INTRODUCTION

The computer simulation of a random phenomenon consists in generation of sequences of random numbers which are statistically equivalent to the sequences obtained in the real experiment with a random outcome which can be repeated many times under the same conditions.

Example 1. An experiment with multiple tossing of a fair coin gives the sequence of random numbers $S = 0$ ("tails") and $S = 1$ ("heads") with probabilities $P(0) = P(1) = 1/2$. In a computer simulation of this experiment such numbers can be obtained by the formula

$$S = [2\gamma_i],$$

γ_i being the random numbers uniformly distributed on the interval $(0, 1)$. The symbol $[q]$ means the integer part of a number q.

Example 2. The random numbers $1, 2, ...6$ with equal probabilities which can be observed in the experiment consisting in rolling of a fair die can be obtained by the formula

$$S = [6\gamma] + 1.$$

Example 3. If the range (a, b) where the random numbers x_i take values is divided into n equal parts then index s_i of the interval where x_i appears can be calculated by the formula

$$S = [n\frac{(x_i - a)}{b - a}] + 1.$$

Example 4. The random number s which is equal to the number of tosses of a fair coin until the first "tail" appears can be obtained by the flowchart and listing are given in Fig. 1.
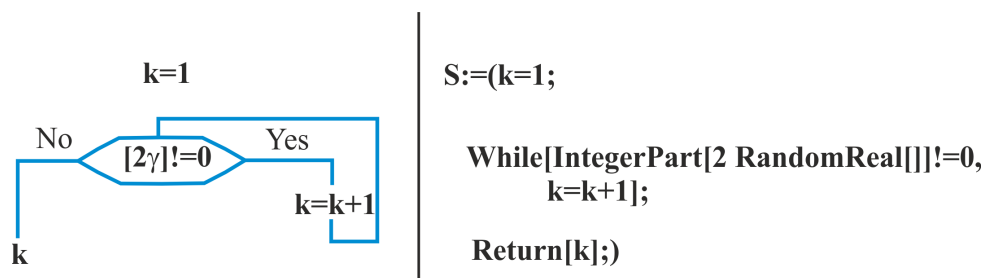

Figure 1: The flowchart and code describing the subroutine for the simulation of a coin tossing until the first "tail" appears.

Example 5. Let the range $(0,1)$ where the uniformly distributed random numbers γ vary is divided into unequal parts with the lengths P_k which are determined by the recurrent formula

$$P_{k+1} = r_k P_k, \ \left(\sum_k P_k = 1\right). \tag{1}$$

The index k of the interval where γ appears can be determined using the flowchart given in Fig. 2

```
                                              P1=?;
                                              r[k_]:=?;
                                              ........................
                  g=γ                         S:=(g=Random[];
                  RP=P1                           RP=P1;
                  RS=P1                           RS=P1;
                  k=1 or 0                        k=1;

    No                     Yes
          RS<g                                  While[RS<g,
                    RP=RP•r_k                        RP*=r[k];
                    RS=RS+RP                          RS+=RP;
                    k=k+1                             k+=1];

    k                                          Return[k];)
```
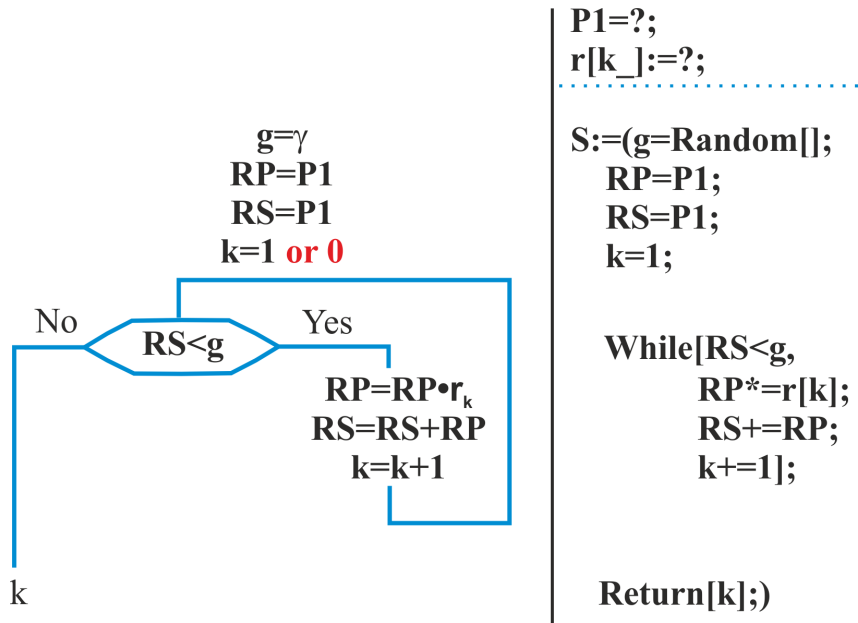
Figure 2: The flowchart and code describing the subroutine for the simulation of the interval index for the probabilities given by the recurrent formula (1).

