“Graph theory”
Course for the master degree program “Geographic Information Systems”

Yulia Burkatovskaya

Department of Information Systems and Technologies

Associate professor

6. Matchings and Covers

- Independent and covering sets
- Independent and covering sets of vertices
- Dominating sets
- Independent and covering sets of edges

6.1. Independent and covering sets

- Covering sets
- Cover numbers
- Independent sets
- Independence numbers
- Cover and independence numbers theorem

Covering sets

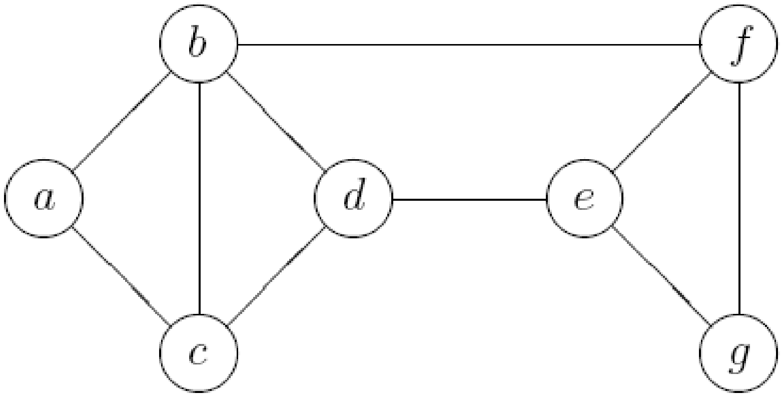
A vertex **covers** an edge if they are incident.

An edge **covers** a vertex if they are incident.

Example.

The vertex b covers the edges ab, bc, bd, bf

The edge ab covers the vertices a and b

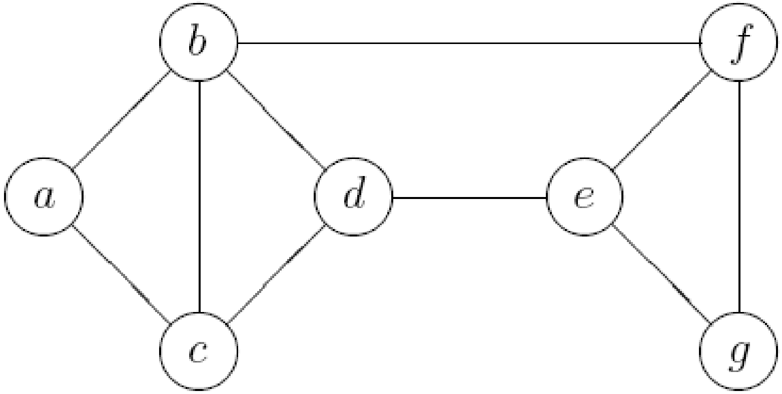


Covering sets

A **vertex covering set (vertex cover)** is a set of vertices of G covering all edges of G.

Example.

{a,b,d,e,f} – a vertex covering set.

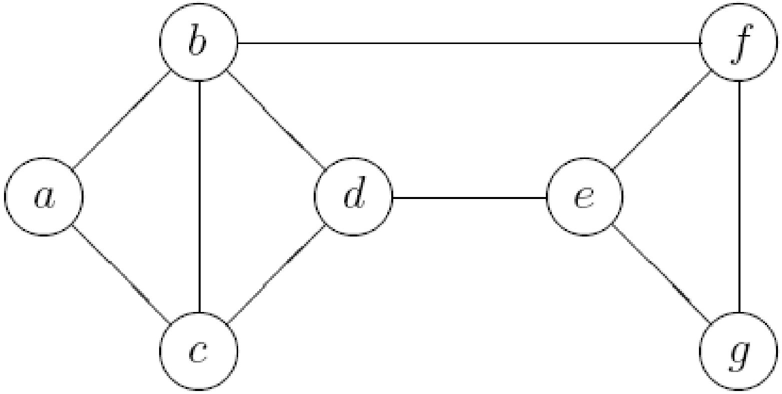


Covering sets

An **edge covering set (edge cover)** is a set of edges of G covering all vertices of G.

Example.

{ab,ac,de,fg} – an edge covering set.



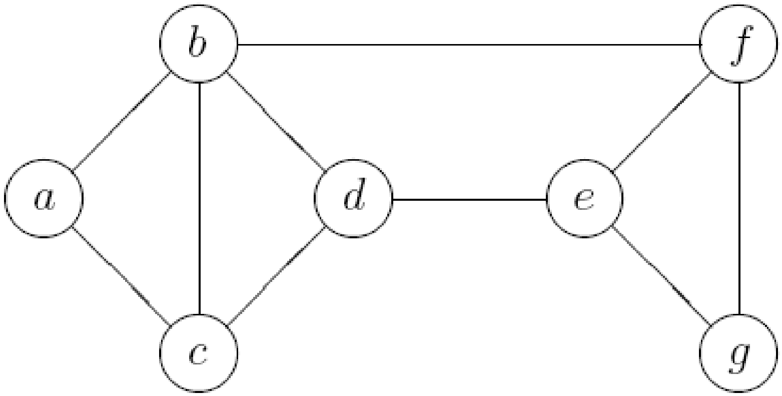
Minimum covering sets

A cover is called **minimum** when it contains the smallest possible number of vertices (edges).

Example.

{a,b,c,d,e,f} is not a minimum vertex cover

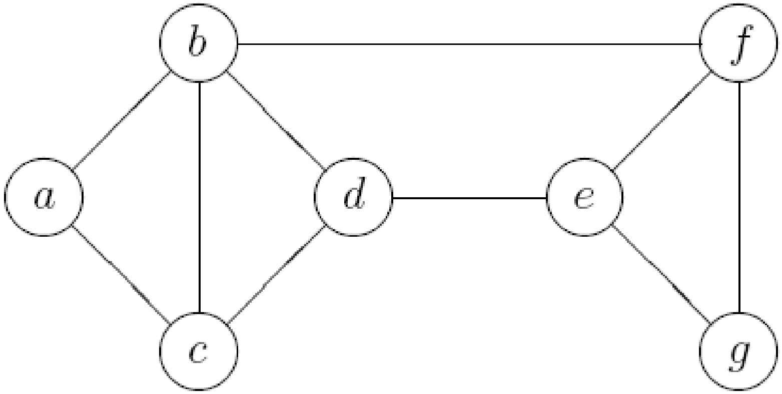
{b,c,e,g} is a minimum vertex cover.



Cover numbers

The **vertex cover number** α_0 of a graph G is the size of a minimum vertex cover in a graph, i.e., the minimum number of vertices covering all edges.

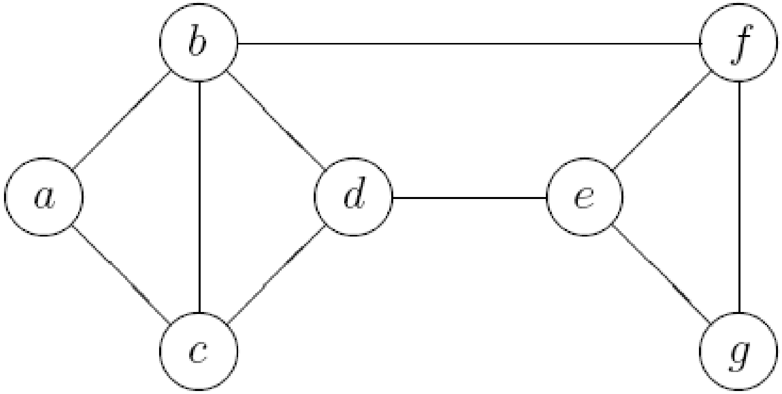
Example. α_0 =4, {b,c,e,f} – minimum vertex cover.



Cover numbers

The **edge cover number** α_1 of a graph G is the size of a minimum edge cover in a graph, i.e., the minimum number of edges covering all vertices.

Example. $\alpha_1 = 4$, {ab,cd,eg,ef} – minimum edge cover.



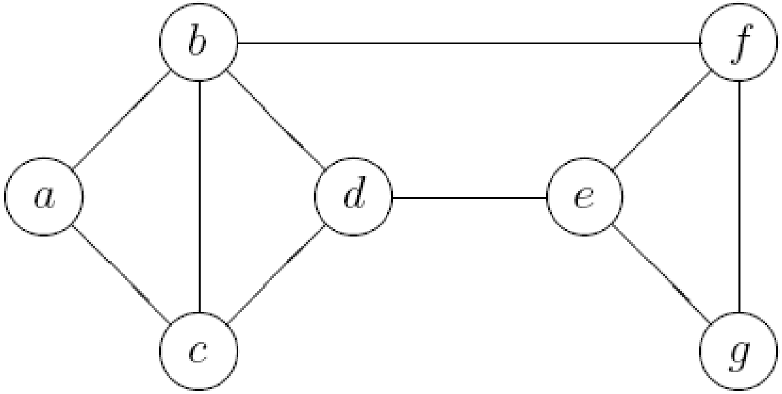
Independent sets

A **vertex (edge) independent set** is a set of vertices (edges) of G so that no two vertices (edges) of the set are adjacent.

Example.

{b,e} – independent vertex set.

{ab,cd,fg} – independent edge set.



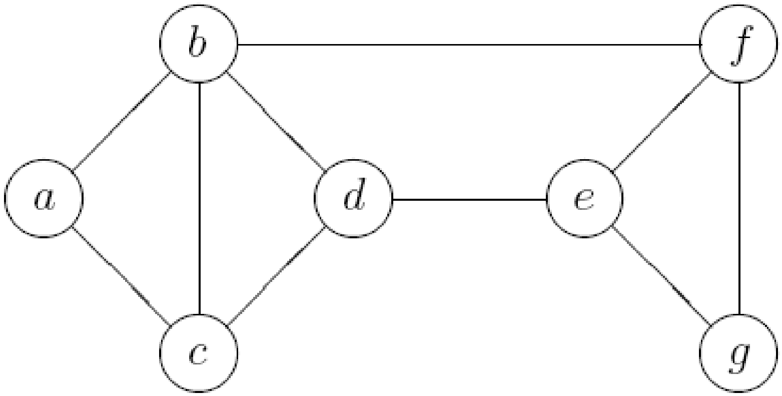
Maximum independent sets

An independent set is called **maximum** when it contains the greatest number of vertices (edges).

Example.

{b,e} is not a maximum vertex independent set.

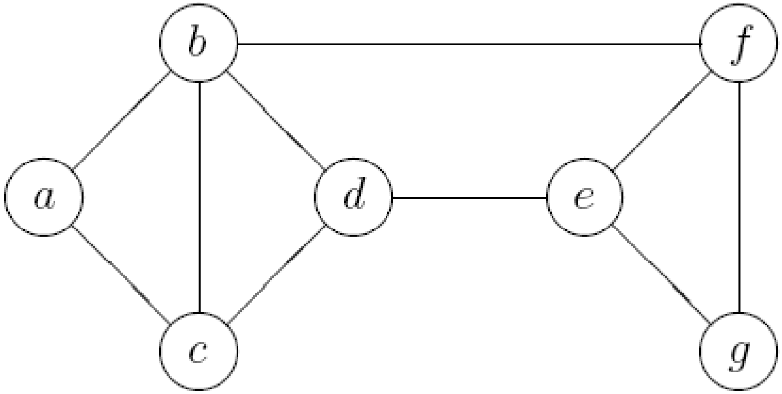
{a,d,f} is a maximum vertex independent set.



Independence numbers

The **vertex independence number** β_0 of a graph G is the maximum number of independent vertices.

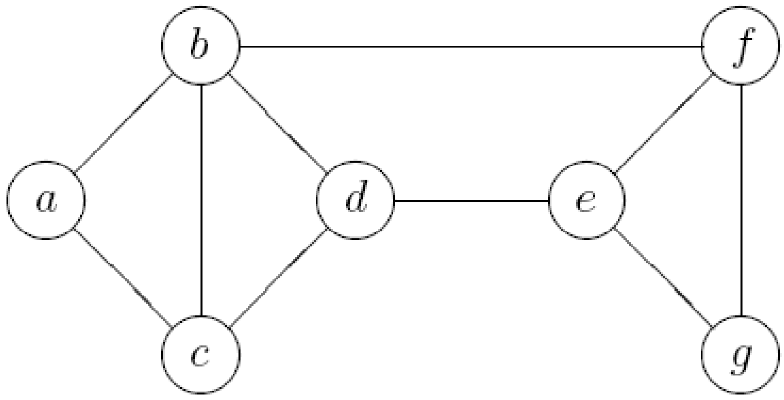
Example. $\beta_0 = 3$, {a,d,f} – independent vertex set.



Independence numbers

The **edge independence number** β_1 of a graph G is the maximum number of independent edges.

Example. β_1 =3, {ab,cd,ef} – independent edge set.



Cover and independence numbers

	α_0	α_1	β_0	β_1
K_p				
$K_{m,n}$				
C_p				
Empty				

Cover and independence numbers

	α_0	α_1	β_0	β_1
K_p	p-1	p/2 (p–even), (p+1)/2 (p–odd)	1	p/2 (p–even), (p-1)/2 (p–odd)
$K_{m,n}$	min(m,n)	max(m,n)	max(m,n)	min(m,n)
C_p	p/2 (p–even), (p+1)/2 (p–odd)	p/2 (p–even), (p+1)/2 (p–odd)	p/2 (p–even), (p-1)/2 (p–odd)	p/2 (p–even), (p-1)/2 (p–odd)
Empty	0	no	p	0

Cover and independence number theorem

For every connected non-trivial graph

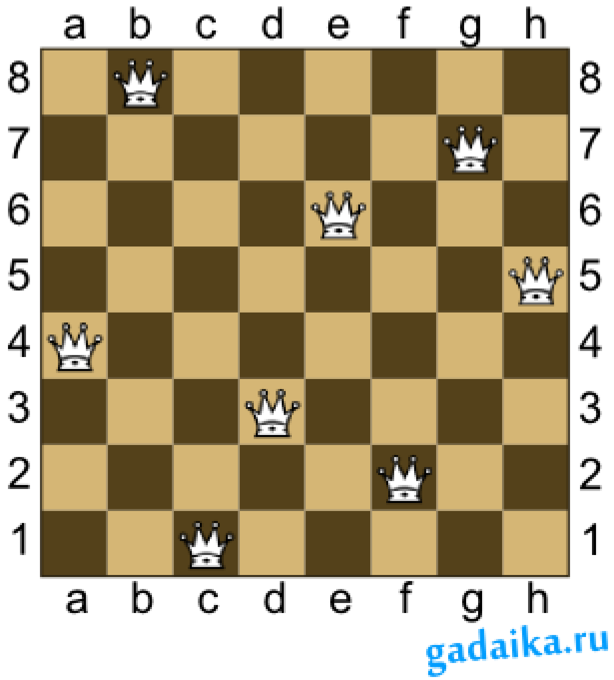
$$\alpha_0 + \beta_0 = \alpha_1 + \beta_1 = p.$$

6.2. Independent and covering sets of vertices

- Construction of independent sets
- Construction of covering sets
- Independent and covering sets
- Dominating sets
- Dominating and independent sets

Construction of independent sets

The **eight queens puzzle** is the problem of placing eight chess queens on an 8×8 chessboard so that no two queens threaten each other.

