SPECIAL CHAPTERS OF HIGH MATHEMATICS

The main topic – **Solving of Integral Equations**

Classes: Lectures and Seminars

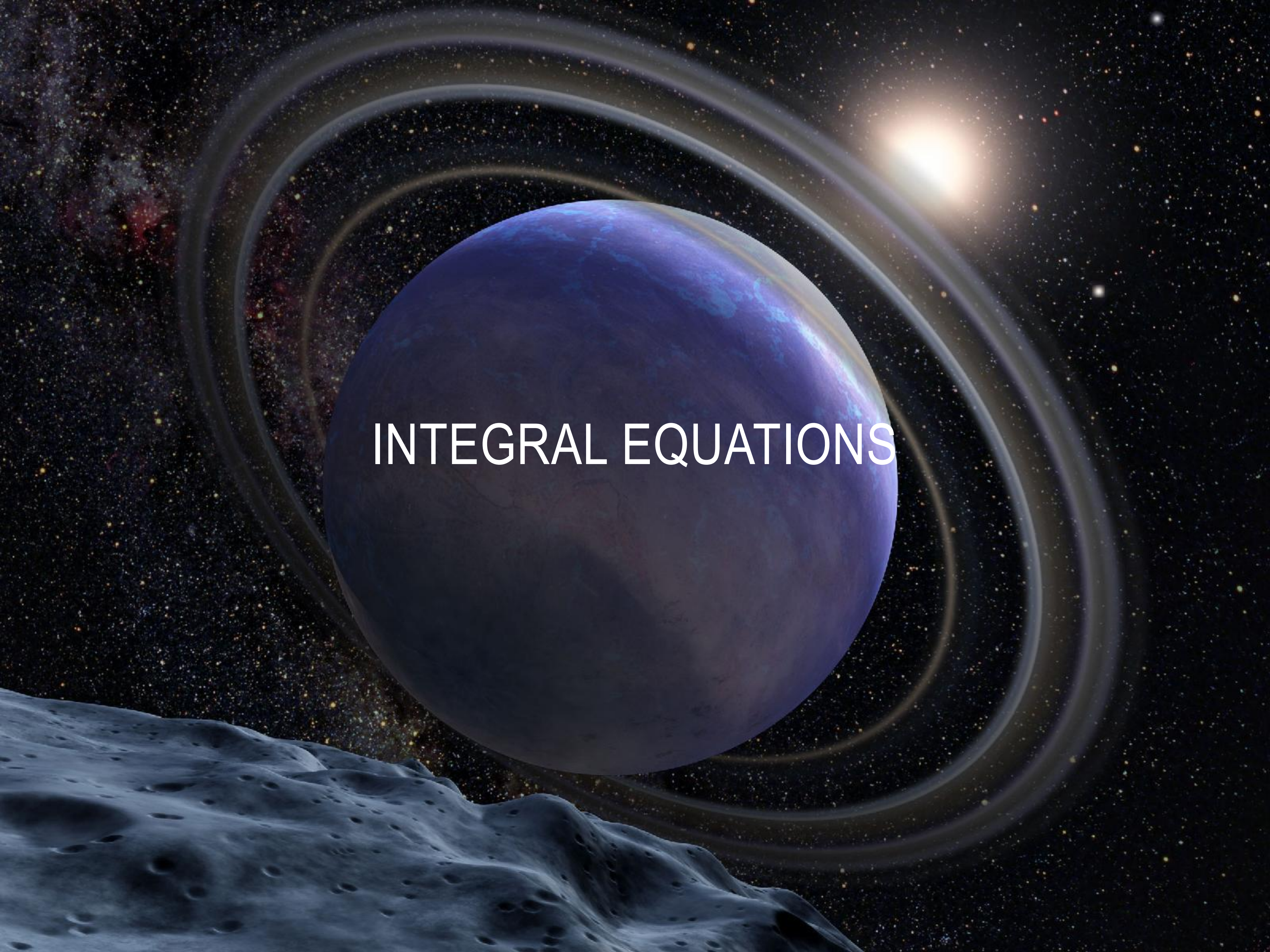
7 Individual tasks should be done till the end of this course.

