The Poisson Distribution

Particle Penetration Through Matter

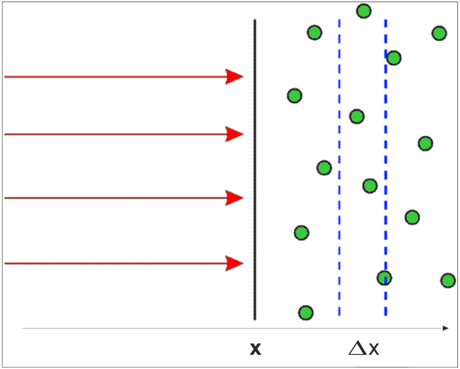


Figure: Particle penetration through matter.

Let imagine a stream of particles with flow density $\Phi_0\ 1/(cm^2 \cdot sec)$ are hitting an absorber made up of randomly distributed atoms with diameter $\sigma\ (cm^2)$ and the mean number of atoms per unit volume equal $n_0\ (1/cm^3)$.

If particles are absorbed when colliding with atoms, the flux density $\Phi(x)$ will be a random depth function.

After each collision a number of particles decreases by a unit and $\Phi(x+\Delta x) = \Phi(x) - Q(\Delta x)$

where $Q(\Delta x)$ is the number of particles absorbed per centimeter squared in a layer Δx per unit time.

Obviously, this equality remains valid after averaging due to lots of experiments:

$$\bar{\Phi}(x+\Delta x) = \bar{\Phi}(x) - \bar{Q}(\Delta x). \tag{1}$$

It is easily seen that the mean number of collisions of the particle flux $\bar{\Phi}(x)$ with a single atom per unit time equals $\sigma\bar{\Phi}(x)$ and $n_0\Delta x$ is the number of atoms per 1 cm in the layer Δx, therefore, $\bar{Q}(\Delta x)$ equals their product:

$$\bar{Q}(\Delta x) = \bar{\Phi}(x)\ \sigma n_0\ \Delta x. \tag{2}$$

The ratio of the mean number of collisions in the layer Δx to the mean number of particles striking this layer

$$\frac{\bar{Q}(\Delta x)}{\bar{\Phi}(x)} = \mu \Delta x$$

is the mean number of collisions on the path Δx for one particle and $\mu = \sigma n_0$ is ***the mean number of collisions per unit path.***

By inserting eq. (2) in (1) and proceeding to the limit $\Delta x \to 0$ one could get a differential equation for $\bar{\Phi}(x)$:

$$\frac{d\bar{\Phi}}{dx} = -\mu\bar{\Phi}.$$

It is necessary to add a boundary condition $\bar{\Phi}(0) = \Phi_0$ to find the solution.

And it is the function

$$\bar{\Phi}(x) = \Phi_0 e^{-\mu x}. \qquad (3)$$

It is seen from formulae (3) that the value $1/\mu$ is numerically equal to the depth in which the mean flux density $\bar{\Phi}$ decreases by e times.

That is why the value μ is called ***the linear attenuation coefficient***.

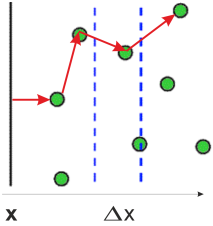
The ratio of the mean number of particles that have travelled the path x without any interaction to the initial number of particles targeted to the absorber

$$P_0(x) = \frac{\bar{\Phi}(x)}{\Phi_0} = e^{-\mu x}$$

is the probability to travel along this path without any interaction.

Let is assume now that particles in matter not only could get absorbed but scatter as well and they change the movement direction when colliding.

In this case the particle path represents a broken line made up of sections of random length.



Let us designate the path length by x travelled by a particle along a trajectory, mentally stretching it to a straight line.

On this path a particle experiences a random number of collisions, therefore, the random number of particles having passed the path x can be written in the form of a sum

$$\bar{\Phi}(x) = \bar{\Phi}_0(x) + \bar{\Phi}_1(x) + \bar{\Phi}_2(x) + \ldots,$$

where $\bar{\Phi}_k(x)$ is the mean number of particles having experienced the k-collisions on the path x.

By dividing the both parts of this equation on $\bar{\Phi}(x)$ we get the normalization condition

$$\sum_{k=0}^{\infty} P_k(x) = 1,$$

where $P_k(x) = \dfrac{\bar{\Phi}_k(x)}{\bar{\Phi}(x)}$ is the probability to experience k collisions.

It is easily seen that as before $P_0(x) = e^{-\mu x}$ and for a small path Δx when one can neglect the probability to experience more than one collision and taking to account eq. (2) we could get

$$P_1(\Delta x) = \mu \Delta x,$$

that is, μ isthe ***collision probability per unit path***.

