Nonuniform Distributed Random Numbers

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Random numbers obtained in an experiment often have a nonuniform distribution.

Several methods based on the transformation of uniformly distributed numbers are used for computer simulation of statistical equivalent sequences.

Some of these methods are discussed below.

The Distribution Function Method

Let random numbers γ uniformly distributed within the interval $(0, 1)$ on the axis Oy are carried over the axis Ox by the monotonous function $F(x)$ meeting the conditions $F(a) = 0$, $F(b) = 1$ (Fig. 1).

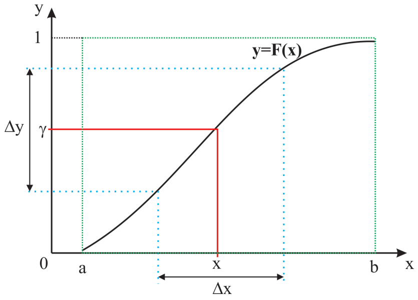

Figure: The simulation of random numbers by the distribution function method.

It is easily seen that the numbers x obtained here have the nonuniform distribution within the interval (a,b) in the axis Ox.

Distribution density of these numbers can be computed in the following way.

Each number γ on the axis Oy corresponds to the number x in the axis Ox, therefore, the amount of numbers that get into the interval δy is equal to the amount of numbers x in the interval δx, the boundaries of which are in agreement with the boundaries Δy:

$$N(x \in \Delta x) = N(\gamma \in \Delta y).$$

But for the uniformly distributed numbers

$$N(\gamma \in \Delta y) \approx N \; \Delta y,$$

where N is the total amount of obtained numbers γ.

Therefore, the probability density for the numbers x is equal to

$$w(x) \approx \frac{N(x \in \Delta x)}{N\ \Delta x} = \frac{N(\gamma \in \Delta y)}{N\ \Delta x} \approx \frac{\Delta y}{\Delta x} \approx \frac{dF}{dx}.$$

That is, to obtain random numbers with the density $w(x)$ as the function $F(x)$ on fig. 1 it is necessary to take such a function for which

$$\frac{dF}{dx} = w(x)$$

or

$$F(x) = \int_a^x w(x)\ dx. \tag{1}$$

Such function $F(x)$ is called ***the distribution function of the random value*** x.

It is seen from fig. 1 that the numbers γ and x are related by the correlation

$$\gamma = F(x),$$

that is

$$x = F^{-1}(\gamma),$$

where F^{-1} is the function reverse to F.

To illustrate it, the simulation results of random points distributed according to the exponential law $w(x) = e^{-x}$, $0 \leq x < \infty$ are presented in fig. 2.

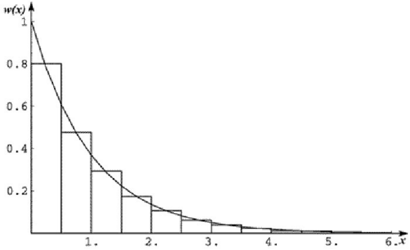

Figure: The results of the exponential distribution simulation.

In this case

$$F(x) = 1 - e^{-x}$$

and

$$x = -\log(1 - \gamma).$$

Note that the random numbers $1-\gamma$ as well as the numbers γ have a uniform distribution over the interval $(0,1)$, that is, they are statistically equivalent to the numbers γ, therefore, when simulating the following formula was used

$$x = -\log\gamma.$$

Once More About Integrals.

Let random numbers x are selected from the distribution $w(x)$ in the segment (a,b), for each of them the value of some function $f(x)$ is calculated and these values are averaged:

$$I = \frac{1}{N}\sum_{i=1}^{N} f(x_i). \tag{2}$$

To calculate the value I, we divide the region (a,b) into small segments Δx_k and transform the sum over i into the sum over these intervals having classified contributions from the points that get into one interval.

If one neglects here the function change $f(x)$ in the interval Δx_k, then

$$I \approx \frac{1}{N} \sum_k f(x_k)\ N(x \in \Delta x_k),$$

where $N(x \in \Delta x_k)$ is the quantity of random numbers x in the interval Δx_k.

The simple transformation of this sum

$$I \approx \frac{1}{N} \sum_k f(x_k)\ \frac{N(x \in \Delta x_k)}{\Delta x_k} \Delta x_k$$

and taking into account the definition of probability density function one can get

$$I \approx \sum_k \; f(x_k)\; w(x_k)\; \Delta x_k.$$

At $\Delta x_k \to 0$ this sum will become an integral $\int\limits_a^b f(x)\; w(x)\; dx$.This means that formula (2) gives the estimate for this integral.

To apply this method for the integral calculation

$$I = \int_a^b f(x)\, dx,$$

with no second factor under the integral we transform the subintegral expression in the following way:

$$I = \int_a^b \tilde{w}(x) \frac{f(x)}{\tilde{w}(x)}\, dx,$$

where the function $\tilde{w}(x)$ possesses the properties of the probability density:

$$\tilde{w}(x) > 0, \quad a \le x \le b,$$

$$\int_a^b \tilde{w}(x)\, dx = 1.$$

In correspondence with (2) the integral assessment of I is the sum

$$I \approx \frac{1}{N} \sum_{i} \frac{f(x_i)}{\tilde{w}(x_i)},$$

where x_i are random numbers with the distribution density $\tilde{w}(x)$.

For the integral assessment

$$I = \int_{0}^{\infty} \exp(-x^2)\ dx$$

the following function was used as$\tilde{w}(x)$

$$\tilde{w}(x) = \exp(-x).$$

The example of such a computation is presented in3.

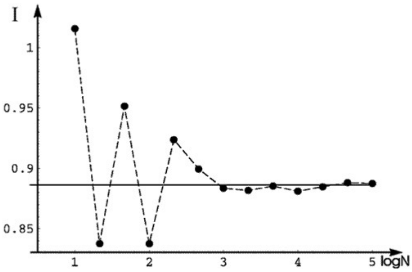

Figure: The results of the integral computation by the modified statistical simulation method.

It is seen from this example that a modified method can be applied to compute an improper integral with infinite boundaries. This can not be done by means of the method considered above where random points were uniformly distributed in the rectangle into which the subintegral function was inscribed.