The problems of statistical simulation considered below can be classified into 3 groups.

1. One needs to get Nt random numbers T_i and print them. The code of the program which solves this problem is given in Fig. 3

```
                                              Nt=?;
                                              S:=?;
                                              ........................

              T[...]=0                        T=Table[0,{Nt}];

               x=Ŝ_γ                          Do[x=S;
               T[i]=x                             T[[i]]=x,

      No    {1≤i≤Nt}  Yes

                                                 {i,1,Nt}];

        T                                     T
```
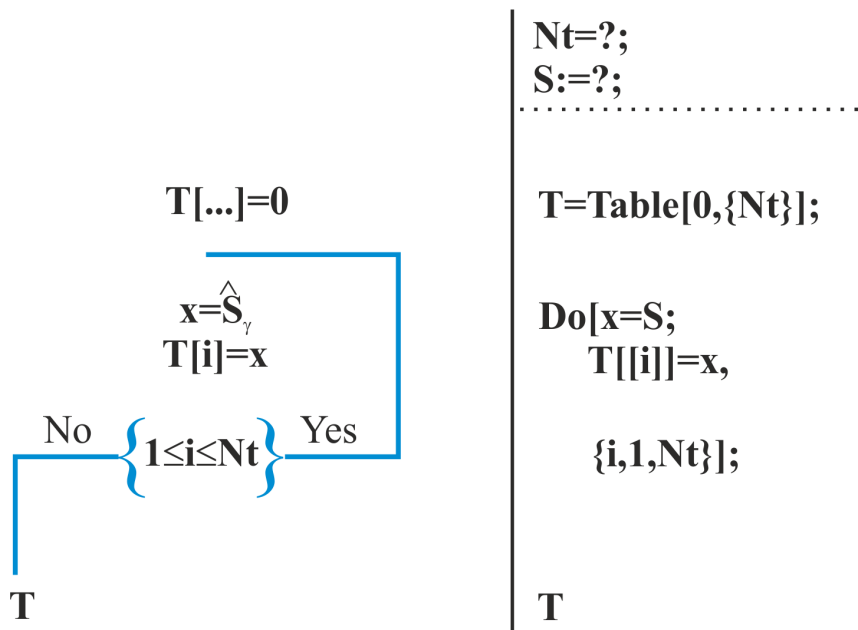
Figure 3: The code describing the program for the simulation of Nt random numbers.

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Or you could use this code:

```
Nt=?
S:=?
T=Table[S,{i,Nt}]
Histogram[T,15,''PDF'']
```

2. One needs to get Nt random numbers T_i from the given distribution and calculate the mean value

$$\bar{s} = \frac{1}{N}\sum_{i=1}^{N} T_i.$$

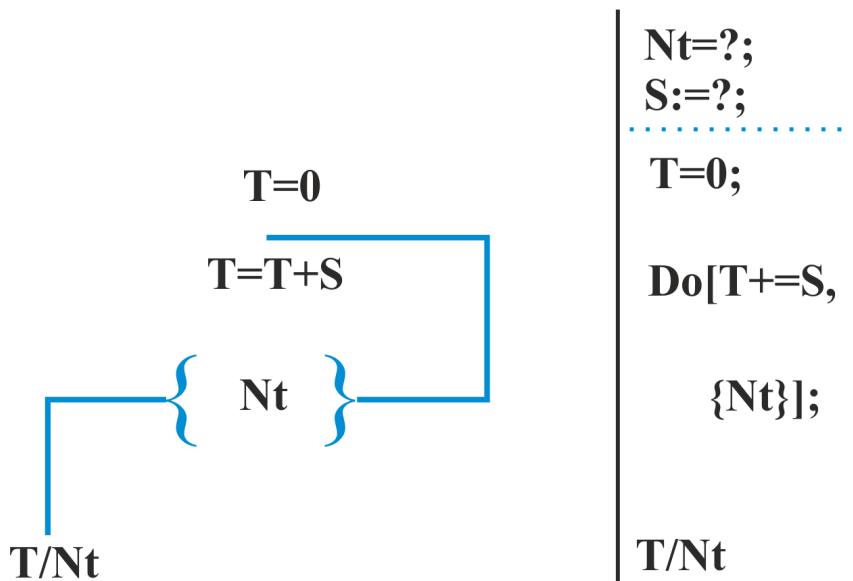The code of the program AVERAGE which solves this problem is given in Fig. 4

T=0

T=T+S

{ Nt }

T/Nt

Nt=?;
S:=?;

T=0;

Do[T+=S,

{Nt}];

T/Nt

Figure 4: The code of the program for the calculation of the mean value of a random quantity.

Or you could use this simple code:

```
Nt=?
S:=?
T=Table[S,{i,Nt}]
Mean[T]
```

UNIT 1

UNIFORMLY DISTRIBUTED RANDOM NUMBERS

QUESTIONS

1. What do the terms regularity, randomness, statistical regularity and statistical equivalence mean?

2. What do the terms probability and probability density function mean? Describe the main probabilistic laws for the discrete and continuous random variables.

