“Graph theory”
Course for the master degree program “Geographic Information Systems”

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6. Matchings and Covers

- Independent and covering sets
- Independent and covering sets of vertices

6.1. Independent and covering sets

- Covering sets
- Cover numbers
- Independent sets
- Independence numbers
- Cover and independence numbers theorem

Covering sets

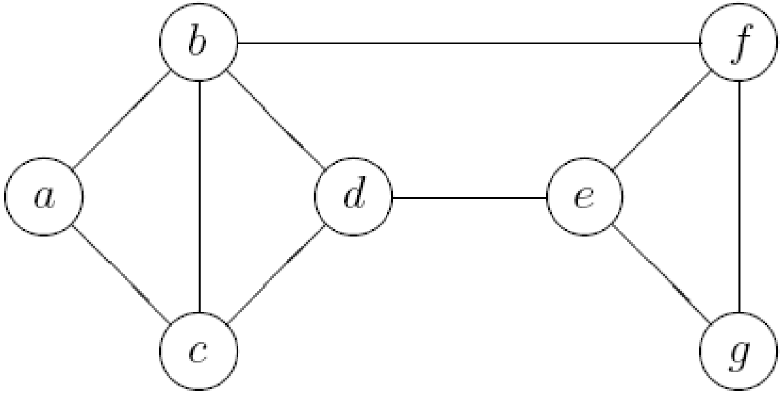
A vertex **covers** an edge if they are incident.

An edge **covers** a vertex if they are incident.

Example.

The vertex b covers the edges ab, bc, bd, bf

The edge ab covers the vertices a and b

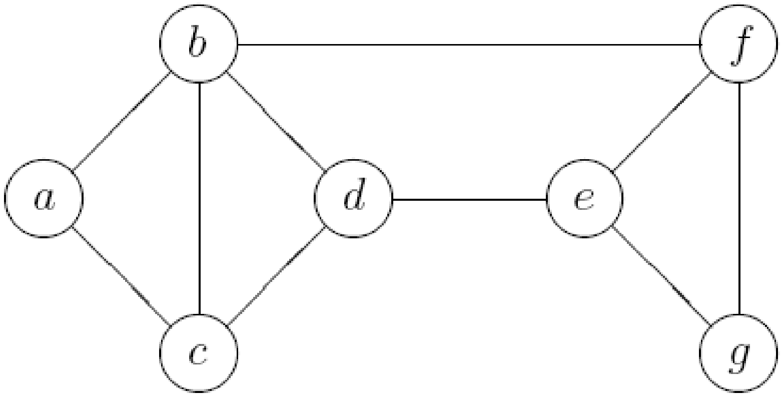


Covering sets

A **vertex covering set (vertex cover)** is a set of vertices of G covering all edges of G.

Example.

{a,b,d,e,f} – a vertex covering set.

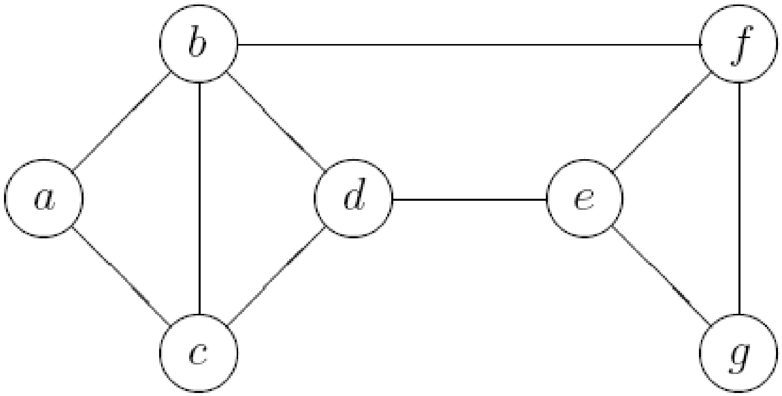


Covering sets

An **edge covering set (edge cover)** is a set of edges of G covering all vertices of G.

Example.

{ab,ac,de,fg} – an edge covering set.



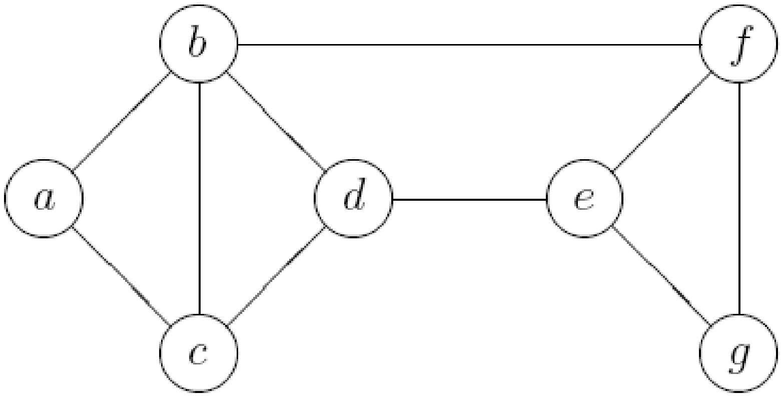
Minimum covering sets

A cover is called **minimum** when it contains the smallest possible number of vertices (edges).

Example.

{a,b,c,d,e,f} is not a minimum vertex cover

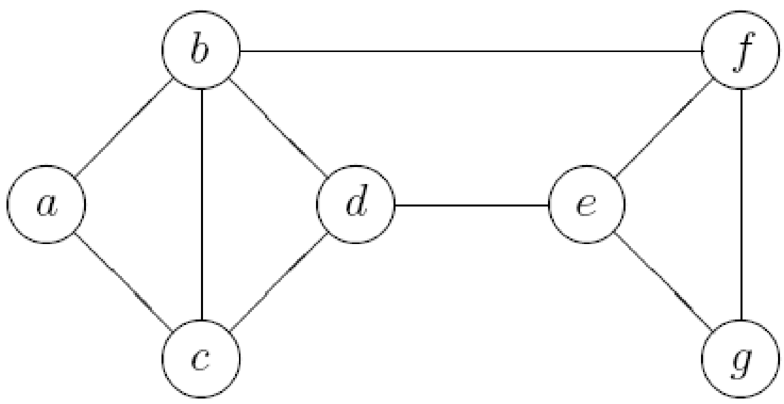
{b,c,e,g} is a minimum vertex cover.



Cover numbers

The **vertex cover number** α_0 of a graph G is the size of a minimum vertex cover in a graph, i.e., the minimum number of vertices covering all edges.

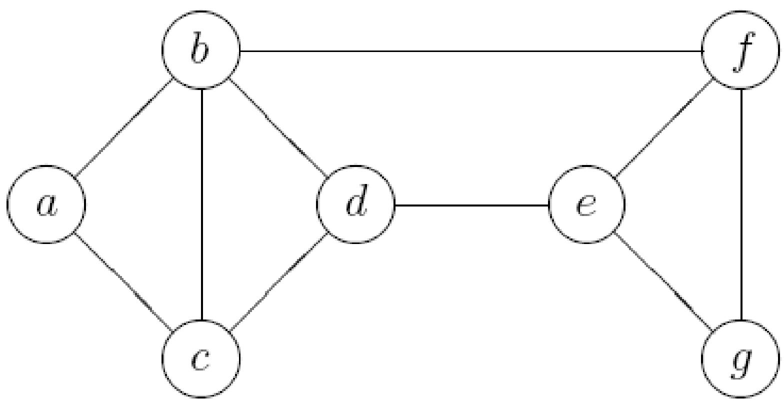
Example. α_0 =4, {b,c,e,f} – minimum vertex cover.



