# Fredholm Integral Equations with Degenerate Kernel

# Integral Equations with Degenerate Kernel

The *Degenerate (or separable) kernel* of 2nd kind Fredholm integral equation is the kernel $K(x,t)$ of a finite sum of products of functions depending only on $x$ and $t$, respectively. Mathematically, this can be written

$$K(x,t) = \sum_{k=1}^{n} a_k(x) \cdot b_k(t). \tag{1}$$

It is assumed in the formula (1) that the functions $a_k(x)$ and $b_k(t)$ $(k = 1, 2, \ldots, n)$ are continuous in the domain $a \leq x, t \leq b$ and are linearly independent among themselves.

In this case, the integral equation with degenerate kernel (1) can be written in the form

$$\varphi(x) - \lambda \int_a^b \left[ \sum_{k=1}^{n} a_k(x) \cdot b_k(t) \right] \varphi(t) dt = f(x). \qquad (2)$$

To obtain a solution of the equation (2) it could be rewritten in form

$$\varphi(x) = f(x) + \lambda \sum_{k=1}^{n} a_k(x) \int_a^b b_k(t) \varphi(t) dt \qquad (3)$$

By introducing the notation

$$\int_a^b b_k(t) \varphi(t) dt = C_k \quad (k = 1, 2, \ldots n), \qquad (4)$$

the formula (3) could be written

$$\varphi(x) = f(x) + \lambda \sum_{k=1}^{n} C_k a_k(x), \tag{5}$$

where $C_k-$ is unknown constants. This is a consequence of the fact that the expressions for $C_k$ include the unknown function $\varphi(x)$.It follows from the calculations that is sufficient to find the $C_k$ $(k = 1, 2, \ldots, 3)$ in order to obtain the solution to integral equations with degenerate kernel.To do this, one may substitute the expression (5) into the equation (2) and after simple transformations one can obtain

$$\sum_{m=1}^{n} \left\{ C_m - \int_a^b b_m(t) \left[ f(t) + \lambda \sum_{k=1}^{n} C_k a_k(t) \right] \mathrm{dt} \right\} a_m(x) = 0$$

Since the coefficients $a_m(x)$ $(m = 1, 2, \ldots, n)$ are linearly independent, the last expression can be written

$$C_m - \int_a^b b_m(t) \left[ f(t) + \lambda \sum_{k=1}^{n} C_k a_k(t) \right] \mathrm{dt} = 0,$$

or

$$C_m - \lambda \sum_{k=1}^{n} C_k \int_a^b a_k(t) b_m(t) \mathrm{dt} = \int_a^b b_m(t) f(t) \mathrm{dt} \;\; (m = 1, 2, \ldots, n).$$

By introducing the notation

$$a_{km} = \int_a^b a_k(t) b_m(t) \mathrm{dt}, \qquad f_m = \int_a^b b_m(t) f(t),$$

one can obtain

$$C_m - \lambda \sum_{k=1}^{n} a_{km} C_k = f_m, \quad (m = 1, 2, \ldots, n)$$

It is more convenient to write the last expression in the form of a system of equations

$$\begin{cases} (1-\lambda a_{11})C_1 - \lambda a_{12}C_2 - \cdots - \lambda a_{1n}C_n = f_1 \\ -\lambda a_{21}C_1 + (1-\lambda a_{22})C_2 - \cdots - \lambda a_{2n}C_n = f_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -\lambda a_{n1}C_1 - \lambda a_{n2}C_2 - \cdots + (1-\lambda a_{nn})C_n = f_n \end{cases} \quad (6)$$

To solve the system (6), i.e. finding the coefficients $C_k$ it is necessary to solve a system of $n$ linear equations with $n$ unknowns.

To do this, it is necessary to find out, at first, what the determinant of a given system is equal to. It could be written as

$$\Delta(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & (1-\lambda a_{22}) & \cdots & -\lambda a_{2n} \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ -\lambda a_{n1} & -\lambda a_{n2} & -\cdots & (1-\lambda a_{nn}) \end{vmatrix} \quad (7)$$

If $\Delta(\lambda) \neq 0$ than the system of equation (6) has a unique solution and coefficients $C_k$ can for example be found by using Cramer's rule

$$C_k = \frac{1}{\Delta(\lambda)} \begin{vmatrix} 1-\lambda a_{11} & \cdots & -\lambda a_{1k-1} f_1 - \lambda a_{1k+1} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & \cdots & -\lambda a_{2k-1} f_2 - \lambda a_{2k+1} & \cdots & -\lambda a_{2n} \\ \dots\dots & \cdots & \dots\dots\dots\dots & \cdots & \dots\dots \\ -\lambda a_{n1} & \dots & -\lambda a_{nk-1} f_n - \lambda a_{nk+1} & \cdots & 1-\lambda a_{nn} \end{vmatrix} \quad (8)$$

In this case solution of equation (2) is the function $\varphi(x)$ is described by the expression

$$\varphi(x) = f(x) + \lambda \sum_{k=1}^{n} C_k a_k(x),$$

where $C_k$ $(k = 1, 2, \ldots n)$ is coefficients are coefficients determined by formula (8).