It is necessary to refresh the knowledge of

- Operations on Matrixes;
- Solving of Linear Equations and Systems of Linear Equations;
- Solving the Integrals;
- Series transformation;
- Solving the Ordinary Differential Equations.

LITERATURE

1. Jerri, Abdul J. Introduction to Integral Equations with Applications – 2nd Edition. JOHN WILEY & SONS, 1999
2. Kanwal, R. P. Linear Integral Equations: Theory and Technique. Boston, MA: Birkhauser, 1996.
3. Masujima, M. Applied Mathematical Methods of Theoretical Physics - Integral Equations and Calculus of Variations. Weinheim, Germany: Wiley-VCH, 2005.
4. Polyanin A.D., Manzhirov A.V. Handbook of Interal Equations. 2nd Edition. Chapman & Hall / CRC, 2008.

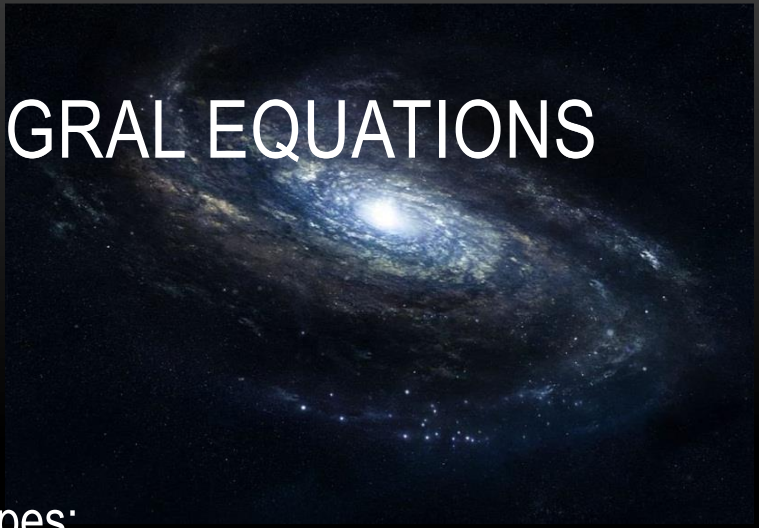

INTEGRAL EQUATIONS

INTEGRAL EQUATIONS: OVERVIEW

The definition of integral equations is very simple:

The Integral equation is an equation in which the unknown function placed under sign of integral.

THE CLASSIFICATION OF INTEGRAL EQUATIONS



Integral equations could be divided on 2 big types

- Linear. In which the unknown function is linear.

Linear integral equation could be classified on 2 types:

1. Integral equation of Volterra - 1 & 2 kinds.
2. Integral equation of Fredholm - 1 & 2 kinds.

- Nonlinear. In this type of equations of unknown function enters into the equation is nonlinear, ie, It has a complex dependence on the parameters of the equation. The classification of non-linear equations is problematic as a consequence of their diversity, but it is possible to allocate the equation: Uryson Hammerstein, Liapunov-Lichtenstein and nonlinear Volterra equation.

PROBLEMS LEADING TO THE INTEGRAL EQUATIONS

We can assume the task of handling the integral

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixy} f(y) dy \qquad (I)$$

as one of the first integral equations (despite the fact that the term "integral equation" was entered latter). There described the finding of a function f(y) from function g(x).

A solution to this problem was obtained by a Fourier and looks like:

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixy} f(x) dx \qquad (II)$$

One can assume that the formula (II) gives the solution of the integral equation (I), where f(y) – unknown function, and g (x) - a given function, and vice versa. As is known, the formula (I) and (II), called as Fourier integral transformation.

Moreover, the solution of the Cauchy problem for linear differential equations can be transformed to the solution of Volterra integral equations of 2nd kind.
These given examples - it is a purely mathematical problems.

