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AUTOMATIC CONTROL SYSTEMS IN SCIENTIFIC RESEARCH AND INDUSTRY

Determining Direct Measures of Performance Based on the Location of Zeros and Pole of the Transfer Function

S. V. Efimov and M. I. Pushkarev

Institute of Cybernetics,
National Research Polytechnical University
ul. Lenina 30, Tomsk, 634050 Russia
E-mail: efimov@tpu.ru

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Abstract—An analysis is made of the effect of zeros and poles of a closed-loop automatic control system on its direct measures of performance: overcontrol and control time. The necessity of determining the dominant poles in the root-locus plane is shown. A criterion for the ε-dominance of the components of the transient function of automatic control system is presented. A method for calculating direct measures of performance based on the location of zeros and poles of the system is developed. Numerical examples are considered.

Keywords: automatic control system, direct measures of performance, overcontrol, control time, dominant pole, root-locus plane.

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INTRODUCTION

In numerous papers devoted to the root-locus analysis of linear automatic control systems, direct measures of performance of transient processes are determined using indirect root-locus measures of performance based on the values of the dominant poles [1, 2]. However, the root-locus measures do not account for the influence of zeros of the system on the transient process, which makes these measures approximate and, in some cases, erroneous. It is believed that the dominant poles are the poles the closest to the imaginary axis of the root-locus plane [1, 2]. Nevertheless, we estimate the validity of the dominance criterion for poles based on their proximity to the imaginary axis and, hence, the correctness of the direct measures of performance: overcontrol and control time.

PROBLEM STATEMENT

We consider a closed-loop linear automatic control system with the transfer function $W(s) = G(s)/H(s)$, where $G(s)$ and $H(s)$ are polynomials in powers of s. Let the transfer function of this system have no multiple poles.

It is necessary to calculate the direct measures of performance based on the location of poles and zeros of the transfer function of the closed-loop system. Since the determination of the indirect measures of performance is largely dependent on the dominant poles, it is required to find the poles of the transfer function of the closed-loop system that are dominant in the root-locus plane of the system.

DOMINANCE OF POLES

The root-locus analysis of system stability is based on the location of poles of the closed-loop automatic control system. The formulas defining the transition from indirect to direct measures of performance include the values of the dominant poles that characterize the dynamic properties of the system [2]:

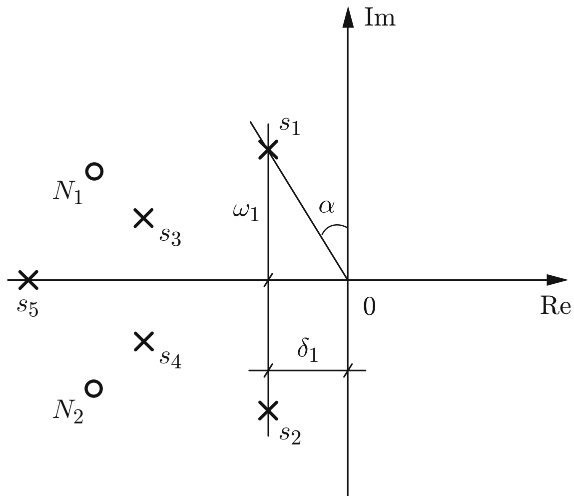

Fig. 1. Root-locus plane (circles represent zeros, and crosses represent poles).

$$\sigma = \mathrm{e}^{\tan(\alpha)}, \tag{1}$$

$$t_\mathrm{c} = 3/\mathrm{Re}(s_1) = 3/\delta_1, \tag{2}$$

where σ is the overcontrol, α is the angle formed by the complex root-locus and the imaginary axis of the root-locus plane, t_c is the control time of the system, and $s_1 = \delta_1 + \omega_1 j$ is the dominant pole (Fig. 1).