Cover numbers

The **edge cover number** α_1 of a graph G is the size of a minimum edge cover in a graph, i.e., the minimum number of edges covering all vertices.

Example. $\alpha_1 = 4$, {ab,cd,eg,ef} – minimum edge cover.



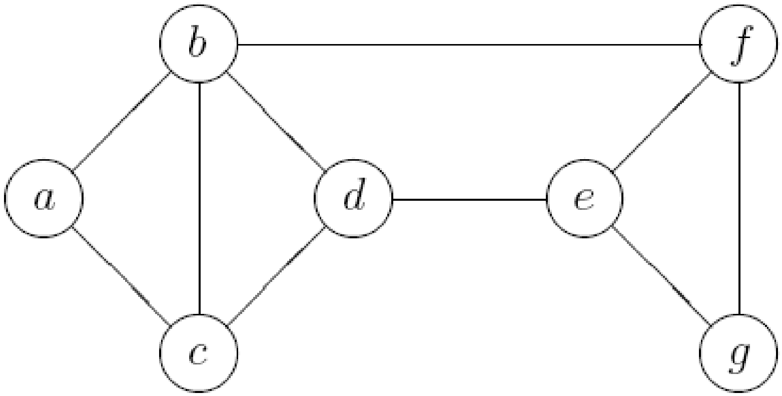
Independent sets

A **vertex (edge) independent set** is a set of vertices (edges) of G so that no two vertices (edges) of the set are adjacent.

Example.

{b,e} – independent vertex set.

{ab,cd,fg} – independent edge set.



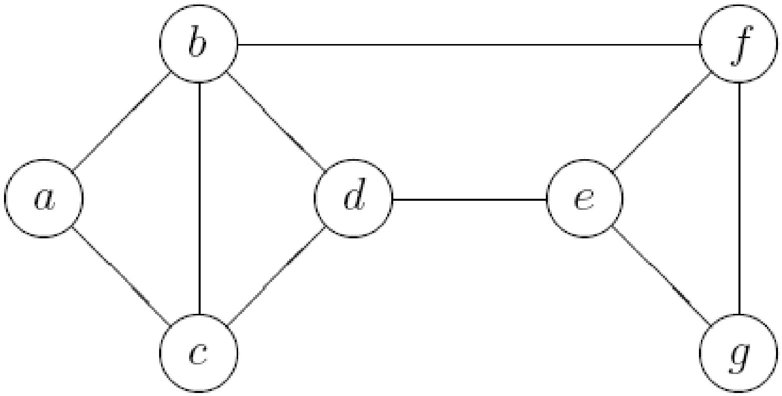
Maximum independent sets

An independent set is called **maximum** when it contains the greatest number of vertices (edges).

Example.

{b,e} is not a maximum vertex independent set.

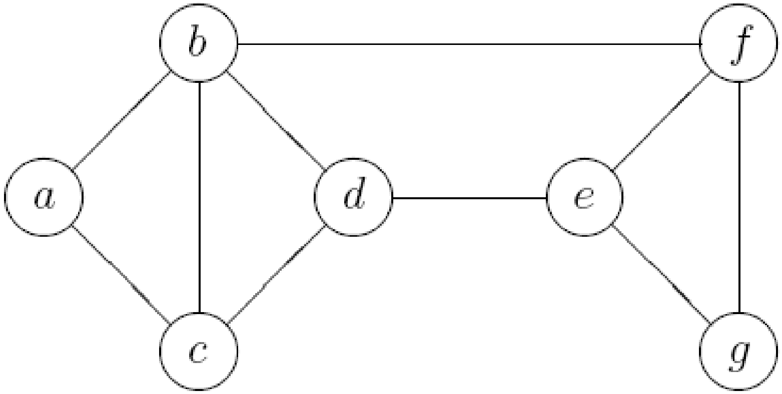
{a,d,f} is a maximum vertex independent set.



Independence numbers

The **vertex independence number** β_0 of a graph G is the maximum number of independent vertices.

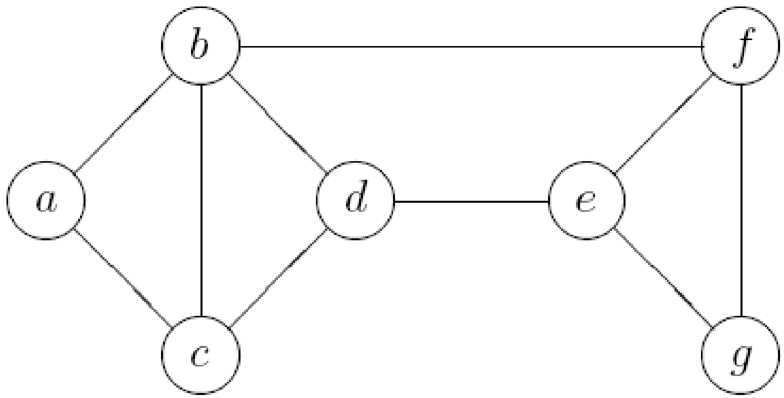
Example. $\beta_0 = 3$, {a,d,f} – independent vertex set.



Independence numbers

The **edge independence number** β_1 of a graph G is the maximum number of independent edges.

Example. $\beta_1 = 3$, {ab,cd,ef} – independent edge set.



Cover and independence numbers

	α_0	α_1	β_0	β_1
K_p				
$K_{m,n}$				
C_p				
Empty				

Cover and independence numbers

	α_0	α_1	β_0	β_1
K_p	p-1	p/2 (p–even), (p+1)/2 (p–odd)	1	p/2 (p–even), (p-1)/2 (p–odd)
$K_{m,n}$	min(m,n)	max(m,n)	max(m,n)	min(m,n)
C_p	p/2 (p–even), (p+1)/2 (p–odd)	p/2 (p–even), (p+1)/2 (p–odd)	p/2 (p–even), (p-1)/2 (p–odd)	p/2 (p–even), (p-1)/2 (p–odd)
Empty	0	no	p	0

Cover and independence number theorem

For every connected non-trivial graph

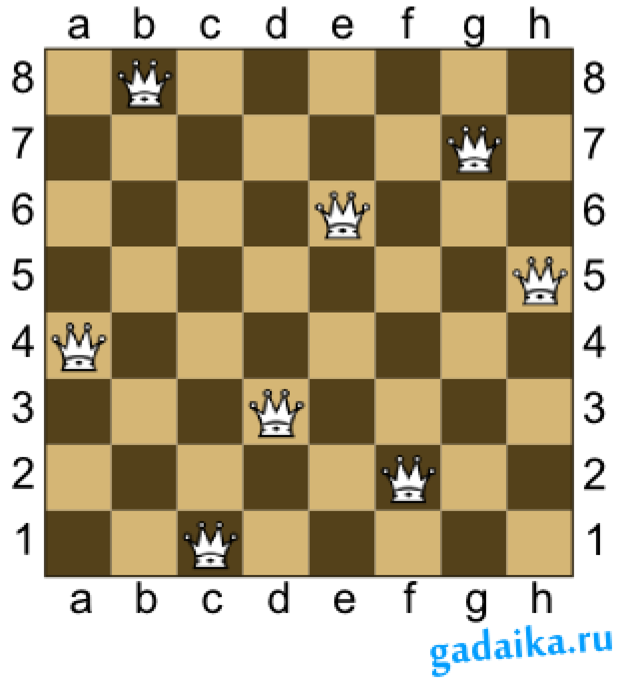
$$\alpha_0 + \beta_0 = \alpha_1 + \beta_1 = p.$$

6.2. Independent and covering sets of vertices

- Construction of independent sets
- Construction of covering sets
- Independent and covering sets
- Dominating sets
- Dominating and independent sets

Construction of independent sets

The **eight queens puzzle** is the problem of placing eight chess queens on an 8×8 chessboard so that no two queens threaten each other.

