Maximizing Stability Degree of Control Systems under Interval Uncertainty Using a Coefficient Method*

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Abstract

For linear automatic control systems, many synthesis methods have been developed that exercise options of the controller structure and parameters to provide the stated requirements to the system quality. Coefficient methods can compute approximate, but rather simple, correlations that link the automatic control system quality indices of a random order and the desired controller parameters. One of the most widely used criteria when designing an automatic control system is the system stability maximum degree. In real systems, the object parameters usually are rough or can be changed within certain limits. Such parameters are called interval parameters, and such control systems are called interval control systems. It seems very interesting to provide the maximum degree of robust stability in the system. The approach is based on coefficient assessment of the stability of interval systems' indices and allows maximizing the robust stability degree when using unsophisticated algebraic associations.

Keywords: interval system, robust, stability degree, coefficient method
AMS subject classifications: 93C99, 65G40

1 Introduction

Even for stationary control systems, parameters can vary with time due to aging or for other reasons. Moreover, the parameters of the controlled object could be uncertain or change within some limits in the process of non-stationary system controller

*Submitted: December 24, 2012; Revised: September 12, 2013; Accepted: December 23, 2013.

development. In such cases, it is necessary to design control systems in such a way that they are stable not only for specified and constant parameter values, but for all possible values within certain ranges. In the last case, a system is called *robust stable*.

Much work is devoted to research on stability and synthesis of the control systems under uncertainty, when parameters of the system can vary within specified intervals [1-45]. Robust theory is a rather new approach whose main results were obtained only recently. These include a variety of methods and approaches, such as graphical criterion of polynomial robust stability [1, 2, 3, 4], edge theorem [5, 6, 7, 8, 9, 10, 11], and polynomial methods based on Kharitonov theorem and its modifications [12, 13, 14, 15, 16, 17, 18].

The controller synthesis problem is the most complicated when developing interval systems. It requires providing a desired robust system operation quality in any possible operation mode by means of linear controller settings. Perturbation theory [19, 20, 21], root approaches [22, 23, 24, 25, 26], μ-analysis [27, 28, 29], and probabilistic approaches [30, 31, 32] all can be used to solve the synthesis problem.

2 Problem Formulation

The maximum stability degree criterion is one of the most widely used criteria for the design of robust control systems. The systems constructed according to this criterion are known to have a higher operation speed, a smaller system overshoot, and a higher stability margin [39, 40] than conventional control systems. It is also important that the systems with maximal stability degree produce a low response to parametric perturbations in controlled objects. In this context, there is a great interest in solving the problem of maximizing a linear system stability degree by appropriate controller settings when an interval uncertainty is present in the controlled object.

We solve the above problem using coefficient methods, which obtain approximate, but rather simple, correlations that link the desired controller parameters and coefficient performance indices of an automatic control system [41]. These indices are determined through coefficients patterns of a system characteristic polynomial.

Pushkarev et al. [44] includes an example of a stationary system synthesis on the basis of coefficient assessments of system quality indices. In this article, we consider an interval extension of the synthesis technique of [44] based on interval analysis techniques [42, 45].

3 Maximizing the Stability Degree of Stationary Systems

We consider a linear stationary control system with a characteristic polynomial

$$A(s) = \sum_{i=0}^{n} a_i s^i = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0, \qquad a_n > 0. \tag{1}$$

For such systems, it makes sense to define stability indices λ_i [41, 43], which are formed by quadruples of nearby polynomial coefficients from (1)

$$\lambda_i = \frac{a_{i-1} a_{i+2}}{a_i a_{i+1}}, \qquad i = \overline{1,\, n-2}. \tag{2}$$

The sufficient stability conditions of a stationary linear systems have been obtained on the basis of the coefficient stability indices (2) in [41]:

$$\lambda_i < \lambda^* \approx 0.465, \qquad \text{for every} \;\; i = \overline{1,\, n-2}. \tag{3}$$

The conditions (3) can be used to choose controller parameters which provide stability for an automatic control system. Their simplicity allows one to form constructive and easily-coded synthesis procedures, while their redundancy on the safe side contributes to stability, which is always necessary when designing real automatic control systems.