The view of probabilities $P_k(x)$ for arbitrary k and x will be discussed latter.

Let x is the path travelled by a particle in matter till the first collision is random and the probability of the collision to occur within the interval $(x, x + \Delta x)$ should be found.

This probability can be found if we take into account that for a particle outgoing from the beginning of coordinates the first collision in the interval $(x, x + \Delta x)$ can be considered as composition of two random events:"the absence of interactions on the path" (the probability of this event is equal to $e^{-\mu x}$)

and "the collision in Δx" (the corresponding probability equals $\mu \Delta x$).

Therefore, $P(x \in \Delta x)$ equals the product of the two given factors:

$$P(x \in \Delta x) = e^{-\mu x} \mu \Delta x.$$

The probability density of the fact that the collision has occurred in the point x is found by formula

$$w(x) = \mu \ e^{-\mu x}, \ 0 \leq x < \infty. \tag{4}$$

The function $w(x)$ describes the distribution of particle paths.

Generating random numbers x from distribution (4) we could simulate the mean free path of a particle. They can be employed to compute the mean free path of a particle:

$$< l >= \frac{1}{N} \sum_{i=1}^{N} x_i.$$

In previous lectures we found that such a mean can be written in the form of an integral $\int_0^\infty x \ \mu \ e^{-\mu x} \ dx$ that equals $1/\mu$, hence, the value $1/\mu$ is***the mean particle path*** till a collision.

Radioactive Decay

Suppose we have N_0 of radioactive atoms.

The lifetime of each of them is random, therefore, the number of disintegrated atoms $N(t)$ is the random function of time:after each decay the number of atoms decreases on 1.

Likewise the problem on particle absorption for a mean due to lots of experiments the following equality is true

$$\bar{N}(t+\Delta t) = \bar{N}(t) - \bar{Q}(\Delta t),$$

where $\bar{Q}(\Delta t)$ is the mean number of atoms disintegrated per time Δt.

It is proportional to the number of non-decayed atoms in the moment t and the interval size Δt:

$$\bar{Q}(\Delta t) = \mu \ \Delta t \ \bar{N}(t),$$

As in the preceding example

$$\frac{d\bar{N}}{dt} = -\mu\bar{N},$$

$$\bar{N}(t) = N_0 e^{-\mu t}.$$

The value μ is called ***a decay constant***.
Its physical meaning is analogous to that of what was spoken in the previous example: μ is the mean number of disintegration per unit time and the disintegration probability per unit time and $1/\mu$ is the time at which the number of atoms decreases in e times and the mean lifetime of an atom.
The atom disintegration moment can be pictured by a point on the time axis and the point hit the interval $(t, t+\Delta t)$ is considered to be the composition of the two random events: "atom survival per time $(0,t)$" (the probability of this event equals $\frac{\bar{N}(t)}{N_0} = e^{-\mu t}$) and "disintegration per time Δt" (the corresponding probability equals $\mu\Delta t$).

Therefore, if an atom existed at $t=0$, the disintegration probability in the moment $t \in \Delta t$ equals the product of two given probabilities:

$$P(t \in \Delta t) = e^{-\mu t}\ \mu \Delta t,$$

and the disintegration probability density in the moment t equals

$$w(t) = \mu e^{-\mu t}.$$

Simulating t from this distribution one can find random atom lifetime.
In the system composed of several radioactive atoms the moment of the first decay is random and the probability of the fact that this disintegration will occur within the interval $(t, t+\Delta t)$ can be found as a composition of probabilities of two random events:
"survival of all atoms in the interval $(0,t)$" and "decay of one of atoms per time Δt".

If there were N radioactive atoms in the initial moment, $(e^{-\mu t})^N$ is the probability of the fact that by the moment t all of them had survived (neither the first nor the second nor N-atoms decayed) and $N\mu\Delta t$ is the probability of the fact that per Δt one of them will decay (either the fist or the second or $\ldots N$).
Therefore,

$$w_N(t) = \mu N \, e^{-\mu N t}$$

is the probability density to simulate the first decay moment in the system composed of N atoms. Decreasing N after each disintegration by unit one can simulate a random $N(t)$.

The result of simulation of the radioactive decay is represented in next figure.

