2$, $d^-(f)=1$, $d(f)=3$



$d^+(a)=1$, $d^-(a)=1$, $d(a)=2$;
$d^+(f)=4$, $d^-(f)=1$, $d(f)=5$

Graph invariants

Minimum degree

$$\delta(G) = \min_{v \in V} d(v)$$

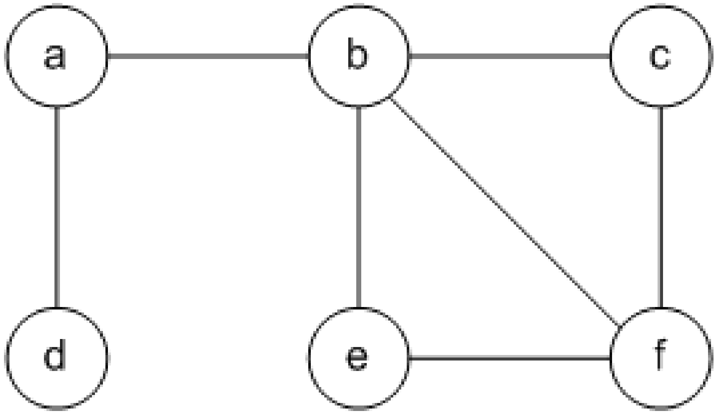
Maximum degree

$$\Delta(G) = \max_{v \in V} d(v)$$

Example

$\delta(G)$=1

$\Delta(G)$=4



Particular cases

- A vertex with degree 0 is called an **isolated vertex**.
- A vertex with degree 1 is called a **leaf vertex** or **end vertex**. This terminology is common in the study of **trees** in graph theory.
- A vertex with degree $n - 1$ in a graph on n vertices is called a **dominating vertex**.

Handshaking lemma (Leonhard Euler)

Lemma 1. The doubled number of edges of a finite undirected graph is equal to the sum of the degrees of vertices:

$$\sum_{v=1}^{p} d(v) = 2q.$$

Lemma 2. Every finite undirected graph has an even number of vertices with odd degree.

1.3. Isomorphism

Graphs $G(V,E)$ and $G'(V',E')$ are **isomorphic** if there exists a bijection $\varphi: V \to V'$ such as for all $x, y \in V$:

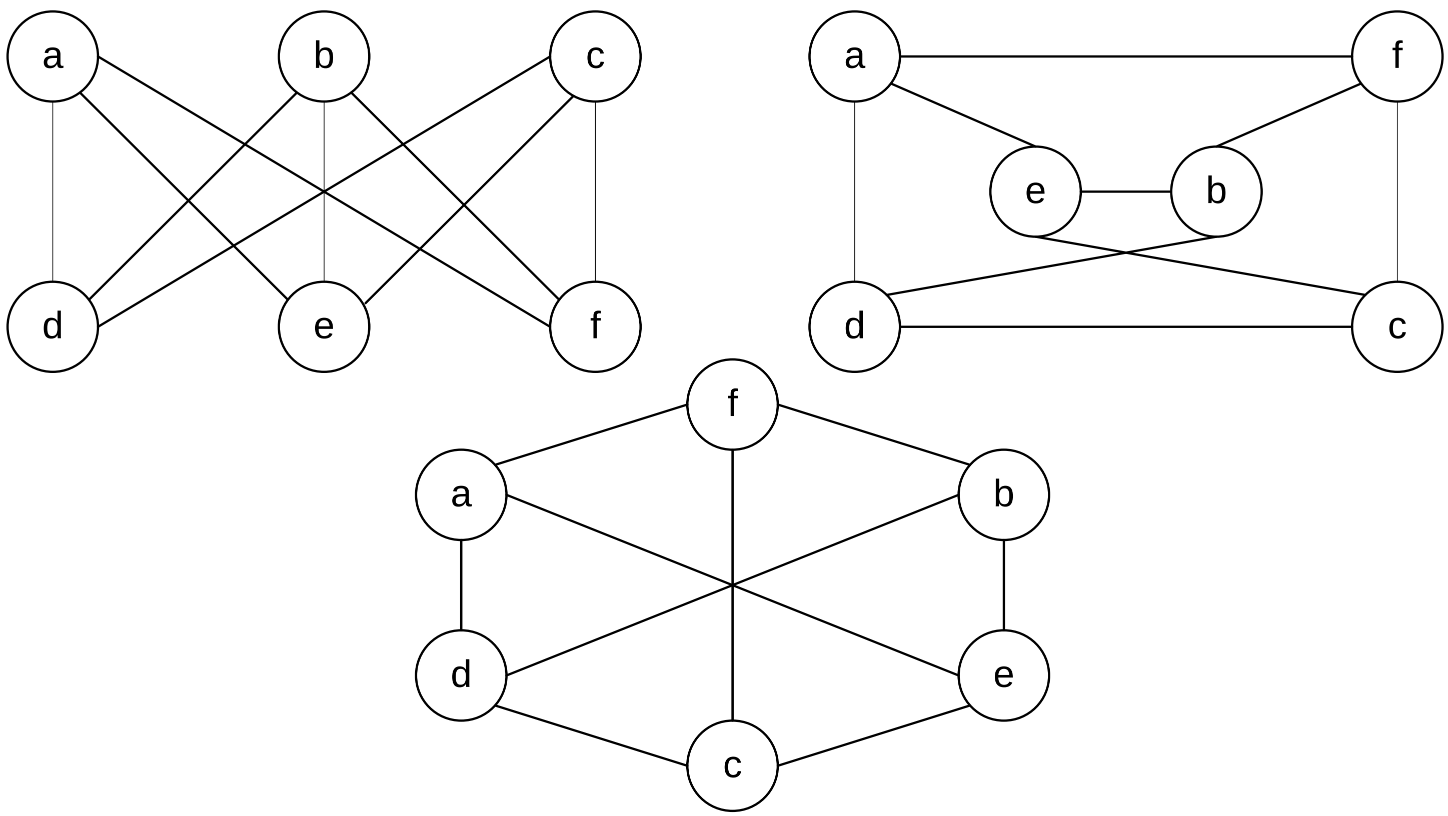
$$xy \in E \text{ if and only if } \varphi(x)\varphi(y) \in E'.$$

Isomorphic graphs are not distinguished.

To prove that graphs are isomorphic it is *necessary* to find a bijection φ.

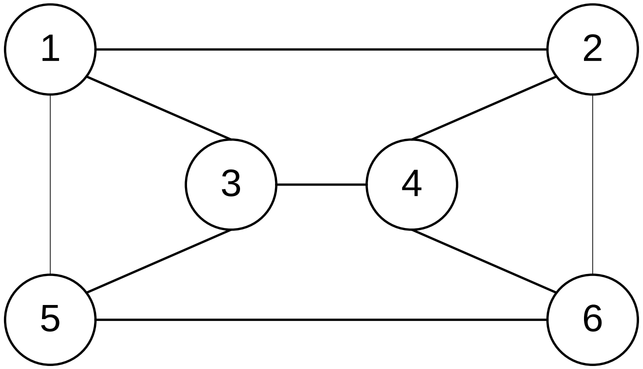
To prove that graphs are not isomorphic it is *sufficient* to prove that one graph has a certain property and another graph has not the property.

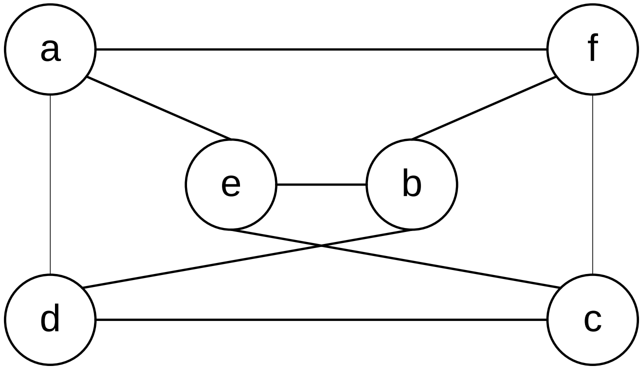
Example: isomorphic graphs



Example: nonisomorphic graphs

All the vertices *1, 3* and *5* (and *2, 4* and *6*) of the graph on the left are pairwise adjacent. There are no such three vertices in the graph on the right.



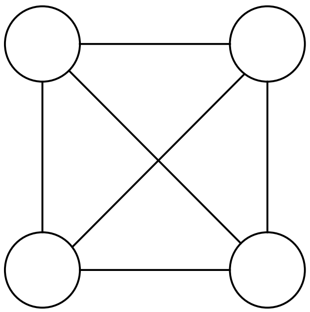


1.4. Types of graphs

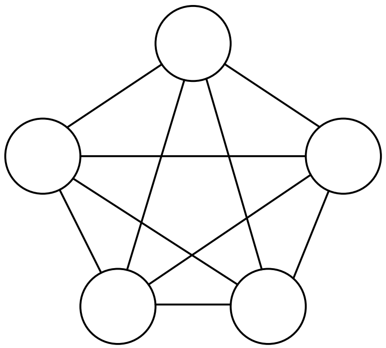
A graph is **complete** if all its vertices are pairwise adjacent.

A complete graph with p vertices is denoted as K_p.

The number of the edges of K_p is equal to $p(p–1)/2$.



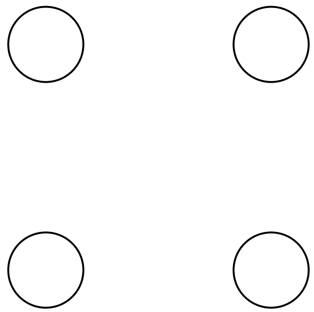
K_4



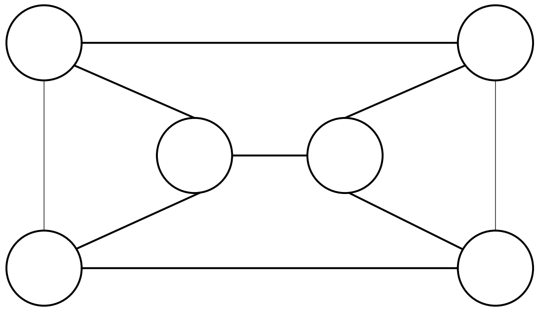
K_5

Types of graphs

A graph is **empty** if any pair of its vertices are not adjacent ($E=\emptyset$).



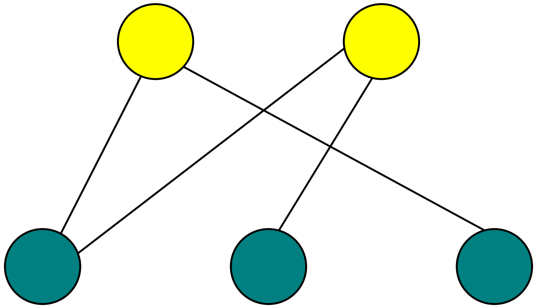
A graph is ***k*-regular** (**regular**) if all its vertices have the same degree *k*.



Types of graphs

A graph is **two-partite (bipartite, bigraph)** if the set of its vertices can be divide into two subsets V_1 and V_2 so that every edge connect vertices from different subsets, i.e.

$$V = V_1 \cup V_2,\ V_1 \cap V_2 = \varnothing, \text{ for all } xy \in E\text{: } x \in V_1,\ y \in V_2.$$

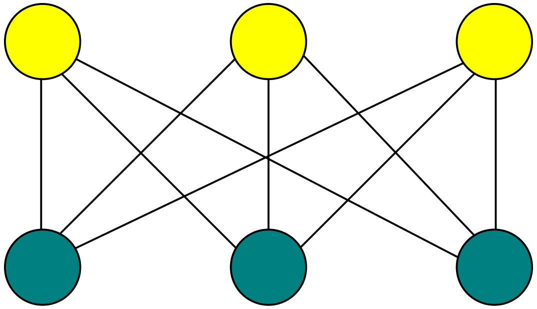


Types of graphs

A graph is **complete two-partite (bipartite, bigraph)** if every vertex from V_1 is adjacent with every vertex from V_2, i.e.

$$V = V_1 \cup V_2,\ V_1 \cap V_2 = \varnothing, \text{ for all } x \in V_1,\ y \in V_{2.:}\ xy \in E.$$

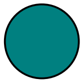
A complete bigraph where $|V_1|=n$, $|V_2|=m$ is denoted as K_{nm}.



$K_{3,3}$

Types of graphs

A graph is **trivial** if |V|=1, |E|=0.

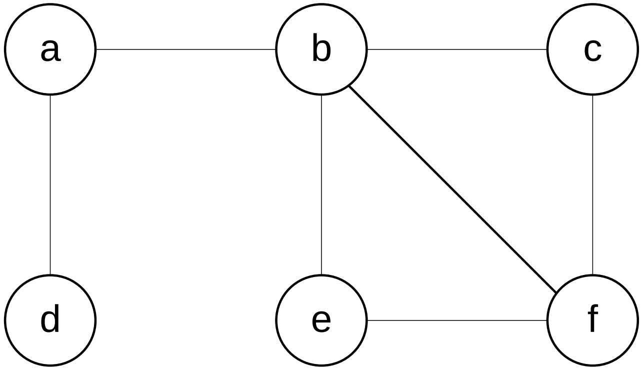


1.5. Subgraphs

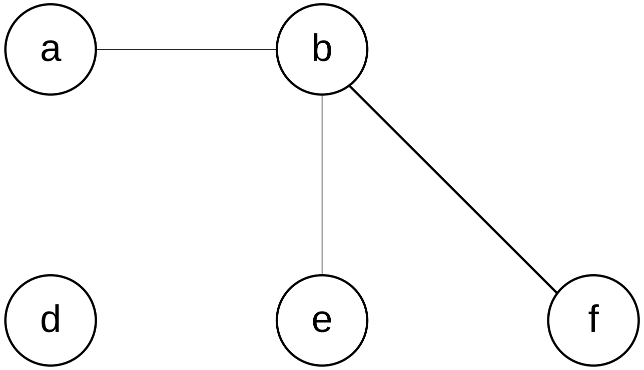
Consider two graphs: $G(V,E)$ and $G'(V',E')$.

If $V' \in V$ and $E' \in E$ then G' is a **subgraph** of G (less formally, G **contains** G, notation $G' \in G$).

Example.



$G(V,E)$



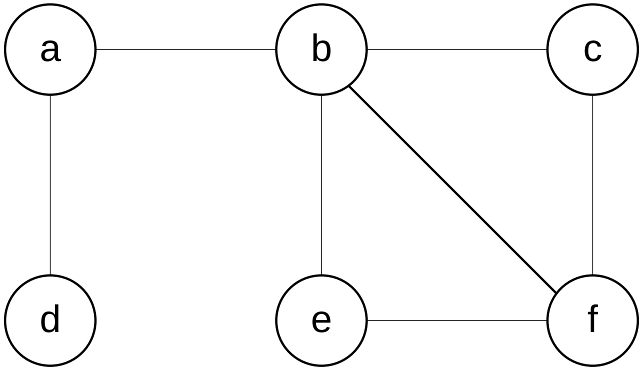
$G'(V',E')$

Subgraphs

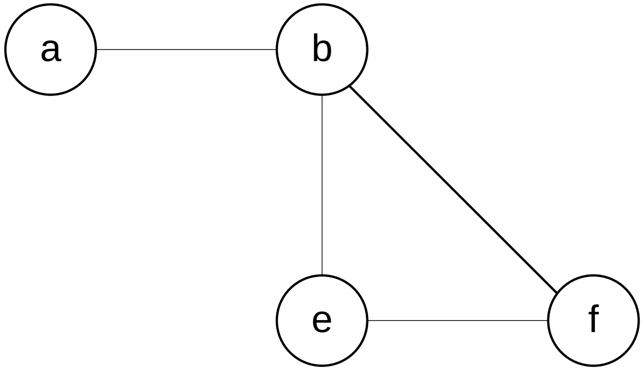
If $G' \in G$ and E' contains *all* the edges $xy \in E$: $x,y \in V'$, then G' is an **induced subgraph** of G.

