Introduction. Statistical Equivalence and Simulation

Stability, Randomness, Statistical Regularity

All phenomena in the world around us are more or less related to each other. Each of them is the cause or the effect of some others and a man studying these phenomena has to be able to answer the question, "What will happen if...".

It was found out that all phenomena could be classified into two groups: phenomena with a ***stable*** outcome (result) and phenomena with a ***random*** outcome.

Stability means the repetition of a result provided that an observation is carried out under the same conditions. For example, water always boils at the temperature of $100^{\circ}C$ under the atmospheric pressure of 760 mm Hg.

When studying such phenomena the researcher's goal is to find experimentally or calculate theoretically this result.

Randomness means that the result is variable and unpredictable in the repeated trials under the same conditions. Thus, random is the result of a coin tossing or the game "Sports lottery", the amount of goods sold in a day, the amount and the mass of fish caught by a fisherman with a rod, the distance travelled by a missile or a pebble thrown and etc.

In problems dealing with the radiation transfer theory the lifetime of a radioactive atom, pathlength of a particle among collisions in radiation protection, a number of particles striking a detector in a certain time and so on are random. Naturally, when studying phenomena with a random outcome, the nature of questions to be answered changes as well.

The analysis of different random events showed that there were those among them for which the ***statistical stability*** was observed. I.e. through the chaos of randomness something repeated and regular is revealed.

For example, the results of a coin tossing or a card drawing from a pack give a random sequence but when repeating the experiment.

However, ***the list of possible results*** remains the same as well as ***how often each of result in quite a long series of trials appears***, that is, the ***probability*** of each result from this list.

This list and probabilities corresponding to it can be called a ***statistically stable portion of a random event*** and sequences having the same statistically stable portion can be called ***statistically equivalent sequences***.

The statistically stable portion definition of the results random sequence is the study purpose of the events with a random outcome.

Some Formulas of the Probability Theory

The mathematical tool used for the analysis of random events and processes is the ***probability theory***.

It, in particular, enables to express some random events probabilities in terms of other events probabilities related to the first ones. Let us give some formulas of such a type that will be used later.

1. Let $\{A_k\}$, $k = 1, 2, \ldots$ be a list of possible results of an experiment and $N(A_k)$ be the number showing how often (on the average) the random result A_k occurs in the set of N trials. Then

$$\sum_k N(A_k) = N.$$

Having divided the both parts of this equality by N, one could get the ***normalization condition***

$$\sum_k P(A_k) = 1, \tag{1}$$

where

$$P(A_k) = \frac{N(A_k)}{N}$$

- ***is the event probability of*** A_k**.**

For instance, the number of particles of cosmic rays striking a detector in a certain time: $n = 0, 1, 2, 3, \ldots.$ is random. The probability of each of these values is calculated below.

If random numbers x assigned to the trial result run a continuous set of values, one can write an obvious equation representing them as points on the number axis

$$\sum_k N(x \in \Delta x_k) = N,$$

where $N(x \in \Delta x_k)$ - is the mean amount of random numbers that got in to the interval Δx_k.

Dividing the both sides of this equation by N we again get the normalization condition

$$\sum_k P(x \in \Delta x_k) = 1,$$

where

$$P(x \in \Delta x_k) = \frac{N(x \in \Delta x_k)}{N}$$

- is the probability of $x \in \Delta x_k$.

The simple transformation of the normalization condition

$$\sum_k \frac{P(x \in \Delta x_k)}{\Delta x_k} \Delta x_k = 1$$

and the fact that limit $\Delta x_k \to 0$ leads this condition to next form

$$\int_a^b w(x)\ dx = 1, \tag{2}$$

where $w(x)-$ ***is the probability density:***

$$w(x) = \lim_{\Delta x \to 0} \frac{P(x \in \Delta x)}{\Delta x}. \tag{3}$$

This function describes the shape of a "hill", which is formed by numbers x if they are marked by points on the number axis, and ***the range of possible values of*** x: $a \leq x \leq b$ are a statistically stable portion of these numbers sequence.

