Fredholm Integral Equations

The Fredholm Determinants Method

By analogy with Volterra equations the solution for Fredholm integral equation of 2nd kind

$$\varphi(x) - \lambda \int_a^b K(x,t)\varphi(t)dt = f(x), \qquad (1)$$

could be written by the expression

$$\varphi(x) = f(x) - \lambda \int_a^b \mathrm{R}(x,t;\lambda) f(t)dt, \qquad (2)$$

where function $R(x,t;\lambda)$ called *Fredholm resolvent* for equation (1) and defined as

$$\mathrm{R}(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} \qquad (3)$$

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One can assume that in the expression (3) should be satisfied by $D(\lambda) \neq 0$. The functions in the numerator and the denominator isexpressions of series in powers of λ and described by following equations:

$$D(x,t;\lambda) = K(x,t) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} B_n(x,t)\lambda^n, \tag{4}$$

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} C_n \lambda^n. \tag{5}$$

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The coefficient $B_n(x,t)$ in equation (4) described by next formula

$$B_0(x,t) = K(x,t)$$

$$B_n(x,t) = \underbrace{\int_a^b \cdots \int_a^b}_{n} \begin{vmatrix} K(x,t) & K(x,t_1) & \dots & K(x,t_n) \\ K(t_1,t) & K(t_1,t_1) & \dots & K(t_1,t_n) \\ K(t_2,t) & K(t_2,t_1) & \dots & K(t_2,t_n) \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ K(t_n,t) & K(t_n,t_1) & \dots & K(t_n,t_n) \end{vmatrix} dt_1 \dots dt_n \tag{6}$$

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The coefficient C_n in equation (4) described by next formula

$$C_n = \underbrace{\int_a^b \cdots \int_a^b}_{n} \begin{vmatrix} K(t_1,t_1) & K(t_1,t_2) & \ldots & K(x,t_n) \\ K(t_2,t_1) & K(t_2,t_2) & \ldots & K(t_2,t_n) \\ K(t_3,t_1) & K(t_3,t_1) & \ldots & K(t_3,t_n) \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ K(t_n,t_1) & K(t_n,t_2) & \ldots & K(t_n,t_n) \end{vmatrix} dt_1 \ldots dt_n \tag{7}$$

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Previously described functions are called: $D(x,t;\lambda)-$ *Fredholm minor*, a $D(\lambda)-$ *Fredholm determinant*.
It should be noted that if the kernel of Fredholm equation $K(x,t)$ bounded or has gaps, but such that the integral

$$\int_a^b \int_a^b |K(x,t)|^2 dxdt$$

is finite, than the series (4) and (5) converges for all values of λ, that is will be integer analytic functions on λ.

This means that resolvent

$$\mathrm{R}(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)}$$

is analytical function on λ, except in cases when $D(\lambda)$ turns into 0.
From all this it follows the algorithm for finding the resolvent: consistently find the coefficients $B_n(x,t)$ and C_n until they become zero, or until the total dependence of the formulas (4) и (5) could be fiunded. Upon receipt of expressions for the determinant and minor Fredholm construct a resolution of (3). Further, according to the formula (2) is the final solution of the integral equation (1) could be found.

Direct calculation of Fredholm's determinants and minors is complicated and should be used only for simple cases. The determinant calculation of matrix greater than 3rd order is complicated.
For calculation coefficients $B_n(x,t)$ and C_n in formulas (4) and (5) for other cases is better to use next recurrence formulae:

$$\mathrm{B}_n(x,t) = C_n K(x,t) - n\int_a^b K(x,s)B_{n-1}(s,t)\mathrm{d}s, \qquad (8)$$

$$\mathrm{C}_n = \int_a^b B_{n-1}(s,s)\mathrm{d}s, \qquad (9)$$

with start conditions $B_0(x,t) = K(x,t)$ and $C_0 = 1$, respectively.