We say that V' induces G' in G and write $G'=G[V']$.

Example.



$G(V,E)$

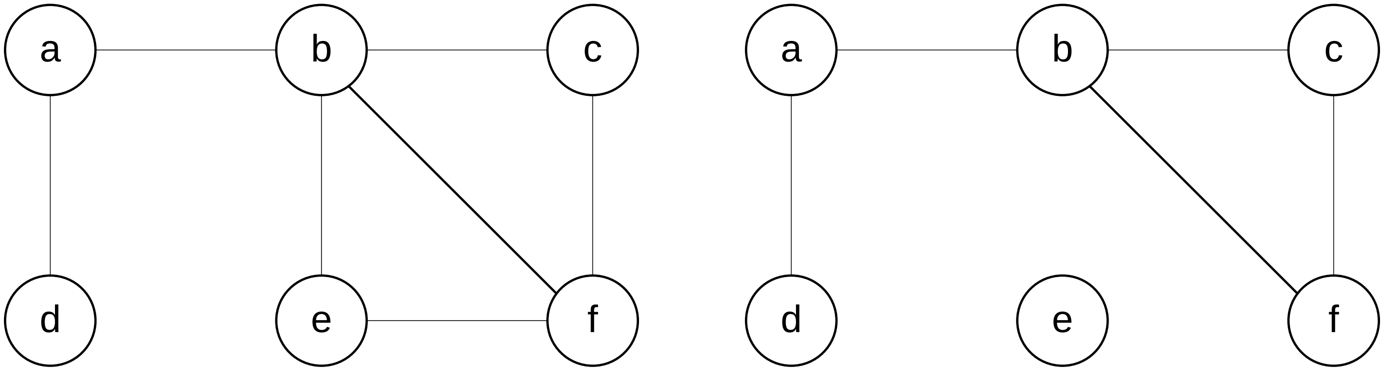


$G'=G[\{a,b,e,f\}]$

Subgraphs

If *V'*=*V* then *G'* is a **spanning subgraph** of *G*.

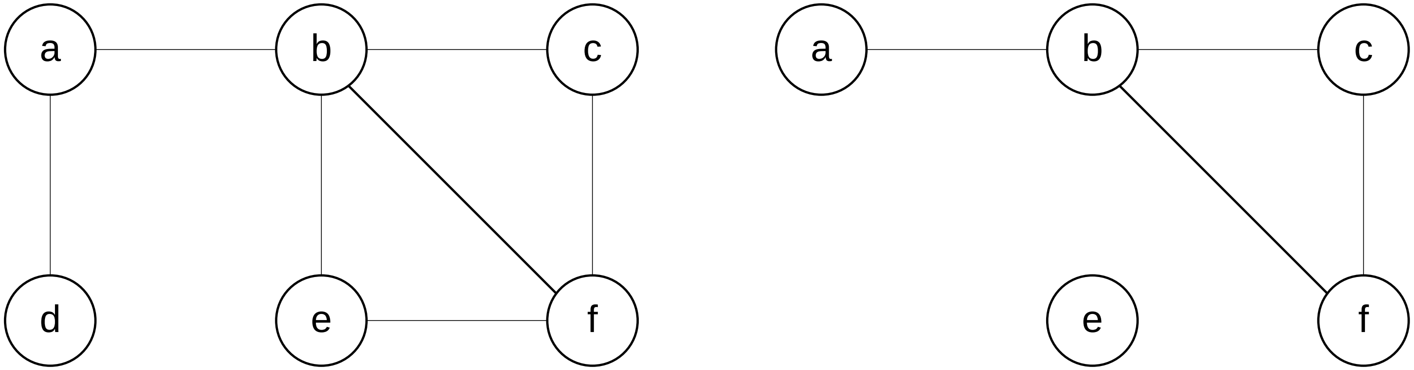
Example.



G(*V*,*E*) *G'*(*V*,*E'*)

Subgraphs

If *V'*≠*V* and *E'*≠*E* then *G'* is a **proper subgraph** of *G*.



G(*V*,*E*) *G'*(*V'*,*E'*)

2. Connectivity

- Walks
- Distances
- Connectivity of simple graphs
- Connectivity of directed graphs

2.1. Walks

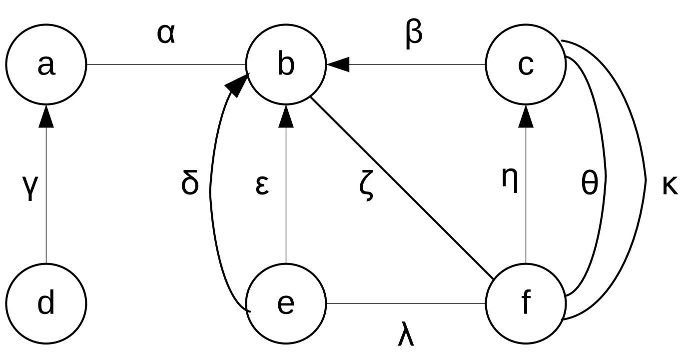
A **walk** is a sequence of vertices and edges

$$<v_0, v_n>=v_0e_1v_1\ldots v_{i-1}e_iv_i\ldots v_{n-1}e_nv_n,$$

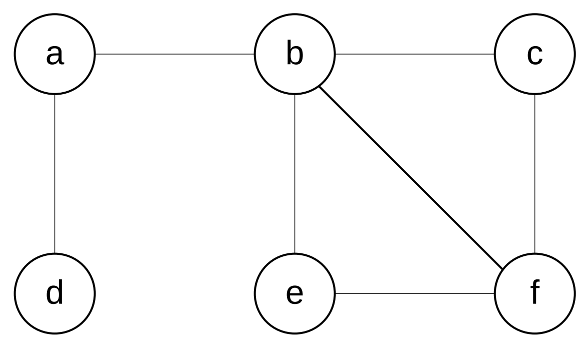
where $e_i=v_{i-1}v_i$. A walk is **closed** if its first and last vertices are the same, and **open** if they are different.

If there are no multiple edges then it is possible to omit edges

Examples.



<a,f> = a α b ζ f η c θ f κ c β b ζ f



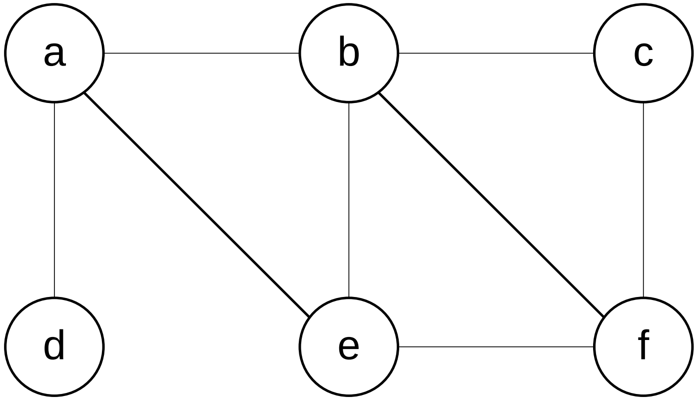
<a,f> = a b f c f c b f

Trail and tour

A **trail** is an open walk in which all the edges are different.

A **tour** (or a **circuit**) is a closed walk in which all the edges are different.

Examples.



Trail <a,f> = a b f c b e f

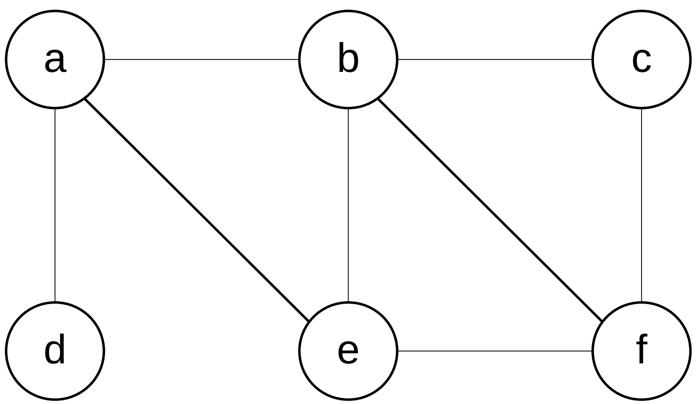
Tour <a,a> = a b f c b e a

Path and cycle

A **path** (or a **chain**) is an open walk in which all the vertices (and hence the edges) are different.

A **cycle** (or a **circuit**) is a closed walk in which all the vertices are distinct.

Examples.



Path <a,f> = a b e f

Cycle <a,a> = a b f e a

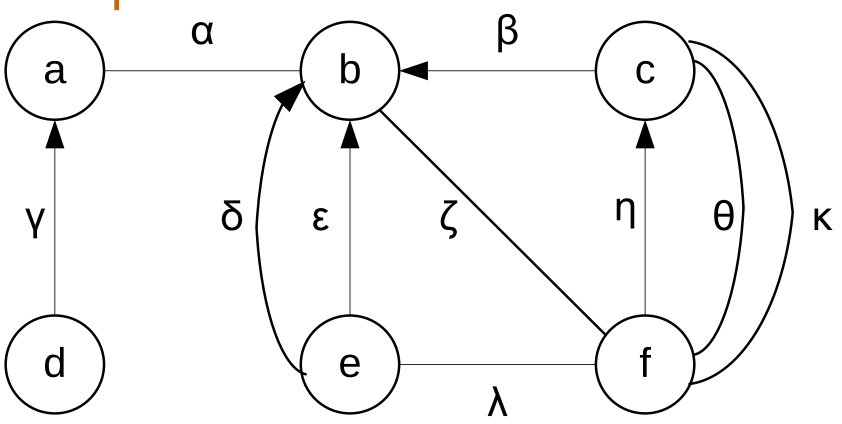
2.2. Distances

The **length** of a walk is the number of edges that it uses.

The **shortest** path <u,v> is a path of minimal length | <u,v> |.

The **distance** between two vertices d(u,v) is the length of a shortest path <u,v>, if one exists, and otherwise the distance is infinity.

Examples.



<a,f> = a α b ζ f η c θ f κ c β b ζ f

|<a,f>| = 7

Shortest path <a,f> = a α b ζ f

d(a,f) = 2

Distances

The **eccentricity** ε(*v*) of a vertex *v* is the maximum distance from *v* to any other vertex.

$$\varepsilon(v) = \max_{u \in V} d(u, v).$$

The **diameter** *D*(*G*) of a graph *G* is the maximum distance between two vertices in a graph or the maximum eccentricity over all vertices in a graph.

$$D(G) = \max_{v \in V} \varepsilon(v) = \max_{u,v \in V} d(u, v).$$

The **radius** *R*(*G*) is the minimum eccentricity over all vertices in a graph.

.

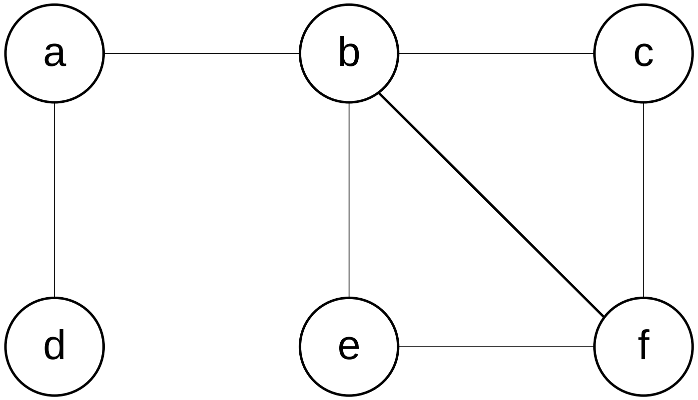
$$R(G) = \min_{v \in V} \varepsilon(v) = \min_{v \in V} \max_{u \in V} d(u, v).$$

Distances

Vertices with maximum eccentricity are called **peripheral vertices**.

Vertices of minimum eccentricity form the **center**.

Examples.



$\varepsilon(a)=\varepsilon(b)=2$

$\varepsilon(c)=\varepsilon(d)=\varepsilon(e)=\varepsilon(f)=3$

$R(G)=2$

$D(G)=3$

Peripheral vertices c, d, e, f

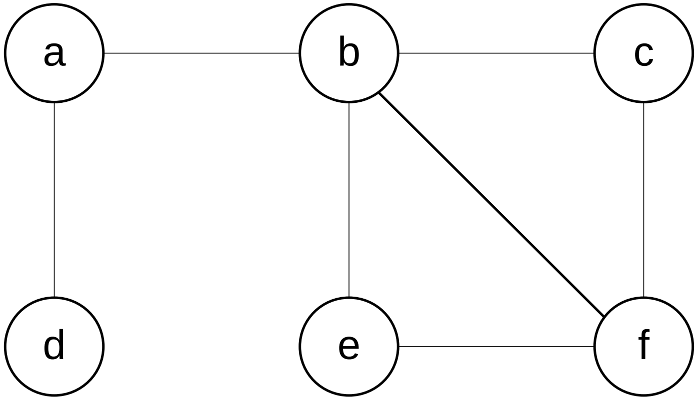
Centre a, b

2.3. Connectivity of simple graphs

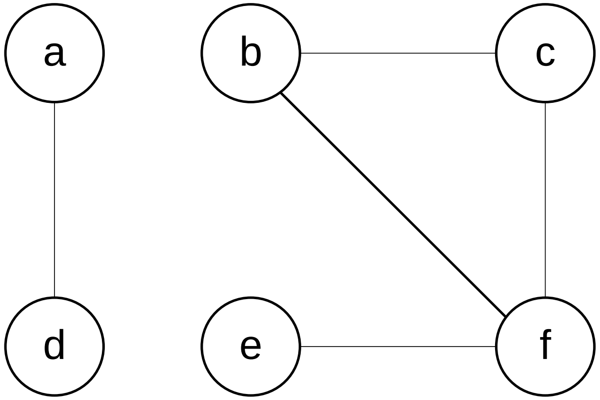
If it is possible to establish a path <u,v> from vertex u to other vertex v, the vertices u and v are **connected**.

If all the pairs of vertices are connected, the graph is said to be **connected**; otherwise, the graph is **disconnected**.

Examples.



Connected graph

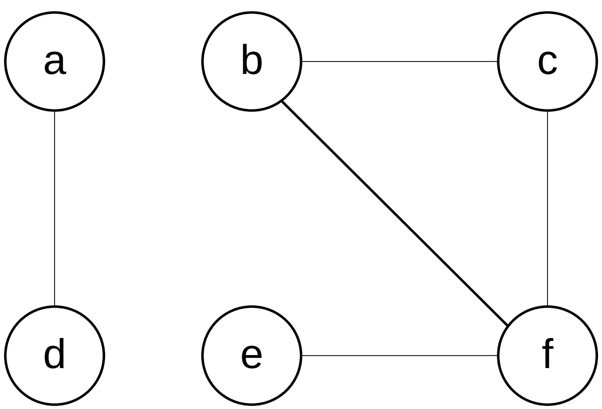


Disconnected graph

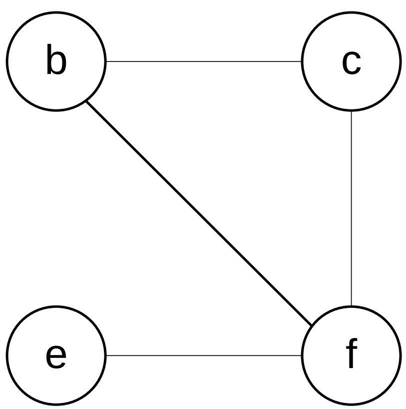
Connected component

A **connected component** of a graph G(V,E) is any its maximally connected subgraph, i.e. an induced subgraph which is not a proper subgraph of any other connected subgraph of G(V,E) .