We consider an automatic control system with the transfer function

$$W(s) = \frac{1.79 \cdot 10^6 s + 2.703 \cdot 10^7}{s^4 + 55s^3 + 9.11 \cdot 10^4 s^2 + 3.16 \cdot 10^6 s + 2.703 \cdot 10^7}. \tag{3}$$

This system has a zero $N_1 = -15.1$ and poles $s_1 = -20$, $s_2 = -15$, and $s_{3,4} = -10 \pm 300j$ (Fig. 2). According to the figure, $s_{3,4}$ are the dominant poles. We estimate the effect of each pole on the transient process using the Heaviside formula [3, 4]

$$h(t) = \frac{G(0)}{H(0)} + \sum_{k=1}^{n} \frac{G(s_k)\mathrm{e}^{s_k t}}{s_k \prod\limits_{i=1,\, i\neq k}^{n} (s_k - s_i)} = \frac{G(0)}{H(0)} + \frac{\prod\limits_{i=1}^{n} |s_i|}{\prod\limits_{i=1}^{m} |N_i|} \sum_{k=1}^{n} \frac{\prod\limits_{i=1}^{m} (s_k - N_i)}{s_k \prod\limits_{i=1,\, i\neq k}^{n} (s_k - s_i)}\, \mathrm{e}^{s_k t}. \tag{4}$$

Using formula (4), whose terms are components of the transient function, it is found that the dominant influence factor in the transient is the pole $s_{3,4}$ rather than the pair of complex conjugate poles $s_1 = -20$ which are the closest to the imaginary axis (Fig. 3).

Thus, the choice of the dominant poles $s_{3,4}$ of the system in the initial stage of the analysis was made incorrectly, resulting in erroneous values of the performance measures. Therefore, an important problem in root-locus analysis is to choose poles that determine the dynamic properties of the system — the dominant poles.

CRITERION FOR THE ε-DOMINANCE OF POLES

A component $A_i \mathrm{e}^{s_i t}$ (A_i is the amplitude of the action of the pole) of the transient function (4) is ε-dominant in the interval of time $t_1 \le t \le t_2$, if for $t \in [t_1, t_2]$, the following inequality is satisfied [3]:

$$\Big| \sum_{k=1,\, k\neq i}^{n} A_k \mathrm{e}^{s_k t} \Big| \Big/ \left| A_i \mathrm{e}^{s_i t} \right| < \varepsilon.$$

In [3], sufficient conditions were obtained for the ε-dominance of a component of the transient function:

$$\sum_{\lambda,\ \lambda\neq i} \frac{|A_\lambda|}{|A_i|}\, \mathrm{e}^{(s_\lambda - s_i)t} + \sum_{\nu} 2\frac{|A_\nu|}{|A_i|}\, \mathrm{e}^{(\delta_\nu - s_i)t} < \varepsilon, \tag{5}$$

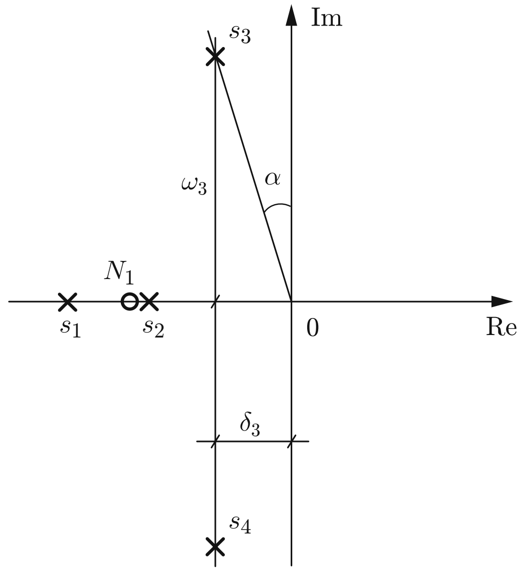

Fig. 2. Location of the zero (circles) and poles (crosses) of system (3).