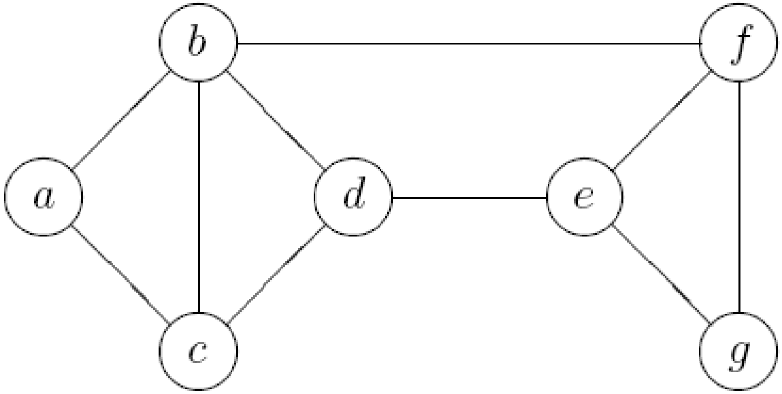


Construction of independent sets

An independent set is **maximal** if it is not a subset of any other independent set.

In other words, there is no vertex outside the independent set that may join it.

Example. {a,d} is not a maximal independent set, {a,d,f} is a maximal independent set.



Construction of independent sets

- **Backtracking** is a general algorithm for finding all (or some) solutions to some computational problems, notably constraint satisfaction problems, that incrementally builds candidates to the solutions, and abandons a partial candidate ("backtracks") as soon as it determines that it cannot possibly be completed to a valid solution.

- https://www.youtube.com/watch?v=kX5frmc6B7c
- https://www.youtube.com/watch?v=xouin83ebxE

Construction of independent sets

Generalized Algorithm:

- Pick a starting point.
- While(Problem is not solved)
- For each path from the starting point.
  - check if selected path is safe,
  - if yes select it and make recursive call to rest of the problem
  - If recursive calls returns true, then return true. else undo the current move and return false.
- End For
- If none of the move works out, return false, NO SOLUTON.

Construction of independent sets

- S_k – obtained independent set of the cardinality k;
- Q_k – set of vertices that can be added to S_k ($\Gamma(S_k) \cap Q_k = \varnothing$);
- Q_k^- – vertices that have been used already to expand S_k;
- Q_k^+ – vertices that have not been used yet to expand S_k;

- Start: $k=0$, $S_k=\varnothing$, $Q_k^+ = \varnothing$, $Q_k^- = \varnothing$.
- End:
  - if $Q_k^+ = V$, $Q_k^- = \varnothing$ then the set can not be expand;
  - if there exists $u \in Q_k^-$ such as $\Gamma(u) \cap Q_k^+ = \varnothing$ then the obtaining set is not maximal as u can not be removed.

Construction of independent sets

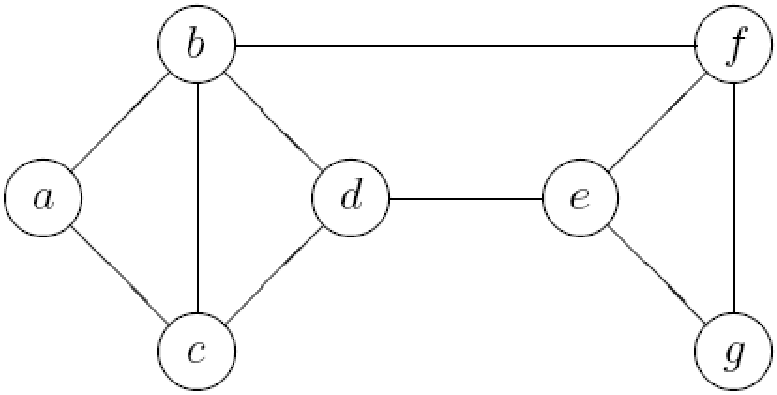
- Going ahead (from k to k+1):

$$S_{k+1} = S_k \cup \{v\};$$
$$Q^-_{k+1} = Q^-_k \setminus \Gamma(v);$$
$$Q^+_{k+1} = Q^+_k \setminus \{\Gamma(v) \cup v\};$$

- Going back (from k+1 to k):

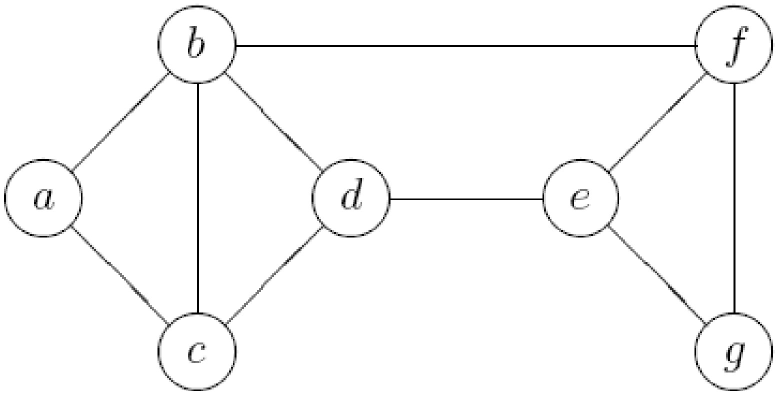
$$S_k = S_{k+1} \setminus \{v\};$$
$$Q^-_k = Q^-_k \cup \{v\};$$
$$Q^+_k = Q^+_k \setminus \{v\}.$$

Construction of independent sets



k	S_k	Q_k^+	Q_k^-
0	Ø	abcdefg	Ø
1	a	defg	Ø
2	ad	fg	Ø
3	adf	Ø	Ø
2	ad	g	f
3	adg	Ø	Ø
2	ad	Ø	fg
1	a	efg	d
2	ae	Ø	Ø

Construction of independent sets



k	S_k	Q_k^+	Q_k^-
2	ae	Ø	Ø
1	a	fg	ed
0	Ø	bcdefg	a
1	b	eg	Ø
2	be	Ø	Ø
1	b	g	e
2	bg	Ø	Ø
1	b	Ø	eg
0	Ø	cdefg	ab

Construction of covering sets

- ξ_j = 1 if and only if the vertex j belongs to the covering set;
- *I* is the incidence matrix;