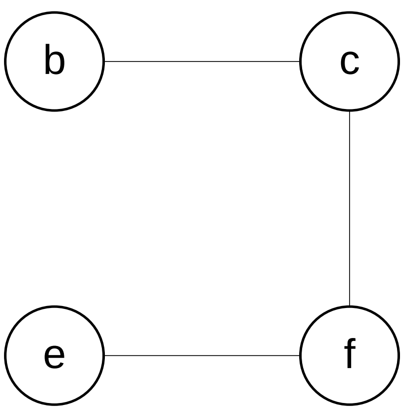
Examples.



Graph with two components



Component



Not a component

Articulation point and bridge

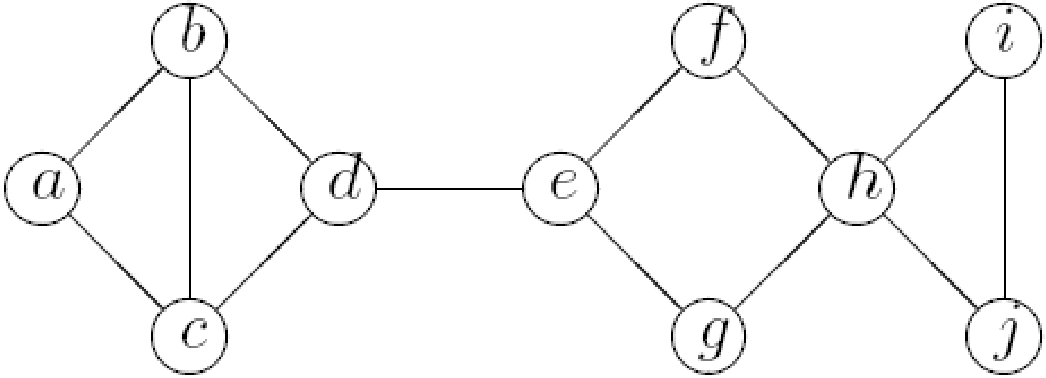
An **articulation point** (or **separating vertex**) of a graph is a vertex whose removal from the graph increases its number of connected components.

A **bridge**, or (**cut edge**) is an analogous edge.

Examples.

de – bridge

d, e, h – articulation points

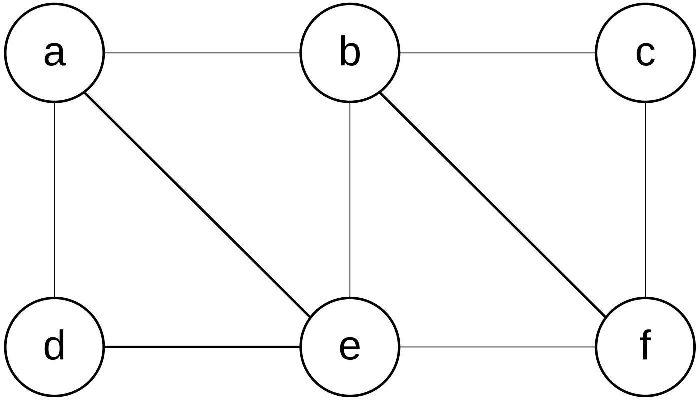


Cuts

A **vertex cut**, (or **separating set**) of a connected graph *G* is a set of vertices whose removal makes *G* disconnected or trivial.

Analogous concept can be defined for edges.

Examples.



{ b, e } – vertex cut

{ ab, be, ef } – edge cut

Graph invariants

k(G) – the number of connected components

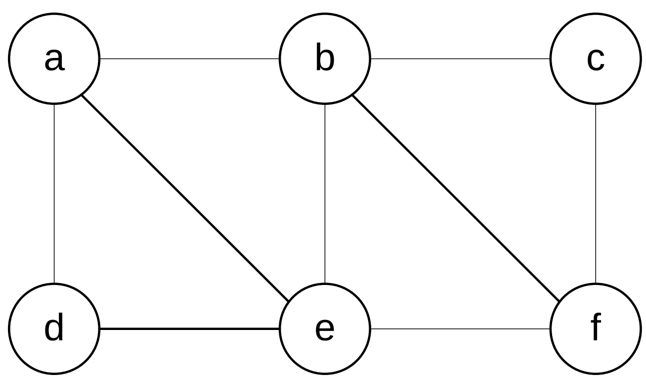
The **vertex connectivity** κ(G) is the size of a minimal vertex cut.

The **edge connectivity** λ(G) is the size of a smallest edge cut.

A graph is called **n-vertex-connected (n-edge-connected)** if its vertex (edge) connectivity is n or greater.

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

Examples.



κ = 2 (vertices d, e)

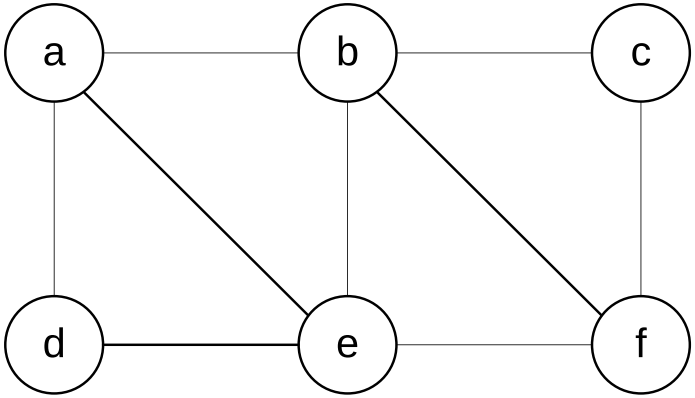
λ = 2 (edges ad, de)

Cuts for a pair of vertices

A **vertex cut S(u,v)**, (or **separating set**) for two connected vertices u and v is a set of vertices whose removal mekes the vertices u and v disconnected.

Analogous concept can be defined for edges.

Examples.



Vertex cut S(a,f)={b,d,e}

Edge cut S(a,f)={ab,ae,ef}

Menger theorem

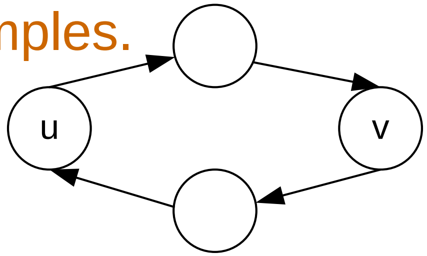
2.4. Connectivity of directed graphs

If it is possible to establish a path <u,v> and a path <v,u> in a digraph, the vertices u and v are **strongly connected**.

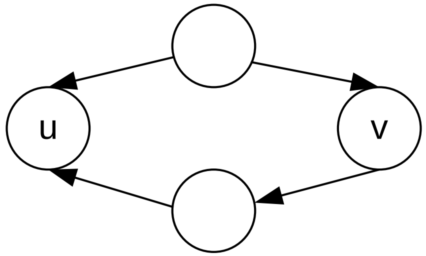
If it there exists either a path <u,v> or a path <v,u> in a digraph, the vertices u and v are **unilaterally connected**.

If it there exists a path <u,v> in a graph obtained from a digraph by canceling of edges direction, the vertices u and v are **weakly connected**.

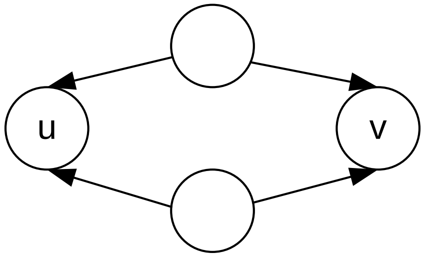
Examples.



Strongly connected



Unilaterally connected

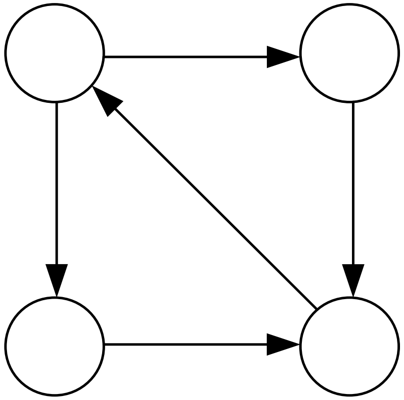


Weakly connected

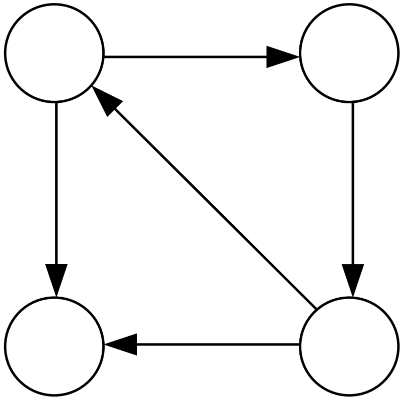
Connectivity of directed graphs

If all the pairs of vertices of a digraph are strongly / unilaterally / weakly connected, the digraph is **strongly / unilaterally / weakly connected**.

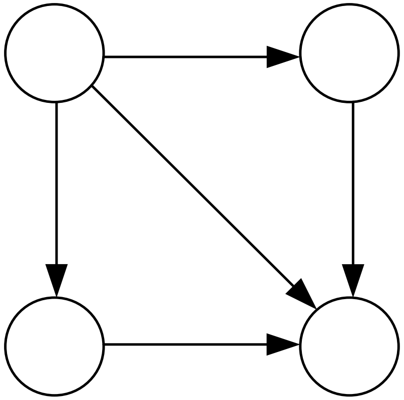
Examples.



Strongly connected



Unilaterally connected

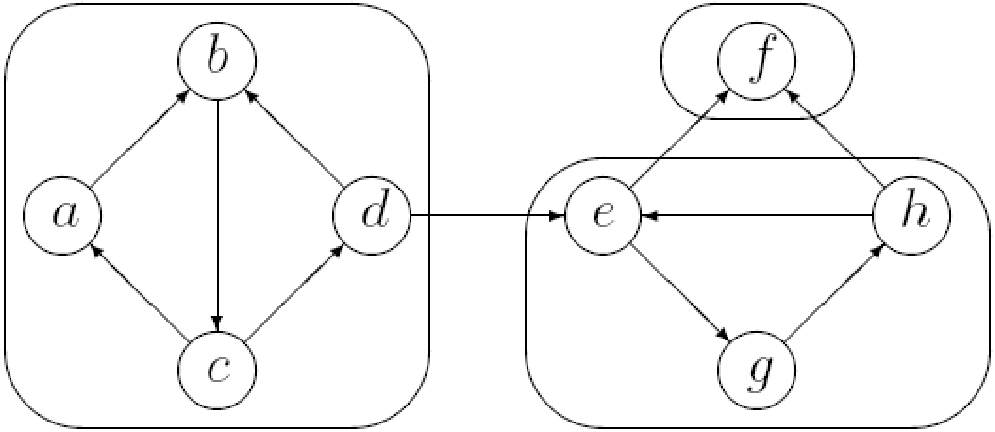


Weakly connected

Strongly connected component

A **strongly connected component** of a digraph G(V,E) is any its maximally strongly connected subgraph, i.e. an induced subgraph which is not a proper subgraph of any other strongly connected subgraph of G(V,E) .

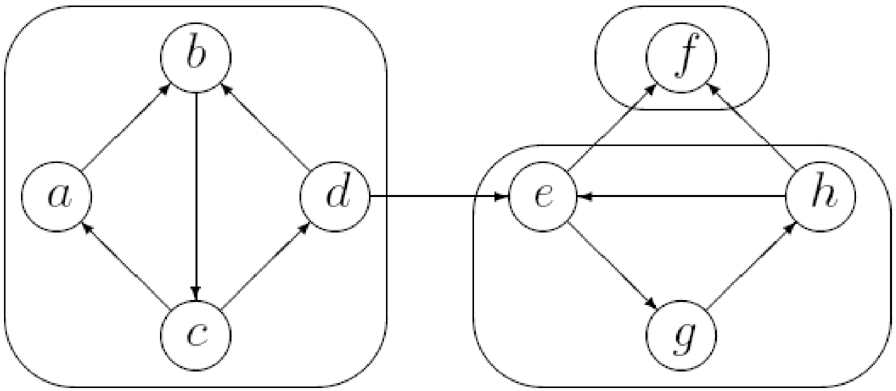
Example.



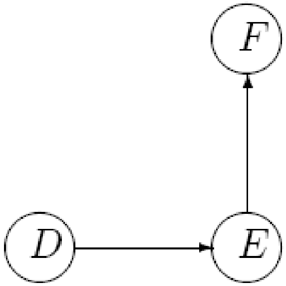
Quotient graph

The **quotient graph** of a digraph D(V,E) with k strongly connected components induced by sets of vertices $V_1,\ldots,V_k$ is a graph D'(V',E') where $V'=\{v_1,\ldots,v_k\}$, $v_i v_j \in E'$ if there is an edge $u_i u_j \in E$: $u_i \in V_i$, $u_j \in V_j$.

Example.



Digraph



Quotient graph

3. Paths

- Graph traversal
- Shortest path

3.1. Graph traversal

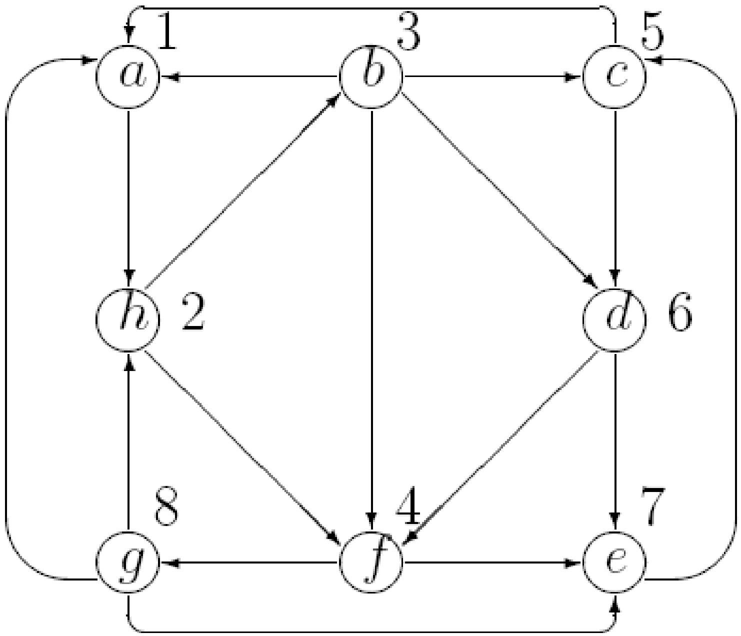
Graph traversal is the problem of visiting all the vertices in a graph, updating and/or checking their values along the way.

Breadth-first search (**BFS**) is a graph traversal algorithm that begins at a start vertex and explores all its neighbors (out-neighbors for a digraph). Then for each of those nearest vertices, it explores their unexplored neighbors, and so on, until all the vertices are visited.

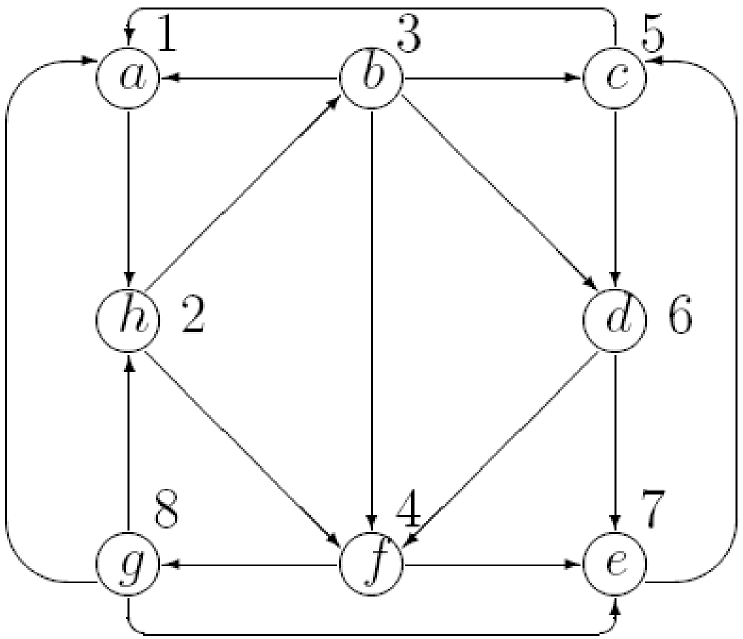
Depth-first search (**DFS**) is a graph traversal algorithm that begins at a start vertex, explores its not visited neighbor and then considers that neighbor as a start vertex. If all the neighbors are visited then "backtracking" is used, i.e. the previous vertex is considered as a start vertex.