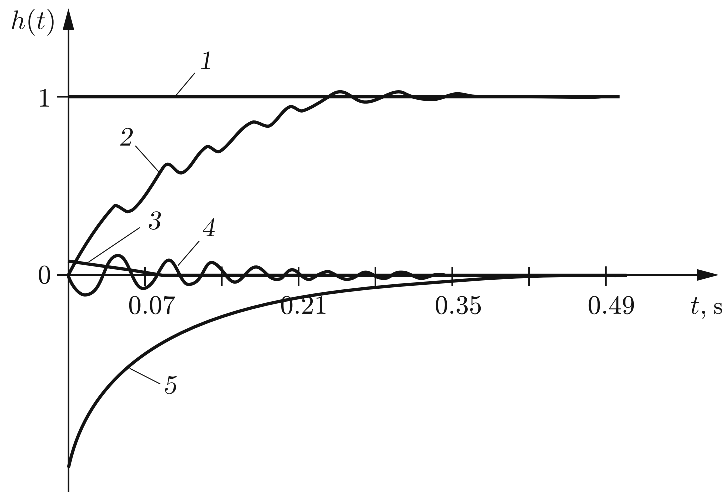

Fig. 3. Plots of the components of the transient process: curve *1*) refers to $W(0)$, curve *2*) to resulting plot of system (3), curve *3*) to component from the pole s_2, curve *4*) to component from the poles $s_{3,4}$, and curve *5*) to component from the pole s_1.

$$\sum_{\lambda} \frac{|A_\lambda|}{2|A_m|}\, \mathrm{e}^{(s_\lambda - \delta_m)t} + \sum_{\nu,\, \nu \neq m} \frac{|A_\nu|}{|A_m|}\, \mathrm{e}^{(\delta_\nu - \delta_m)t} < \varepsilon, \tag{6}$$

where s_λ are real poles, $s_{\nu,\,\nu+1} = \delta_{\nu,\,\nu+1} \pm \omega_{\nu,\,\nu+1} j$ are complex conjugate poles, and A is the amplitude of the component of the transient function; the time t varies in the range of $t_1 \leq t \leq t_2$.

In the analysis of the dynamic properties of the system, conditions (5) or (6) for the component of the transient function that is a candidate for ε-dominance is an accuracy criterion for existing estimates of control performance measures based on the distribution of poles and zeros of the transfer function that are derived from the presence of a certain dominant component in the transient function of the system.

If condition (5) is satisfied in the interval $0 \leq t \leq t_i$, the time of decrease t_i of the aperiodic component to a set point Δ is equal to $\ln(|A_i|/\Delta)/|s_i|$ [3].

If condition (6) is satisfied in the interval $0 \leq t \leq t_m$, the time of decrease t_m of the amplitude of the oscillatory component to a set point Δ is equal to $\ln(2|A_m|/\Delta)/|\delta_m|$ [3].

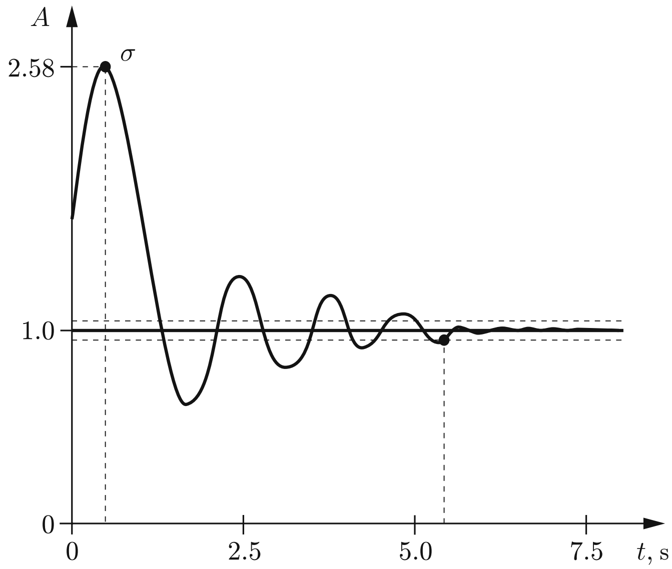

Fig. 4. Plot of the transient process.