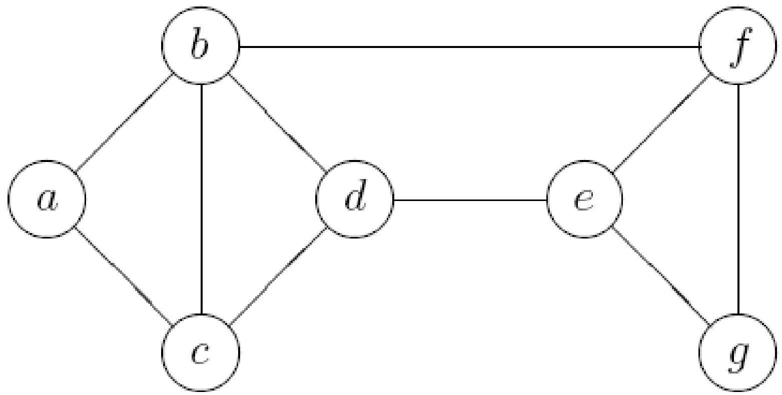
The problem can be converted to the search of the shortest cover for the incidence matrix.

$$\sum_{j=1}^{p} c_j \xi_j \to \min;$$

$$\sum_{j=1}^{p} I_{jk} \xi_j \geq 1, \quad \forall k = 1, \ldots, q;$$

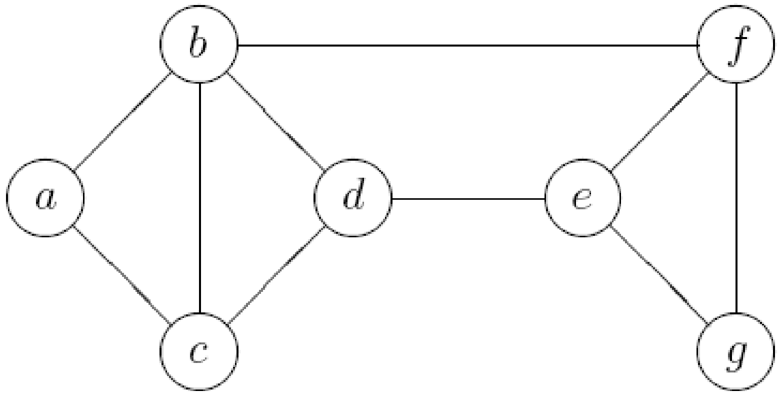
$$\xi_j \in \{0, 1\}.$$

Construction of covering sets



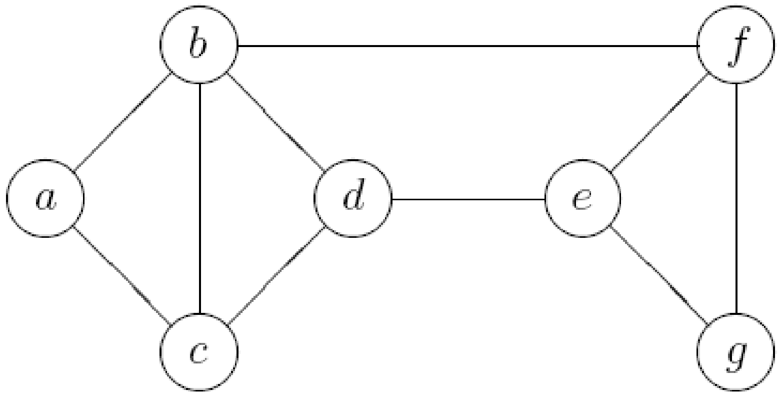
a	1	1								
b	1		1	1		1				
c		1	1		1					
d				1	1		1			
e							1	1	1	
f						1		1		1
g									1	1

Construction of covering sets



a	1	1								
c		1	1		1					
d				1	1		1			
e							1	1	1	
f						1		1		1
g									1	1

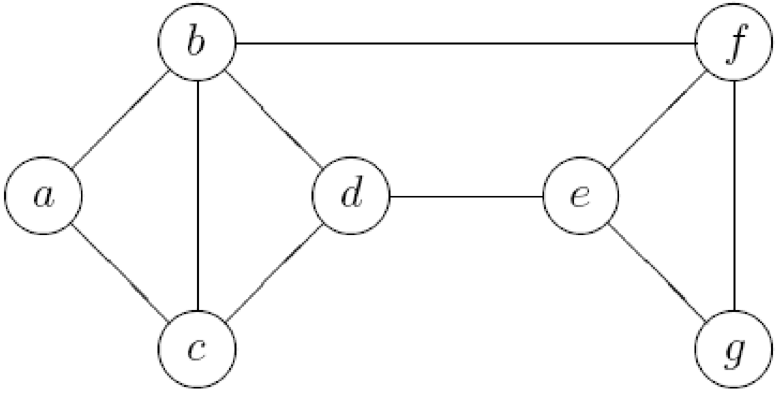
Construction of covering sets



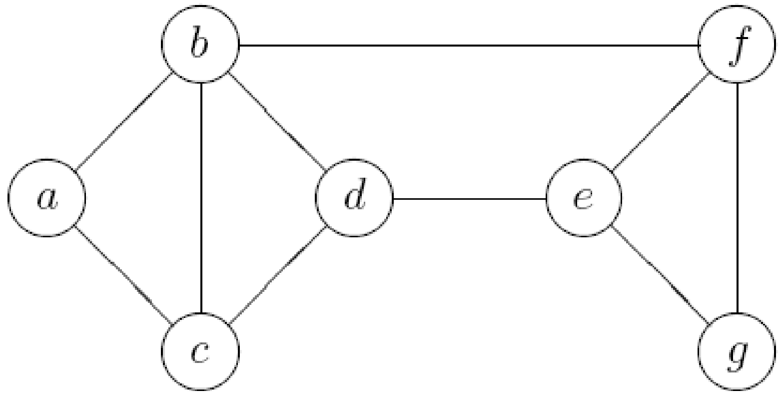
e	1
g	1

Construction of covering sets

- Cover: acdef

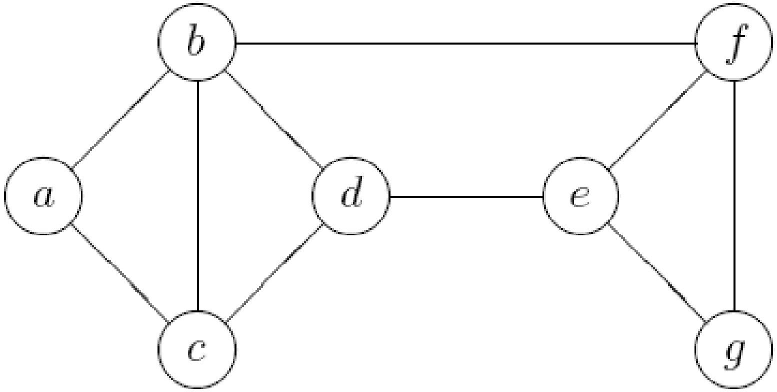


Construction of covering sets



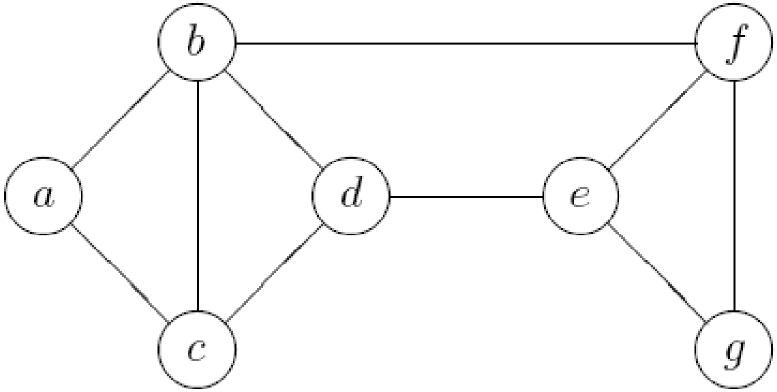
a	1	1								
b	1		1	1		1				
c		1	1		1					
d				1	1		1			
e							1	1	1	
f						1		1		1
g									1	1