Graph traversal examples

BFS



DFS

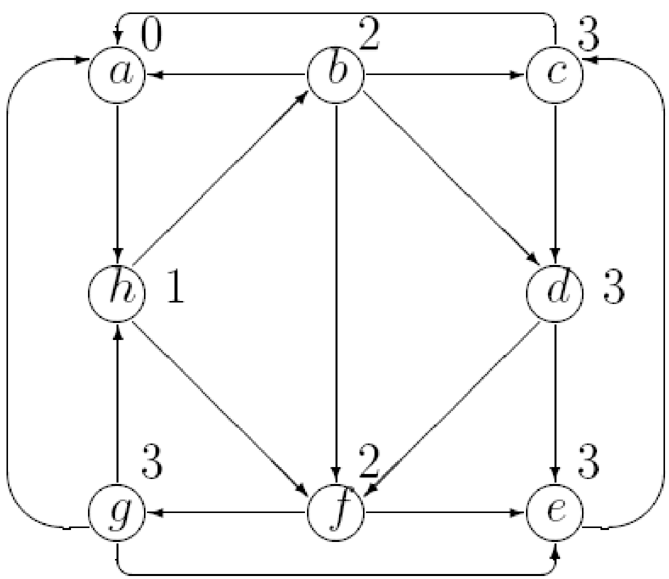


3.2. Shortest path

The **shortest** path <u,v> is a path of minimal length |<u,v>|.

Lee algorithm (based on the DFS) is usually used to find the shortest path.

Example.



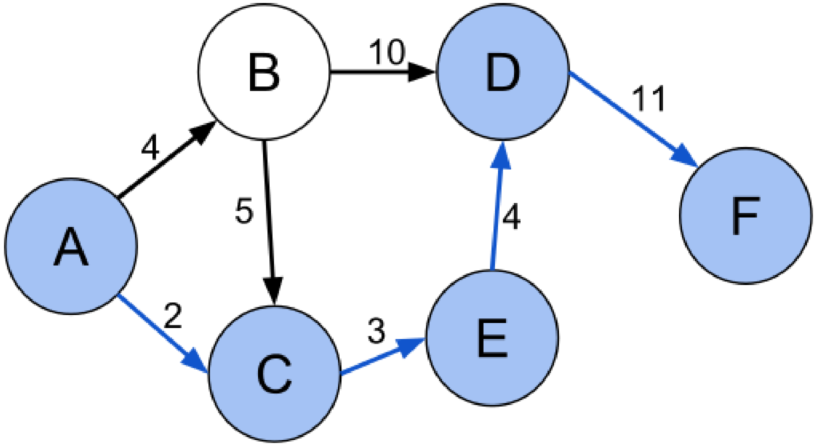
Shortest path

A **weighted graph** associates a label (**weight**) with every edge in the graph.

The **weight** of a path W(<u,v>) is the sum of weights of the edges included in the path.

The **shortest** path <u,v> in a weighted graph is a path of minimal weight W(<u,v>) .

Example.



Shortest paths problems

- The **single-pair shortest path problem**, in which we have to find shortest paths from a source vertex *v* to a single destination vertex *u*.
- The **single-source shortest path problem**, in which we have to find shortest paths from a source vertex *v* to all other vertices in the graph.
- The **single-destination shortest path problem**, in which we have to find shortest paths from all vertices in the directed graph to a single destination vertex *v*.
- The **all-pairs shortest path problem**, in which we have to find shortest paths between every pair of vertices *v*, *u* in the graph.

Shortest paths algorithms

- **Dijkstra's** algorithm solves the single-source shortest path problem.
- **Bellman–Ford** algorithm solves the single-source problem if edge weights may be negative.
- **Floyd–Warshall** algorithm solves all pairs shortest paths.

4. Location problems

- Distances in a weighted graph
- Centre
- Median
- Extencions
- Absolute P-centre
- P-median

4.1. Distances in a weighted graph

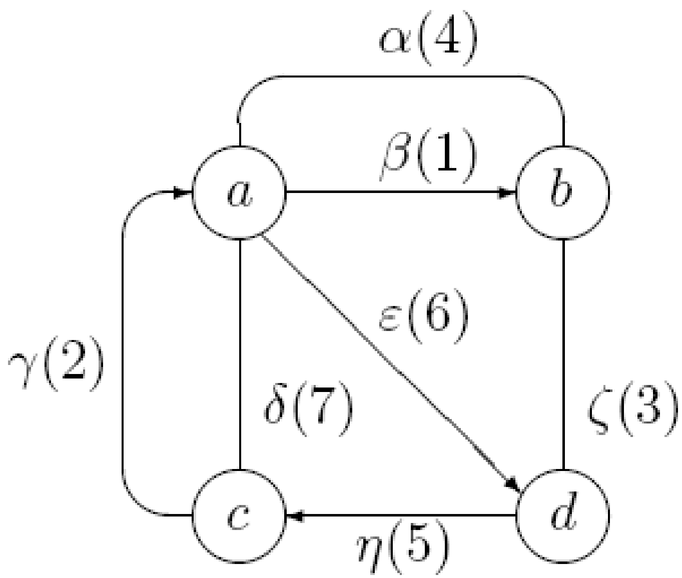
- Vertex-vertex distance
- Point-vertex distance
- Vertex-point distance
- Vertex-edge distance

Vertex-vertex distance

The **vertex-vertex distance** between vertices *i* and *j* (notation *d*(*i*,*j*)) is the weight of the shortest path <*i*,*j*>.

It can be found by the Floyd–Warshall algorithm.

Example.

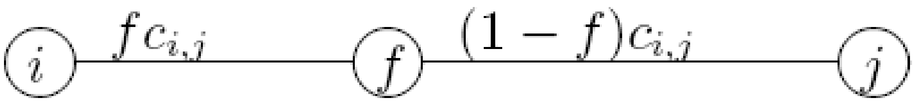


	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

F-point

Consider an edge $e=(i,j)$ with the weight $c_{ij}>0$ and a parameter f : $0 \leq f \leq 1$.

The point at the edge which divide the edge in proportion f : $(1–f)$ is called the ***f*-point** (notation $f_{(i,j)}$).



The weight of the edge part if is equal to fc_{ij}, the weight of the part fj is equal to $(1–f)c_{ij}$.

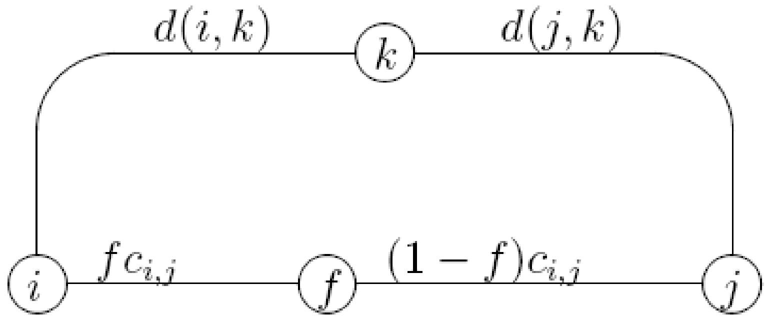
The vertex i is 0-point, the vertex j is 1-point.

The other points are **interior**.

Point-vertex distance

The **point-vertex distance** between a point $f_{(i,j)}$ and a vertex k (notation $d(f_{(i,j)},k)$) is the weight of the minimal path $< f_{(i,j)},k>$.

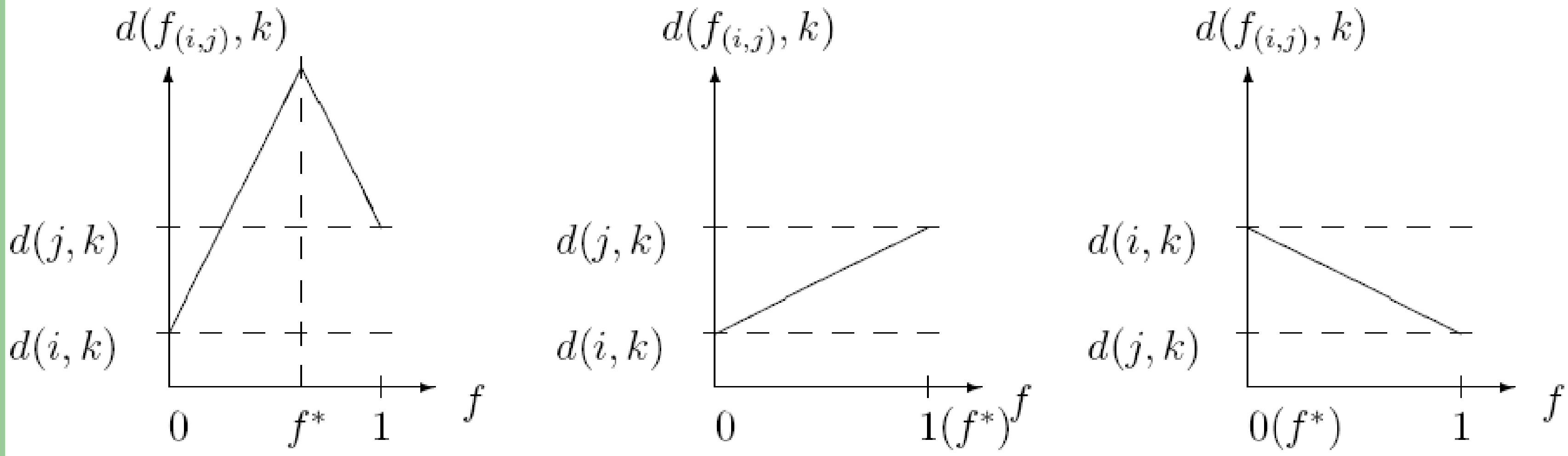
For an undirected edge (i,j):



$$d(f_{(i,j)}, k) = \min\{fc_{i,j} + d(i,k), (1-f)c_{i,j} + d(j,k).\}$$

Point-vertex distance

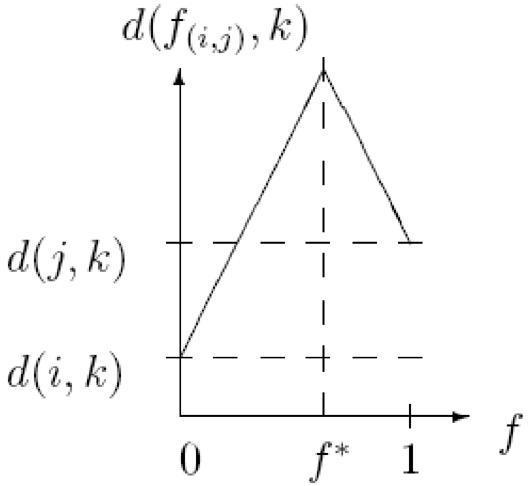
The dependence $d(f_{(i,j)},k)$) of f can be one of three types.



Point-vertex distance

The maximum point f^* is the point of the lines intersection:

$$f c_{i,j} + d(i,k) = (1-f)c_{i,j} + d(j,k)$$



$$f^* = \frac{d(j,k) - d(i,k) + c_{i,j}}{2c_{i,j}}$$

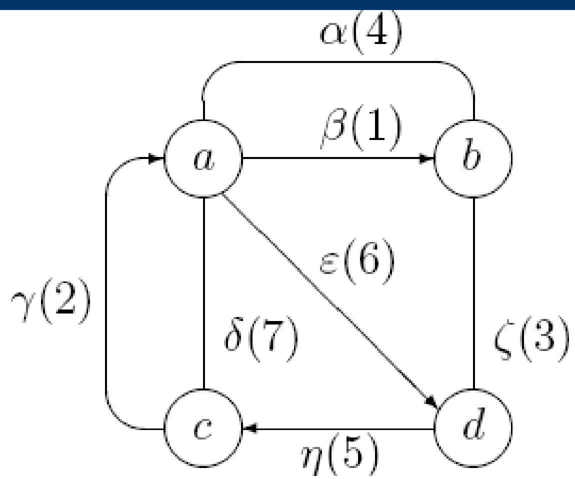
Since

$$d(i,k) \leq c_{i,j} + d(j,k);$$
$$d(j,k) \leq c_{i,j} + d(i,k),$$

so $f^* \in [0,1]$.

Point-vertex distance

Example

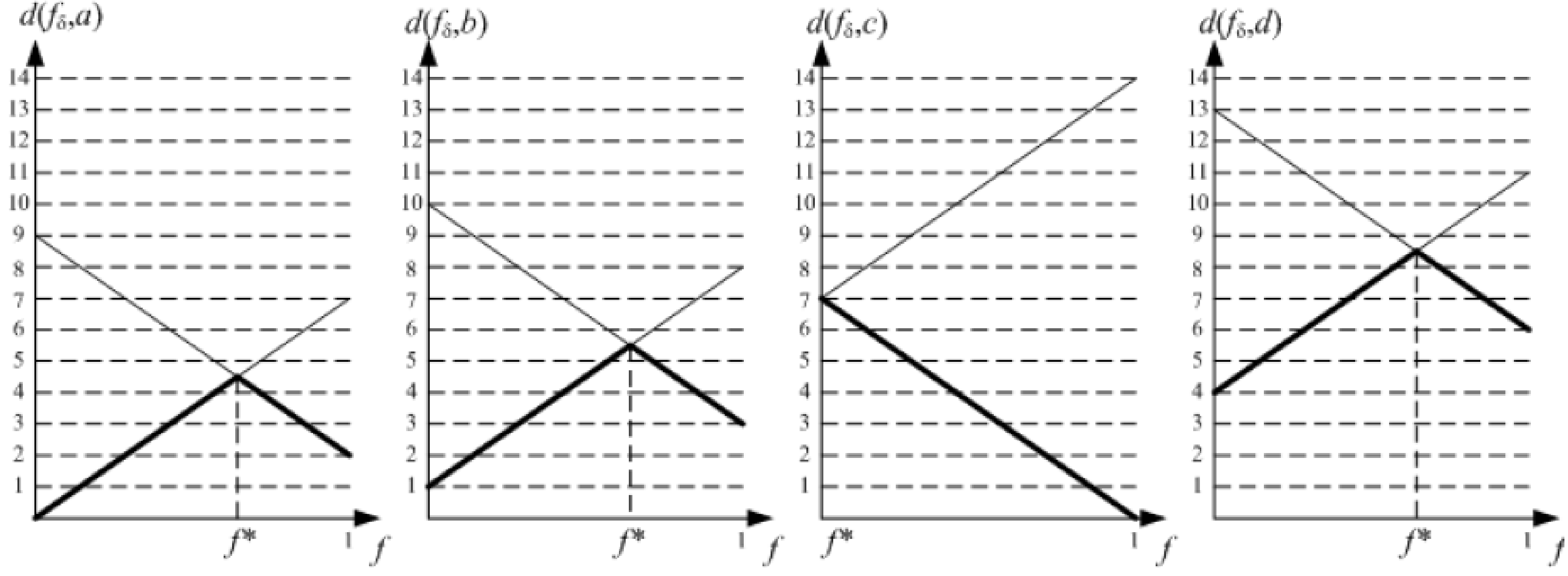


	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$d(f_\delta, a) = \min\{f c_\delta + d(a,a), (1-f)c_\delta + d(c,a)\} = \min\{7f + 0, 7(1-f) + 2\}\,;$$
$$d(f_\delta, b) = \min\{f c_\delta + d(a,b), (1-f)c_\delta + d(c,b)\} = \min\{7f + 1, 7(1-f) + 3\}\,;$$
$$d(f_\delta, c) = \min\{f c_\delta + d(a,c), (1-f)c_\delta + d(c,c)\} = \min\{7f + 7, 7(1-f) + 0\}\,;$$
$$d(f_\delta, d) = \min\{f c_\delta + d(a,d), (1-f)c_\delta + d(c,d)\} = \min\{7f + 4, 7(1-f) + 6\}\,.$$

