The Simulation of Random Points in Curvilinear Coordinates

Point position in a plane can be plotted not only by Decartesian coordinates x, y but also by polar ones ρ, φ

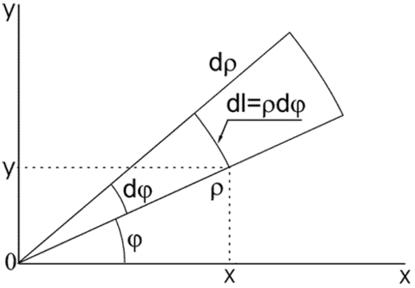

Figure: For transformation to polar coordinates.

The simulation of polar coordinates of random points in some cases might prove to be preferable. As an example let us consider the problem on simulation of points uniformly distributed over the circle with radius R discussed before.

If there are N random points uniformly distributed in the circle, their mean density equals $\dfrac{N}{\pi\ R^2}$, the number of points on the ground ds equals $N(\vec{r} \in ds) = \dfrac{N}{\pi\ R^2}\ ds$, and the hit probability into ds equals

$$P(\vec{r} \in ds) = \frac{N(\vec{r} \in ds)}{N} = \frac{ds}{\pi\ R^2}.$$

Substituting the expression for ds in polar coordinates in to this formula

$$ds = d\rho\ dl = \rho\ d\rho\ d\varphi,$$

we can see that the probability $P(\vec{r} \in ds)$ can be presented in the form

$$P(\vec{r} \in ds) = w_\rho(\rho)d\rho \; w_\varphi(\varphi)d\varphi,$$

where

$$w_\rho(\rho) = \frac{2\rho}{R^2}, \; 0 \le \rho \le R$$

is the probability density for the polar radius ρ, and

$$w_\varphi(\varphi) = \frac{1}{2\pi}, \; 0 \le \varphi \le 2\pi$$

the probability density for the azimuth φ.

If a random point falls to the ground ds that means that its coordinate ρ belongs to the interval $d\rho$ and the coordinate φ to the interval $d\varphi$. Therefore, the probability $P(\vec{r} \in ds)$ turned out to be equal to the product of the two factors depending upon ρ and φ.

Random numbers ρ and φ with given distributions can be obtained by the distribution function method from uniformly distributed numbers γ according to these formulas:

$$\rho = R\ \sqrt{\gamma},$$

$$\varphi = 2\pi\ \gamma.$$

Random Points Uniformly Distributed over the Sphere Surface and in the Sphere Volume.

If N random points are uniformly distributed over the sphere surface with radius R their mean density is equal to $\frac{N}{4\pi\ R^2}$, the number of points on the ground ds equals$N(\vec{r} \in ds) = \frac{N}{4\pi\ R^2}\ ds$, and the hit probability in ds equals

$$P(\vec{r} \in ds) = \frac{N(\vec{r} \in ds)}{N} = \frac{ds}{4\pi\ R^2}.$$

The size of the elementary ground ds formed by close parallels and meridians in spherical coordinates equals

$$ds = R^2\ \sin\vartheta\ d\vartheta\ d\varphi,$$

hence

$$P(\vec{r} \in ds) = w_\vartheta(\vartheta)d\vartheta \; w_\varphi(\varphi)d\varphi,$$

where

$$w_\vartheta(\vartheta) = \frac{1}{2}\sin\vartheta, \; 0 \le \vartheta \le \pi$$

is the probability density for the polar angle $\vartheta$and

$$w_\varphi(\varphi) = \frac{1}{2\pi}, \; 0 \le \varphi \le 2\pi$$

is the probability density for the azimuth φ.Here as in the previous example the random point getting in to the ground ds means that its polar angle ϑ belongs to the interval $d\vartheta$ and the azimuth φ to the interval $d\varphi$.

Random numbers ϑ and φ with given distributions can be easily obtained by the distribution function method from uniformly distributed γ.