Problems leading to the integral equations

It is possible to in light some physical problems as examples:

- The problem of determination of potential energy of the field where particle oscillates from the known dependence of particle period oscillation from its energy can be transformed to the Volterra equation of the 1st kind.
- Problems associated with the phenomena of aftereffects, such as the transient processes in electric circuits usually can be reduced to the Volterra equation of the 2nd kind.
- Data obtained from the indirect experiments when the direct observation is not possible, for example to find the planets in other solar systems or the presence of minerals by gravimetric or reconstruction tasks shot out of focus images and etc can be reduced to solving the Fredholm equation of the 1st kind.
- The problem of finding the profile of the string in the free harmonic vibrations, for example, could be transformed to Fredholm integral equations. Moreover, the problems described by the Laplace equation can be reduced to Fredholm integral equations.

VOLTERRA INTEGRAL EQUATIONS

OVERVIEW

The equation

$$\varphi(x) = f(x) + \lambda \int_a^x K(x,t)\varphi(t)dt, \qquad (1)$$

where f(x), K(x,t) – known functions, but φ(x) – unknown (seeking) function, λ – numerical parameter, called as linear Volterra integral equation. Function K(x,t) often called as kernel of Volterra equation.

If we assume that f(x)≡0, than eq. (1) transforms into

$$\varphi(x) = \lambda \int_a^x K(x,t)\varphi(t)dt, \qquad (2)$$

called homogeneous Volterra equation of the 2nd kind.

It is worth mentioning that, in the solution of many problems the lower limit of integration in the eq. (1) and (2) can be replaced by zero.

VOLTERRA INTEGRAL EQUATIONS

It should be noted that the integral Volterra equations occur in cases where there is a preferred direction of change in some of the independent variable, such as time, direction of movement of the radiation, its energy, etc.

Consider the problem of passing X-rays through matter in the direction of the axis 0X, and the scattering of the radiation beam will maintain this trend. Assume that the beam includes rays of a known wavelength. Choose a part of them at a predetermined wavelength, and consider what happens when they pass through a layer of thickness dx. Naturally, some rays change wavelength due to scattering, and part of absorbed. This will lead to a drop in the number of beams with a wavelength under consideration. On the other hand, this is augmented with a set of rays that originally had more energy (or shorter wavelength λ) , but lost it due to scattering. Thus, if the function f (λ,x)d given set of rays with wavelengths in the range of up to λ λ+dλ, then

$$\frac{\partial \mathrm{f}(\lambda, x)}{\partial x} = -\mu \cdot \mathrm{f}(\lambda, x) + \int_0^\lambda P(\lambda, \tau)\mathrm{f}(\tau, x)\,d\tau,$$

Volterra integral equations

wher μ – is absorption coefficient, $P(\lambda,\tau)$ – the probability that a beam with a wavelength of , after passing through the layer of length, will determine the wavelength range $(\lambda, \lambda+d\lambda)$.
The result is a so-called integral-differential equation, ie, an equation in which the unknown function f $f(\lambda,x)$ is included as both a sign of the integral, and the derivative.

If we assume that

$$f(\lambda, x) = \int_0^\infty e^{-px}\psi(\lambda, p)dp,$$

where $\psi(\lambda,p)$ – new unknown function, than we can show that $\psi(\lambda,p)$ will satisfy the Volterra integral equation of the 2nd kind

$$\psi(\lambda, p) = \frac{1}{\mu - p}\int_0^\lambda P(\lambda,\tau)\psi(\tau, p)d\tau$$

THE CONNECTION BETWEEN THE LINEAR DIFFERENTIAL EQUATIONS AND VOLTERRA INTEGRAL EQUATIONS

Let we consider the linear differential equation

$$\frac{d^{\mathrm{n}} y}{dx^{n}} + \mathrm{a}_1(\mathrm{x}) \frac{d^{\mathrm{n-1}} y}{dx^{n-1}} + \ldots + a_{\mathrm{n}}(\mathrm{x})\mathrm{y} = \mathrm{F}(\mathrm{x}) \qquad (1)$$

with continuous coefficients $a_i(x)$ if $(i=1,2,\ldots,n)$ and the initial conditions

$$y(0) = \mathrm{C}_0, \quad y'(0) = \mathrm{C}_1, \; \ldots, \; y^{n-1}(0) = \mathrm{C}_{\mathrm{n-1}} \qquad (2)$$