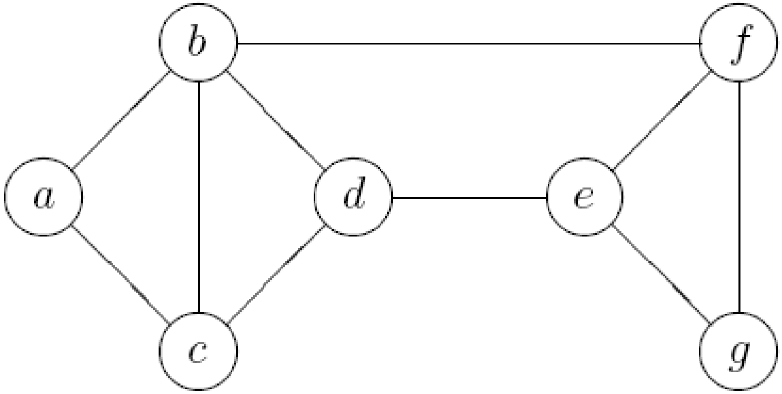


Construction of independent sets

An independent set is **maximal** if is not a subset of any other independent set.

In other words, there is no vertex outside the independent set that may join it.

Example. {a,d} is not a maximal independent set, {a,d,f} is a maximal independent set.



Construction of independent sets

- **Backtracking** is a general algorithm for finding all (or some) solutions to some computational problems, notably constraint satisfaction problems, that incrementally builds candidates to the solutions, and abandons a partial candidate ("backtracks") as soon as it determines that it cannot possibly be completed to a valid solution.

- https://www.youtube.com/watch?v=kX5frmc6B7c
- https://www.youtube.com/watch?v=xouin83ebxE

Construction of independent sets

Generalized Algorithm:

- Pick a starting point.
- While(Problem is not solved)
- For each path from the starting point.
  - check if selected path is safe,
  - if yes select it and make recursive call to rest of the problem
  - If recursive calls returns true, then return true. else undo the current move and return false.
- End For
- If none of the move works out, return false, NO SOLUTON.

Construction of independent sets

- S_k – obtained independent set of the cardinality k;
- Q_k – set of vertices that can be added to S_k ($\Gamma(S_k) \cap Q_k = \varnothing$);
- Q_k^- – vertices that have been used already to expand S_k;
- Q_k^+ – vertices that have not been used yet to expand S_k;

- Start: $k=0$, $S_k=\varnothing$, $Q_k^+=V$, $Q_k^-=\varnothing$.
- End:
  - if $Q_k^+ = \varnothing$, $Q_k^- = \varnothing$ then the set can not be expand;
  - if there exists $u \in Q_k^-$ such as $\Gamma(u) \cap Q_k^+ = \varnothing$ then the obtaining set is not maximal as u can not be removed.

Construction of independent sets

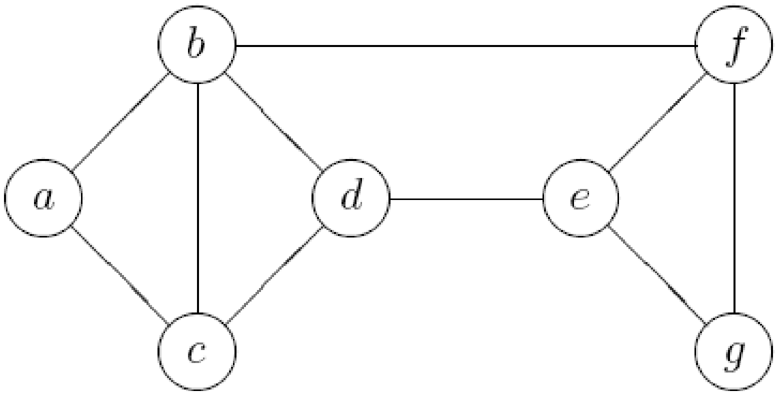
- Going ahead (from k to k+1):

$$S_{k+1} = S_k \cup \{v\};$$
$$Q_{k+1}^- = Q_k^- \setminus \Gamma(v);$$
$$Q_{k+1}^+ = Q_k^+ \setminus \{\Gamma(v) \cup v\};$$

- Going back (from k+1 to k):

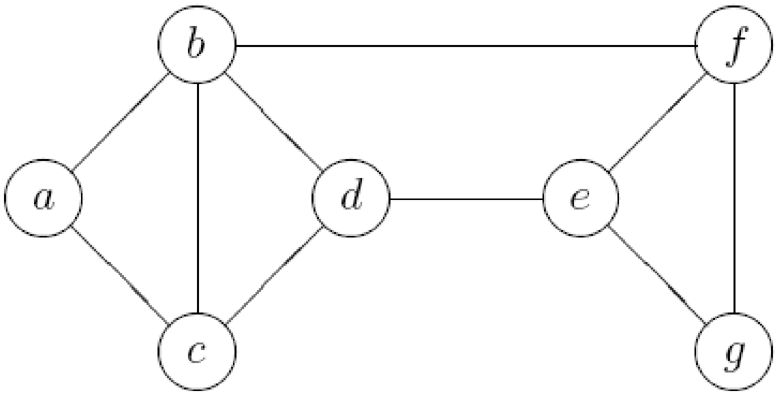
$$S_k = S_{k+1} \setminus \{v\};$$
$$Q_k^- = Q_k^- \cup \{v\};$$
$$Q_k^+ = Q_k^+ \setminus \{v\}.$$

Construction of independent sets



k	S_k	Q_k^+	Q_k^-
0	Ø	abcdefg	Ø
1	a	defg	Ø
2	ad	fg	Ø
3	adf	Ø	Ø
2	ad	g	f
3	adg	Ø	Ø
2	ad	Ø	fg
1	a	efg	d
2	ae	Ø	Ø

Construction of independent sets



k	S_k	Q_k^+	Q_k^-
2	ae	∅	∅
1	a	fg	ed
0	∅	bcdefg	a
1	b	eg	∅
2	be	∅	∅
1	b	g	e
2	bg	∅	∅
1	b	∅	eg
0	∅	cdefg	ab

Construction of covering sets

- $\xi_j = 1$ if and only if the vertex j belongs to the covering set;
- I is the incidence matrix;

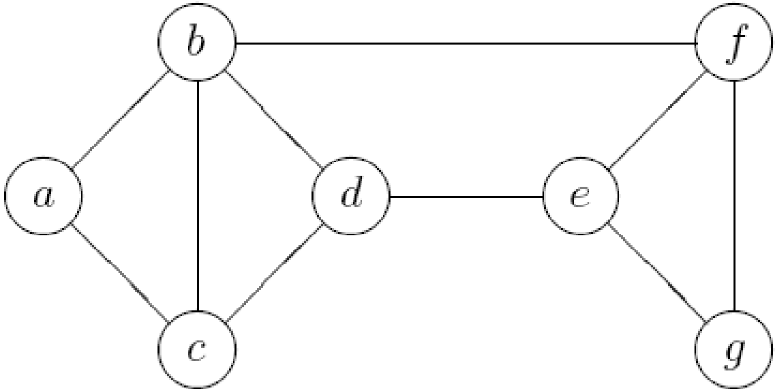
The problem can be converted to the search of the shortest cover for the incidence matrix.

$$\sum_{j=1}^{p} c_j \xi_j \to \min;$$

$$\sum_{j=1}^{p} I_{jk} \xi_j \geq 1, \quad \forall k = 1, \ldots, q;$$

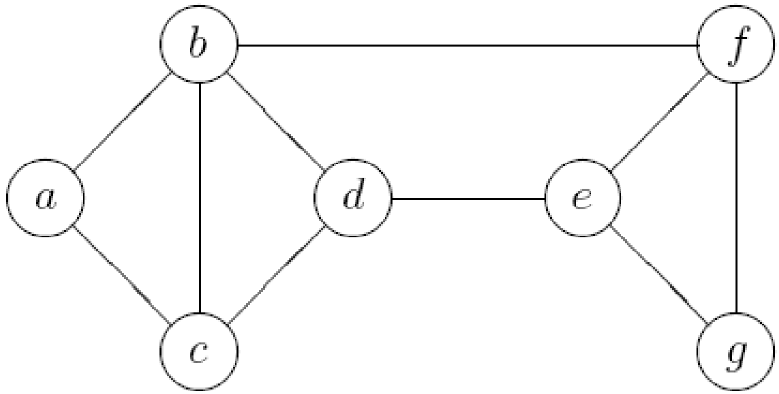
$$\xi_j \in \{0, 1\}.$$

Construction of covering sets



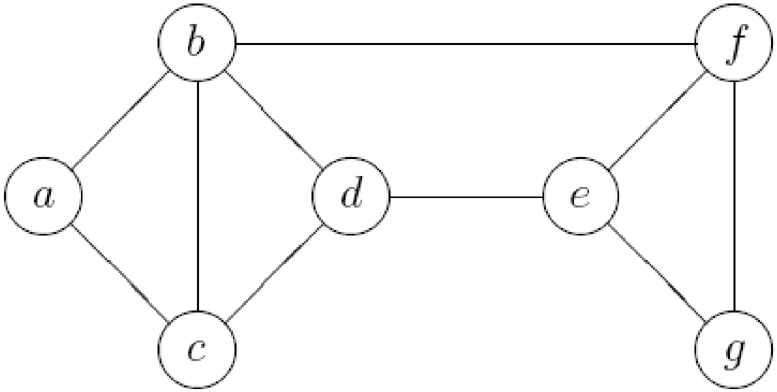
a	1	1								
b	1		1	1		1				
c		1	1		1					
d				1	1		1			
e							1	1	1	
f						1		1		1
g									1	1

Construction of covering sets



a	1	1								
c		1	1		1					
d				1	1		1			
e							1	1	1	
f						1		1		1
g									1	1

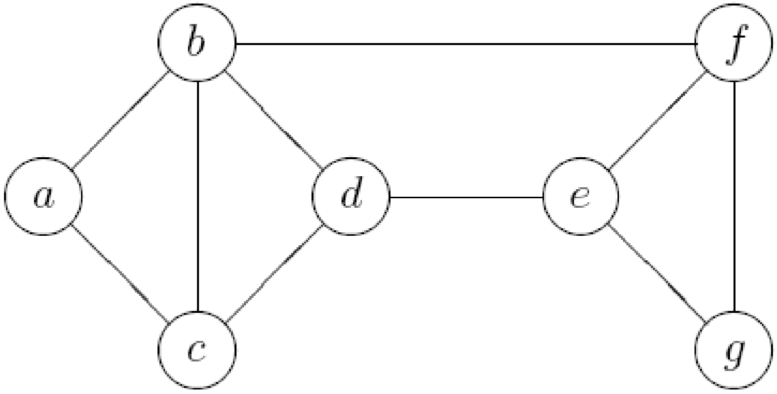
Construction of covering sets



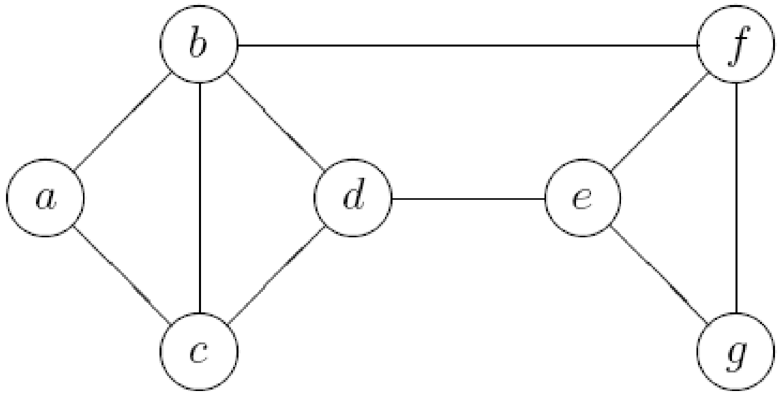
e	1
g	1

Construction of covering sets

- Cover: acdef

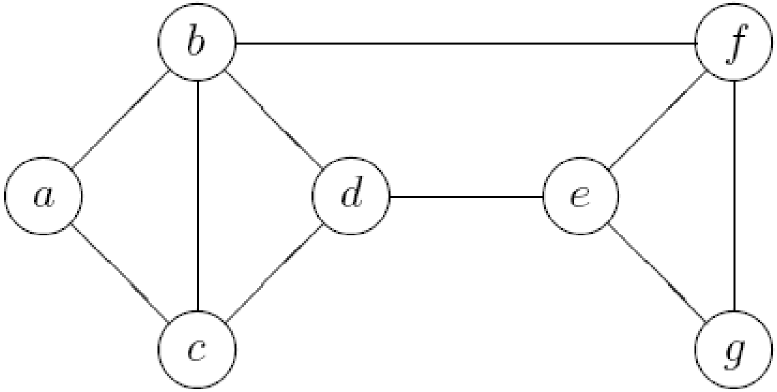


Construction of covering sets



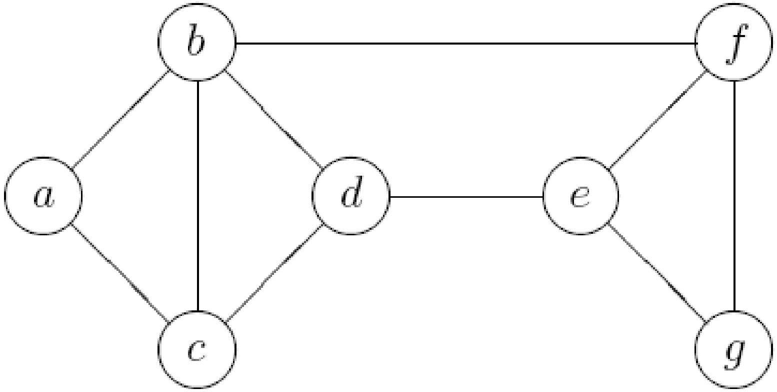
a	1	1								
b	1		1	1		1				
c		1	1		1					
d				1	1		1			
e							1	1	1	
f						1		1		1
g									1	1