3. Describe the concept of statistical equivalence and the algorithm for the simulation of random events.

4. Describe the algorithm for the simulation of the random numbers uniformly distributed on the intervals $(0, 1)$ and (a, b).

5. Describe the algorithm for simulation of the simplest random events: a coin tossing and a dice rolling.

6. Describe the algorithm for simulation of random events with unequal probabilities.

7. Describe the probabilistic characteristics of the breeding processes and the possibility to simulate the population evolution in generations.

EXERCISES

1. Obtain the recurrent formula

$$P_{k+1} = r_k P_k, \ \left(\sum_k P_k = 1 \right)$$

and make up the subroutines for the simulation of the interval index k where the random number γ uniformly distributed on $(0, 1)$ appears for the distributions

$$P_k = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, ...n$$

-the binomial distribution,

$$P_k = p(1-p)^k, \quad k = 0, 1, 2, ...$$

-the geometric distribution,

$$P_k = \frac{a^k}{k!} e^{-a}, \quad k = 0, 1, 2, ...$$

-the Poisson distribution. Use the subroutines to obtain a few values of k and plot histograms of this distributions and find mean value using Mean[] function.

PROBLEMS

1. Write the program VIEW for the simulation of N random numbers x uniformly distributed on the interval (a, b). Present the simulation results.

2. Write the program C-DISTRIBUTION for the simulation of random numbers x uniformly distributed on the interval (a, b) and construction of the distribution graph $w(x) \approx \dfrac{N(x \in \Delta x)}{N}$. Compare and discuss the simulation results for a few values of N.

3. Write the program VIEW for the simulation of a fair coin tossing. Present the simulation results of N values S $(S = 0, 1)$.

4. Write the program D-DISTRIBUTION for the simulation of a fair coin tossing and estimation of the probabilities P_0 and P_1. Compare and discuss the calculation results for a few values of N.

5. Write the program VIEW for the simulation of a fair die rolling. Present the simulation results of N values of s $(s = 1, 2, ...6)$.

6. Write the program D-DISTRIBUTION for the simulation of the a fair die rolling and estimation of the probabilities $P_k, (k = 1, 2, ...6)$. Compare the calculation results for a few values of N.

7. Write the program D-DISTRIBUTION for the simulation of a pair of dice rolling with recording the score sum each time and estimation of the probabilities of getting the possible sum values. Compare and discuss the calculation results for a few values of N.

8. Write two programs for the simulation of a fair die rolling until 6 points appears:

   a) the program D-DISTRIBUTION for the estimation of the probability P_k that this occurs in $k-$th rolling,

   b) the program AVERAGE for the simulation of the mean number of rollings. Discuss the simulation results.

9. Write the program VIEW for the simulation of tossing of M fair coins with recording the score sum each time. Present the results of simulation of N values of the sum.

10. Write the program D-DISTRIBUTION for the simulation of M fair coins tossing with recording the score sum each time and estimation of the probabilities of getting the possible sum values. Discuss the calculation results.

11. Write the program AVERAGE for the simulation of the prize index in Exercise 11 and calculation of the mean number of chocolate eggs which you would purchase in order to win all of the prizes.

12. Write the program D-DISTRIBUTION for the simulation of the prize index in Exercise 11 and calculation of the probabilities P_n, n being the number of chocolate eggs which you would purchase in order to win all of the prizes.

13. Write the program D-DISTRIBUTION for the calculation of probabilities P_n in Exercise 12, n being the number of your friends who has the same birthday as you.

14. Write the program VIEW for the simulation of a random quantity with possible values $k = 1, 2, ...$ if corresponding probabilities P_k are determined by the recurrent formula

$$P_{k+1} = r_k P_k, \quad \left(\sum_k P_k = 1\right).$$

Present the simulation results of N random numbers for the binomial, geometric and Poisson distributions.

15. Write two programs for the simulation of N values of a random quantity with possible values $k = 1, 2, ...$ if corresponding probabilities P_k are determined by the recurrent formula

$$P_{k+1} = r_k P_k, \quad \left(\sum_k P_k = 1\right):$$

a) the program D-DISTRIBUTION for the estimation of the probabilities P_k,

b) the program AVERAGE for the estimation of the mean value $\bar{k} = \dfrac{1}{N}\sum_{i=1}^{N} k_i$.

Present the simulation results for the binomial, geometric and Poisson distributions. Study the dependence of $\bar{k}$ on the distributions parameters.

16. Write two programs for the simulation of the matching pennies:

a) the program AVERAGE for the estimation of the mean time of gambling until one of the gamblers ruins,

b) the program D-DISTRIBUTION for the estimation of the ruining probability for each gambler.

Study the dependence of these quantities on the initial amounts of money $\$Q_1$ and $\$Q_2$ of the gamblers.