Point-vertex distance

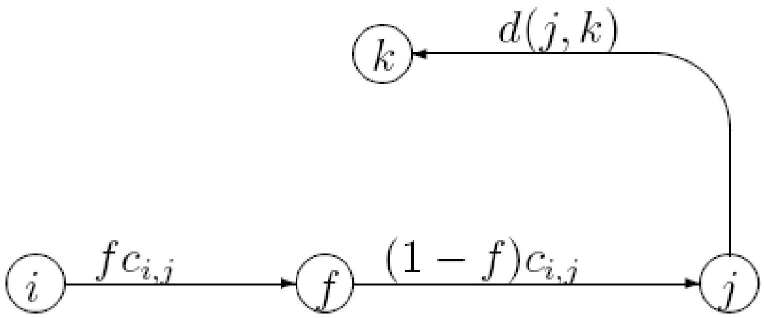
Example:

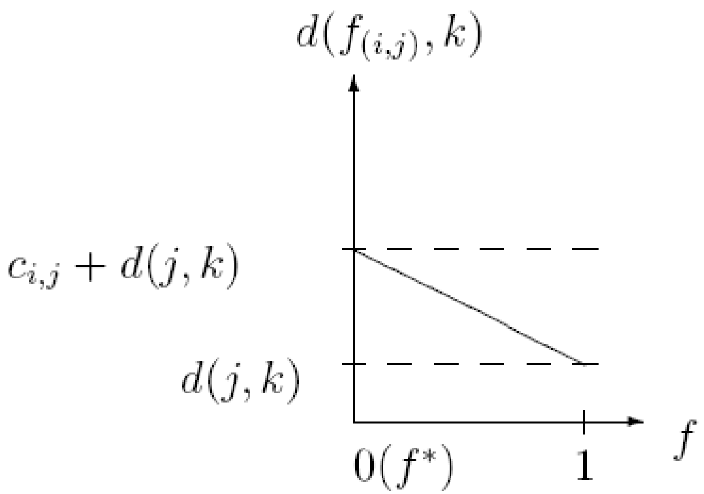


Point-vertex distance

For a directed edge (*i*,*j*):

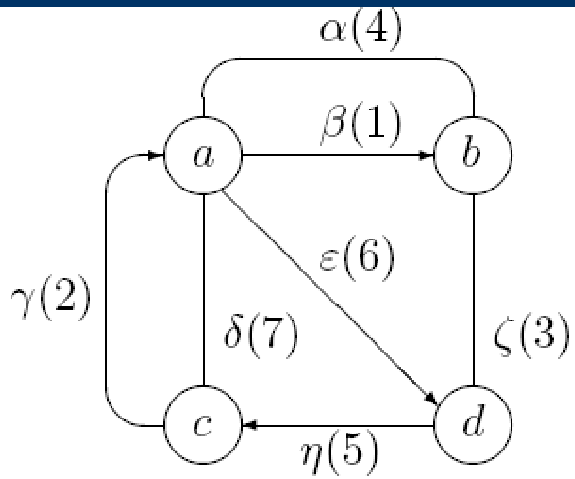
$$d(f_{(i,j)}, k) = (1 - f)c_{i,j} + d(j,k).$$





Point-vertex distance

Example

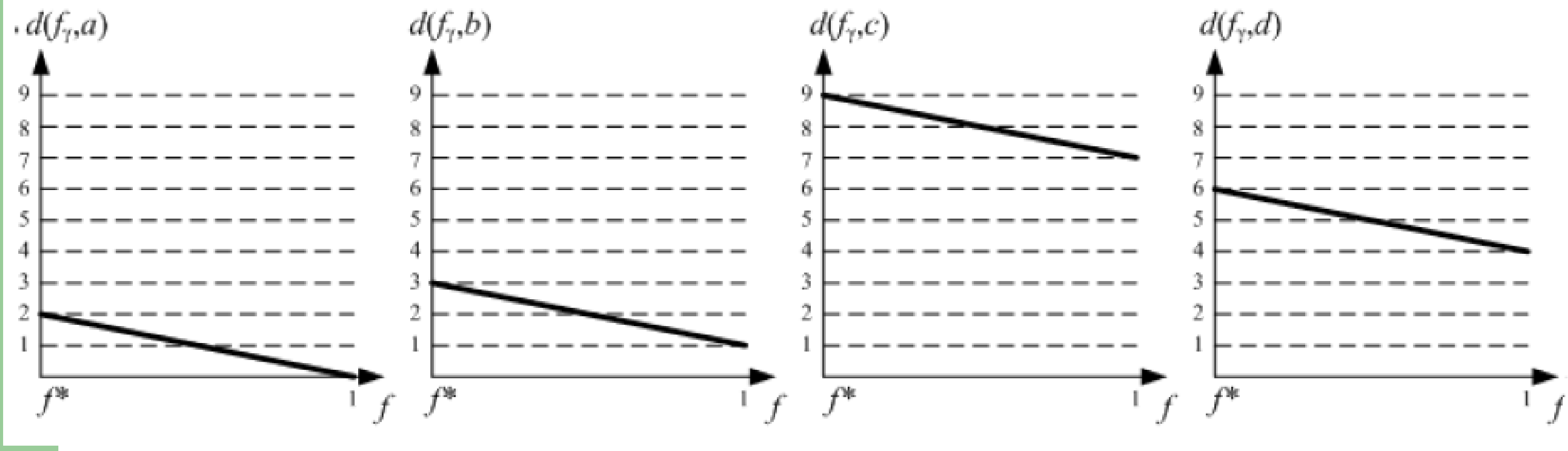


	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$d(f_\gamma, a) = (1-f)c_\gamma + d(a,a) = 2(1-f) + 0 = 2 - 2f;$$
$$d(f_\gamma, b) = (1-f)c_\gamma + d(a,b) = 2(1-f) + 1 = 3 - 2f;$$
$$d(f_\gamma, c) = (1-f)c_\gamma + d(a,c) = 2(1-f) + 7 = 9 - 2f;$$
$$d(f_\gamma, d) = (1-f)c_\gamma + d(a,d) = 2(1-f) + 4 = 6 - 2f;$$

Point-vertex distance

Example:



Vertex-point distance

The **vertex-point distance** between a vertex k and a point $f_{(i,j)}$ (notation $d(k, f_{(i,j)})$) is the weight of the minimal path $<k, f_{(i,j)}>$.

For an undirected edge ij:

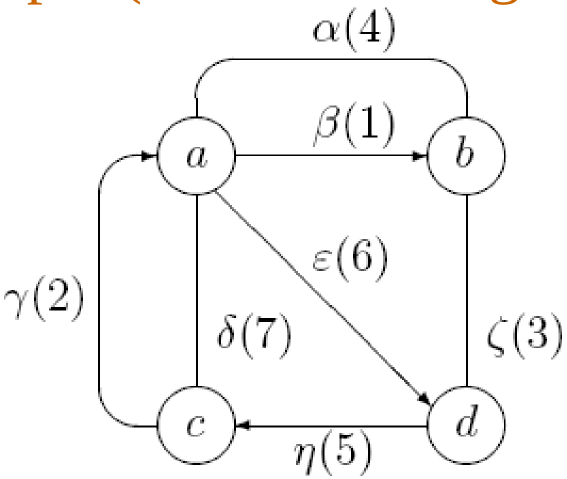
$$d(k, f_{(i,j)}) = \min\{d(k,i) + fc_{i,j}, d(k,j) + (1-f)c_{i,j}.\}$$

For a directed edge ij:

$$d(k, f_{(i,j)}) = d(k,i) + fc_{i,j}.$$

Vertex-point distance

Example (undirected edges):

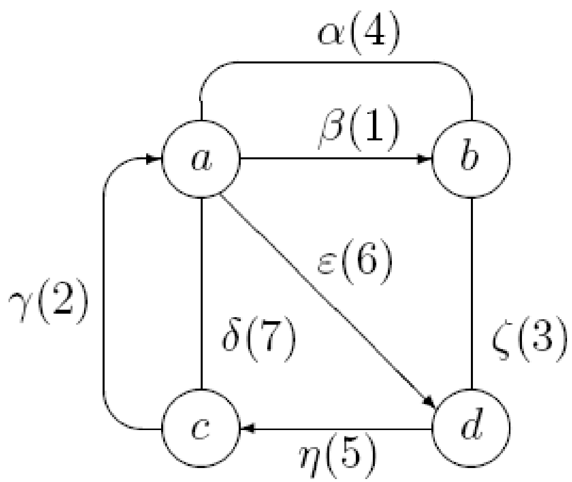


	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$\alpha = (a,b): \quad d(a, f_\alpha) = \min\{d(a,a) + fc_\alpha, d(a,b) + (1-f)c_\alpha\} = \min\{0 + 4f, 1 + 4(1-f)\};$$
$$\delta = (a,c): \quad d(a, f_\delta) = \min\{d(a,a) + fc_\delta, d(a,c) + (1-f)c_\delta\} = \min\{0 + 7f, 7 + 7(1-f)\};$$
$$\zeta = (b,d): \quad d(a, f_\zeta) = \min\{d(a,b) + fc_\zeta, d(a,d) + (1-f)c_\zeta\} = \min\{1 + 3f, 1 + 3(1-f)\}.$$

Vertex-point distance

Example (directed edges):



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$\beta = (a, b) : \quad d(a, f_\beta) = d(a, a) + fc_\beta = 0 + f;$$

$$\gamma = (c, a) : \quad d(a, f_\gamma) = d(a, c) + fc_\gamma = 7 + 2f;$$

$$\varepsilon = (a, d) : \quad d(a, f_\varepsilon) = d(a, a) + fc_\varepsilon = 0 + 6f;$$

$$\eta = (d, c) : \quad d(a, f_\eta) = d(a, d) + fc_\eta = 4 + 5f;$$

Vertex-edge distance

The **vertex-edge distance** between a vertex k and an edge ij (notation $d(k,(i,j))$) is the maximum vertex-point distance $d(k, f_{(i,j)})$:

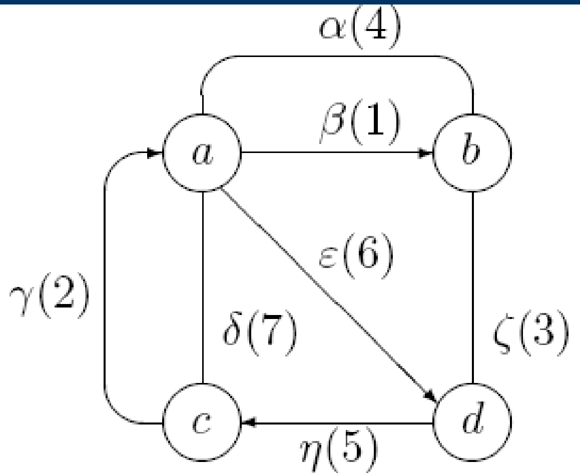
$$d(k,(i,j)) = \max_{f \in [0,1]} d(k, f_{(i,j)}).$$

For a directed edge (i,j) the maximum point $f^*=1$ and the vertex-edge distance

$$d(k,(i,j)) = d(k,i) + c_{i,j}.$$

Vertex-edge distance

Example (directed edges):

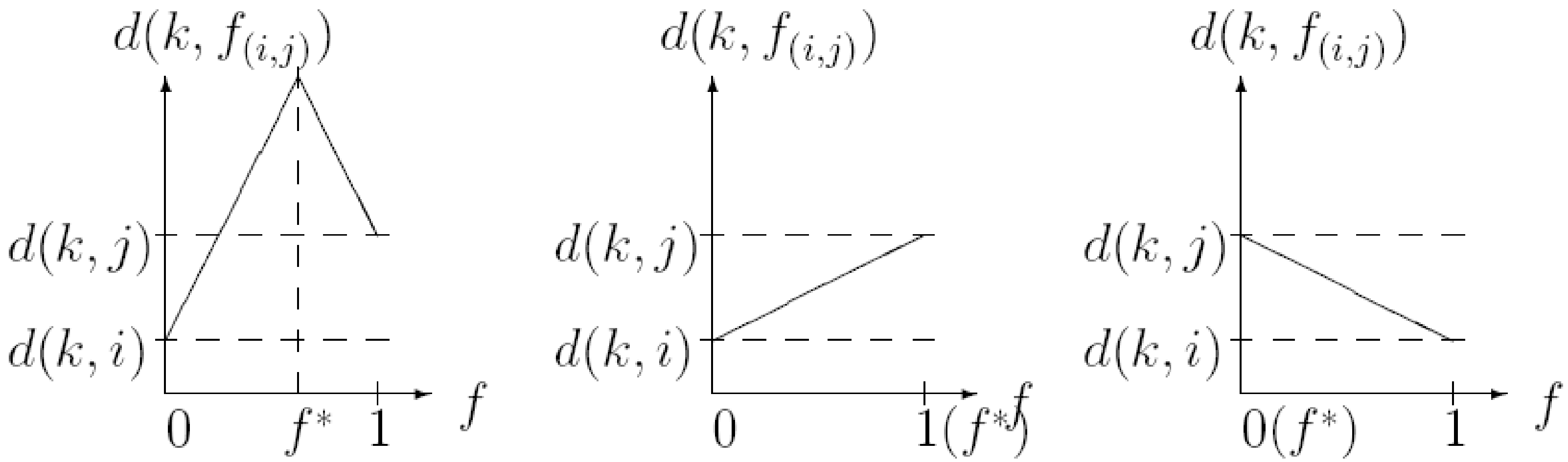


	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$\beta = (a, b) : \quad d(a, \beta) = d(a, a) + c_\beta = 0 + 1 = 1;$$
$$\gamma = (c, a) : \quad d(a, \gamma) = d(a, c) + c_\gamma = 7 + 2 = 9;$$
$$\varepsilon = (a, d) : \quad d(a, \varepsilon) = d(a, a) + c_\varepsilon = 0 + 6 = 6;$$
$$\eta = (d, c) : \quad d(a, \eta) = d(a, d) + c_\eta = 4 + 5 = 9.$$

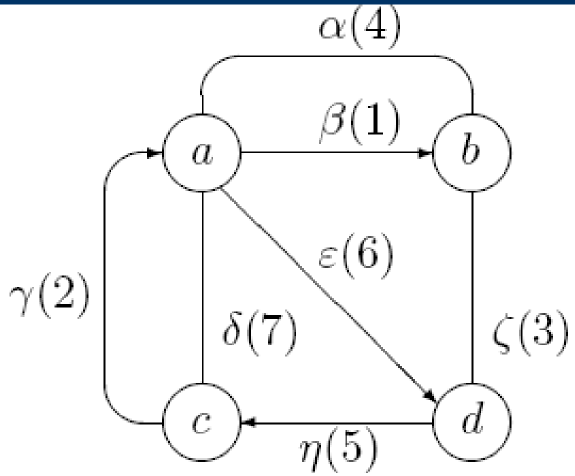
Vertex-edge distance

For an undirected edge (i,j) the dependence $d(k,f_{(i,j)})$ of f can be one of three types.