Construction of covering sets



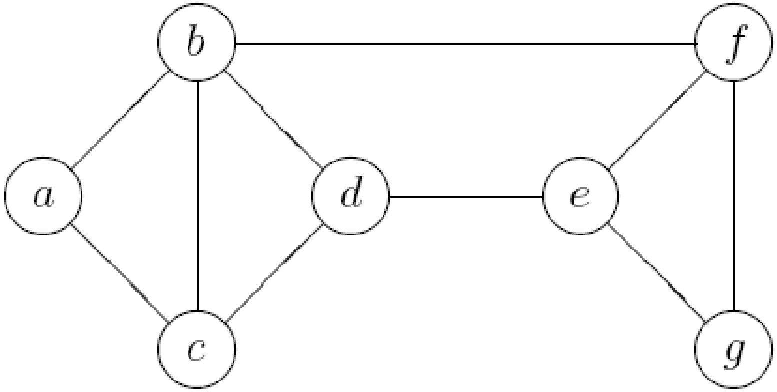
a	1					
c	1	1				
d		1	1			
e			1	1	1	
f				1		1
g					1	1

Construction of covering sets



a	1		
c	1	1	
d		1	
f			1
g			1

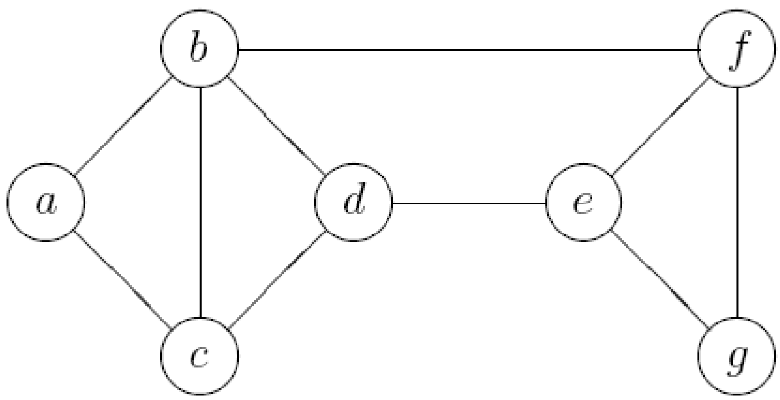
Construction of covering sets



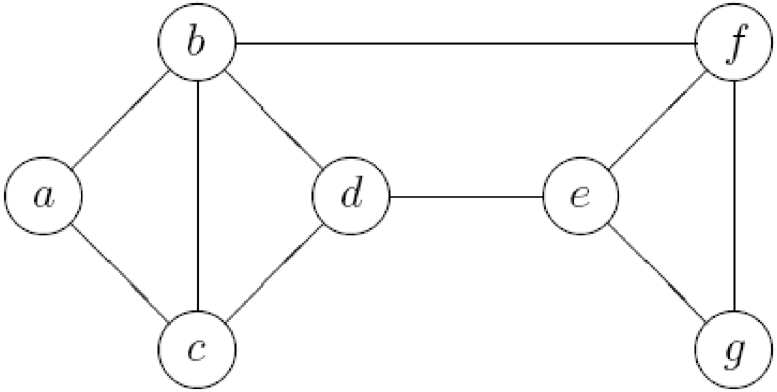
c	1	1	
f			1

Construction of covering sets

- Cover: bcef

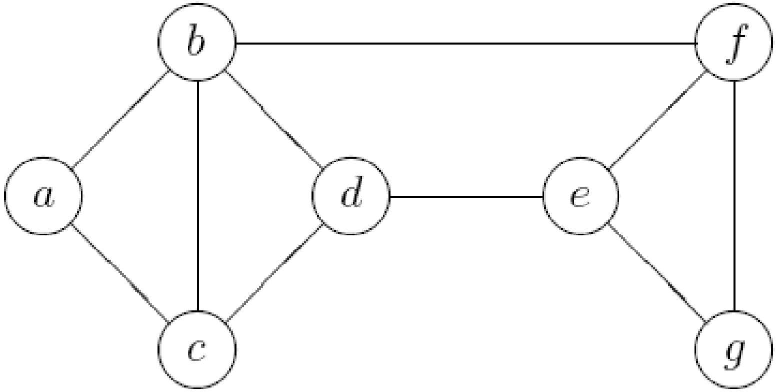


Construction of covering sets



a	1					
c	1	1				
d		1	1			
f				1		1
g					1	1

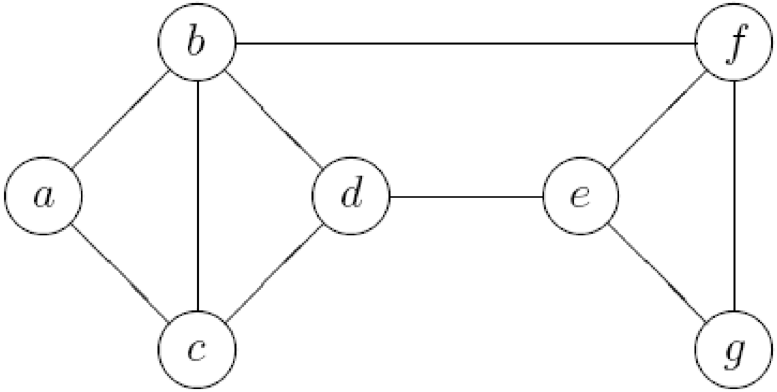
Construction of covering sets



a	1
c	1

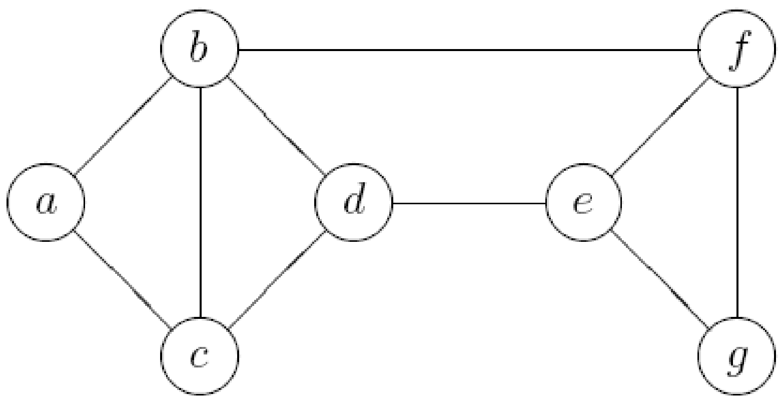
Construction of covering sets

- Cover: abdfg

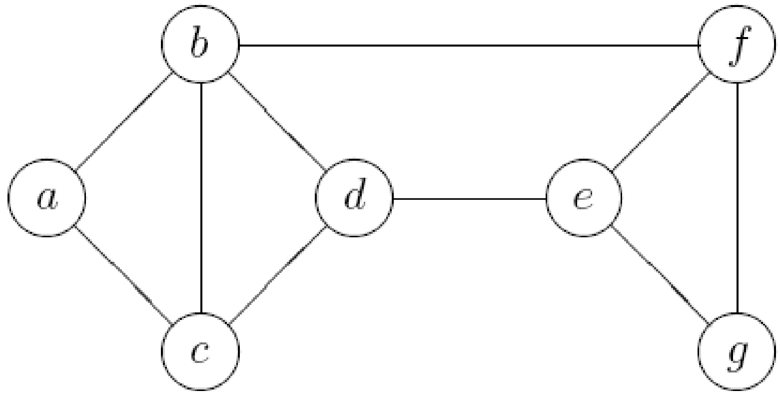


Construction of covering sets

- Shortest cover: bcef



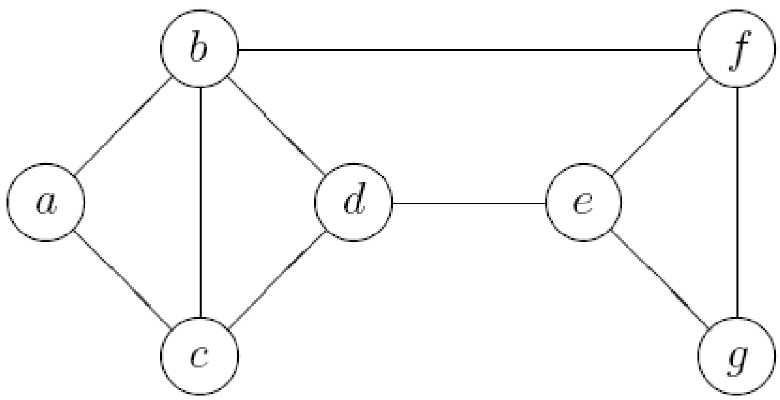
Construction of covering sets



a	1	1								
b	1		1	1		1				
c		1	1		1					
d				1	1		1			
e							1	1	1	
f						1		1		1
g									1	1

Construction of covering sets

- Shortest cover: bcef

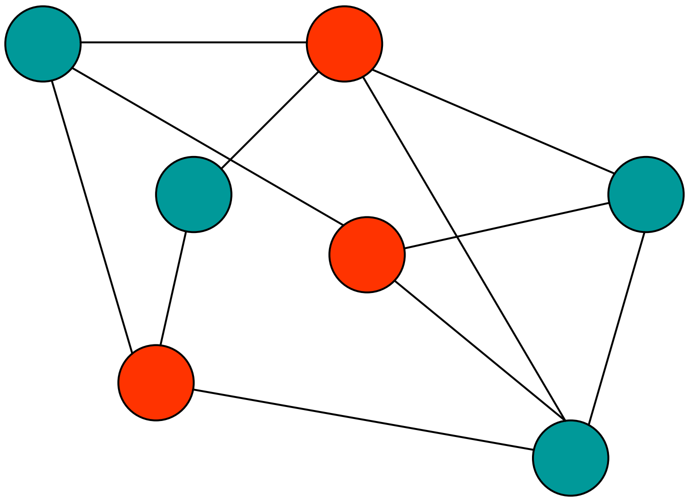


Independent and covering sets

A set is independent if and only if its complement is a vertex cover.

A set is covering if and only if its complement is an independent set.

Example. Red – independent, blue – covering.

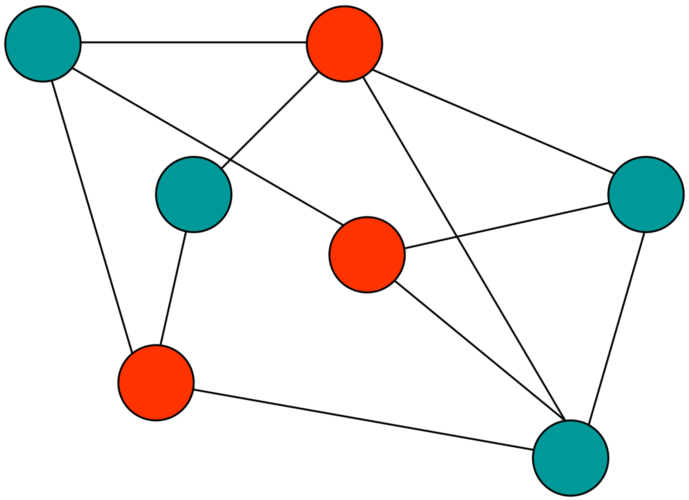


Independent and covering sets

The complement of a maximum independent set is a minimum vertex cover.

The complement of a minimum vertex cover is a maximum independent set.

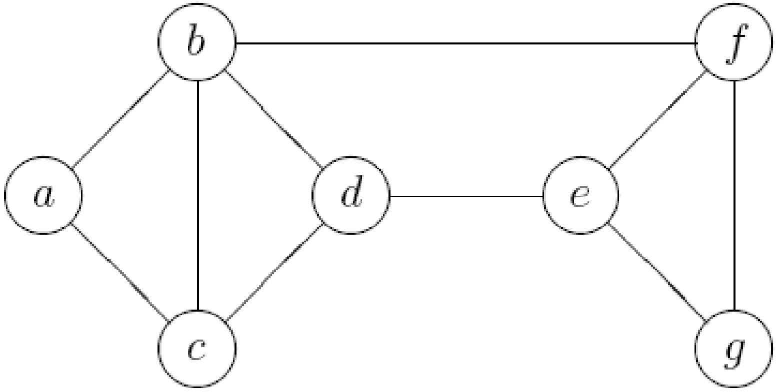
A solution of one problem gives a solution of another problem.



6.3. Dominating sets

A **dominating set** for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D.

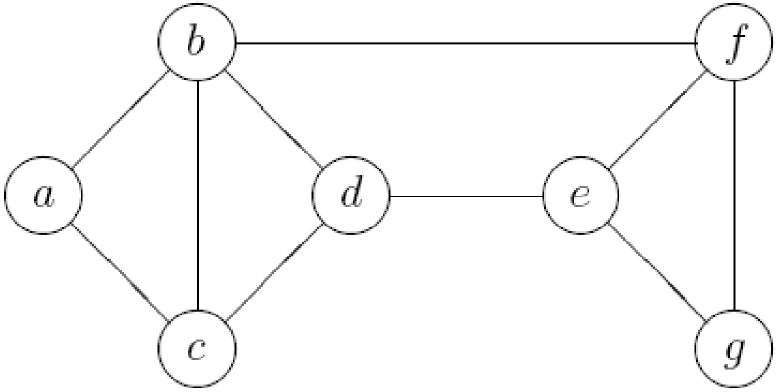
Example. {a,d,f} – dominating set.