Construction of covering sets



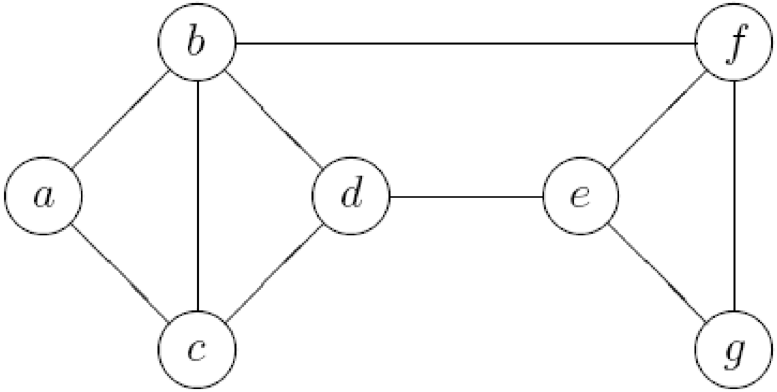
a	1					
c	1	1				
d		1	1			
e			1	1	1	
f				1		1
g					1	1

Construction of covering sets



a	1		
c	1	1	
d		1	
f			1
g			1

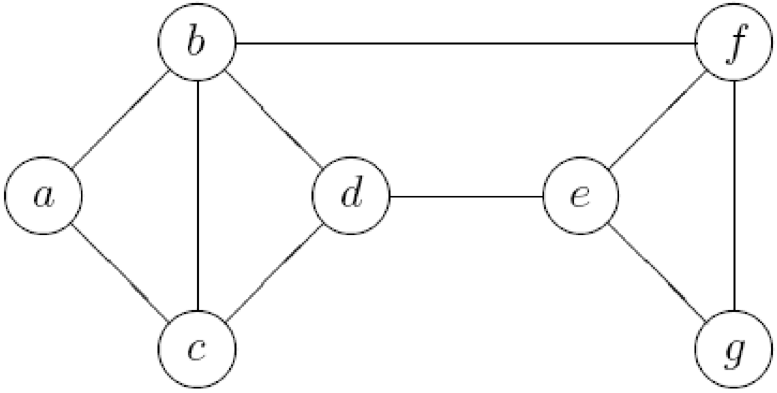
Construction of covering sets



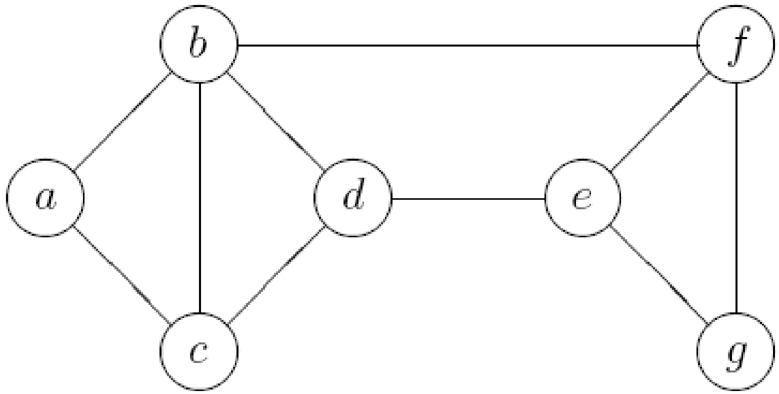
c	1	1	
f			1

Construction of covering sets

- Cover: bcef

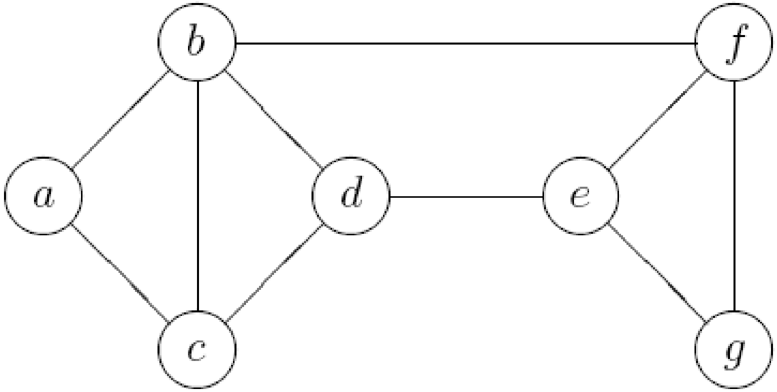


Construction of covering sets



a	1					
c	1	1				
d		1	1			
f				1		1
g					1	1

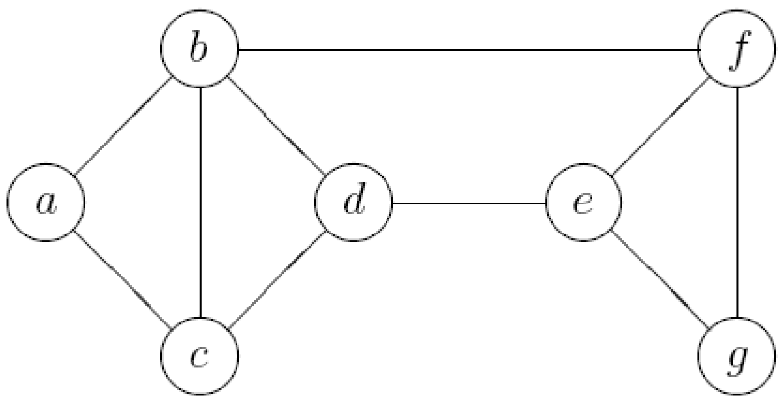
Construction of covering sets



a	1
c	1

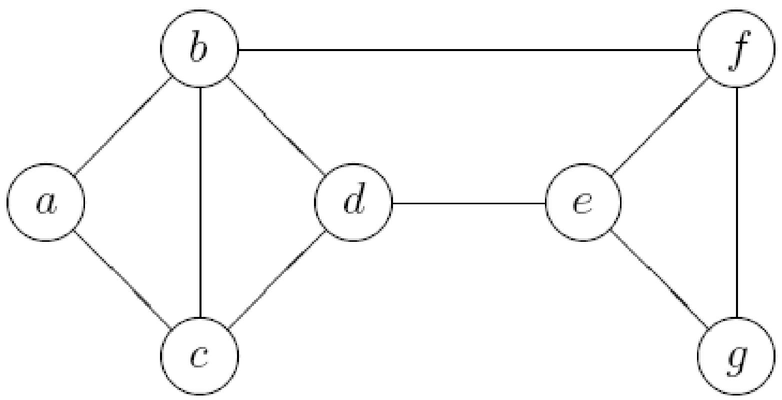
Construction of covering sets

- Cover: abdfg

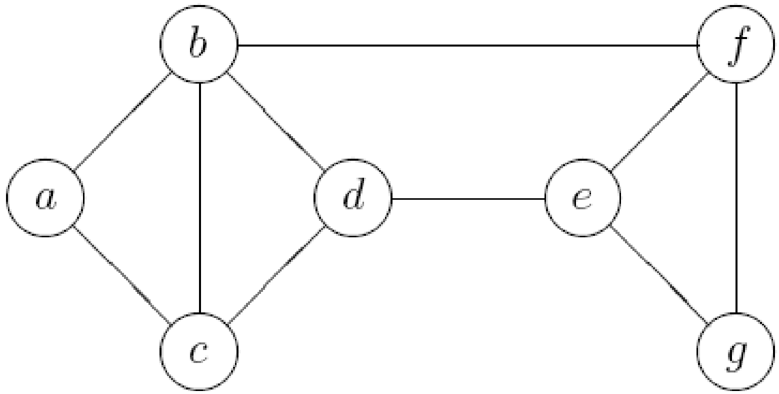


Construction of covering sets

- Shortest cover: bcef



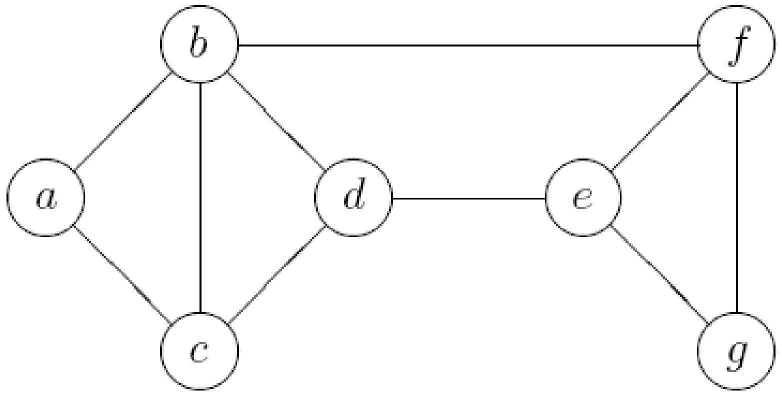
Construction of covering sets



a	1	1								
b	1		1	1		1				
c		1	1		1					