Vertex-edge distance

Example (undirected edges):



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$\alpha = (a,b): \quad d(a,\alpha) = \frac{d(a,a)+d(a,b)+c_\alpha}{2} = \frac{0+1+4}{2} = 2,5;$$

$$\delta = (a,c): \quad d(a,\delta) = \frac{d(a,a)+d(a,c)+c_\delta}{2} = \frac{0+7+7}{2} = 7;$$

$$\zeta = (b,d): \quad d(a,\zeta) = \frac{d(a,b)+d(a,d)+c_\zeta}{2} = \frac{1+4+3}{2} = 4.$$

Point-edge distance

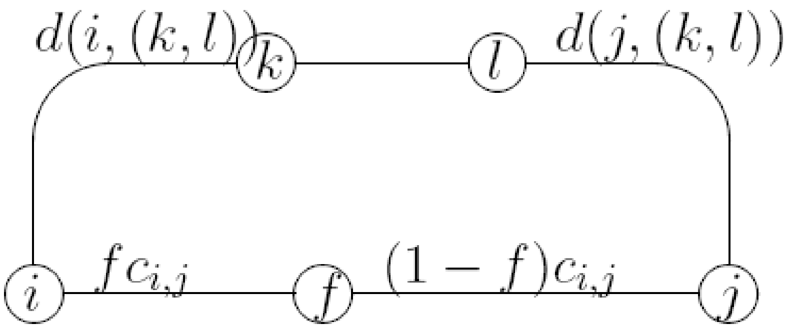
The **point-point distance** between a point $f_{(i,j)}$ and a point $g_{(k,l)}$ (notation $d(f_{(i,j)},g_{(k,l)})$) is the weight of the minimal path $<f_{(i,j)},g_{(k,l)}>$.

The **point-edge distance** between a point $f_{(i,j)}$ and an edge (k,l) (notation $d(f_{(i,j)},(k,l))$) is the maximum point-point distance $d(f_{(i,j)},g_{(k,l)})$:

$$d(f_{(i,j)},(k,l)) = \max_{g\in[0,1]} d(f_{(i,j)},g_{(k,l)}).$$

Point-edge distance

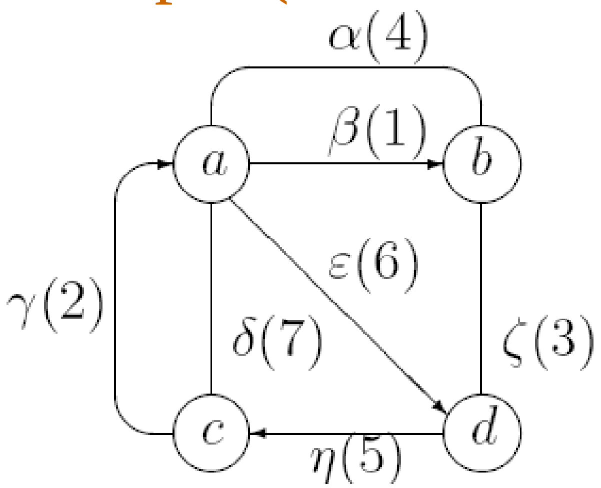
For an undirected edge $(i,j) \neq (k,l)$ the minimal path can pass through the vertex i or the vertex j:



$$d(f_{(i,j)},(k,l)) = \min\{fc_{i,j} + d(i,(k,l)), (1-f)c_{i,j} + d(j,(k,l))\}.$$

Point-edge distance

Example (undirected edge):



	α	β	γ	δ	ε	ζ	η
a	2, 5	1	9	7	6	4	9
b	4	5	10	9, 5	10	3	8
c	4, 5	3	2	4, 5	8	6	11
d	7	8	7	9, 5	13	3	5

$$\begin{aligned} d(f_\delta, \eta) &= \min\{f c_\delta + d(a,\eta), (1-f)c_\delta + d(c,\eta)\} = \\ &= \min\{7f + 9, 7(1-f) + 11\}\,. \end{aligned}$$

$$7f + 9 = 18 - 7f, \qquad f^* = 9/14.$$

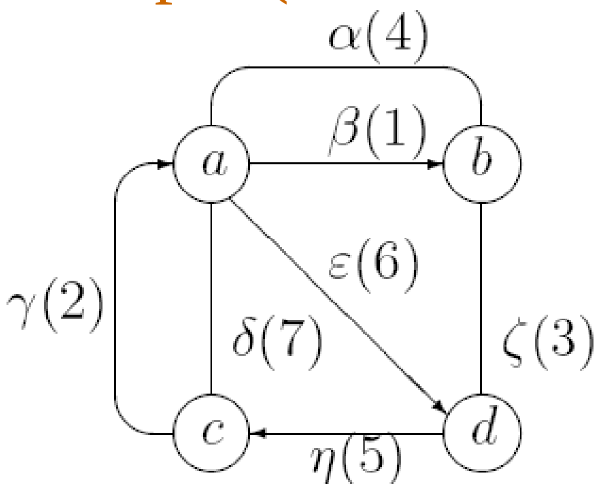
Point-edge distance

For a directed edge $(i,j) \neq (k,l)$ the minimal path can pass only through the vertex j:

$$d(f_{(i,j)}, (k,l)) = (1-f)c_{i,j} + d(j, (k,l)).$$

Point-edge distance

Example (directed edge):

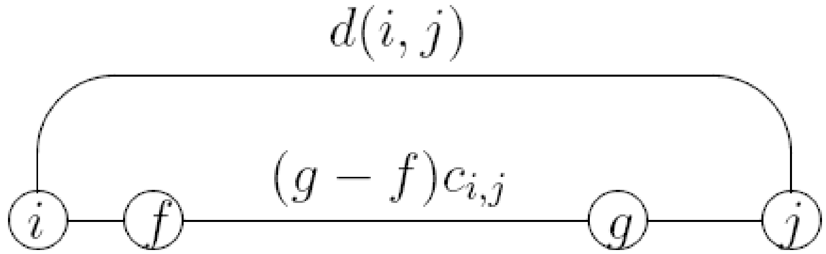


	α	β	γ	δ	ε	ζ	η
a	2, 5	1	9	7	6	4	9
b	4	5	10	9, 5	10	3	8
c	4, 5	3	2	4, 5	8	6	11
d	7	8	7	9, 5	13	3	5

$$d(f_\gamma, \eta) = (1-f)c_\gamma + d(a,\eta) = 2(1-f) + 9 = 11 - 2f.$$

Point-edge distance

For an undirected edge $(i,j)=(k,l)$ and $f<1/2$ the most distant points g are close to the vertex j. If $d(i,j)<c_{i,j}$ then the minimal path $<f_{(i,j)},g_{(i,j)}>$ can pass through the vertex i:



$$d(f_{(i,j)}, g_{(k,l)}) = \min\{(g-f)c_{i,j}, d(i,j) + (1-g+f)c_{i,j}\}$$

Point-edge distance

The maximum point g^* is the point of the lines intersection:

$$(g - f)c_{i,j} = d(j,i) + (1 - g + f)c_{i,j}.$$

Hence $$\max_g d(f_{(i,j)}, g_{(i,j)}) = \frac{d(i,j) + c_{i,j}}{2}$$

Point-edge distance

If the minimal path $<f_{(i,j)},g_{(i,j)}>$ passes only through the edge (i,j) then:

$$d(f_{(i,j)}, g_{(i,j)}) = (g - f)c_{i,j}.$$

The maximum point $g^{*}=1$.

$$\max_{g} d(f_{(i,j)}, g_{(i,j)}) = (1 - f)c_{i,j}.$$

Point-edge distance

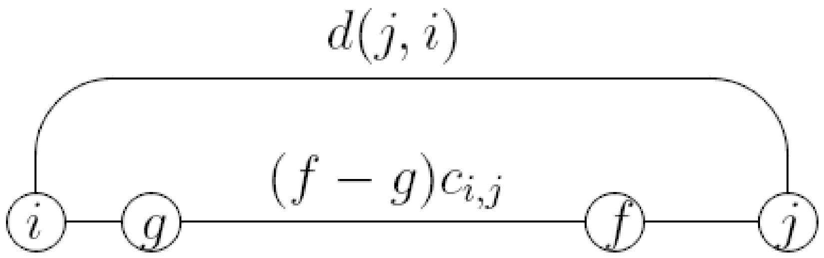
Hence the point-edge distance for $f<1/2$

$$d(f_{(i,j)},(i,j)) = \min\left\{(1-f)c_{i,j}, \frac{d(i,j)+c_{i,j}}{2}\right\}$$

This distance is maximum for $f=0$ and minimum for $f=1/2$. The minimum distance is equal to $c_{i,j}/2$.

Point-edge distance

For an undirected edge $(i,j)=(k,l)$ and $f>1/2$ the most distant points g are close to the vertex i. If $d(j,i)<c_{j,i}$ then the minimal path $<f_{(i,j)},g_{(i,j)}>$ can pass through the vertex j:



$$d(f_{(i,j)}, g_{(k,l)}) = \min\{(f-g)c_{i,j}, d(j,i) + (1-f+g)c_{i,j}\}$$

Point-edge distance

The maximum point g^* is the point of the lines intersection:

$$(f-g)c_{i,j} = d(j,i) + (1-g+f)c_{i,j}.$$

Hence $$\max_g d(f_{(i,j)}, g_{(i,j)}) = \frac{d(j,i)+c_{i,j}}{2}$$

Point-edge distance

If the minimal path <$f_{(i,j)}$,$g_{(i,j)}$> passes only through the edge (i,j) then:

$$d(f_{(i,j)}, g_{(i,j)}) = (f - g)c_{i,j}.$$

The maximum point $g^*=0$.

$$\max_g d(f_{(i,j)}, g_{(i,j)}) = f c_{i,j}.$$

Point-edge distance

Hence the point-edge distance for $f>1/2$

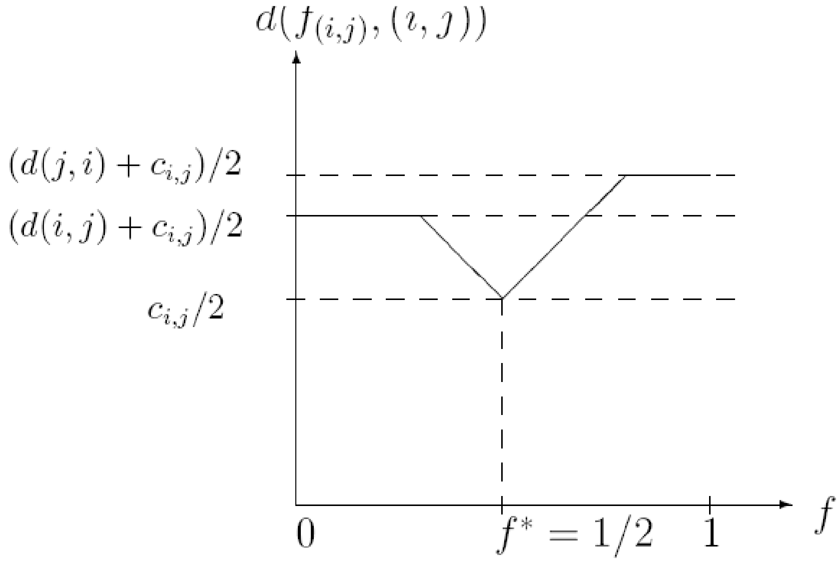
$$d(f_{(i,j)},(i,j)) = \min\left\{ f c_{i,j}, \frac{d(j,i)+c_{i,j}}{2} \right\}$$

This distance is maximum for $f=1$ and minimum for $f=1/2$. The minimum distance is equal to $c_{i,j}/2$.

Point-edge distance

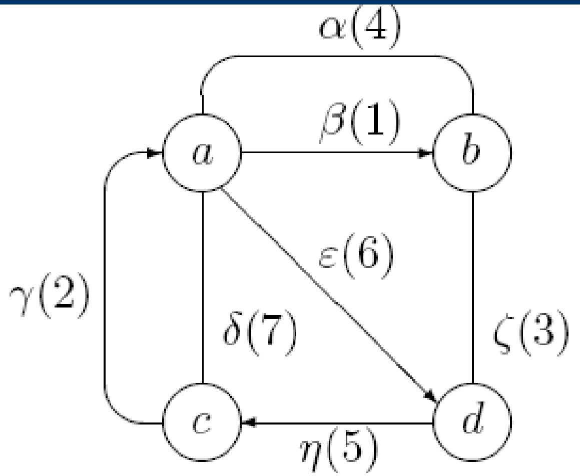
Finally, the point-edge distance is

$$\max\left\{\min\left\{(1-f)c_{i,j}, \frac{d(i,j)+c_{i,j}}{2}\right\}, \min\left\{fc_{i,j}, \frac{d(j,i)+c_{i,j}}{2}\right\}\right\}.$$



Point-edge distance

Example (undirected edges):



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$d(f_\delta, \delta) =$$

$$= \max\left\{\min\left\{(1-f)c_\delta, \frac{d(a,c)+c_\delta}{2}\right\}, \min\left\{fc_\delta, \frac{d(c,a)+c_\delta}{2}\right\}\right\} =$$

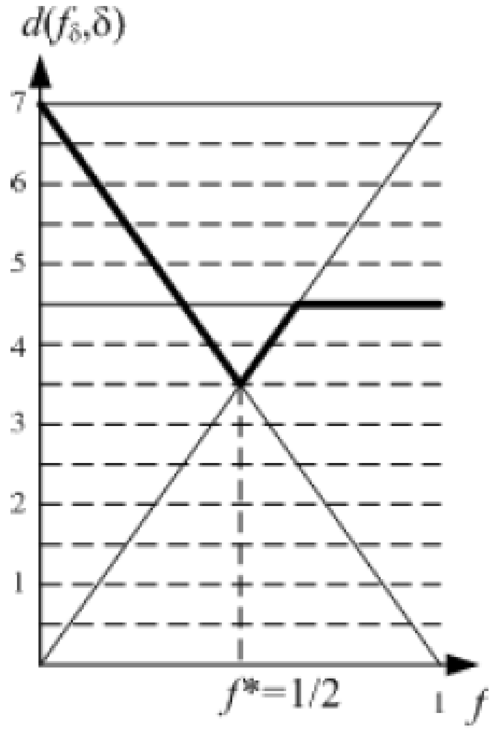
$$= \max\left\{\min\left\{7(1-f), \frac{7+7}{2}\right\}, \min\left\{7f, \frac{2+7}{2}\right\}\right\} =$$

$$= \max\left\{\min\left\{7-7f, 7\right\}, \min\left\{7f, 4, 5\right\}\right\}.$$

Point-edge distance

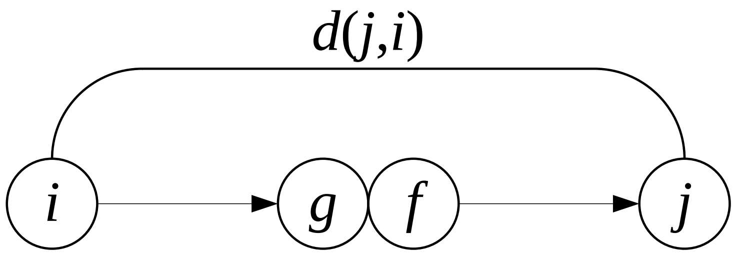
Example (undirected edges):

$$\max\{\min\{7-7f, 7\}, \min\{7f, 4, 5\}\}$$



Point-edge distance

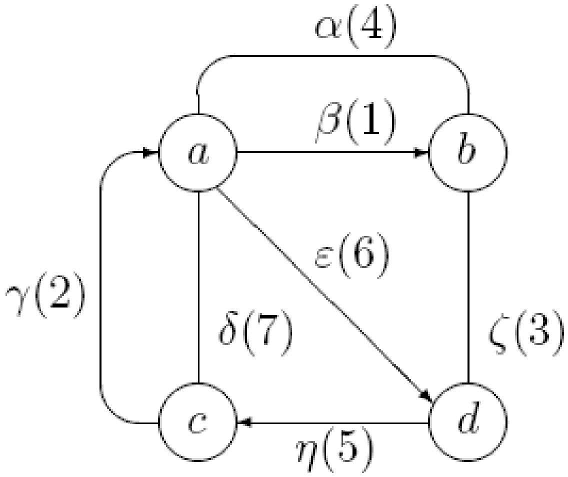
For a directed edge $(i,j)=(k,l)$ the most distant points g are situated between the vertex i and the point f close to the point f .