Dominating sets

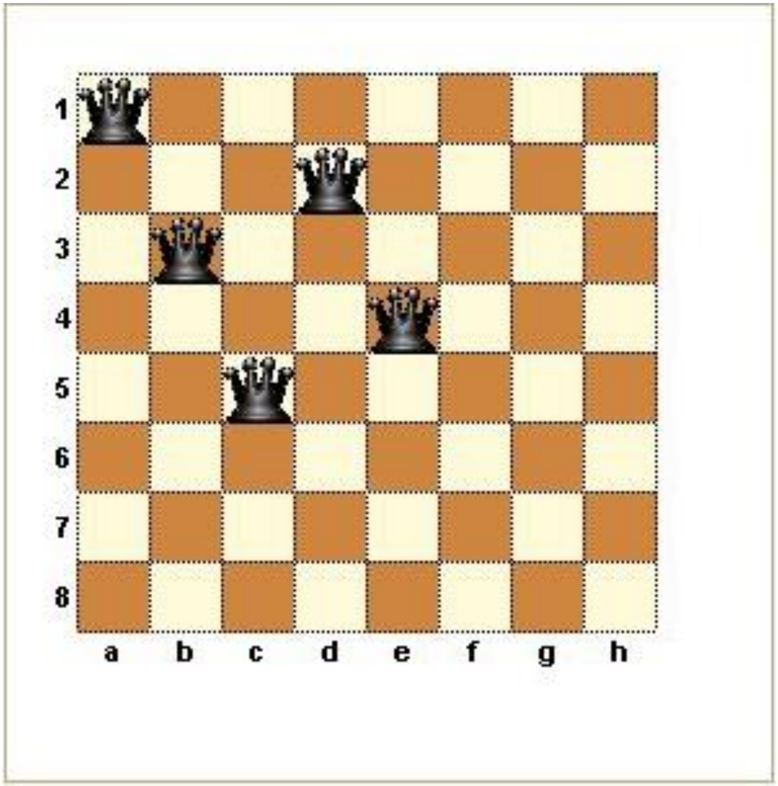
The **domination number** γ(*G*) is the number of vertices in a smallest dominating set for *G*. The set is called as **minimum dominating set**.

Example. {b,f} – a minimum dominating set, γ(*G*)=2.



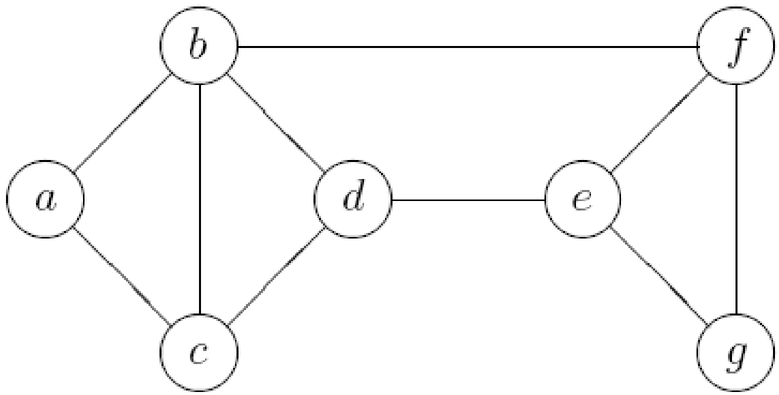
Dominating sets

The **five queens puzzle** is the problem of placing five chess queens on an 8×8 chessboard so that the queens can attack all the board.



Dominating sets

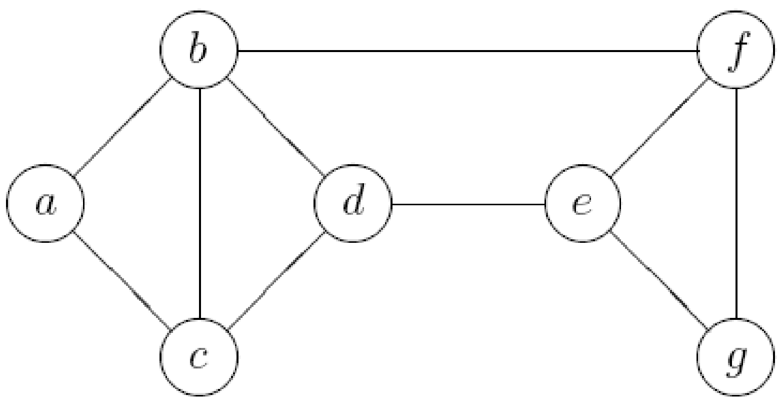
The **domination matrix** for a graph G(V,E) is its adjacency matrix where all elements of the main diagonal are equal to unity.



	a	b	c	d	e	f	g
a	1	1	1				
b	1	1	1	1		1	
c	1	1	1	1			
d		1	1	1	1		
e				1	1	1	1
f		1			1	1	1
g					1	1	1

Dominating sets

A **minimum dominating set** correspond to the shortest cover of the domination matrix.

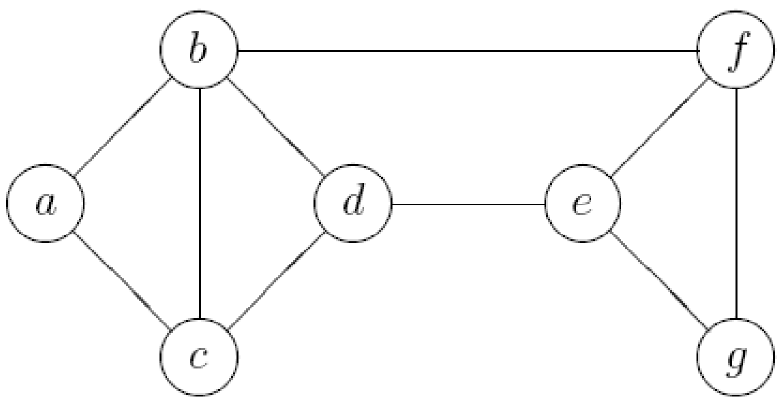


	a	b	c	d	e	f	g
a	1	1	1				
b	1	1	1	1		1	
c	1	1	1	1			
d		1	1	1	1		
e				1	1	1	1
f		1			1	1	1
g					1	1	1

Dominating sets

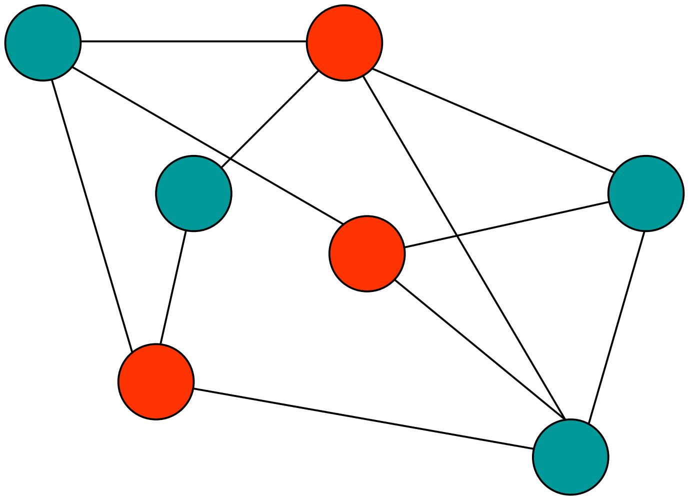
A **minimal dominating set** is a dominating set that does not contain any other dominating set.

Example. {b,e,f} is not a minimal dominating set, {b,f} is a minimal dominating set.



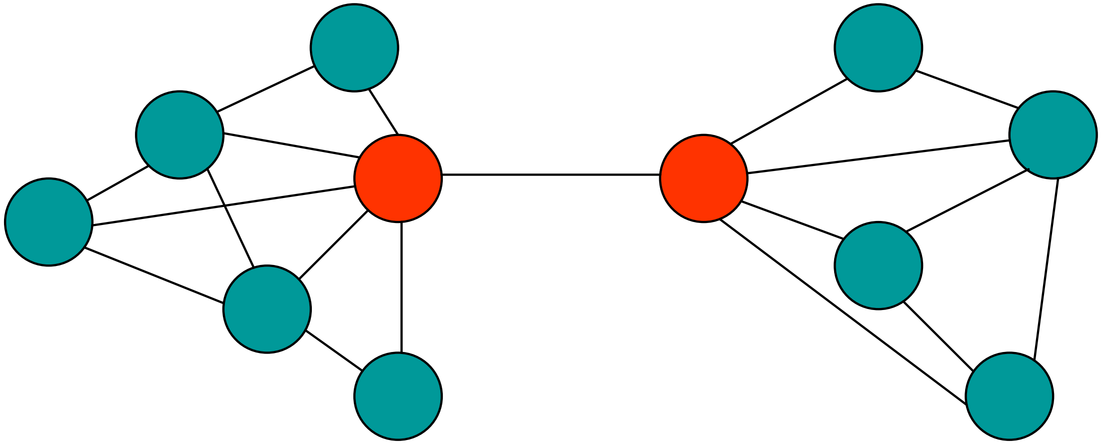
Dominating and independent sets

An **independent set** is also a **dominating set** if and only if it is a **maximal independent set**, so any **maximal independent set** in a graph is also a **minimal dominating set**.



Dominating and independent sets

A **dominating set** is not necessary an **independent set**.



6.4. Independent and covering sets of edges

- Matching problem statement
- Cover problem statement
- Matchings and covering sets
- Maximum-cardinality matching
- Maximum-weight matching

Matching problem statement

Matching is an independent set of edges.

Let M be a matching in G(V,E).

Two ends of an edge in M are **matched under** M.

A matching M **saturates** a vertex v (and v is **M-saturated**) if some edge of M is incident with v; otherwise, v is **M-unsaturated**.

Matching problem statement

If every vertex of G is M-saturated, the matching M is **perfect**.

M is a **maximum matching** in G, if $|M|=\beta_1$.

Every perfect matching is a maximum one. A perfect matching does not always exist.

Matching problem statement

- ξ_j = 1 if and only if the edge j belongs to the matching;
- c_j is the weight of the edge j;
- I is the incidence matrix.

The problem can be stated as a discrete linear programming problem.

$$\sum_{j=1}^{q} c_j \xi_j \to \max;$$

$$\sum_{j=1}^{q} I_{kj} \xi_j \leq 1, \quad \forall k = 1, \ldots, p;$$

$$\xi_j \in \{0, 1\}.$$

Cover problem statement

- ξ_j = 1 if and only if the edge j belongs to the cover;
- c_j is the weight of the edge j;
- I is the incidence matrix.

The problem can be stated as a discrete linear programming problem (the shortest cover of the transposed incidence matrix).

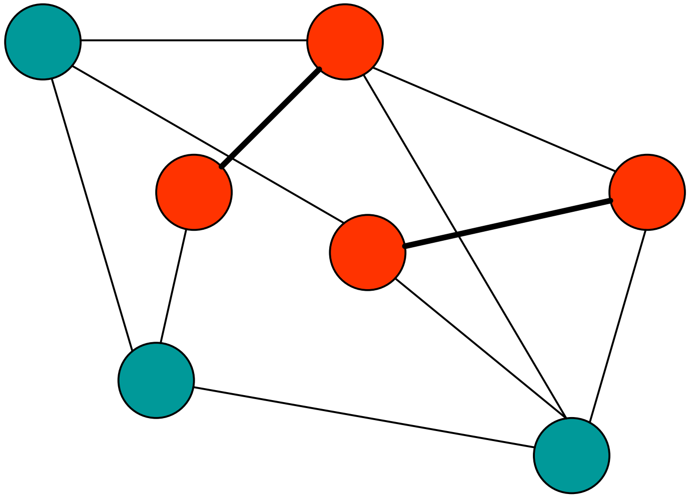
$$\sum_{j=1}^{q} c_j \xi_j \to \min;$$

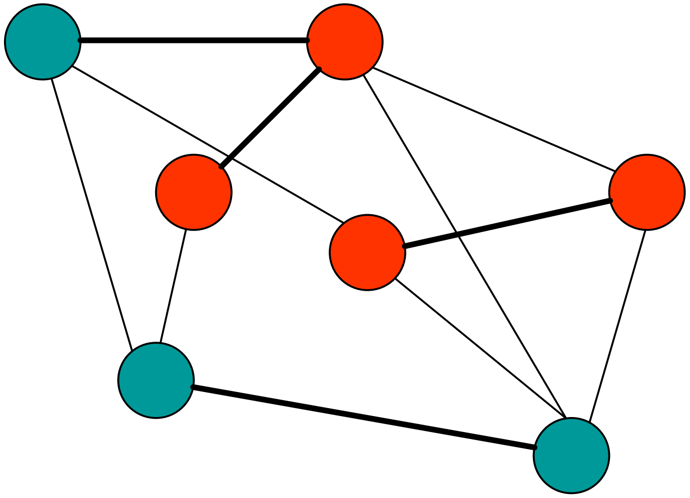
$$\sum_{j=1}^{q} I_{kj} \xi_j \geq 1, \quad \forall k = 1, \ldots, p;$$

$$\xi_j \in \{0, 1\}.$$

Matchings and covering sets

- A solution of the **maximum matching problem** provides a solution of the **minimum cover problem**.
- From matching to cover: let M be a matching. Choose vertex v that is not covered by M. Add to M an edge incident to v. Repeat until there are no non-covered vertices, as a result get a cover C.