The uniform distribution of points over the sphere means that the unit vector $\vec{\Omega} = \vec{r}/r$ with projections

$$\Omega_x = \sin\vartheta \; \cos\varphi,$$

$$\Omega_y = \sin\vartheta \; \sin\varphi,$$

$$\Omega_z = \cos\vartheta$$

has an isotropic distribution.Directions of the particle ejection in a radioactive decay, for example, have such a distribution and these formulas can be used for simulating the initial direction of the particle motion.

Analogous considerations for random points uniformly distributed in the sphere volume give

$$P(\vec{r} \in dV) = w_r(r)dr\ w_\vartheta(\vartheta)d\vartheta\ w_\varphi(\varphi)d\varphi,$$

where

$$w_r(r) = \frac{3r^2}{R^3},\ 0 \le r \le R,$$

$$w_\vartheta(\vartheta) = \frac{1}{2}\sin\vartheta,\ 0 \le \vartheta \le \pi,$$

$$w_\varphi(\varphi) = \frac{1}{2\pi},\ 0 \le \varphi \le 2\pi$$

are probability densities for spherical coordinates, r, ϑ, φ or simulation of which the distribution function method can be used.

In problems dealing with the transport theory points can have uniform distribution where particles are produced in a source. One can simulate their spherical coordinates with the aid of the method described.

The Simulation of the Normal (Gaussian) Distribution

Let us imagine that small pieces of paper are falling down on the floor from some height.

The trajectory and the fall place of each piece will be obviously random, however, the shape of "the hill" formed by them on the floor possesses the statistical stability.

Probability densities describing distributions of x and y-coordinates of random points in such problems are commonly considered to be the Gauss ones and, if the beginning of coordinates is in the center of the "hill" then

$$w_x(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{x^2}{2\sigma^2}), \ -\infty < x < \infty,$$

$$w_y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{y^2}{2\sigma^2}), \ -\infty < y < \infty,$$

where σ^2 is **the dispersion** - the parameter characterizing the "hill width".

Let us select the ground $ds = dx\,dy$ in a plane. Then the probability that the point will get into ds equals

$$P(\vec{r} \in \ ds) = P(x \in dx) \ P(y \in dy) = \frac{1}{2\pi\sigma^2} \exp(-\frac{x^2+y^2}{2\sigma^2}) \ dx \ dy.$$

In polar coordinates $x^2+y^2=\rho^2, \;\; ds=\rho\ d\rho\ d\varphi$,therefore,

$$P(\vec{r}\in\ ds)=\frac{1}{2\pi\sigma^2}\exp(-\frac{\rho^2}{2\sigma^2})\rho\ d\rho\ d\varphi.$$

This means that probability density of random quantities ρ and φ have the given by

$$w_\rho(\rho)=\frac{1}{\sigma^2}\exp(-\frac{\rho^2}{2\sigma^2})\ \rho,\ 0\le\rho<\infty,$$

$$w_\varphi(\varphi)=\frac{1}{2\pi},\ 0\le\varphi\le 2\pi.$$

With such probability densities the distribution function method gives

$$\rho=\sigma\sqrt{-2\log\gamma},$$

$$\varphi=2\pi\gamma,$$

If from polar coordinates ρ, φ obtained with the help of simulation we proceed to Decartesian ones, random numbers

$$x = x_0 + \rho \ \cos\varphi,$$
$$y = y_0 + \rho \ \sin\varphi$$

will have the normal distribution with mean values x_0, y_0 and the dispersion σ^2.

The result of the normal distribution simulation is presented in next figure.

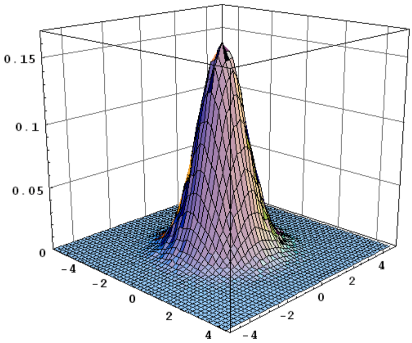

Figure: The results of simulation of the two-dimension normal distribution.

Note that the normal distribution is common in solving scientific and applied problems where the described method can be used for simulation.