$$d(f_{(i,j)}, (i,j)) = d(j,i) + c_{i,j}.$$

Point-edge distance

Example (directed edges):



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

$$d(f_\gamma, \gamma) = d(a, c) + c_\gamma = 7 + 2 = 9.$$

Maximum distances

Maximum vertex-vertex: $\mathrm{MVV}(i) = \max_j \{d(i, j)\}.$

Maximum point-vertex: $\mathrm{MPV}(f_{(i,j)}) = \max_k \{d(f_{(i,j)}, k)\}.$

Maximum vertex-edge: $\mathrm{MVE}(i) = \max_{(k,l)} \{d(i, (k, l))\}.$

Maximum point-edge: $\mathrm{MPE}(f_{(i,j)}) = \max_{(k,l)} \{d(f_{(i,j)}, (k, l))\}.$

Total distances

Total vertex-vertex: $$\mathrm{TVV}(i) = \sum_j \{d(i,j)\}.$$

Total point-vertex: $$\mathrm{TPV}(f_{(i,j)}) = \sum_k \{d(f_{(i,j)}, k)\}.$$

Total vertex-edge: $$\mathrm{TVE}(i) = \sum_{(k,l)} \{d(i,(k,l))\}.$$

Total point-edge: $$\mathrm{TPE}(f_{(i,j)}) = \sum_{(k,l)} \{d(f_{(i,j)},(k,l))\}.$$

4.2. Centers of a graph

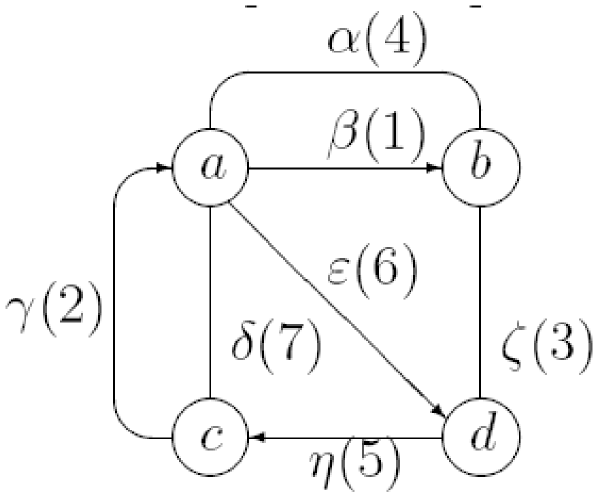
- Center
- General center
- Absolute center
- General absolute center

Center

A **center** of graph G is any vertex v of graph G such that

$$\mathrm{MVV}(v) = \min_j \mathrm{MVV}(j).$$

Example. Vertex *c* is the center.



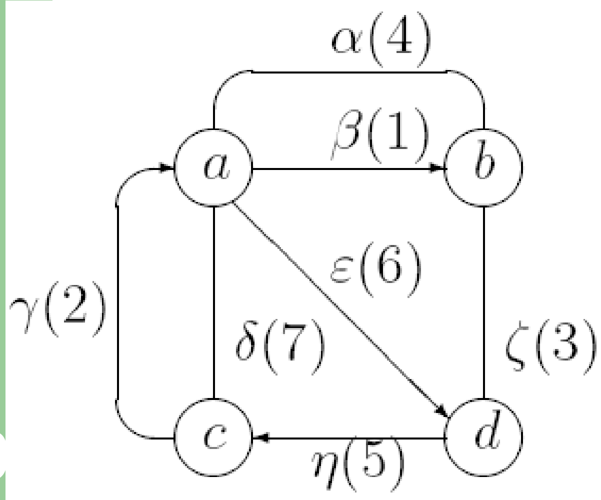
	a	b	c	d	$\mathrm{MVV}(v)$	
a	0	1	7	4	7	
b	4	0	8	3	8	
c	2	3	0	6	6	min
d	7	3	5	0	7	

General center

A **general center** of graph G is any vertex v of graph G such that

$$\mathrm{MVE}(v) = \min_j \mathrm{MVE}(j).$$

Example. Vertex *a* is the general center.



	α	β	γ	δ	ε	ζ	η	$\mathrm{MVE}(v)$	
a	$2,5$	1	9	7	6	4	9	9	min
b	4	5	10	$9,5$	10	3	8	10	
c	$4,5$	3	2	$4,5$	8	6	11	11	
d	7	8	7	$9,5$	13	3	5	13	

Absolute center

An **absolute center** of graph *G* is any point *g* of graph *G* such that

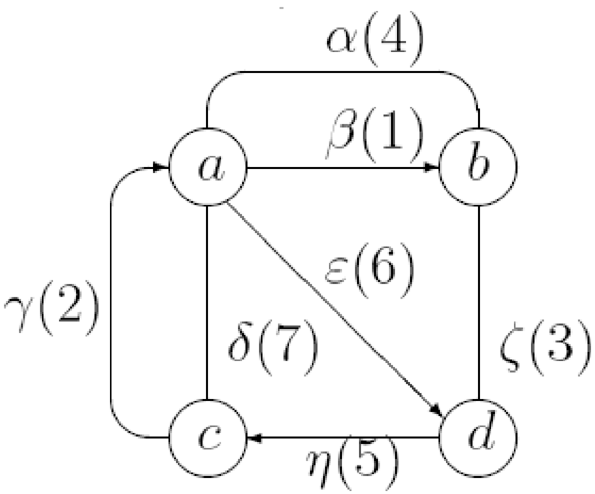
$$\mathrm{MPV}(g_{(v,u)}) = \min_{f_{(i,j)}} \mathrm{MPV}(f_{(i,j)}).$$

Theorem. No interior point of a directed edge can be an absolute center.

Point *f** of an undirected edge can be a candidate for absolute center if it is gives the minimal value of the upper portion of the point-vertex distance from point *f** to all the vertices.

Absolute center

Example.



	a	b	c	d	MVV(v)
a	0	1	7	4	7
b	4	0	8	3	8
c	2	3	0	6	6
d	7	3	5	0	7

Absolute center

Example. Edge δ=(a,c).

$$
\begin{aligned}
d(f_\delta, a) &= \min\{fc_\delta + d(a,a), (1-f)c_\delta + d(c,a)\} = \\
&= \min\{7f+0, 7(1-f)+2\}\,; \\
d(f_\delta, b) &= \min\{fc_\delta + d(a,b), (1-f)c_\delta + d(c,b)\} = \\
&= \min\{7f+1, 7(1-f)+3\}\,; \\
d(f_\delta, c) &= \min\{fc_\delta + d(a,c), (1-f)c_\delta + d(c,c)\} = \\
&= \min\{7f+7, 7(1-f)+0\}\,; \\
d(f_\delta, d) &= \min\{fc_\delta + d(a,d), (1-f)c_\delta + d(c,d)\} = \\
&= \min\{7f+4, 7(1-f)+6\}\,.
\end{aligned}
$$

Absolute center

Example. Edge α=(a,b).

$$\begin{aligned} d(f_\alpha, a) &= \min\{f c_\alpha + d(a,a), (1-f)c_\alpha + d(b,a)\} = \\ &= \min\{4f + 0, 4(1-f) + 4\}; \\ d(f_\alpha, b) &= \min\{f c_\alpha + d(a,b), (1-f)c_\alpha + d(b,b)\} = \\ &= \min\{4f + 1, 4(1-f) + 0\}; \\ d(f_\alpha, c) &= \min\{f c_\alpha + d(a,c), (1-f)c_\alpha + d(b,c)\} = \\ &= \min\{4f + 7, 4(1-f) + 8\}; \\ d(f_\alpha, d) &= \min\{f c_\alpha + d(a,d), (1-f)c_\alpha + d(b,d)\} = \\ &= \min\{4f + 4, 4(1-f) + 3\}. \end{aligned}$$

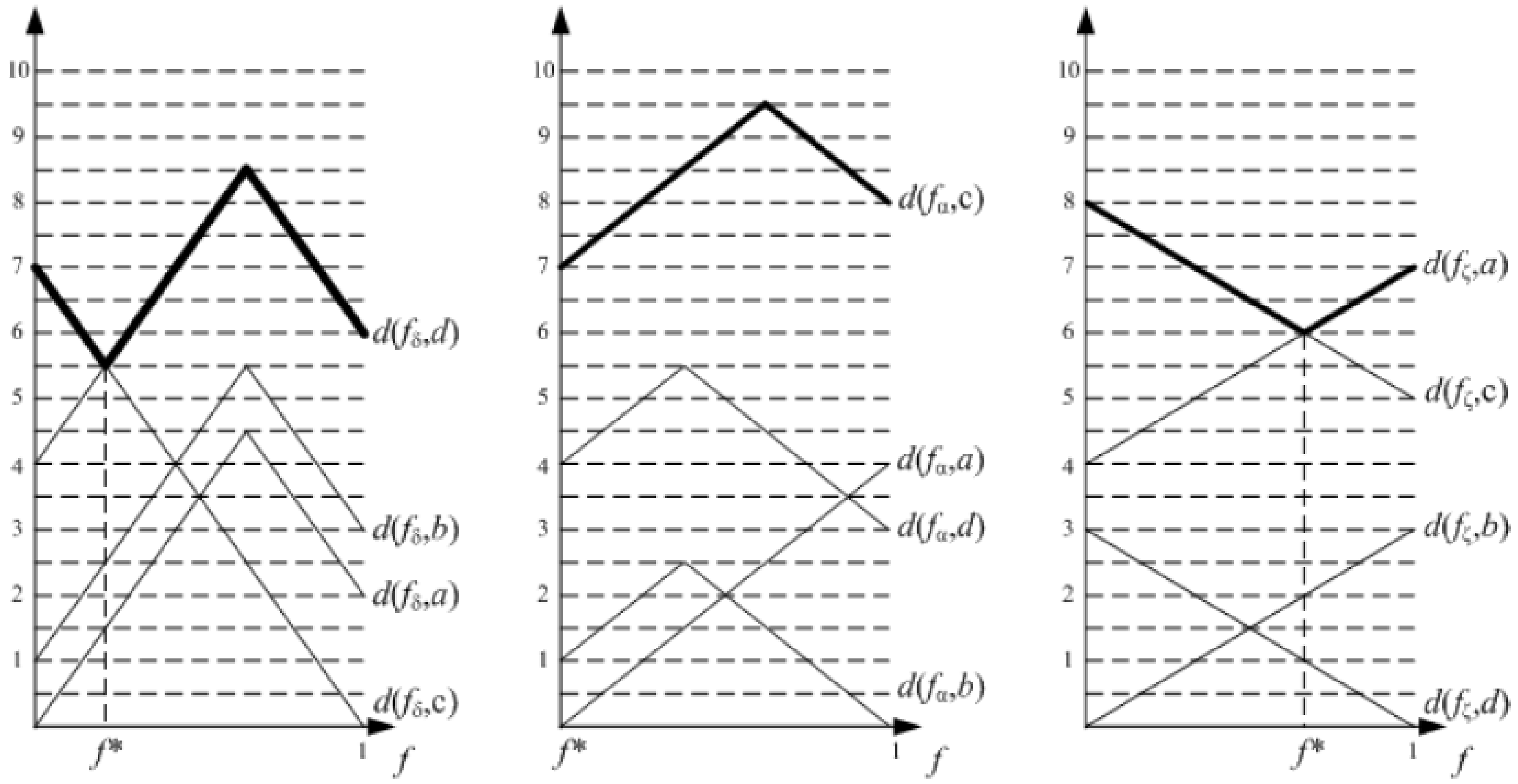
Absolute center

Example. Edge ζ=(b,d).

$$\begin{aligned}
d(f_\zeta, a) &= \min\{fc_\zeta + d(b,a), (1-f)c_\zeta + d(d,a)\} = \\
&= \min\{3f + 4, 3(1-f) + 7\}\,; \\
d(f_\zeta, b) &= \min\{fc_\zeta + d(b,b), (1-f)c_\zeta + d(d,b)\} = \\
&= \min\{3f + 0, 3(1-f) + 3\}\,; \\
d(f_\zeta, c) &= \min\{fc_\zeta + d(b,c), (1-f)c_\zeta + d(d,c)\} = \\
&= \min\{3f + 8, 3(1-f) + 5\}\,; \\
d(f_\zeta, d) &= \min\{fc_\zeta + d(b,d), (1-f)c_\zeta + d(d,d)\} = \\
&= \min\{3f + 3, 3(1-f) + 0\}\,.
\end{aligned}$$

Absolute center

Example. Plots of point-vertex distances.



Absolute center

Example.

For edge δ=(a,c):

$$7 - 7f = 4 + 7f,$$

$$f^* = 3/14, \quad \mathrm{MPV}(f^*_\delta) = 7 - 7 \times 3/14 = 5,5.$$

For edge α=(a,b): f*=0 (vertex a).

For edge ζ=(b,d):

$$8 - 3f = 4 + 3f,$$

$$f^* = 2/3, \quad \mathrm{MPV}(f^*_\delta) = 8 - 3 \times 2/3 = 6.$$

Absolute center: point $3/14_{\delta}$, MPV($3/14_{\delta}$)=5,5.

General absolute center

An **general absolute center** of graph *G* is any point *g* of graph *G* such that

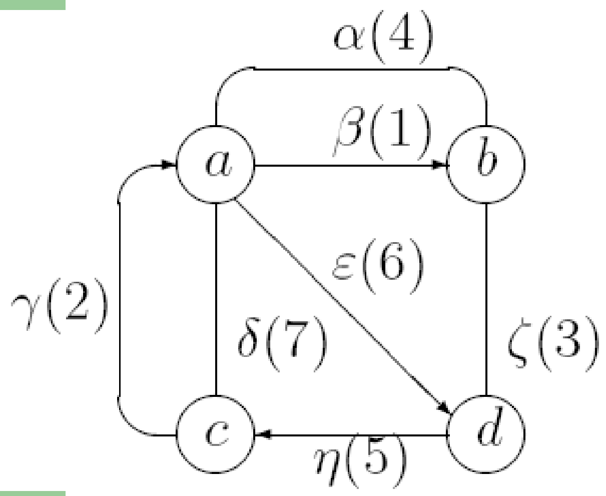
$$\mathrm{MPE}(g_{(v,u)}) = \min_{f_{(i,j)}} \mathrm{MPE}(f_{(i,j)}).$$

Theorem. If an interior point of a directed edge is a general absolute center then its end is also a general absolute center.

Point *f** of an undirected edge can be a candidate for general absolute center if it is gives the minimal value of the upper portion of the point-edge distance from point *f** to all the edges.