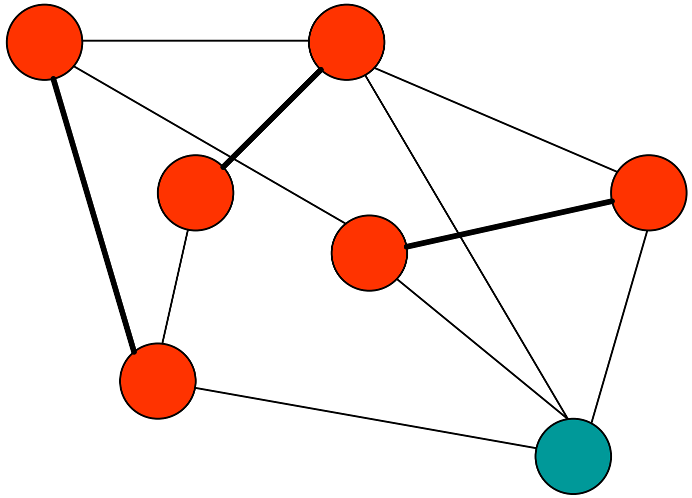


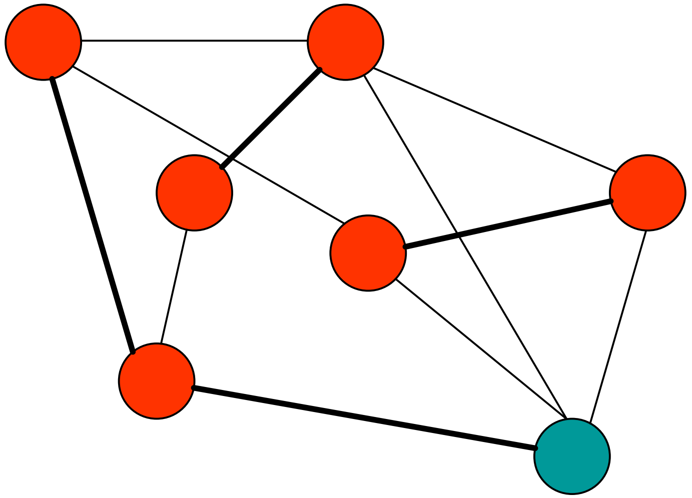


Matchings and covering sets

If M is a maximum matching then C is a minimum cover.

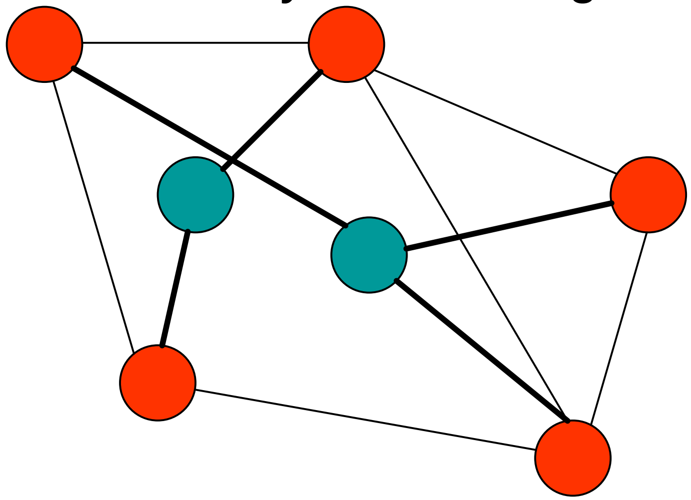
- M covers 2|M| vertices.
- |C|=|M|+(p–2|M|), because if M is a maximum matching then there are no edges connecting vertices non-covered by M; hence, to cover the vertices we need V–2|M| edges.
- If $|M|=\beta_1$ then $|C|=\beta_1+(p-2\,\beta_1)= p-\beta_1 = \alpha_1$.

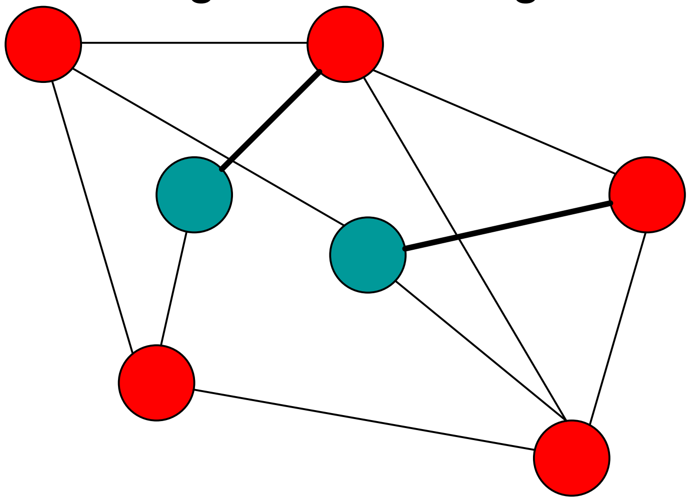




Matchings and covering sets

- A solution of the **minimum cover problem** provides a solution of the **maximum matching problem**.
- From cover to matching: let C be a cover. Choose vertex v that is incident to more then one edge of C. Remove from C any edge incident to v. Repeat until there are no vertices covered by several edges, as a result get a matching M.

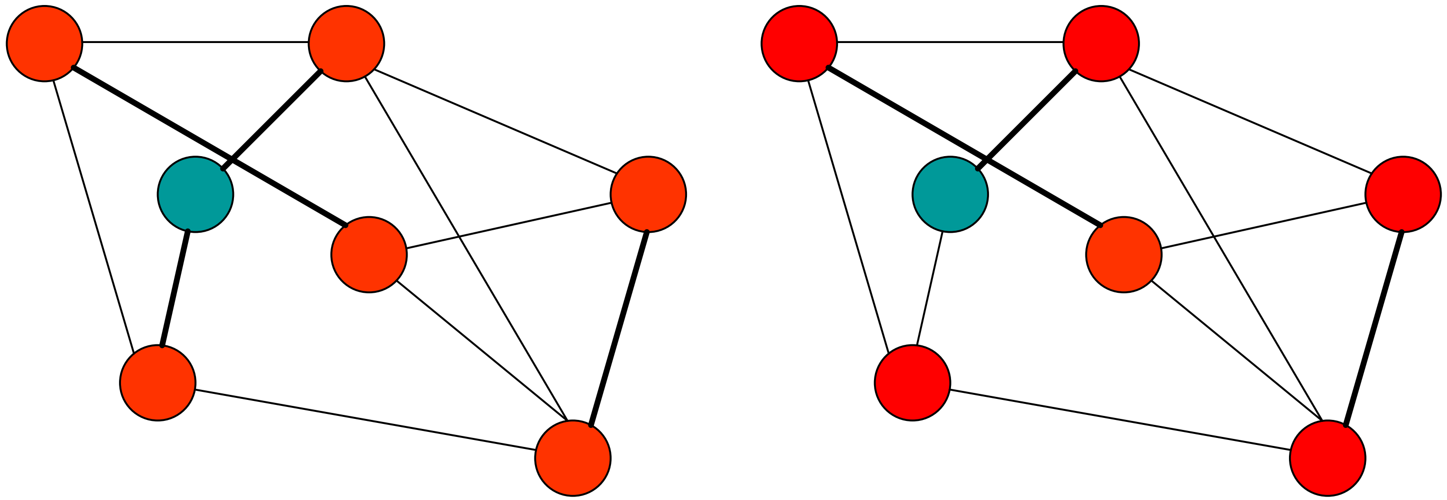




Matchings and covering sets

If C is a minimum cover then M is a maximum matching.

- If C were a matching it would cover 2|C| vertices.
- We remove 2|C|–p edges
- If $|C| = \alpha_1$ then $|M| = \alpha_1 - (2\alpha_1 - p) = p - \alpha_1 = \beta_1$.

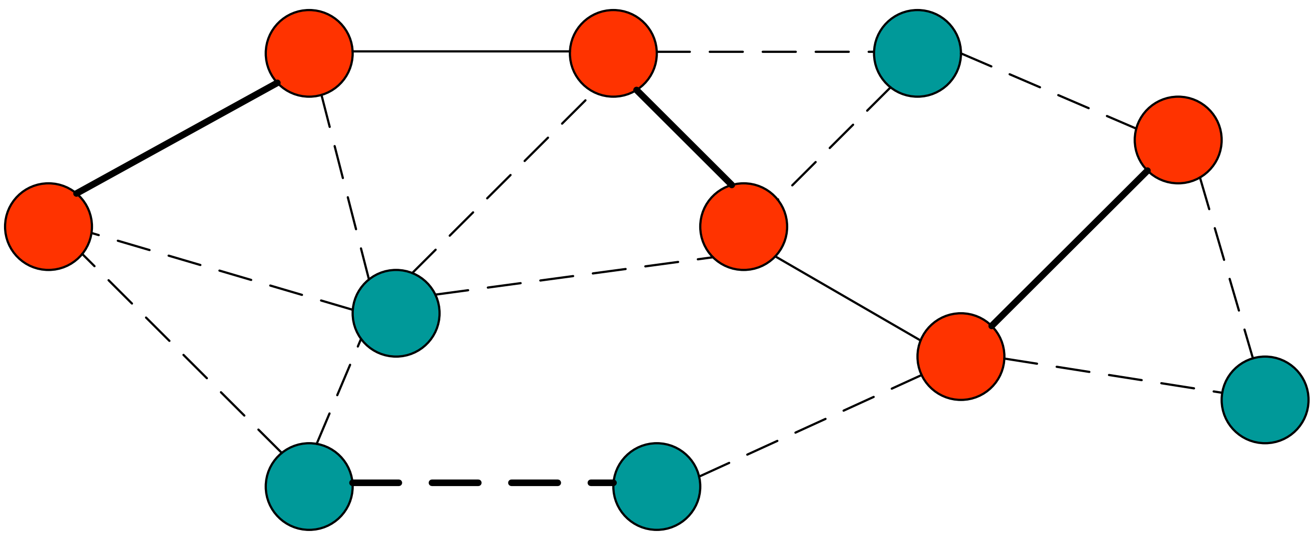


Maximum-cardinality matching

- https://www.youtube.com/watch?v=JpapV5DrBek
- https://www.youtube.com/watch?v=q26mBLtEHfk
- https://www.youtube.com/watch?v=03PUwWef2Dg&index=54&list=PLaLOVNqqD-2H-Ri_EwTR6YHmApSBpinkC

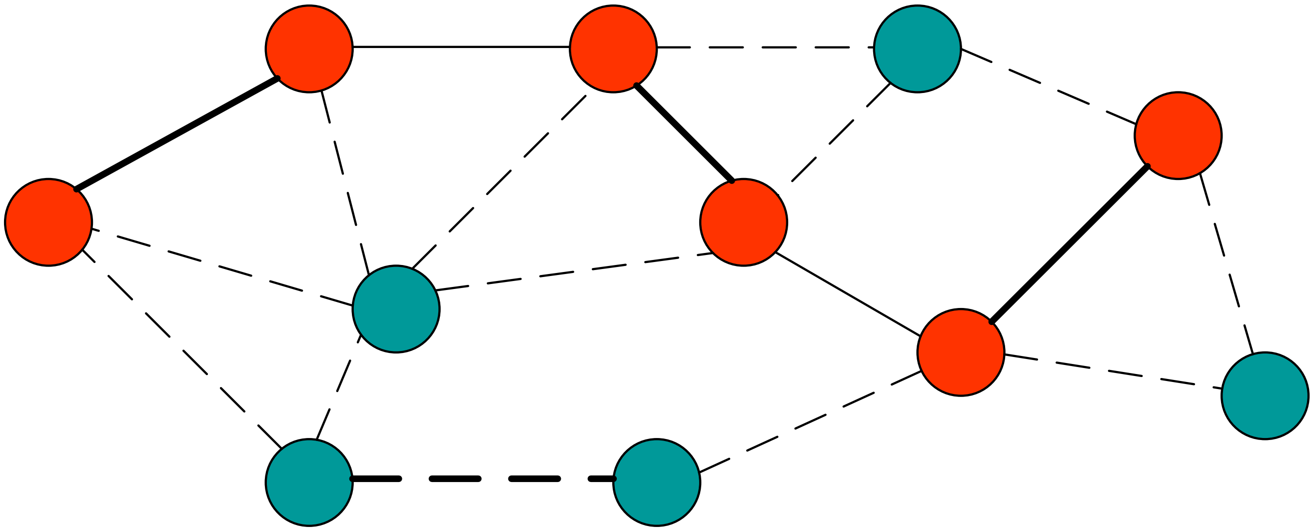
Maximum-cardinality matching

- Given a matching M in graph G, an *alternating path* is a path in which the edges are alternately in and out of matching M.



Maximum-cardinality matching

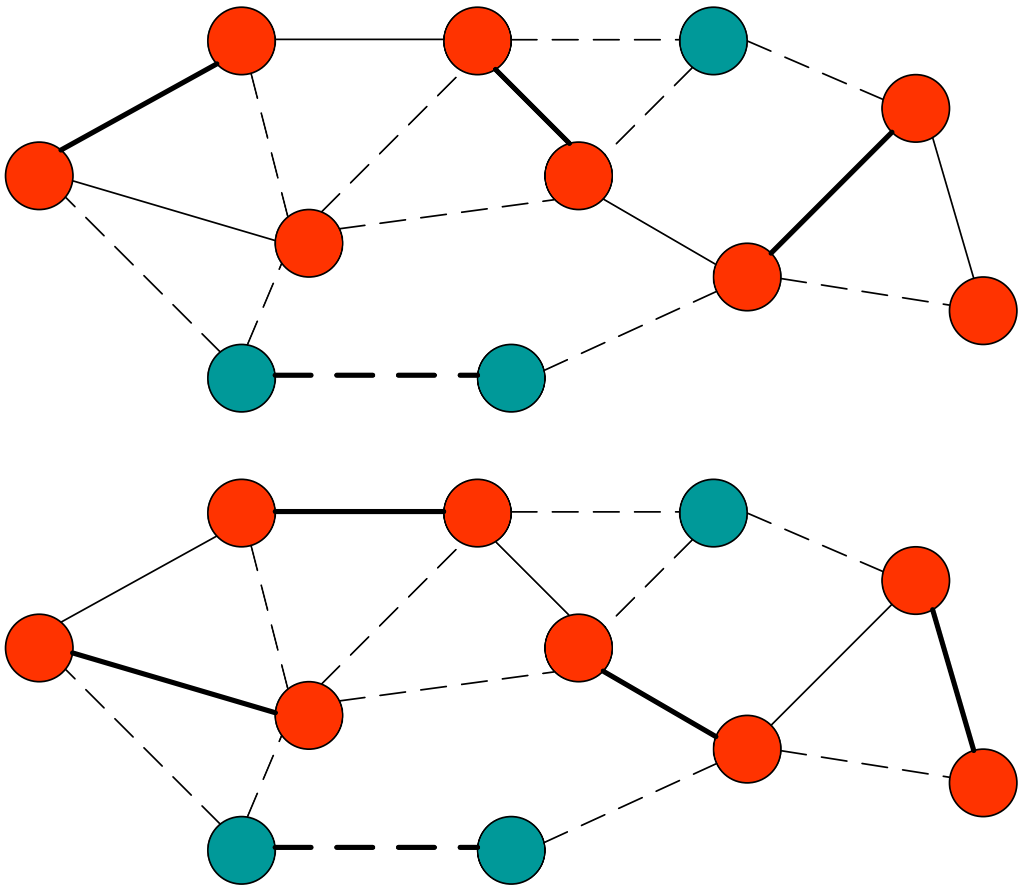
- An *augmenting path* is an alternating path whose first and last vertices are M-unsaturated.