The example of a continuous random value can be the lifetime of a radioactive atom: t $(0 \leq t < \infty)$.

The probability density type of this random value is considered below.

2. If random event A is composed of a few events B_k $(k = 1, 2, \dots)$ combined by the conjunction ***or***, then

$$P(A) = \sum_k P(B_k). \tag{4}$$

For example, when tossing a die your partner has got a "4". To win, you have to get a "5" ***or*** a "6" therefore the successful probability is $P(5\ \mathbf{or}\ 6) = P(5) + P(6)$.

For an "honest" die $P(5) = P(6) = 1/6$ and the successful probability is equal to $1/3$.

3. If random event A is composed of a few events B_k, $(k = 1, 2, \dots)$ combined by the conjunction **and**, then

$$P(A) = P(B_1)P(B_2)\dots \tag{5}$$

For example, the score sum for a die and a coin thrown simultaneously can be any number from 1 to 7.

In this case you will get 7 points if the die shows a "6" **and** the coin turns up "tail".

The probabilities of these events are equal to $1/6$ and $1/2$ respectively therefore $P(7) = 1/12$.

4. The example of the formulas combination (4) and (5) can be ***the total probability formula*** which we are going to illustrate by a simple example.

Some tourists visiting a city want to find and see sightseeing A.

Let suppose they have no map of the city and they do not speak the native language and could not ask the way to it.

Therefore, everyone chooses a street to start the tour randomly. The same random way they decide where to turn at every crossroad.

Let N be the number of tourists and $N(k)$ be the number of those who chose the road with index k at the beginning of the way. Then

$$\sum_k N(k) = N$$

If we divide both parts of equation by N. Than

$$\sum_k P(k) = 1,$$

where

$$P(k) = \frac{N(k)}{N}$$

- is the corresponding probability.

Let $N(A)$ be the number of the tourists who succeeded to find A in a given time and

$$P(A) = \frac{N(A)}{N}$$

- is the success probability.

Let write the value $N(A)$ in this formula as a sum

$$N(A) = \sum_k N(A;k),$$

where $N(A;k)$ - is the number of the people started the tour from the road with index k and reached their destination A.

In these notations the probability $P(A)$ can be written as

$$P(A) = \frac{N(A)}{N} = \sum_k \frac{N(k)}{N} \frac{N(A;k)}{N(k)} = \sum_k P(k)\ P(A;k), \tag{6}$$

where

$$P(A;k) = \frac{N(A;k)}{N(k)}$$

- is the success probability for those tourists who started the tour from the road with index k (***the conditional probability***).

In some problems "the road index" can take on a continuous set of values. In this case the summing procedure in the normalization condition and in the total probability formula is replaced by integration.

It will be shown later that if a statistically regular portion of a random event is known then the sequence of numbers statistically equivalent to those received experimentally can be generated with the help of computers. Such a procedure is referred to as statistical simulation of the given event.

The methods of statistical simulation are often called ***Monte Carlo methods***.

This name originates from the name of the city Monte Carlo in the Monaco principality known for its casinos where a roulette is played. That gambling device can be the simplest way to generate the random numbers.

During the World War II the method was used to solve problems concerning the atom bomb creation.

The name "the Monte Carlo method" was proposed in 1949 in the work "The Monte Carlo method" by Metropolis N., Ulam S.M., J. Amer. Statist. Assoc., 1949, 44, No 247, p.335-341.

From that time its systematic development and application was started.

The creators of the method are considered to be Heiman J., Ulam S. and Metropolis N.

But the first who applied this method for solving problems dealing with the moderation of neutrons in the early 30-s of the previous century was presumably E. Fermy.

The method is effective and useful in solving probability problems , however, at present it is successfully used as a research tool in the problems with no probabilistic content.