In view of formulas (4)–(6), for the poles of system (3) s_1, s_2, and $s_{3,4}$, the values of ε are equal to 0.36, 49.44, and 5.73, respectively. For $\varepsilon > 1$, the component cannot be considered dominant in the specified time interval. Thus, according to the values of ε, it is established that the pole $s_1 = -20$ is dominant in this interval.

ROOT-LOCUS MEASURES OF PERFORMANCE

We consider a closed-loop system with the transfer function

$$W(s) = \frac{1.897s^3 + 15.18s^2 + 28.94s + 37}{s^3 + 5s^2 + 13.25s + 37}. \tag{7}$$

System (7) has zeros $N_{1,2} = -1 \pm 1.5j$ and $N_3 = -6$ and poles $s_1 = -4$ and $s_{2,3} = -0.5 \pm 3j$.

According to the ε-dominance criterion for the poles of system (7) s_1 and $s_{2,3}$, the values of ε are 474.3 and 0.01, respectively, and $s_{2,3}$ are therefore the dominant poles.

According to [1, 2], the overcontrol σ and the control time t_c of system (7) can be determined based on indirect measures of performance [degree of stability δ and oscillation $\tan(\alpha)$] using formulas (1) and (2). As a result, we have $\sigma = 84\%$ and $t_c = 6$ s. However, the performance measures derived from Fig. 4 are $\sigma = \dfrac{2.58 - 1}{1} \cdot 100\% = 158\%$ and $t_c = 5.77$ s.

Thus, the performance measures determined in the first and second cases are different, suggesting inaccuracy of formulas (1) and (2) relating the direct and indirect measures of performance of the systems.

EFFECT OF ZEROS AND POLES OF THE SYSTEM ON PERFORMANCE MEASURES

Based on [4], the peak of the oscillating system is

$$\xi \approx 2A_1 \frac{\omega_1}{\omega_0} \exp\left[\frac{\delta_1}{\omega_1}\left(\pi + \sum_3^n \varphi_k - \sum_1^m \Phi_j\right)\right] + \sum_3^n B_k e^{s_k t_m}, \tag{8}$$

where m and n are the order of the numerator and denominator of its transfer function, $s_1 = \delta_1 + \omega_1 j$ is the dominant pole, φ_k is the angle formed by the dominant pole and any other pole, Φ_j is the angle formed by the dominant pole and the zero of the system, and

$$\omega_0 = \sqrt{\delta_1^2 + \omega_1^2}; \quad A_1 = \left|\frac{G(s_1)}{s_1 H'(s_1)}\right|; \quad B_k = \left|\frac{G(s_k)}{s_k H'(s_k)}\right|; \quad t_m = \frac{1}{\omega_1}\left(\pi + \sum_3^n \varphi_k - \sum_1^m \Phi_j\right).$$

The overcontrol of the system can be found from the overshoot ξ:

$$\sigma = \frac{\xi}{A_{\rm st}} \cdot 100\%, \tag{9}$$

where $A_{\rm st}$ is the steady-state value [5].

For the control time, which is determined by the time interval from the time of applying an action to the system to the time of establishment of a deviation of the controlled variable that does not exceed a predetermined threshold $\Delta = 0.05$, there are also estimation formulas that take into account the amplitude of the dominant pole [3]. For real dominant poles, we have

$$t_{\rm c} = \frac{3 + \ln(A_1)}{\delta_1}, \tag{10}$$

for complex conjugate dominant poles,

$$t_{\rm c} = \frac{3 + \ln(2A_1)}{\delta_1}. \tag{11}$$

Formulas (10) and (11) include the amplitude of the dominant pole but these relations ignore the effect of the other poles and zeros of the transfer function of the system.