When designing an automatic control system, it is important not only to achieve stability, but also to provide a specified system performance quality. From this viewpoint, the sufficient conditions of the stability degree η offered in [41] could be useful:

$$\begin{cases} \dfrac{a_{i-1}a_{i+2}}{\big(a_i - a_{i+1}(n-i-1)\eta\big)\big(a_{i+1} - a_{i+2}\,(n-i-2)\,\eta\big)} < \lambda^*, \qquad i = \overline{1, n-2}; \\ a_l - a_{l+1}(n-l-1)\eta \geq 0, \quad l = \overline{1,\, n-1}; \\ a_0 - a_1\eta + \dfrac{2a_2\eta^2}{3} \geq 0. \end{cases} \tag{4}$$

The fulfillment of the conditions (4) guarantees that the roots of the characteristic polynomial (1) are positioned at the left-hand side from the vertical straight line through the point $(-\eta, j0)$. Increasing η within the above conditions allows one to find its maximum value that can be considered as a maximum estimate of the system stability degree. Let us denote it by η^*.

The controller synthesis problem is to choose such controller parameters $\overline{k}^*$ that provide the maximum value of η^*. We denote this maximum as $\eta^*_{\max}$. Therefore, $\eta^*_{\max} = \max\limits_{\overline{k}} \eta^*$, where $\eta^*_{\max}$ is an estimate of the maximum stability degree. It can serve as a quasi-maximal stability degree for an automatic control system.

We introduce the notation

$$\begin{cases} \lambda_i(\overline{k},\eta) = \dfrac{a_{i-1}(\overline{k})a_{i+2}(\overline{k})}{\big(a_i(\overline{k}) - a_{i+1}(\overline{k})(n-i-1)\eta\big)\big(a_{i+1}(\overline{k}) - a_{i+2}(\overline{k})(n-i-2)\eta\big)}, \\ \qquad\qquad i = \overline{1, n-2}; \\ f_l\left(\overline{\mathrm{k}},\eta\right) = a_l(\overline{k}) - a_{l+1}(\overline{k})(n-l-1)\eta, \quad l = \overline{1,\, n-1}; \\ g(\overline{k},\eta) = a_0(\overline{k}) - a_1(\overline{k})\eta + \dfrac{2a_2(\overline{k})\eta^2}{3}. \end{cases} \tag{5}$$

Proposition 1. A linear controller with the settings $\overline{k}^*$ provides the quasi-maximal stability degree $\eta^*_{\max}$ in the system with a characteristic polynomial (1) if

$$\begin{cases} \lambda_i\big(\overline{k}^*, \eta^*_{\max}\big) = \lambda^*, \qquad i = \overline{1,\; n-2}; \\ \lambda_j\big(\overline{k}^*, \eta^*_{\max}\big) < \lambda^*, \qquad j = \overline{1,\, n-2}, \;\; j \neq i; \\ f_l\big(\overline{k}^*, \eta^*_{\max}\big) \geq 0, \qquad l = \overline{1,\;\, n-1}; \\ g\big(\overline{k}^*, \eta^*_{\max}\big) \geq 0. \end{cases} \tag{6}$$

Proof: The growth of η in each expression $\lambda_i(\overline{\mathrm{k}}, \eta)$ from (5) is possible up to the value when $\lambda_i(\overline{k}^*, \eta^*_{\max}) = \lambda^*$, $i = \overline{1,\ n-2}$ by means of changing the controller settings. Thereby, to determine the quasimaximal stability degree and corresponding controller settings $\overline{k}^*$, it is sufficient to solve $(n-2)$ times the system of equations (6) defining $\eta^*_{\max}$ for every system, and then to choose the maximum among them. ■

4 Maximizing the Stability Degree of Interval Systems

Under interval uncertainty of the system parameters, the characteristic polynomial (1) of the system reduces to

$$A(s) = \sum_{i=0}^{n} \boldsymbol{a}_i s^i = \boldsymbol{a}_n s^n + \boldsymbol{a}_{n-1} s^{n-1} + \ldots + \boldsymbol{a}_0, \qquad \boldsymbol{a}_n > 0, \tag{7}$$

$$\underline{a_i} \le \boldsymbol{a}_i \le \overline{a_i}, \qquad i = \overline{0,\ n},$$

where $\underline{\boldsymbol{a}_i}$ and $\overline{\boldsymbol{a}_i}$ are a priori specified interval bounds.

Kharitonov's Theorem [12] is known to be the origin of extremum point-based approaches to robust stability testing for the systems under uncertainty. The theorem gives necessary and sufficient conditions for interval polynomials' robust stability. However, the study of the robust stability of many polynomials is brought to the stability check of a maximum of four of them, regardless of the polynomial degree n [42]. This theorem is correct if characteristic polynomial coefficients change independently of each other in the stated intervals. Kharitonov's Theorem does not support analyzing the robust quality of interval systems, including the robust stability degree.