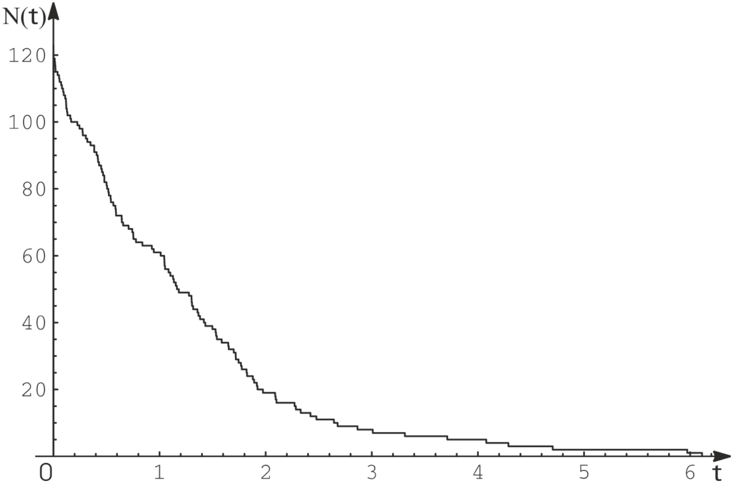


Figure: The result of simulating the radioactive decay curve.

For large values of N one can neglect statistical nature of the decay process and assume that the number of radioactive atoms decreases according to the exponential law.

For small N the exponential law is true only for a mean value due to a great number of experiments.

The disintegration of radioactive nuclei results in the formation of daughter nuclei - decay product and their accumulation can be automatically calculated during simulation.

Multi-Channel and Complex Radioactive Decay

For atoms with several ways (***channels***) of decay a decay constant μ equals the sum of partial constants μ_i corresponding to various channels:

$$\mu = \sum_i \mu_i,$$

and $P_i = \mu_i/\mu$ is the decay probability over i-channel.

By knowing these probabilities one can find a decay type by means of simulation.

For this one needs to plot probabilities P_i in the segment $(0, 1)$ and determine a channel number according to a uniformly distributed random number γ.
The joint simulation of a random function $N(t)$ and a decay type is simulation of such a decay and accumulation of its products.

If some decay products are radioactive themselves (***complex radioactive decay***) the decay moment of the next daughter nucleus is simulated in the same way taking into account how many nuclei of the given element were there before that decay.
In this case the dynamics of the process has a more complex nature since some nuclei can not only decay but also be produced as a result of preceding decay.
The calculation of radioactive decay is carried out in the following way.
A list of all substances that can be formed at different stages of decay is made up and an initial quantity of each of these elements is set up.
For each of them the first decay moment is calculated in the way described above. They are compared among each other to define which decay exactly will be the first.
A number of atoms of a disintegrated element decreases by unit whereas a number of corresponding daughter atoms increases by unit.
After that the process of simulation is repeated. The calculation is performed as long as there are radioactive atoms in the system.

Simulation of Birth and Death Process

In problems of biology and medicine living organisms can not only born but also be die.
Let $\mu\Delta t$ be the probability of death per Δt, and $\nu\Delta t$ be the probability of birth in the same time.
Then $(\mu+\nu)\Delta t$ is the probability of birth and death per Δt.
Let us designate $\Sigma=\mu+\nu$, then the ratios $\frac{\mu}{\Sigma}$ and $\frac{\nu}{\Sigma}$ define relative probabilities of death and birth.
Successive simulation of a random moment when a number of living organisms is changed in the system along with a random type of the process (birth and death) makes it possible to simulate the evolution of population.
In those cases when one organism is able to give a birth to several twins , a simulation algorithm is supplemented by sampling of a number of descendants from known probabilities to have k descendants as it was carried out in lecture 3.

The results of population evolution simulation with various ratios between birth and death probabilities of organisms are presented in next figure.

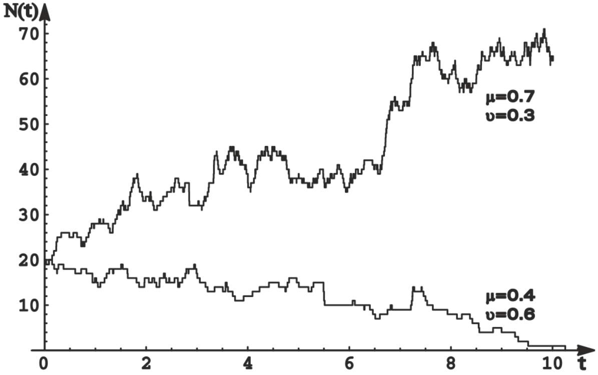


Figure: The results of simulation of population evolution over time.

It is seen from the figure that if $\mu > \nu$, the quantity of the population on the average increases and if $\mu < \nu$, it decreases.

Analogous problems are typical for physics of cosmic rays where one has to consider cascades of high energy particles, where each particle interacting with a substance and can be absorbed or produce a random number of secondary particles.