d				1	1		1			
e							1	1	1	
f						1		1		1
g									1	1

Construction of covering sets

- Shortest cover: bcef

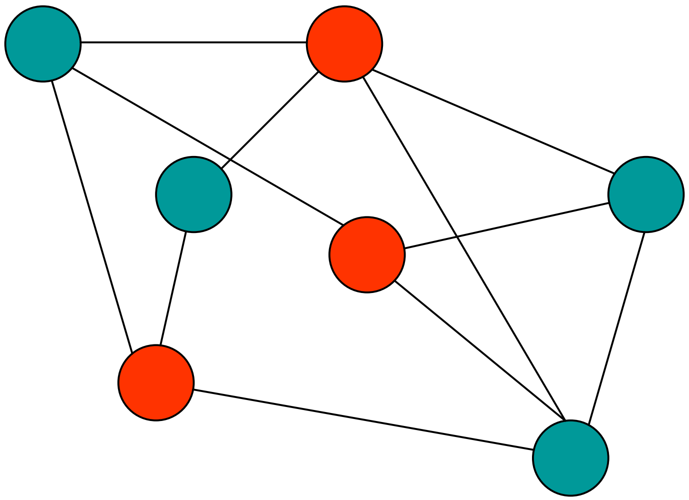


Independent and covering sets

A set is independent if and only if its complement is a vertex cover.

A set is covering if and only if its complement is an independent set.

Example. Red – independent, blue – covering.

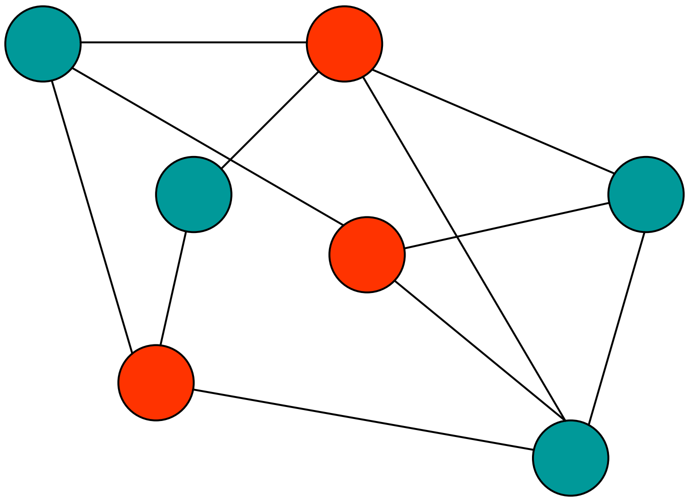


Independent and covering sets

The complement of a maximum independent set is a minimum vertex cover.

The complement of a minimum vertex cover is a maximum independent set.

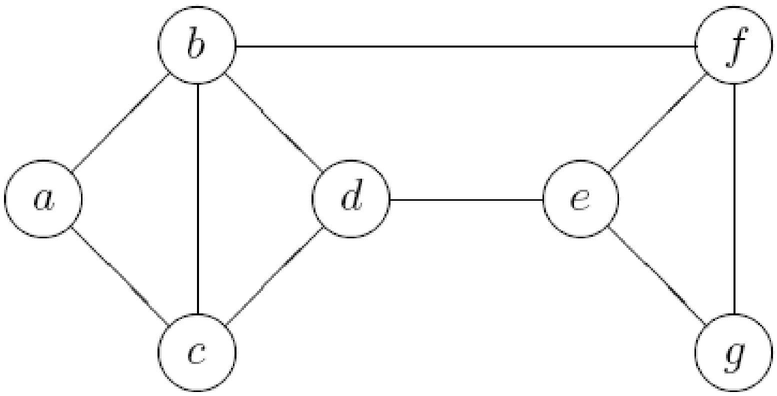
A solution of one problem gives a solution of another problem.



Dominating sets

A **dominating set** for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D.

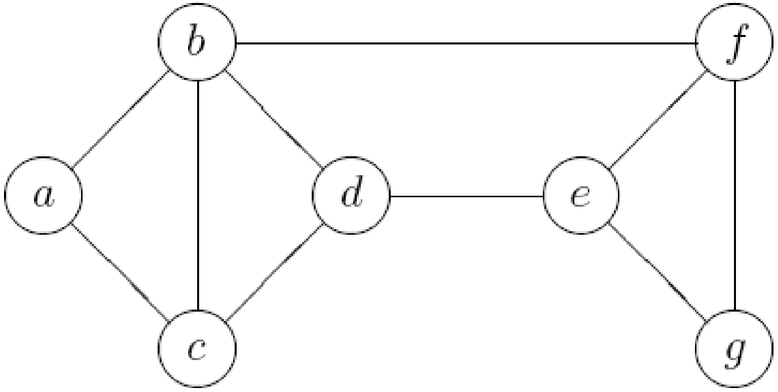
Example. {a,d,f} – dominating set.