Figure 1 shows an example of interval system roots localization area $\Re(\boldsymbol{a})$. The figure suggests that the interval system robust stability degree $\alpha_r(\boldsymbol{a})$ is determined by its stability degree in the worst operation mode to which a certain set of interval parameters corresponds.

Proposition 2. For the interval polynomial (7) to have robust stability, he following conditions are sufficient:

$$\frac{\overline{a_{i-1}}\,\overline{a_{i+2}}}{\underline{a_i}\,\underline{a_{i+1}}} < 0.465, \qquad i = \overline{1, n-2}. \tag{8}$$

Proof: Conditions (3) on the interval polynomial have the form

$$\boldsymbol{\lambda}_i = \frac{\boldsymbol{a}_{i-1}\,\boldsymbol{a}_{i+2}}{\boldsymbol{a}_i\,\boldsymbol{a}_{i+1}} < 0.465, \qquad i = \overline{1, n-2}. \tag{9}$$

To check (4), it is necessary to find the maximum possible values of λ_i, assuming that the polynomial coefficient can change arbitrarily in the prescribed intervals. Let $\boldsymbol{a}_{i-1}\boldsymbol{a}_{i+2} = \boldsymbol{c}$, $\boldsymbol{a}_i\boldsymbol{a}_{i+1} = \boldsymbol{d}$. Then λ_i has the maximum values for $c_{\max}$ and $d_{\min}$. Insofar as $c_{\max} = \overline{a_{i-1}a_{i+2}}$ and $d_{\min} = \underline{a_i}\underline{a_{i+1}}$, then, if the conditions (4) hold true for $\overline{a_{i-1}}$, $\underline{a_i}$, $\underline{a_{i+1}}$, $\overline{a_{i+2}}$, $i = \overline{1, n-2}$, they should be satisfied for any other values of the interval coefficients, ensuring the robust stability of the corresponding interval polynomial. ■

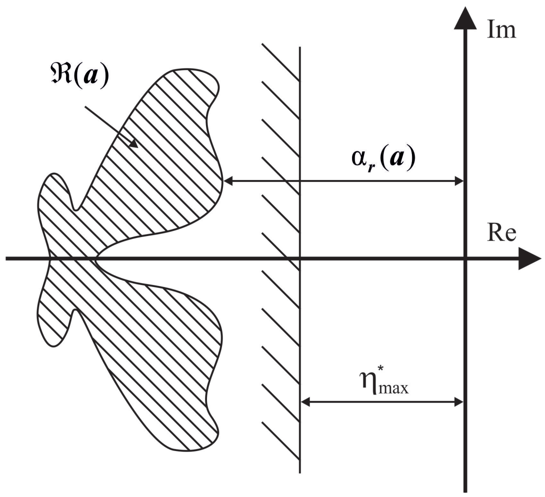

Figure 1: Root domain

Proposition 3. To make the robust stability degree of an interval polynomial (7) higher than a specified robust stability degree η_{sp}, it is sufficient to satisfy

$$\begin{cases} \lambda_{iz} = \dfrac{\overline{a_{i-1}a_{i+2}}}{\left(\underline{a_i} - \overline{a_{i+1}}(n-i-1)\eta_{sp}\right)\left(\underline{\overline{a_{i+1}}} - \overline{a_{i+2}}(n-i-2)\eta_{sp}\right)} < \lambda^*, \\ \qquad i = \overline{1,\ n-2},\ z = 0 \text{ at } \underline{a_{i+1}}; z = 1 \ \text{ at } \overline{a_{i+1}}; \\ \underline{a_l} - \overline{a_{l+1}}(n-l-1)\eta_{sp} \geq 0, \qquad l = \overline{1,\ n-1}; \\ \underline{a_0} - \overline{a_1}\eta_{sp} + \dfrac{2\underline{a_2}\eta_{sp}^2}{3} \geq 0. \end{cases} \tag{10}$$

