# Iterated Kernels. The Resolvent Construction by Means of Iterated Kernels

# Iterated Kernels. The Resolvent Construction by Means of Iterated Kernels

By analogy with Volterra equations for the Fredholm equations of the 2nd kind

$$\varphi(x)-\lambda\int_a^b K(x,t)\varphi(t)dt=f(x), \tag{1}$$

it is possible to find solution using successive substitutions method. For this purpose we will assume that

$$\varphi(x)=f(x)+\sum_{n=1}^{\infty}\psi_n(x)\lambda^n, \tag{2}$$

where $\psi_n(x)-$ described by formulas

$$\psi_1 = \int_a^b K(x,t) f(t) \mathrm{d}t,$$

$$\psi_2 = \int_a^b K(x,t) \psi_1(t) \mathrm{d}t = \int_a^b K_2(x,t) f(t) \mathrm{dt},$$

$$\psi_2 = \int_a^b K(x,t) \psi_1(t) \mathrm{d}t = \int_a^b K_2(x,t) f(t) \mathrm{dt},$$

$$\psi_3 = \int_a^b K(x,t) \psi_2(t) \mathrm{d}t = \int_a^b K_3(x,t) f(t) \mathrm{dt}, \ \cdots$$

In shown formulas introduced notation that

$$\begin{aligned} &K_1(x,t) \equiv K(x,t) \\ &K_n(x,t) = \int_a^b K(x,z) K_{n-1}(z,t) \mathrm{dz}, \quad (n = 2, 3, \dots). \end{aligned} \tag{3}$$

It should be noted that functions $K_n(x,t)$, described by equation (3) named as *Iterated kernels*. For them, we have the relation

$$K_n(x,t) = \int_a^b K_m(x,s)K_{n-m}(s,t)\mathrm{ds}, \tag{4}$$

where $m-$ is any integer less than $n$.
The resolvent of the integral equation (1) is obtained through iterated kernels (3) using the formula

$$R(x,t;\lambda) = \sum_{n=1}^{\infty} K_n(x,t)\lambda^{n-1}. \tag{5}$$

The right side of the formula (5) represents definition of *Neumann series of kernel* $K(x,t)$.

This series converges if it is satisfy by the condition

$$|\lambda| < \frac{1}{B}, \tag{6}$$

where

$$B = \sqrt{\int_a^b \int_a^b K^2(x,t)\mathrm{dxdt}}.$$

Finaly, the solution of Fredholm integral equation of 2nd kind (1) using resolvent could be written as

$$\varphi(x) = f(x) + \lambda \int_a^b R(x,t;\lambda) f(t)\mathrm{dt}. \tag{7}$$

# Some Properties of Iterated Kernels

It is necessary to notice that eq. (6) is essential to convergence of a series (5). However, there are cases when the solution of the equation (1) could be founded on $|\lambda| > 1/B$.For example, let's consider the solution of equations of the form

$$\varphi(x) - \lambda \int_0^1 \varphi(x,t)\mathrm{dt} = 1. \tag{8}$$

One cane see from it that $K(x,t) \equiv 1$, and hence

$$B^2 = \int_0^1 \int_0^1 K^2(x,t)\mathrm{dxdt} = \int_0^1 \int_0^1 \mathrm{dxdt} = 1$$

Thus, the condition (6) provides that a series (5) converges when $|\lambda| < 1$.However, if you solve the equation (8) using a method of solution for the equations with a degenerate kernels, we get that $(1-\lambda)C = 1$, where $C = \int_0^1 \varphi(t)\mathrm{dt}$. It is easy to see that if $\lambda = 1$ the last equation is unsolvable, or in this case, the equation (8) has no solution. Consequently, in a circle of radius greater than 1, the successive approximations for the equation (8) can not converge.

But with $\lambda > 1$ one can get a solution of equation (8). Indeed, with $\lambda \neq 1$ the solution of the equation will be a function $\varphi(x) = \dfrac{1}{1-\lambda}$. This is easily verified by substituting this function in equation (8).

In some cases, the resolvent described by Neumann series (5), converges for all values of $\lambda$. This can be illustrated by the example of two kernels $K(x,t)$ и $L(x,t)$ for which two conditions are satisfied:

$$\int_a^b K(x,z)L(z,t)\mathrm{dz} = 0, \qquad \int_a^b L(x,z)K(z,t)\mathrm{dz} = 0, \tag{9}$$

for any values $x$ и $t$.
Conditions (9), called orthogonality conditions and kernel satisfying these conditions are named *orthogonal kernels*. An example of orthogonal kernels may be the following two kernels $K(x,t) = xt$ and $L(x,t) = x^2t^2$. They are orthogonal in the range $[-1,1]$.

Indeed, if we use eq. (9), we get:

$$\int_{-1}^{1}(xz)(z^2t^2)\mathrm{dz} = xt^2\int_{-1}^{1}z^3\mathrm{dz} = 0,$$
$$\int_{-1}^{1}(x^2z^2)(zt)\mathrm{dz} = x^2t\int_{-1}^{1}z^3\mathrm{dz} = 0.$$

In addition, there are kernels which may be orthogonal to itself. For such kernels the condition is satisfied $K_2(x,t) \equiv 0$, where $K_2(x,t)-$ 2nd iterated kernel. In this case, obviously, all subsequent kernels will also be zero.Then from (3) and (5) it follows that the resolvent will be the same as the kernel itself $K(x,t)$.

Rule of orthogonality may be useful in cases when the kernel of the integral equation $K(x,t)$ is the sum of orthogonal functions $M(x,t)$ and $N(x,t)$, in other words $K(x,t) = M(x,t) + N(x,t)$. In this case resolvent $R(x,t;\lambda)$ will be equal the sum of resolvents $R_1(x,t;\lambda)$ and $R_2(x,t;\lambda)$ of the corresponding kernels $M(x,t)$ and $N(x,t)$. This rule can easily be generalized to the case of the sum of more orthogonal functions.