Maximum-cardinality matching

- A matching is a maximum-cardinality matching if, and only if, it does not contain an augmenting path.
- If an augmenting path is found, the roles in the matching of the edges in this path are reversed. This creates a matching with greater cardinality.

Maximum-cardinality matching

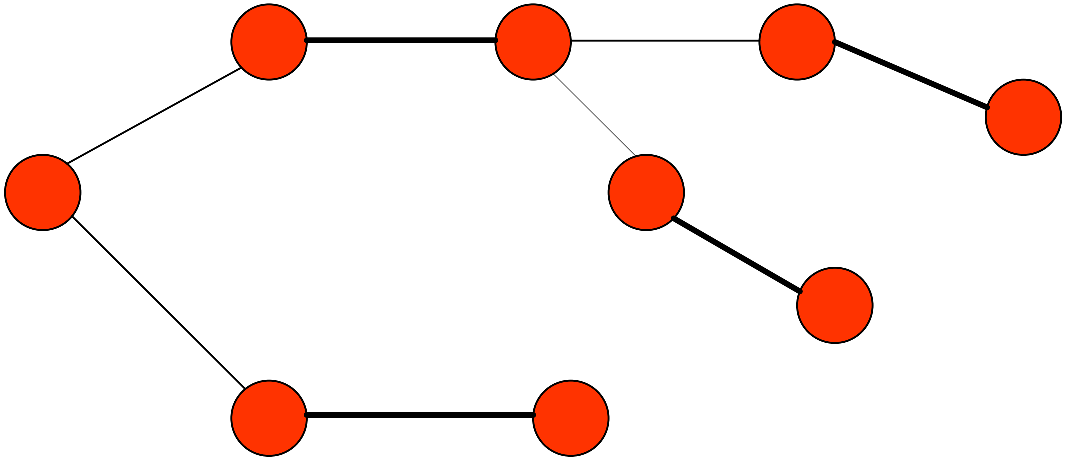


Maximum-cardinality matching

An *alternating tree* relative to a given matching M is a tree T for which:

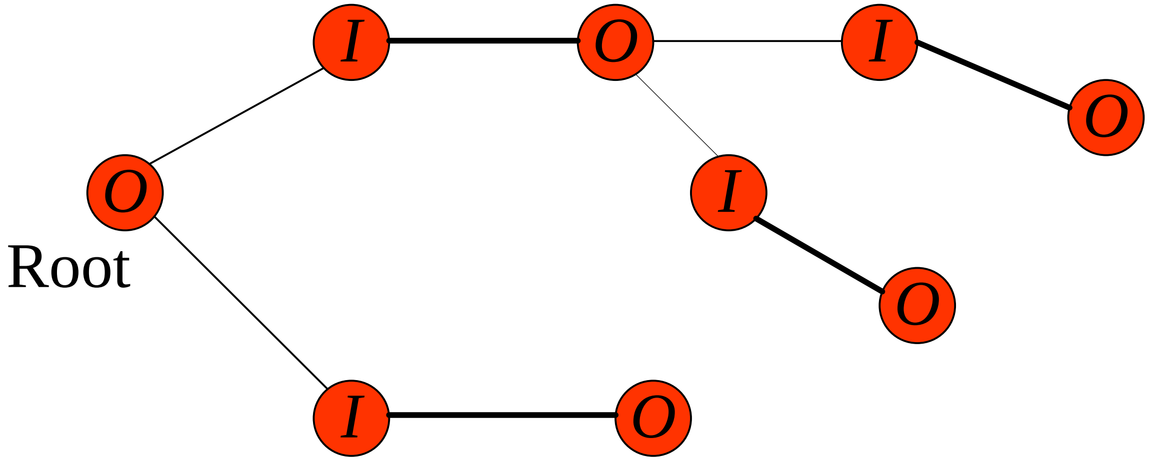
- One vertex of T is M-unsaturated and is called the *root* of T.
- All paths starting at the root are alternating paths.
- All maximal paths from the root of T are of even cardinality, i.e. contain an even number of edges.

Root



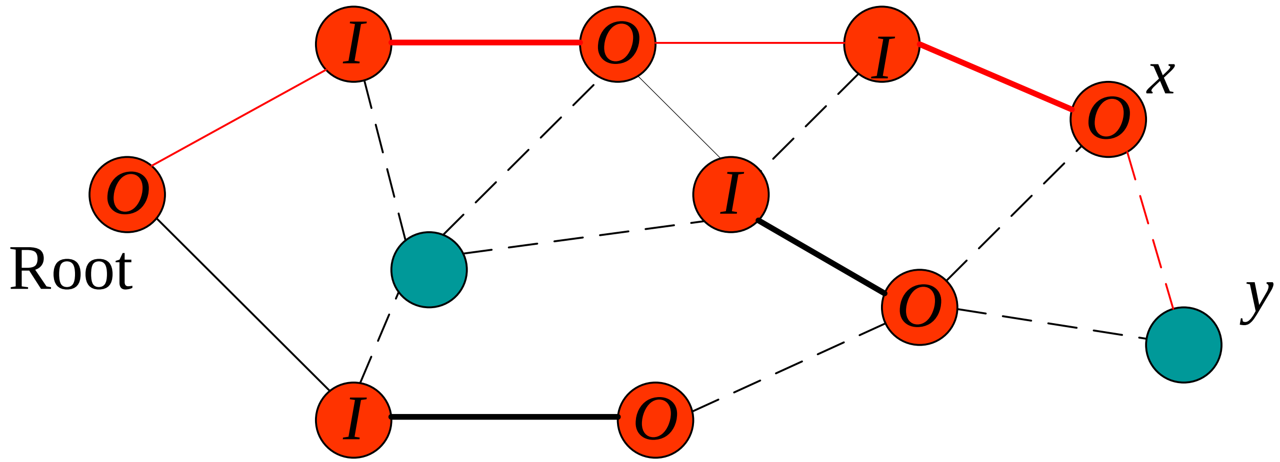
Maximum-cardinality matching

- Starting from the root of the tree and labeling it outer the vertices along any path starting from the root are labeled alternately *inner* and *outer*.
- Degree of all inner vertices is exactly 2 whereas the degree of an outer vertex can be any integer greater than or equal to 1.

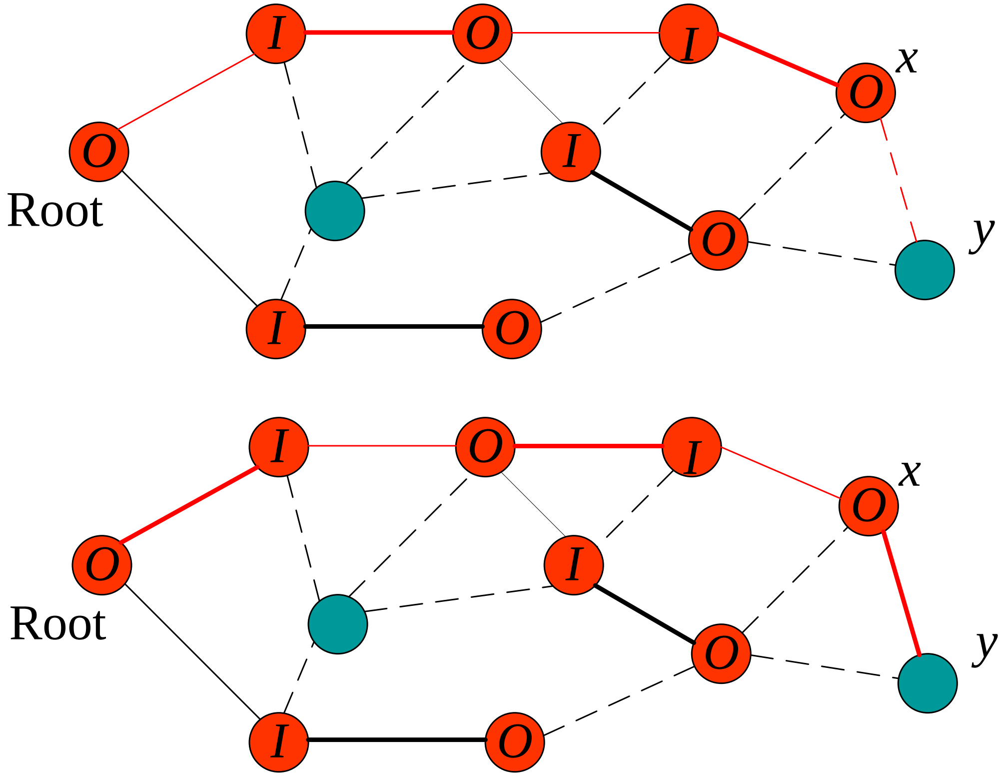


Maximum-cardinality matching

- An *augmenting tree* is an alternating tree relative to a given matching M whenever an edge exists from an outer vertex x of the tree to a M-saturated vertex y not in the tree.
- The unique path from the root of the tree to x plus link (x, y) is then an augmenting path.

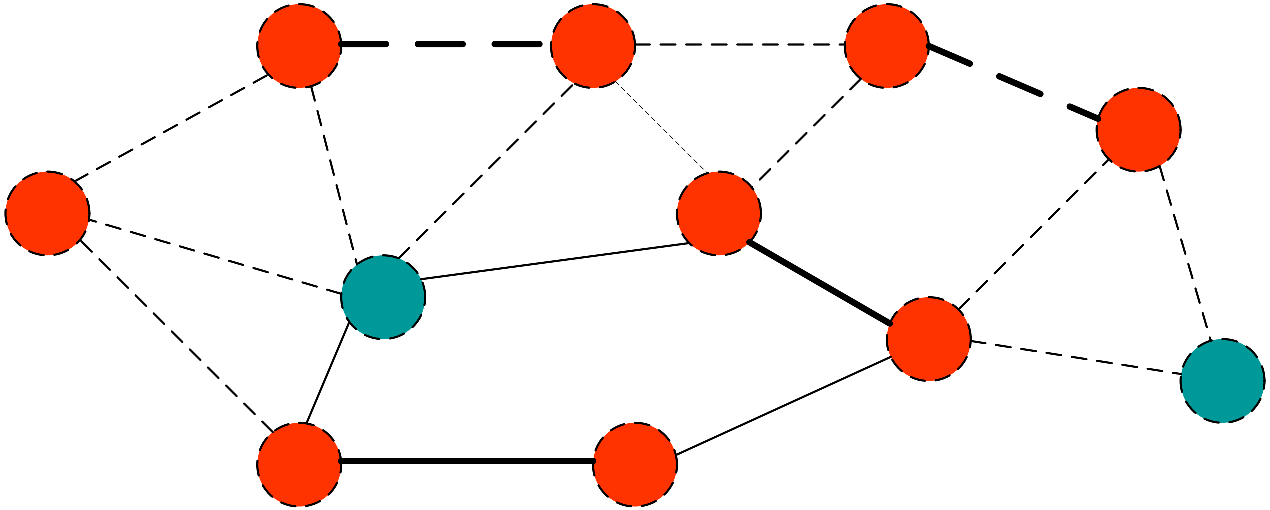


Maximum-cardinality matching



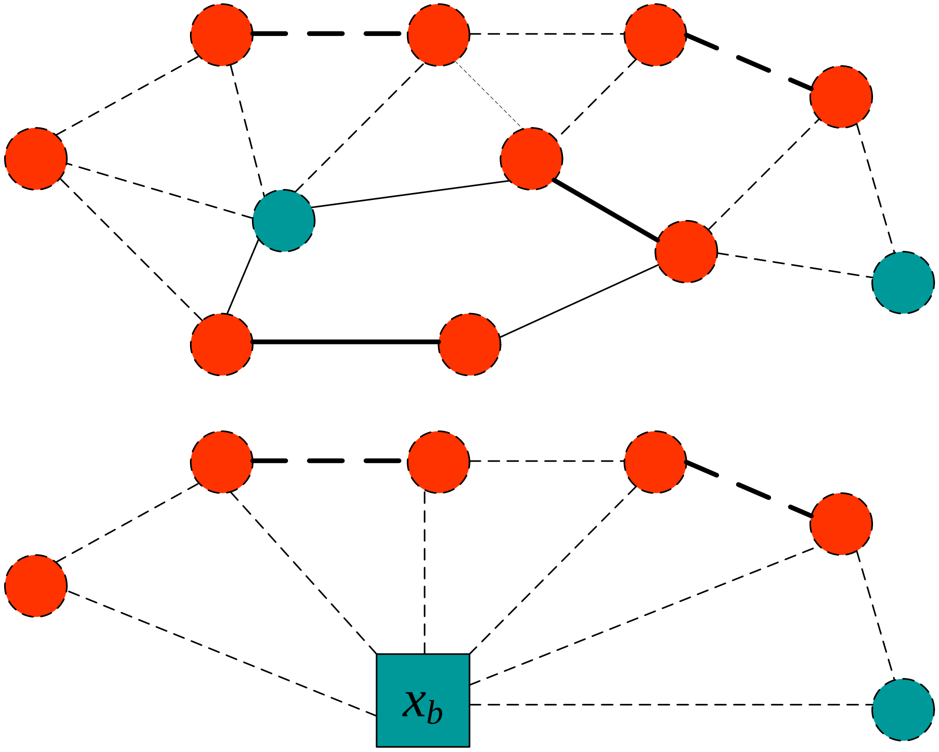
Maximum-cardinality matching

- A **blossom** with respect to a matching M is an augmenting path for which the initial and final exposed vertices are identical—i.e. the path forms a circuit—and the number of edges (or vertices) of the circuit is odd.