Dominating sets

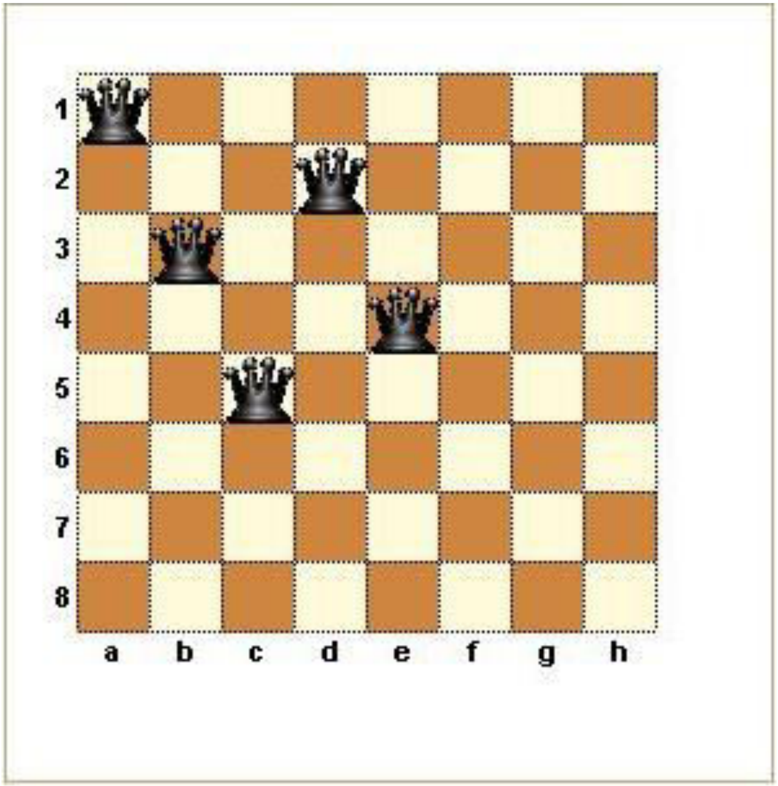
The **domination number** γ(*G*) is the number of vertices in a smallest dominating set for *G*. The set is called as **minimum dominating set**.

Example. {b,f} – a minimum dominating set, γ(*G*)=2.



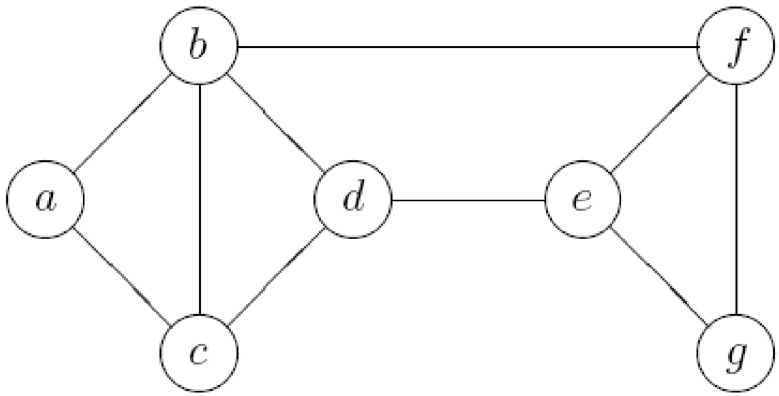
Dominating sets

The **five queens puzzle** is the problem of placing five chess queens on an 8×8 chessboard so that the queens can attack all the board.



Dominating sets

The **domination matrix** for a graph G(V,E) is its adjacency matrix where all elements of the main diagonal are equal to unity.

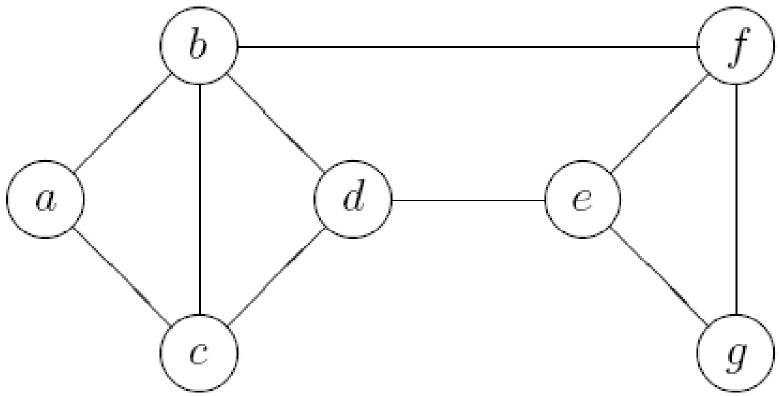


	a	b	c	d	e	f	g
a	1	1	1				
b	1	1	1	1		1	
c	1	1	1	1			
d		1	1	1	1		
e				1	1	1	1
f		1			1	1	1
g					1	1	1

Dominating sets

The **minimum dominating set** correspond to the shortest cover of the domination matrix.

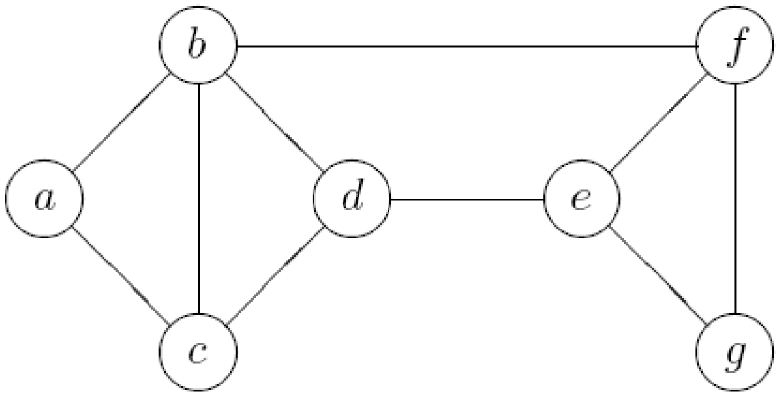
a≤b, a≤b, g≤f



	a	b	c	d	e	f	g
a	1	1	1				
b	1	1	1	1		1	
c	1	1	1	1			
d		1	1	1	1		
e				1	1	1	1
f		1			1	1	1
g					1	1	1

Dominating sets

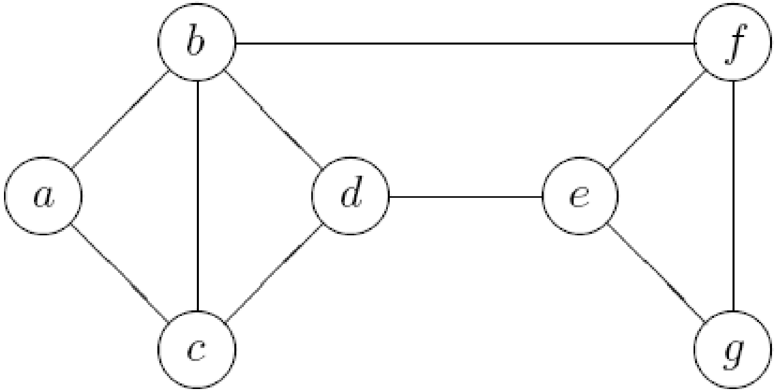
The row b is an essential row.