General absolute center

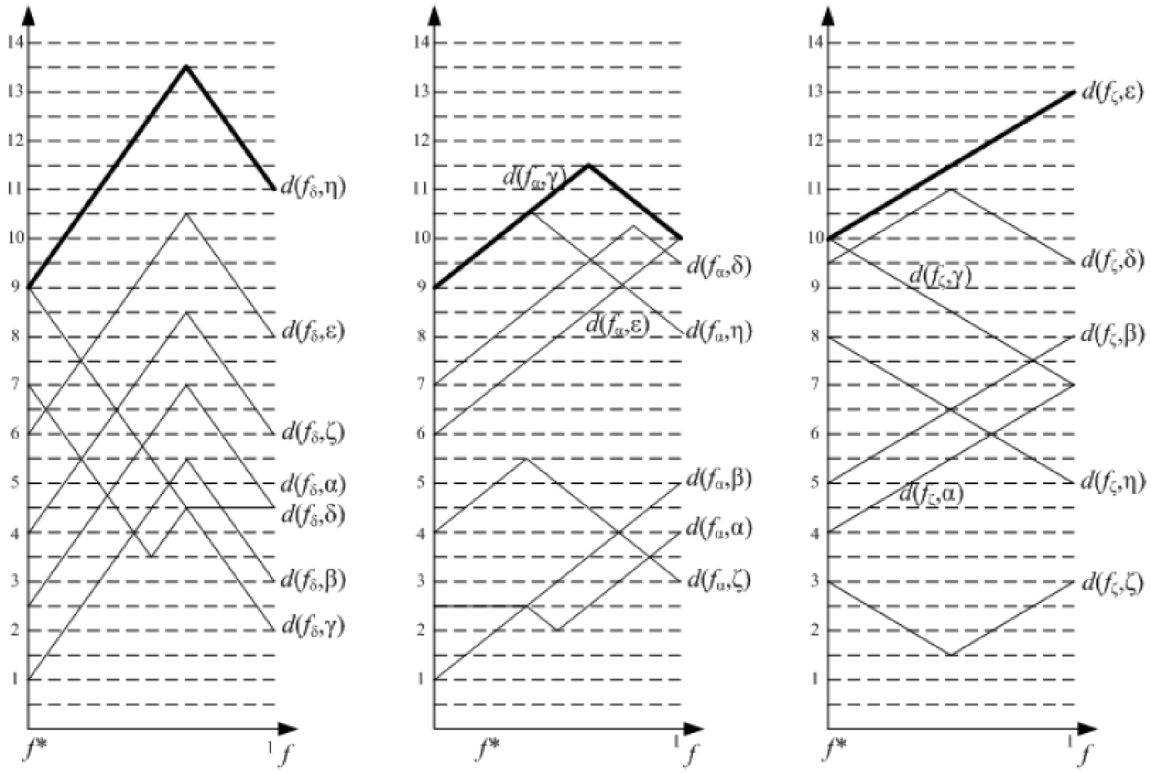
Example.



	α	β	γ	δ	ε	ζ	η	MVE(v)	
a	2, 5	1	9	7	6	4	9	9	min
b	4	5	10	9, 5	10	3	8	10	
c	4, 5	3	2	4, 5	8	6	11	11	
d	7	8	7	9, 5	13	3	5	13	

General absolute center

Example. Plots of point-edge distances. Vertex *a* is the general absolute center.



4.3. Medians of a graph

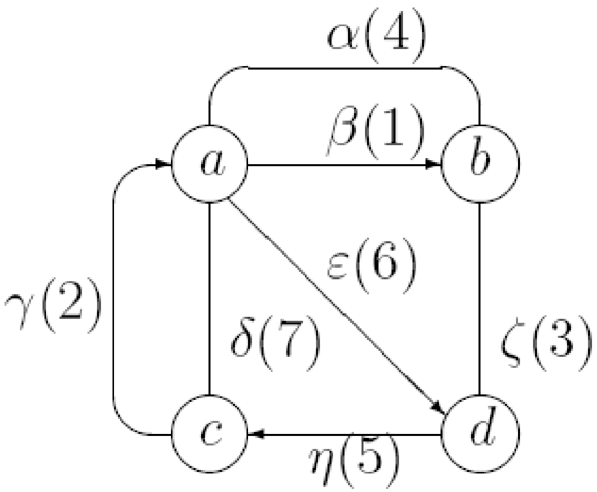
- Median
- General median
- Absolute median
- General absolute median

Median

A **median** of graph G is any vertex v of graph G such that

$$\mathrm{TVV}(v) = \min_j \mathrm{TVV}(j).$$

Example. Vertex *c* is the median.



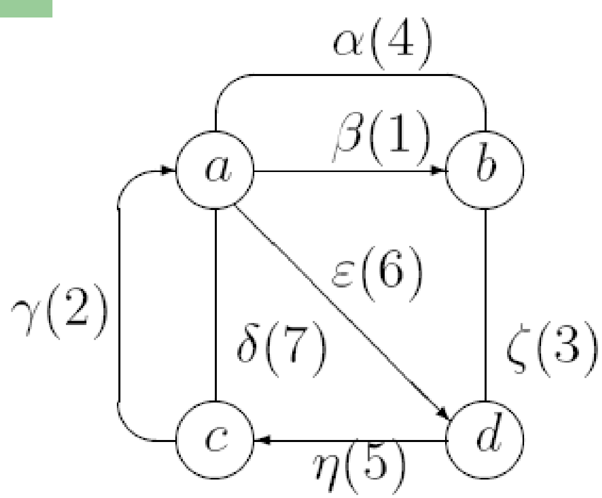
	a	b	c	d	TVV(v)	
a	0	1	7	4	12	
b	4	0	8	3	15	
c	2	3	0	6	11	min
d	7	3	5	0	15	

General median

A **general median** of graph G is any vertex v of graph G such that

$$\mathrm{TVE}(v) = \min_j \mathrm{TVE}(j).$$

Example. Vertex *a* is the general median.



	α	β	γ	δ	ε	ζ	η	TVE(v)	
a	2,5	1	9	7	6	4	9	38,5	min
b	4	5	10	9,5	10	3	8	49,5	
c	4,5	3	2	4,5	8	6	11	39	
d	7	8	7	9,5	13	3	5	52,5	

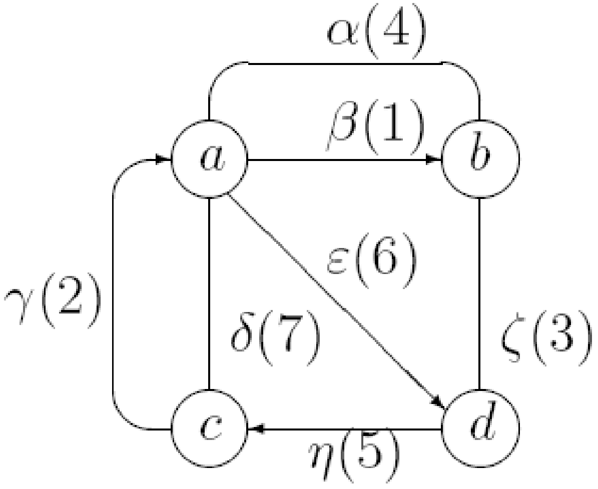
Absolute median

An **absolute median** of graph G is any point g of graph G such that

$$\mathrm{TPV}(g_{(v,u)}) = \min_{f_{(i,j)}} \mathrm{TPV}(f_{(i,j)}).$$

Theorem. There is always a vertex that is an absolute median.

Example. Vertex *c* is the median and the absolute median.



	a	b	c	d	$\mathrm{TVV}(v)$	
a	0	1	7	4	12	
b	4	0	8	3	15	
c	2	3	0	6	11	min
d	7	3	5	0	15	

General absolute median

A **general absolute median** of graph G is any point g of graph G such that

$$\mathrm{TPE}(g_{(v,u)}) = \min_{f_{(i,j)}} \mathrm{TPE}(f_{(i,j)}).$$

Theorem. No interior point of a directed edge can be a general absolute median.

Theorem. There is always a vertex or the middle point of an undirected edge that is a general absolute median.

General absolute median

$$(i,j) \neq (k,l)$$

$$d((1/2)_{(i,j)},(k,l)) = \frac{1}{2}c_{i,j} + \min\{d(i,(k,l)), d(j,(k,l))\}.$$

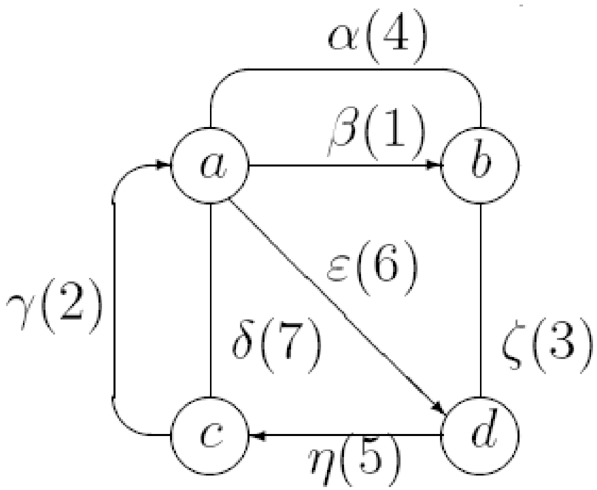
$$(i,j) = (k,l)$$

$$d((1/2)_{(i,j)},(i,j)) = \frac{1}{2}c_{i,j}.$$

$$\mathrm{TPE}\left(\frac{1}{2}_{(i,j)}\right) = \frac{q}{2}c_{i,j} + \sum_{(k,l)\neq(i,j)} \min\{d(i,(k,l)), d(j,(k,l))\}.$$

General absolute median

Example.



	α	β	γ	δ	ε	ζ	η	TVE(v)	
a	2,5	1	9	7	6	4	9	38,5	min
b	4	5	10	9,5	10	3	8	49,5	
c	4,5	3	2	4,5	8	6	11	39	
d	7	8	7	9,5	13	3	5	52,5	

General absolute median

Example.

$$\mathrm{TPE}\left(\frac{1}{2}_\alpha\right) = \frac{7}{2}c_\alpha + \sum_{e\neq\alpha}\min\{d(a,e),d(b,e)\} =$$

$$= \frac{7}{2}4 + \min\{1,5\} + \min\{9,10\} +$$

$$+ \min\{7,\ 9,5\} + \min\{6,10\} + \min\{4,3\} + \min\{9,8\} =$$

$$= 14 + 1 + 9 + 7 + 6 + 3 + 9 = 49;$$

$$\mathrm{TPE}\left(\frac{1}{2}_\delta\right) = \frac{7}{2}c_\delta + \sum_{e\neq\delta}\min\{d(a,e),d(c,e)\} =$$

$$= \frac{7}{2}7 + \min\{2,5,4,5\} + \min\{1,3\} +$$

$$+ \min\{9,2\} + \min\{6,8\} + \min\{4,6\} + \min\{9,11\} =$$

$$= 24,5 + 2,5 + 1 + 2 + 6 + 4 + 9 = 49;$$

General absolute median

Example

$$\mathrm{TPE}\left(\frac{1}{2}_{\zeta}\right) = \frac{7}{2}c_{\zeta} + \sum_{e \neq \zeta} \min\{d(b,e), d(d,e)\} =$$
$$= \frac{7}{2}3 + \min\{4,7\} + \min\{5,8\} +$$
$$+ \min\{10,7\} + \min\{9,5,\ 9,5\} + \min\{10,13\} + \min\{8,5\} =$$
$$= 10,5 + 4 + 5 + 7 + 9,5 + 10 + 5 = 51.$$

Vertex *a* is the general absolute median.

4.4. Extensions

- Weighted location
- Multicentres and multimedians

Weighted location

Suppose that different weights *W*(*j*) (*W*(*i*,*j*)) are associated with vertex *j* (edge (*i*,*j*)). This weights can be considered as probabilities or frequencies of visiting the vertex or the edge.

Vertex-vertex distance:

$$d^*(i,j) = W(j)d(i,j)$$

Vertex-edge distance:

$$d^*(i,(k,l)) = W(k,l)d(i,(k,l)).$$

Multicentres and multimedians

Let X_r be a subset of points of graph $G(V,E)$ containing r points.

Set-vertex distance d(X_r,j) is the minimum distance between any one of the points in set X_r and vertex j; i.e.

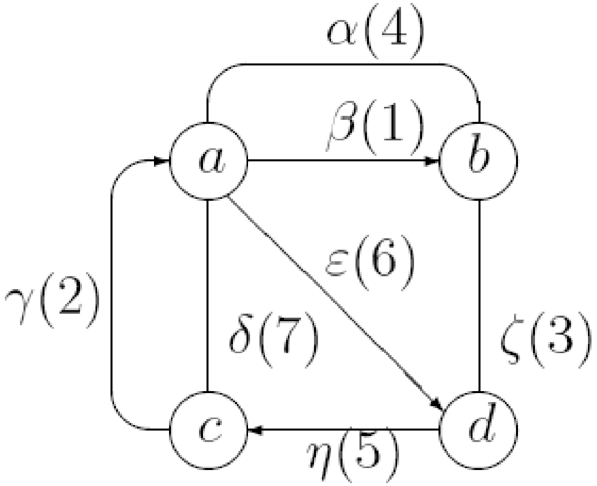
$$d(X_r, j) = \min_{i \in X_r} d(i, j).$$

Set-edge distance d(X_r,(k,l)) is the minimum distance between any one of the points in set X_r and edge (k,l), i.e.

$$d(X_r, (k, l)) = \min_{i \in X_r} d(i, (k, l)).$$

Multicentres and multimedians

Example. $X_3=\{c,(2/7)_\delta,(1/2)_\alpha\}$



	a	b	c	d
a	0	1	7	4
b	4	0	8	3
c	2	3	0	6
d	7	3	5	0

Multicentres and multimedians

Example.

$$d(c,d) = 6$$

$$\begin{aligned} d\left(\frac{2}{7}\delta, d\right) &= \min\left\{\tfrac{2}{7}c_\delta + d(a,d), \left(1-\tfrac{2}{7}\right)c_\delta + d(c,d)\right\} = \\ &= \min\left\{\frac{2}{7}7+4, \frac{5}{7}7+6\right\} = 6; \\ d\left(\frac{1}{2}\alpha, d\right) &= \min\left\{\tfrac{1}{2}c_\alpha + d(a,d), \left(1-\tfrac{1}{2}\right)c_\alpha + d(b,d)\right\} = \\ &= \min\left\{\frac{1}{2}4+4, \frac{1}{2}4+3\right\} = 5. \end{aligned}$$

$$d(X_3, d) = \min\left\{d(c,d), d\left(\frac{2}{7}\delta, d\right), d\left(\frac{1}{2}\alpha, d\right)\right\} = \min\{6,6,5\} = 5.$$

Multicentres and multimedians

Example.

$$d(c,\eta) = 11$$

$$\begin{aligned} d\left(\frac{2}{7}\delta,\eta\right) &= \min\left\{\tfrac{2}{7}c_\delta + d(a,d\eta), \left(1-\tfrac{2}{7}\right)c_\delta + d(c,\eta)\right\} = \\ &= \min\left\{\frac{2}{7}7+9, \frac{5}{7}7+11\right\} = 11; \\ d\left(\frac{1}{2}\alpha,\eta\right) &= \min\left\{\tfrac{1}{2}c_\alpha + d(a,\eta), \left(1-\tfrac{1}{2}\right)c_\alpha + d(b,\eta)\right\} = \\ &= \min\left\{\frac{1}{2}4+9, \frac{1}{2}4+8\right\} = 10. \end{aligned}$$

$$d(X_3,\eta) = \min\left\{d(c,\eta), d\left(\frac{2}{7}\delta,\eta\right), d\left(\frac{1}{2}\alpha,\eta\right)\right\} = \min\{11,11,10\} = 10.$$

Multicentres and multimedians

Multicenter and **multimedian** problems arise when there is a need to locate a number of facilities in the best possible way. The following distances can be minimize:

- Maximum set-vertex distance (MSV)
- Maximum set-edge distance (MSE)
- Total set-vertex distance (TSV)
- Total set-edge distance (TSE)

4.5. Absolute multicentres

Problems:

- (a) Find the optimal location anywhere on the graph of a given number (say p) of centres so that the distance required to reach the most remote vertex from its nearest centre is a minimum.
- (b) For a given "critical" distance, find the smallest number (and location) of centres so that all the vertices of the graph lie within this critical distance from at least one of the centres.

4.6. Multimedians

Problems:

- (a) Find the optimal location anywhere on the graph of a given number (say p) of medians so that the total distance required to reach all the vertices from its nearest median is a minimum.
- (b) For a given "critical" distance, find the smallest number (and location) of medians so that the total distance required to reach all the vertices from its nearest median lie within this critical distance.

Problem statement

X_p – multimedian (p-median)

$v \in X_p$ – **median vertex**

$v \notin X_p$ – **non-median vertex**

Vertex *j* is **allocated** to vertex *i* if vertex *i* is a median vertex and

$$d(Xp,j)=d(i,j).$$

Any median vertex *i* is allocated to vertex *i*.