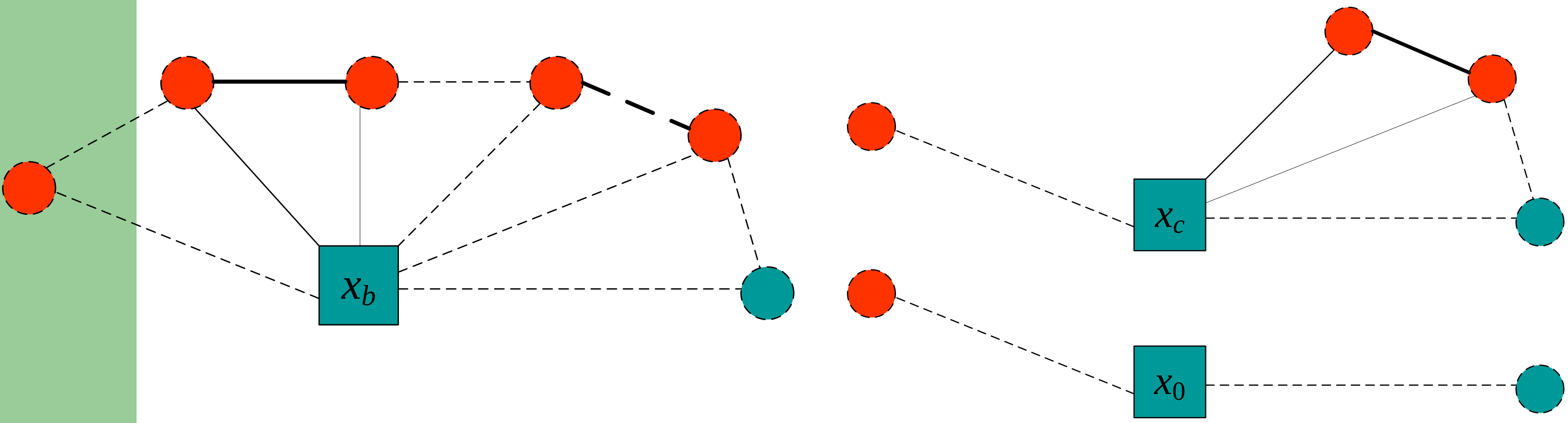
Maximum-cardinality matching

- Blossoms are *shrunk* to derive a new simpler graph. The *shrinking* of a blossom B implies the replacement of all vertices of B (say X_b) by a single new pseudovertex x_b.



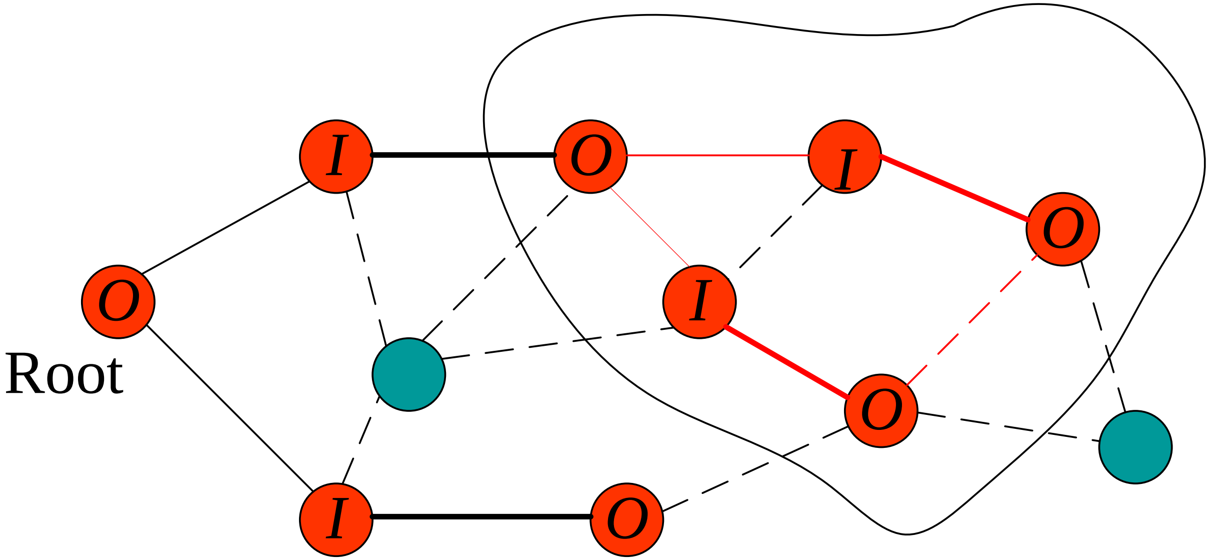
Maximum-cardinality matching

- In the simpler graph resulting from such a shrinking, vertex x_b may form a new blossom which is shrunk again and so on.
- The final blossom B0 which is not contained in any other blossom is called an *outermost blossom*.



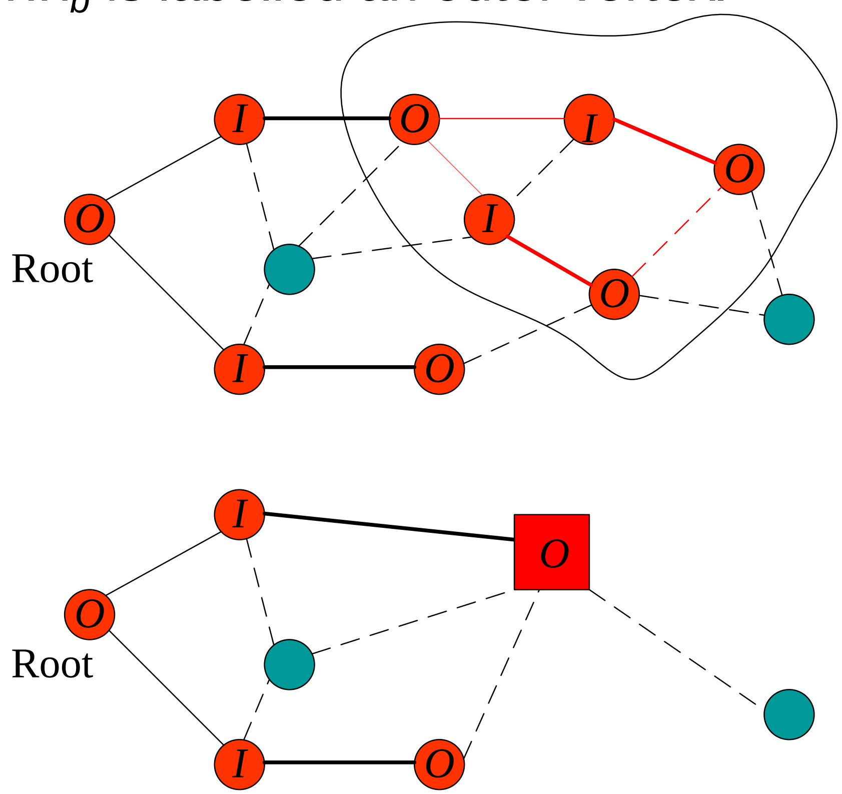
Maximum-cardinality matching

- A *blossomed tree* is an alternating tree relative to a given matching whenever a link exists between two outer vertices of the tree.



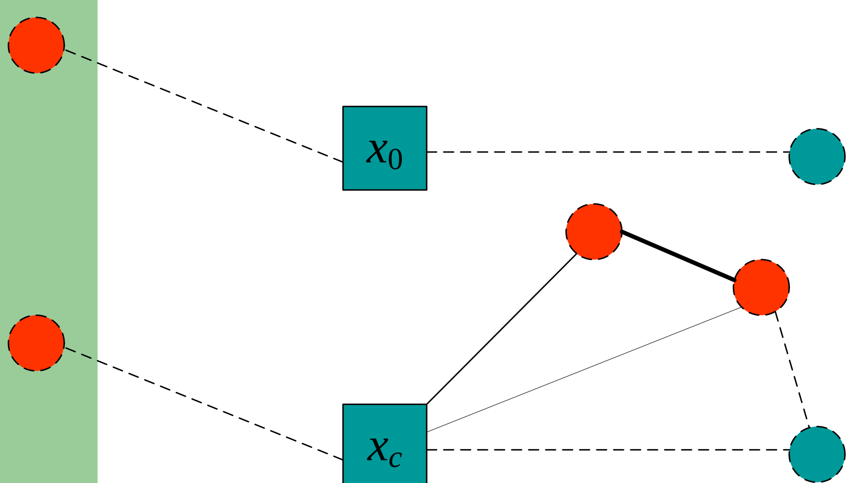
Maximum-cardinality matching

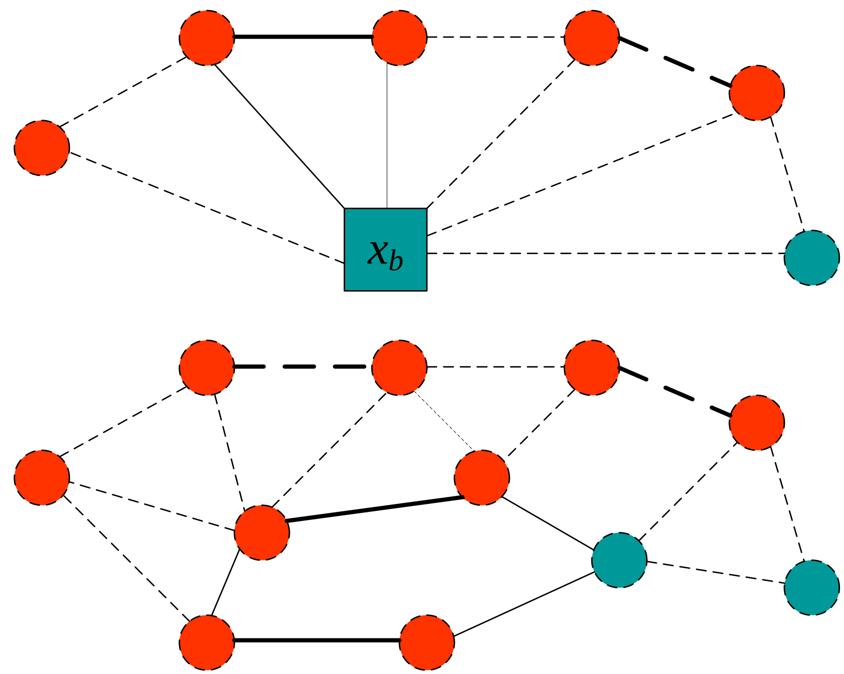
- Whenever a blossom B is shrunk, the resulting pseudovertex x_b is labelled an outer vertex.



Maximum-cardinality matching

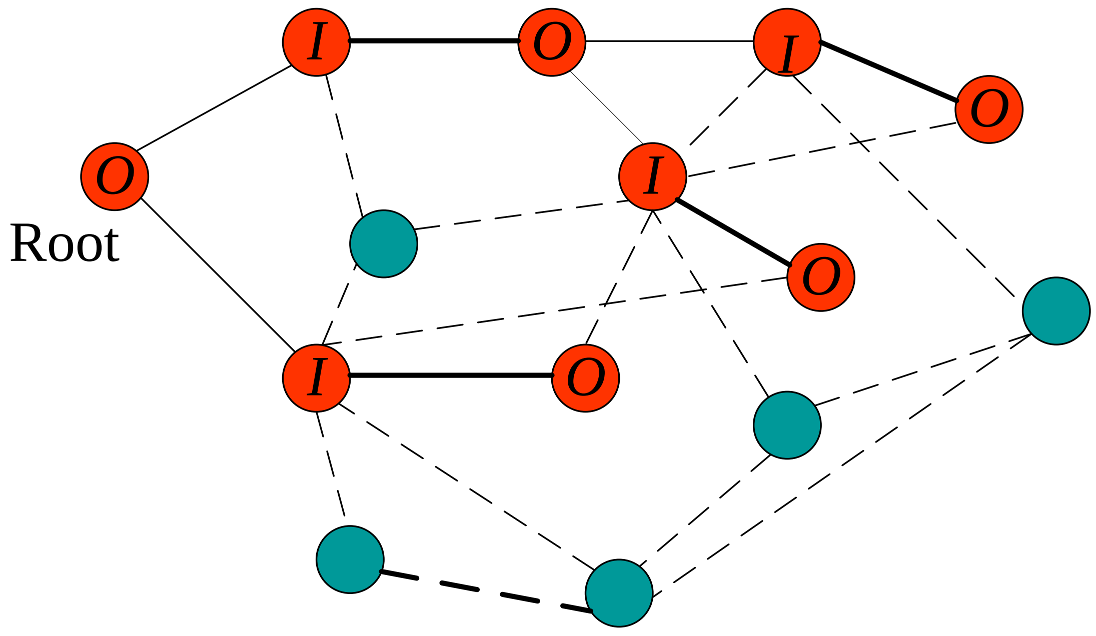
- If B is a blossom based on the odd vertex set Xb, and if *x* is any vertex in Xb, then there exists a maximum cardinality matching in the subgraph induced by Xb which leaves *x* unsaturated.