	a	b	c	d	e	f	g
b	1	1	1	1		1	
d		1	1	1	1		
e				1	1	1	1
f		1			1	1	1

Dominating sets

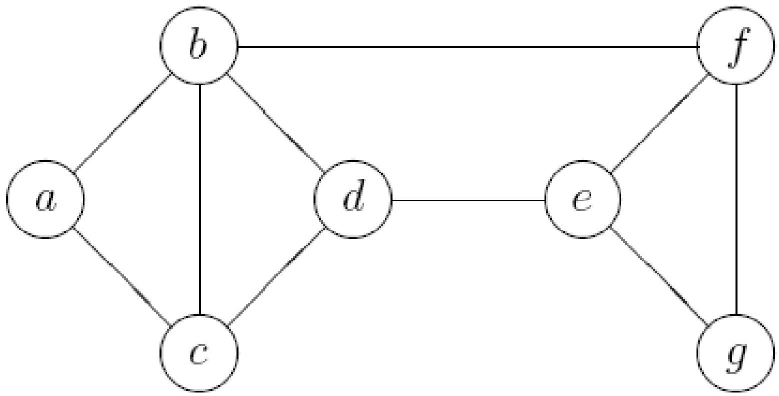
The row f covers all columns.



	e	g
d	1	
e	1	1
f	1	1

Dominating sets

The rows {b,e} form the shortest cover of the domination matrix.

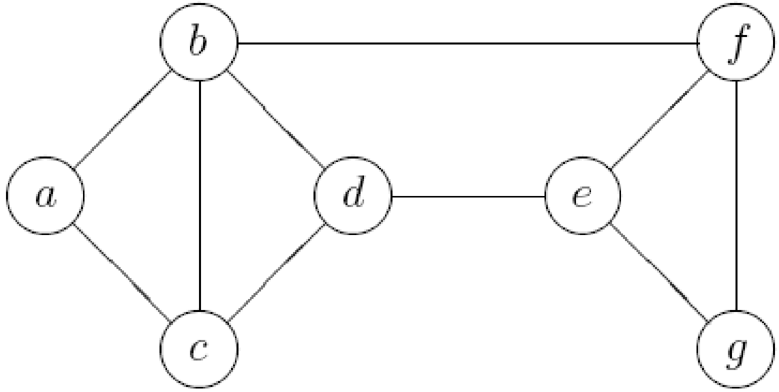


	a	b	c	d	e	f	g
a	1	1	1				
b	1	1	1	1		1	
c	1	1	1	1			
d		1	1	1	1		
e				1	1	1	1
f		1			1	1	1
g					1	1	1

Dominating sets

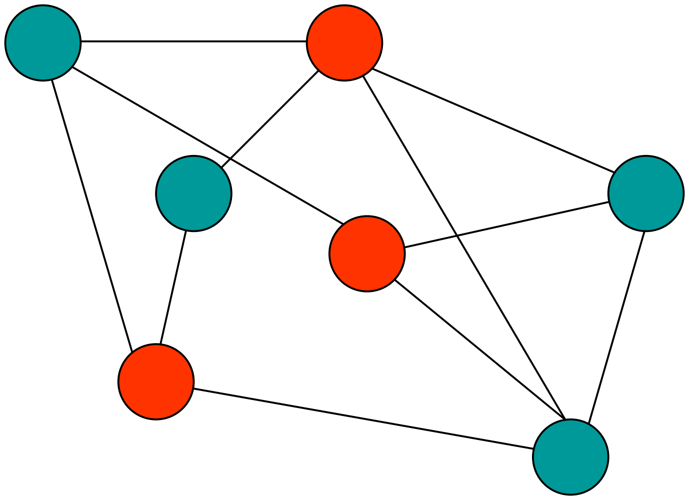
The **minimal dominating set** is a dominating set that does not contain any other dominating set.

Example. {b,e,f} is not a minimal dominating set, {b,f} is a minimal dominating set.



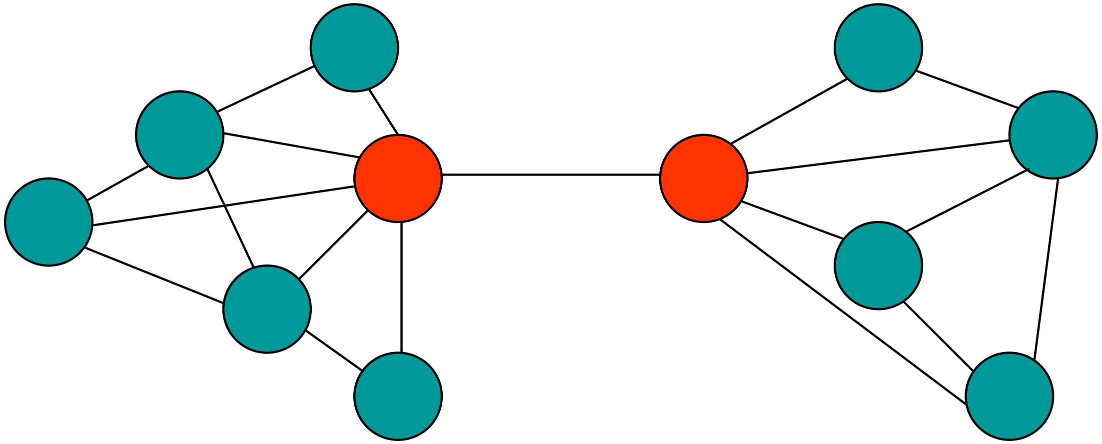
Dominating and independent sets

An **independent set** is also a **dominating set** if and only if it is a **maximal independent set**, so any **maximal independent set** in a graph is also a **minimal dominating set**.



Dominating and independent sets

A **dominating set** is not necessary an **independent set**.



6.3. Independent and covering sets of edges

- Matching problem statement
- Cover problem statement
- Matchings and covering sets

Matching problem statement

Matching is an independent set of edges.

Let M be a matching in G(V,E).

Two ends of an edge in M are **matched under** M.

A matching M **saturates** a vertex v (and v is **M-saturated**) if some edge of M is incident with v; otherwise, v is **M-unsaturated**.

Matching problem statement

If every vertex of G is M-saturated, the matching M is **perfect**.

M is a **maximum matching** in G, if $|M|=\beta_1$.

Every perfect matching is a maximum one. A perfect matching does not always exist.

Matching problem statement

- ξ_j = 1 if and only if the edge j belongs to the matching;
- c_j is the weight of the edge j;
- I is the incidence matrix.

The problem can be stated as a discrete linear programming problem.

Covering set problem statement

- $\xi_j = 1$ if and only if the edge j belongs to the cover;
- c_j is the weight of the edge j;
- *I* is the incidence matrix.

The problem can be stated as a discrete linear programming problem (the shortest cover of the transposed incidence matrix).

Matchings and covering sets

- A solution of the **minimum cover problem** provides a solution of the **maximum matching problem**.
- From matching to cover: