Resolvent of Volterra's Integral Equations. Solving the Integral Equations with Help of Resolvent.

Lets assume that we have the Volterra integral equation of 2nd kind

$$\varphi(x) = f(x) + \lambda \int_0^x K(x,t)\varphi(t)dt, \qquad (1)$$

where $K(x,t)$ is continuous function on $0 \le x \le a$, $0 \le t \le x$ and $f(x)$ is continuous on $0 \le x \le a$.
Let's find the solution of this integral equation (1) in the form of an infinite power series in powers of λ.

$$\varphi(x) = \varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \cdots + \lambda^n\varphi_n(x) + \dots \qquad (2)$$

If we substitute the expression (2) to (1) than we obtain

$$\varphi_0(x)+\lambda\varphi_1(x)+\lambda^2\varphi_2(x)+\cdots+\lambda^n\varphi_n(x)+\cdots=f(x)+$$
$$+\lambda\int_0^x K(x,t)[\varphi_0(t)+\lambda\varphi_1(t)+\lambda^2\varphi_2(t)+\cdots+\lambda^n\varphi_n(t)+\ldots]dt$$

By comparing the coefficients of identical powers of λ, one could find that

$$\begin{aligned}
&\varphi_0(x)=f(x),\\
&\varphi_1(x)=\int_0^x K(x,t)\varphi_0(t)dt=\int_0^x K(x,t)f(t)dt, \qquad (3)\\
&\varphi_2(x)=\int_0^x K(x,t)\varphi_1(t)dt=\int_0^x K(x,t)\int_0^t K(t,t_1)f(t_1)dt_1dt,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Formulas (3) define a way of iterative evaluation for the functions $\varphi_n(x)$.
It can be shown that under this assumptions with respect to $f(x)$ and $K(x,t)$ the series (2) converges uniformly with respect to x and λ for any λ at $x \in [0,a]$. And sum of this series is the unique solution of equation (1).

By continuing the transformations in formula (3), one may obtain:

$$\varphi_1(x) = \int_0^x K(x,t)f(t)dt,$$

$$\varphi_2(x) = \int_0^x K(x,t)\left[\int_0^t K(t,t_1)f(t_1)dt_1\right]dt =$$

$$= \int_0^x f(t_1)dt_1 \int_{t_1}^x K(x,t)K(t,t_1)dt = \int_0^x K_2(x,t_1)f(t_1)dt_1,$$

where

$$K_2(x,t_1) = \int_{t_1}^x K(x,t)K(t,t_1)dt$$

Next formula can be obtained by continuing further transfomation:

$$\varphi_n(x) = \int_0^x K_n(x,t) f(t) dt \qquad (n = 1, 2, 3, \dots) \tag{4}$$

In this formula the functions $K_n(x,t)$ are known as ***iterative*** or ***sequential*** kernels.
It is not difficult to show that they can be written using the recurrence formulas:

$$K_1(x,t) = K(x,t),$$
$$K_{n+1} = \int_t^x K(x,z) K_n(z,t) dz \qquad (n = 1, 2, 3, \dots). \tag{5}$$

The formula (2) could be rewritten with help of equations (4) and (5) and for determination of unknown function $\varphi(x)$ next equation may be used:

$$\varphi(x) = f(x) + \sum_{\nu=1}^{\infty} \lambda^{\nu} \int_{0}^{x} K_{\nu}(x,t) f(t) dt$$

The function $\mathrm{R}(x,t;\lambda)$ defined with help of series

$$\mathrm{R}(x,t;\lambda) = \sum_{\nu=0}^{\infty} \lambda^{\nu} K_{\nu+1}(x,t), \qquad (6)$$

is called the ***resolvent*** (or resolving kernel) of integral equation (1). The series (6) converges absolutely and uniformly, if kernel $K(x,t)$ is continuous.

It should be noted that the iterative kernels as well as resolvent do not depend on the lower limit of the integral equation. Because the resolvent $\mathrm{R}(x,t;\lambda)$ of integral equation should satisfy the functional equation

$$\mathrm{R}(x,t;\lambda) = K(x,t) + \lambda \int_0^x K(x,s)\mathrm{R}(s,t;\lambda)ds,$$

the solution of equation (1) using the resolvent can be written as

$$\varphi(x) = f(x) + \lambda \int_0^x \mathrm{R}(x,t;\lambda) f(t)dt. \tag{7}$$

Sometimes the kernels of Volterra's integral equations could be more complex. Lets consider that kernel $K(x,t)$ is polynomial series of degree $(n-1)$ relative to t and it could be described by formula:

$$K(x,t) = a_0(x) + a_1(x)(x-t) + \cdots + \frac{a_{n-1}(x)}{(n-1)!}(x-t)^{n-1} \quad (8)$$

where $a_k(x)$ are continuous in the interval $(0,C)$.

One could define a function $g(x,t;\lambda)$ for another way to find the resolvent of integral equation. This function should be found as solution of differential equation

$$\frac{\mathrm{d}^n g}{\mathrm{d}x^n} - \lambda \left[a_0(x) \frac{\mathrm{d}^{n-1} g}{\mathrm{d}x^{n-1}} + a_1(x) \frac{\mathrm{d}^{n-2} g}{\mathrm{d}x^{n-2}} + \cdots + a_{n-1}(x) g \right] = 0 \quad (9)$$

with initial conditions

$$g\Big|_{x=t} = \frac{\mathrm{d}g}{\mathrm{d}x}\Big|_{x=t} = \cdots = \frac{\mathrm{d}^{n-2} g}{\mathrm{d}x^{n-2}}\Big|_{x=t} = 0, \quad \frac{\mathrm{d}^{n-1} g}{\mathrm{d}x^{n-1}}\Big|_{x=t} = 1. \quad (10)$$

The function $g(x,t;\lambda)$ found from the solution of the equation (9) with initial conditions (10) can be used to resolvent construction

$$R(x,t;\lambda) = \frac{1}{\lambda}\frac{\mathrm{d}^n g(x,t;\lambda)}{\mathrm{d}x^n} \tag{11}$$

In a similar way one can obtain formulas for finding the resolvent $R(x,t;\lambda)$ if the kernel of the integral equation is given in the form

$$K(x,t) = b_0(t) + b_1(t)(t-x) + \cdots + \frac{b_{n-1}(t)}{(n-1)!}(t-x)^{n-1}$$

In this case the resolvent $R(x,t;\lambda)$ can be found from formula

$$R(x,t;\lambda) = -\frac{1}{\lambda}\frac{\mathrm{d}^n g(t,x;\lambda)}{\mathrm{d}t^n}$$

Where function $g(t,x;\lambda)$ is solution of differential equation

$$\frac{\mathrm{d}^n g}{\mathrm{d}t^n} - \lambda \left[b_0(t)\frac{\mathrm{d}^{n-1} g}{\mathrm{d}t^{n-1}} + b_1(t)\frac{\mathrm{d}^{n-2} g}{\mathrm{d}t^{n-2}} + \cdots + b_{n-1}(t) g \right] = 0$$

with initial conditions

$$g\Big|_{t=x} = \frac{\mathrm{d}g}{\mathrm{d}t}\bigg|_{t=x} = \cdots = \frac{\mathrm{d}^{n-2} g}{\mathrm{d}t^{n-2}}\bigg|_{t=x} = 0, \quad \frac{\mathrm{d}^{n-1} g}{\mathrm{d}t^{n-1}}\bigg|_{t=x} = 1.$$