Maximum-cardinality matching

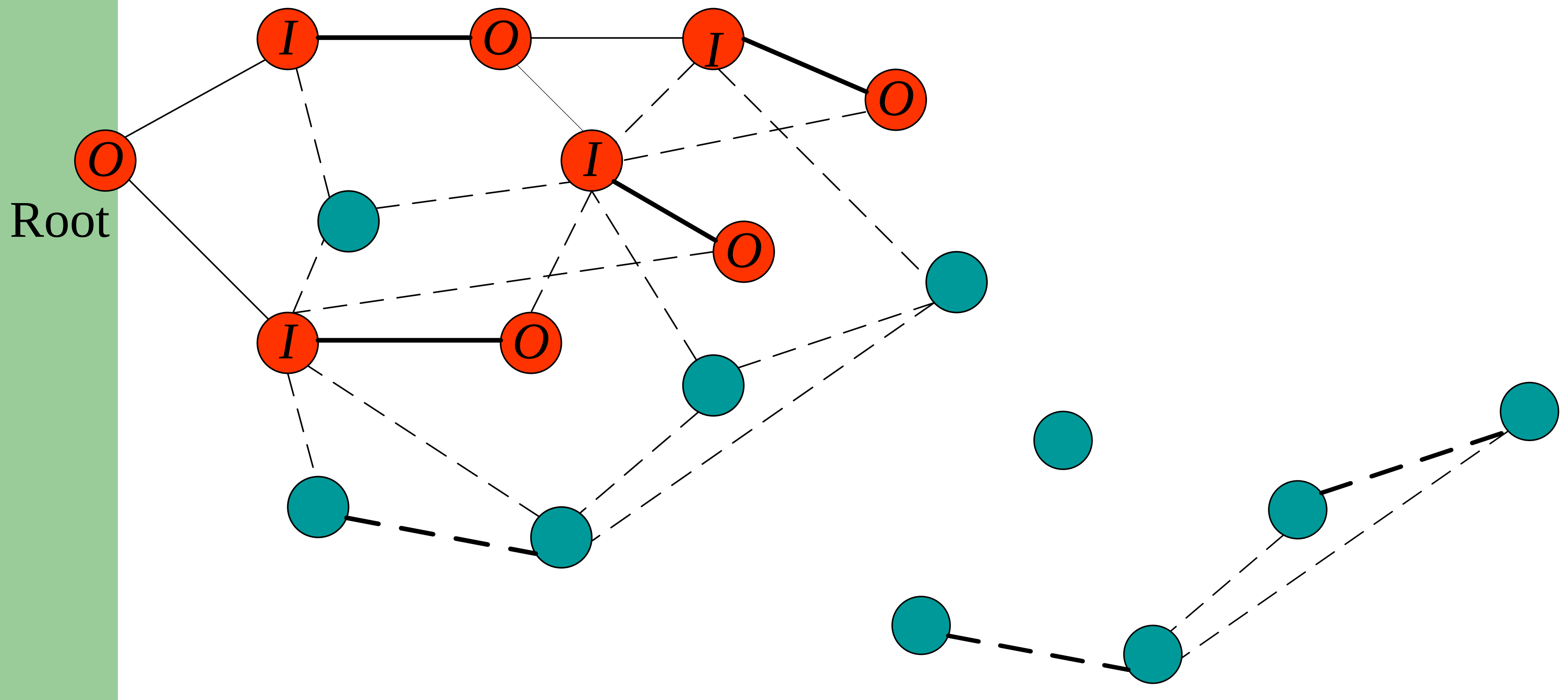
- A *Hungarian* tree is an alternating tree in a graph in which all links having an outer vertex of the tree as one end, have an inner vertex also in the tree as the other end.



Maximum-cardinality matching

- Let H be a Hungarian tree in a graph $G = (V, E)$ and
- $G_0 = (V \setminus V_H, Y)$ be the subgraph of G excluding the set V_H of vertices of H.
- Then, if M_H is the matching in the tree H and M_0 is any maximum cardinality matching in G_0, the set of edge $(M_H \cup M_0)$ is a maximum cardinality matching in G.

Maximum-cardinality matching



Maximum-cardinality matching

An alternating tree is rooted at an exposed vertex and grown by alternately adding links which are in and not in the matching until:

- either (i) The tree becomes augmenting,
- or (ii) The tree blossoms,
- or (iii) The tree becomes Hungarian.

Maximum-cardinality matching

- In case (i) the cardinality of the matching can be increased by one simply by tracing the augmenting path back to the root of the tree and then interchanging those edges of the path that belong to the matching with the ones that do not. After augmentation the tree is discarded, and a new tree is rooted at some remaining unsaturated vertex, if one exists.

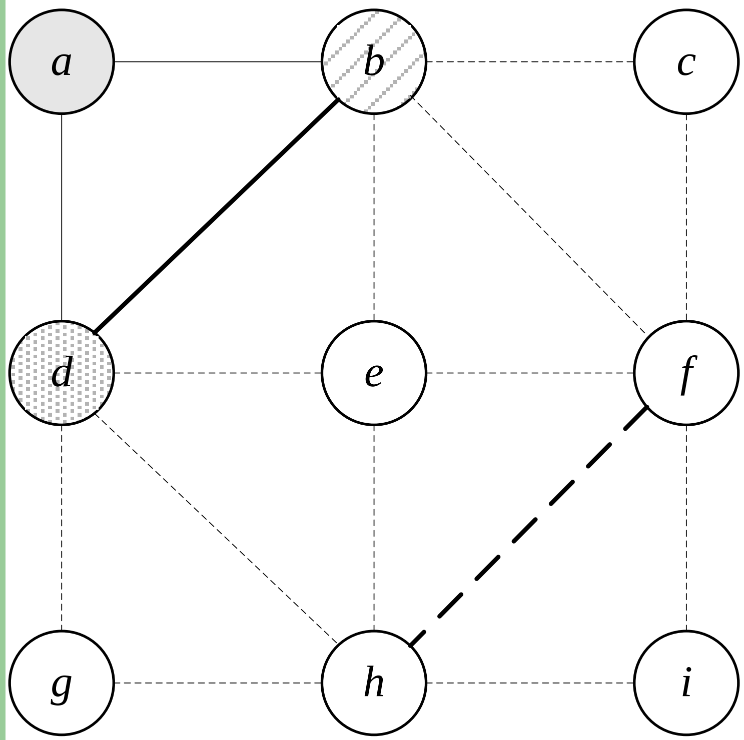
Maximum-cardinality matching

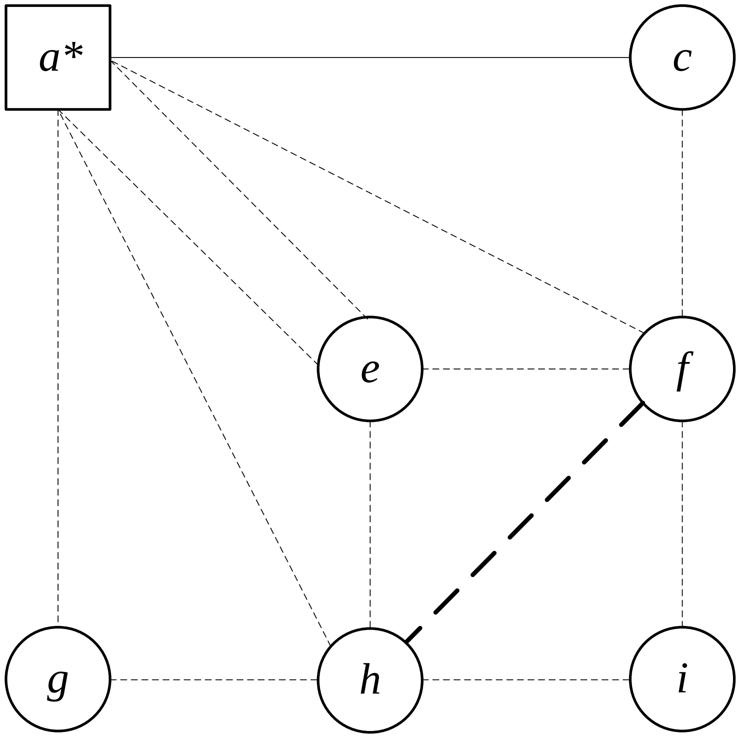
- In case (ii), the resulting blossom is identified, shrunk, and the growing of the tree continued in search for an augmenting path. As far as the computing is concerned, the shrinking of a vertex need not be done explicitly. All that is required is to mark all the vertices of the blossom as outer and set up labels on the vertices to indicate that they all belong to this blossom. The order in which these blossoms have been "shrunk" is important since at the end of the procedure the blossoms must be "expanded" in reverse order.

Maximum-cardinality matching

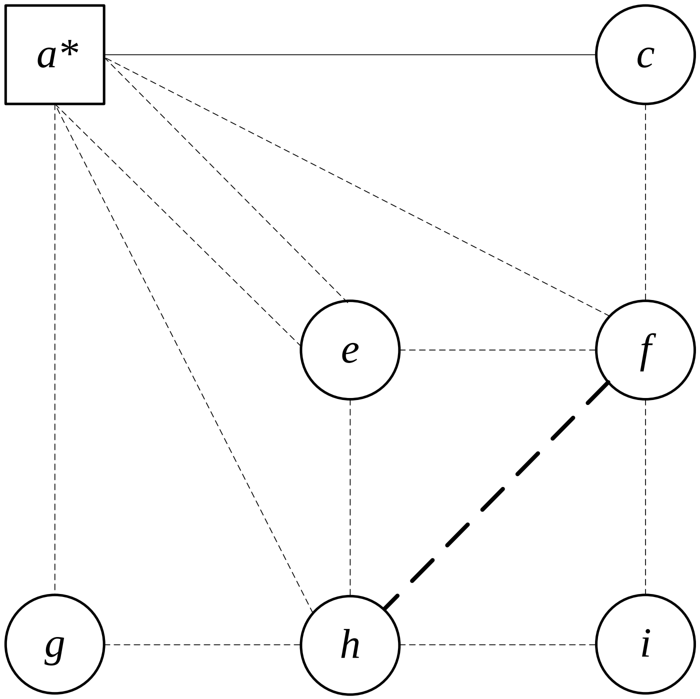
- In case (iii) the vertices of the hungarian tree and their incident links are removed from the graph permantly and the algorithm is reapplied to the remaining subgraph.

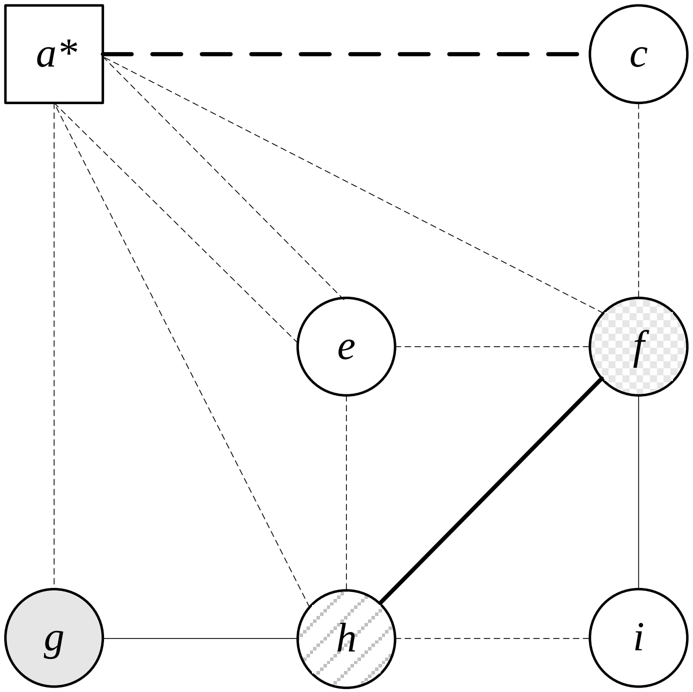
Maximum-cardinality matching



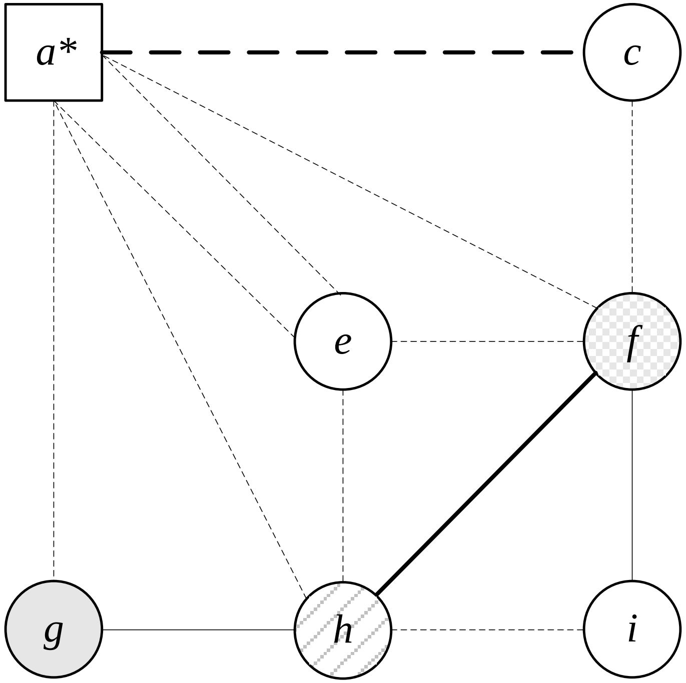


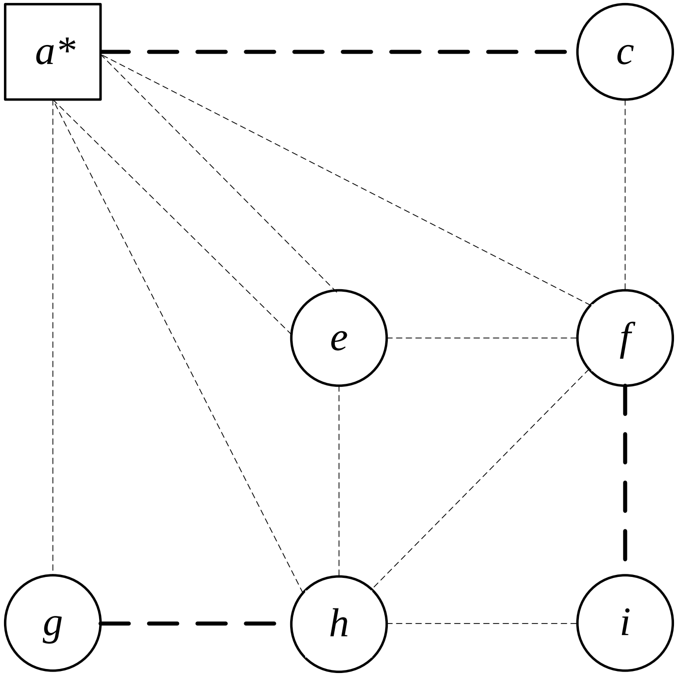
Maximum-cardinality matching





Maximum-cardinality matching





Maximum-cardinality matching