The Poisson Distribution

Let rain drops fall on a selected area of a road (or particles of cosmic rays impinge on a detector) at random moments of time $t_1, t_2, \ldots$.

The number of drops fallen in time Δt is random, it can be equal to $0, 1, 2 \ldots$, and the normalization condition can be written for corresponding probabilities:

$$\sum_{k=0}^{\infty} P_k(\Delta t) = 1.$$

In the same way as in problems on particle absorption in matter or radioactive decay one can suppose that for small Δt the probability $P_1(\Delta t)$ is proportional to Δt:

$$P_1(\Delta t) = \nu \Delta t, \tag{5}$$

whereas $P_2, P_3, \ldots$ have a higher order of smallness and they can be neglected in the normalization condition. In this approximation

$$P_0(\Delta t) = 1 - \nu \Delta t. \tag{6}$$

To find the probability P_0 for a finite time, one can consider the lack of drops in the interval $(0, t+\Delta t)$ as a composition of two random events:

"the absence of drops in $(0,t)$" **and** "the absence of drops in Δt".

Therefore,

$$P_0(t+\Delta t) = P_0(t)\ P_0(\Delta t).$$

Substituting expression (6) for $P_0(\Delta t)$ in to this formula and passing on to the limit $\Delta t \to 0$,we get a formula is similar to obtained in problems on particle penetration through matter and radioactive decay:

$$\frac{dP_0(t)}{dt} + \nu P_0(t) = 0.$$

from (6) it follows that $P_0(0) = 1$, therefore,

$$P_0(t) = e^{-\nu t}.$$

The quantity ν is called ***flux intensity*** and is equal to a mean number of drops per unit time.

The time between two drop falls is random and by analogy with the examples above the probability of this time to be in the interval $(t, t+\Delta t)$ equals

$$P(t \in \Delta t) = e^{-\nu t} \; \nu \Delta t,$$

the probability density for simulation of the time interval between two drops is

$$w(t) = \nu e^{-\nu t},$$

and the mean value of the time interval between two drops equals $1/\nu$.

To find the probability P_1 for a finite time interval we note that the fall of one drop in time $(0, t+\Delta t)$ means that

"one drop has fallen in time $(0,t)$ **and** no drops in Δt" **or** "no drops in time $(0,t)$ **and** one drop has fallen in Δt". Therefore,

$$P_1(t+\Delta t) = P_1(t) \; P_0(\Delta t) + P_0(t) \; P_1(\Delta t).$$

Using formulas (6), (5) for $P_0(\Delta t)$ and $P_1(\Delta t)$ and limiting $\Delta t \to 0$ one can obtain a heterogeneous differential equation

$$\frac{dP_1(t)}{dt} + \nu P_1(t) = \nu P_0(t).$$

The initial condition for the function $P_1(t)$ in accordance with (5) is $P_1(0) = 0$.

Similarly one can show that if $k > 0$

$$\frac{dP_k(t)}{dt} + \nu P_k(t) = \nu P_{k-1}(t),$$

$$P_k(0) = 0.$$

The substitution

$$P_k(t) = e^{-\nu t} \tilde{P}_k(t)$$

leads this equation to the form

$$\frac{d\tilde{P}_k(t)}{dt} = \nu \tilde{P}_{k-1}(t),$$
$$\tilde{P}_k(0) = 0.$$

The successive integration of this equation at $k = 1, 2,$ .. taking into account that $\tilde{P}_0(t) = 1$ gives $\tilde{P}_k(t) = \dfrac{(\nu t)^k}{k!}$,that results in

$$P_k(t) = \frac{(\nu t)^k}{k!} e^{-\nu t}. \tag{7}$$

The solution obtained is called ***the Poisson distribution***.

One can easily check that the probabilities $P_k(t)$ satisfy the normalization condition:

$$\sum_{k=0}^{\infty} P_k(t) = 1,$$

therefore, plotting numbers P_k in the segment $(0,1)$ over a uniformly distributed random number γ one can simulate a random number k.

The sequence of random numbers with the exponential distribution of time intervals between them or with a random k and probabilities P_k defined by formula (7) is called ***the Poisson flow of events***.

It has been found out that the Poisson distribution describes the amount of goods sold, the number of caught fish, the number of strikes or wars, mistakes in a text, calls at a phone station, injuries in a factory, radioactive nuclei decayed or tractors broken down.

Substituting time t by a path travelled it describes the number of punctures in a wheel or the number of collisions of a particle in matter.

And substituting t by $\vec{r}$ it is used to analyze a number of particles impinged on the same grounds ΔS having involved in to interactions in the same volumes ΔV and so on. All these processes can be studied by statistical simulation methods.