We turn to the Heaviside formula

$$h(t) = \frac{G(0)}{H(0)} + \sum_{i=1}^{n} A_i e^{s_i t}. \tag{12}$$

At the time $t_{\rm c}$, the difference between the amplitude of the output signal and the steady-state value is 5% (of the set value) and is written as

$$\left| h(t_{\rm c}) - \frac{G(0)}{H(0)} \right| = \Delta \frac{G(0)}{H(0)};$$

for $\Delta = 0.05$ and $G(0)/H(0) = 1$, it will take the form

$$\left| h(t_{\rm c}) - \frac{G(0)}{H(0)} \right| = 0.05.$$

Then (12), can be written as

$$\sum_{i=1}^{n} A_i e^{s_i t_{\rm c}} = 0.05. \tag{13}$$

However, the form of relation (13) does not allow $t_{\rm c}$ to be expressed analytically. Taylor expansion of the exponential function

$$e^{s_k t_{\rm c}} = 1 + \frac{s_k t_{\rm c}}{1!} + \frac{(s_k t_{\rm c})^2}{2!} + \ldots + \frac{(s_k t_{\rm c})^n}{n!}$$

leads to a significant increase in the order of the equation, which will not simplify the computation of $t_{\rm c}$ and the restriction of the terms of the series to a small number will give a high error. Therefore, it is more reasonable to find $t_{\rm c}$ in (13) by a numerical method. Such methods usually require an initial approximation of the desired quantity: the closer this value to the desired one, the smaller the number of calculations required to determine $t_{\rm c}$. To reduce the computation time $t_{\rm c}$, it is proposed to find the initial approximation from relations (10) and (11). This will provide the most accurate value of the control time by numerical solution of Eq. (13) taking into account that at the time $t_{\rm c}$, the deviation of the amplitude of the output signal from the steady-state signal is 5%.

According to formulas (8)–(13), the performance measures for this example are $\sigma = 162.8\%$ and $t_{\rm c} = 5.97$ s.

Thus, the direct measures of performance of the system obtained from formulas (8)–(13) taking into account that the relative position of the poles and zeros of the system in the root-locus plane are close to the actual measures of performance.

THE METHOD FOR DETERMINING DIRECT MEASURES OF PERFORMANCE

The method for determining direct measures of performance for linear dynamic automatic control systems whose transfer functions do not contain multiple poles consist of calculating the following direct measures of performance:

1) zeros and poles of the automatic control system based on the initial data (transfer function of the system);

2) the amplitudes A_i from the action of each of the poles of the system according to the relation relation $A_i = |G(s_i)/(s_i H'(s_i))|$;

3) the dominant poles of the system by the ε-dominance criterion;

4) the angles Φ_j formed by the dominant pole, the zeroes of the system, and the positive direction of the real axis;

5) the angles φ_i formed by the dominant pole, the other poles of the system, and the positive direction of the real axis;

6) the overcontrol of the automatic control system based on the location of its zeros and poles according to (9);

7) the approximate control time in accordance with (10) or (11);

8) the control time in accordance with (13) taking into account the approximate value obtained in item 7.

CONCLUSIONS

This paper presents a root-locus analysis of automatic control systems based on the location of their zeros and poles. A study was made of the effect of the poles and zeros of the systems on the values of the direct measures of performance: overcontrol and control time. The need to determine the dominant poles was noted and the ε-dominance criterion for the components of the transient functions for the systems was considered since the closeness of the poles to the imaginary axis and the coordinate origin generally cannot be an indicator of their dominance.

The results show that with the establishment of the dominant poles of the system and accounting for the effect of zeros and poles of the system, the calculated values of the performance measures are close to their actual values determined from the plot of the transient process.

The results of these studies were used to develop a method for calculating direct measures of performance from the location of poles and zeros of the system in the root-locus plane.

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