The equations of the form (1) with initial conditions (2) can be transformed to the Volterra integral equation of the second kind

For example, let we transform the following differential equation of 2nd order

$$\frac{d^2 y}{dx^2} + \mathrm{a}_1(\mathrm{x}) \frac{dy}{dx} + a_2(\mathrm{x})\mathrm{y} = \mathrm{F}(\mathrm{x}) \qquad (1^*)$$

THE CONNECTION BETWEEN THE LINEAR DIFFERENTIAL EQUATIONS AND VOLTERRA INTEGRAL EQUATIONS

With initial conditions

$$y(0) = C_0, \quad y'(0) = C_1 \qquad (2^*)$$

If we assume that

$$\frac{d^2y}{dx^2} = \varphi(x) \qquad (3)$$

then, taking into account the initial conditions (2 *) can successively receive

$$\frac{dy}{dx} = \int_0^x \varphi(t)dt + C_1, \qquad y = \int_0^x (x-t)\varphi(t)dt + C_1(x) + C_0 \qquad (4)$$

THE CONNECTION BETWEEN THE LINEAR DIFFERENTIAL EQUATIONS AND VOLTERRA INTEGRAL EQUATIONS

The following transformation are used for finding equation (4)

$$\underbrace{\int_{x_0}^{x} dx \int_{x_0}^{x} dx \ldots \int_{x_0}^{x}}_{n} f(x)\,dx = \frac{1}{(n-1)!}\int_{x_0}^{x} (x-z)^{n-1} f(z)\,dz$$

Than using (3) and (4), equation (1 *) can be rewritten as:

$$\varphi(x) + \int_0^x a_1(x)\varphi(t)\mathrm{dt} + C_1 a_1(x) + \int_0^x a_2(x)(x-t)\varphi(t)\mathrm{dt} + \\ + C_1 \cdot x \cdot a_2(x) + C_0 \cdot a_2(x) = F(x),$$

Let we rewrite this equation by moving to the left side all the expressions of the unknown φ

THE CONNECTION BETWEEN THE LINEAR DIFFERENTIAL EQUATIONS AND VOLTERRA INTEGRAL EQUATIONS

$$\varphi(x) + \int_0^x [a_1(x) + a_2(x)(x-t)]\varphi(t)\mathrm{dt} =$$
$$= F(x) - C_1 \cdot a_1(x) - C_1 \cdot x \cdot a_2(x) - C_0 \cdot a_2(x). \qquad (5)$$

If we introduce the notation

$$K(\mathrm{x}, \mathrm{t}) = -[a_1(x) + a_2(x)(x-t)] \qquad (6)$$

$$\mathrm{f}(x) = F(x) - C_1 \cdot a_1(x) - C_1 \cdot x \cdot a_2(x) - C_0 \cdot a_2(x) \qquad (7)$$

than equation (5) could be presented as

$$\varphi(x) = \int_0^x K(x,t)\varphi(t)dt + \mathrm{f}(t) \qquad (8)$$

The result is a Volterra equation of the 2nd kind.

THE CONNECTION BETWEEN THE LINEAR DIFFERENTIAL EQUATIONS AND VOLTERRA INTEGRAL EQUATIONS

The existence of a unique solution of the equation (8) follows from the existence and uniqueness of solutions of the Cauchy problem (1 *) - (* 2) for linear differential equations with continuous coefficients near zero.

Also converse is also true - solving integral equation (8) with K and f, given by the formula (6) and (7) by substituting the resulting $\varphi(x)$ in the last equation in (4), will obtain a unique solution of equation (1 *) with initial conditions (2 *).

It should be noted that some of Volterra equations of the 1st and 2nd kind it is easier to solve by reducing them to differential equations. The resulting differential equation can solved using already known methods.