Proof: The conditions (4) for an interval polynomial have the form

$$\begin{cases} \lambda_i = \dfrac{\boldsymbol{a}_{i-1}\boldsymbol{a}_{i+2}}{\left(\boldsymbol{a}_i - \boldsymbol{a}_{i+1}(n-i-1)\eta_{sp}\right)\left(\boldsymbol{a}_{i+1} - \boldsymbol{a}_{i+2}(n-i-2)\eta_{sp}\right)} < \lambda^*, \\ \qquad i = \overline{1, n-2}; \\ \boldsymbol{a}_l - \boldsymbol{a}_{l+1}(n-l-1)\eta_{sp} \geq 0, \quad l = \overline{1,\ n-1}; \\ \boldsymbol{a}_0 - \boldsymbol{a}_1\eta_{sp} + \dfrac{2\boldsymbol{a}_2\eta_{sp}^2}{3} \geq 0, \end{cases} \tag{11}$$

where the notation

$$\boldsymbol{a}_{i-1}\boldsymbol{a}_{i+2} = \boldsymbol{c}, (\boldsymbol{a}_i - \boldsymbol{a}_{i+1}(n-i-1)\eta_{sp}) \times (\boldsymbol{a}_{i+1} - \boldsymbol{a}_{i+2}(n-i-2)\eta_{sp}) = \boldsymbol{d}$$

is used in the expression for λ_i in (11). Similar to the proof of Proposition 1, we must satisfy the test (11) for the values $c_{\max}$ and $d_{\min}$. Then, taking into account the specified intervals of polynomial coefficients, $c_{\max} = \overline{a_{i-1}a_{i+2}}$ and $d_{\min} = \left(\underline{a_i} - \overline{a_{i+1}}(n-i-1)\eta_{sp}\right)\left(\underline{\overline{a_{i+1}}} - \overline{a_{i+2}}\,(n-i-2)\,\eta_{sp}\right)$. The symbol $\underline{\overline{a_{i+1}}}$ means that the coefficient a_{i+1} can have both lower and upper endpoints of the corresponding interval.

To check the two remaining inequalities in (11), it is necessary to assign such values of the interval coefficients that provide the minimum values of the left-hand sides of the inequalities, i.e., $\underline{a_l} - \overline{a_{l+1}}(n-l-1)\eta_{sp} \geq 0$, $l = \overline{1,\ n-1}$; $\underline{a_0} - \overline{a_1}\eta_{sp} + \frac{2\underline{a_2}\eta_{sp}^2}{3} \geq 0$.

The specified inequalities (11) hold for any other values of the interval coefficients. Hence, if, for η_{sp}, the conditions (10) are met for the above bounds of the interval coefficients, they are also true for all the other values from the prescribed intervals. Therefor, the localization areas of the roots for the interval polynomial are on the left-hand side of the vertical line passing though the point $(-\eta_{sp}, j0)$. This means that the robust stability degree of the interval polynomial (7) is higher than η_{sp}. ■

Our experience with the solution of the robust stability analysis problem using Proposition 3 showed that it is quite helpful to specify various combinations of bounds of the interval coefficients when each condition is fulfilled in inequalities systems. These combinations form the polynomials

$$V_1(s) = \overline{a_0} + \underline{a_1}s + \underline{a_2}s^2 + \overline{a_3}s^3 + \overline{a_4}s^4 + \underline{a_5}s^5 + \underline{a_6}s^6 + \dots,$$

$$V_2(s) = \overline{a_0} + \underline{a_1}s + \overline{a_2}s^2 + \overline{a_3}s^3 + \underline{a_4}s^4 + \overline{a_5}s^5 + \overline{a_6}s^6 + \dots,$$

$$V_3(s) = \overline{a_0} + \overline{a_1}s + \underline{a_2}s^2 + \underline{a_3}s^3 + \overline{a_4}s^4 + \overline{a_5}s^5 + \underline{a_6}s^6 + \dots,$$

$$V_4(s) = \overline{a_0} + \overline{a_1}s + \underline{a_2}s^2 + \overline{a_3}s^3 + \overline{a_4}s^4 + \underline{a_5}s^5 + \overline{a_6}s^6 + \dots,$$

$$V_5(s) = \underline{a_0} + \overline{a_1}s + \overline{a_2}s^2 + \underline{a_3}s^3 + \underline{a_4}s^4 + \overline{a_5}s^5 + \overline{a_6}s^6 + \dots,$$

$$V_6(s) = \underline{a_0} + \overline{a_1}s + \overline{a_2}s^2 + \underline{a_3}s^3 + \overline{a_4}s^4 + \overline{a_5}s^5 + \underline{a_6}s^6 + \dots,$$

$$V_7(s) = \underline{a_0} + \underline{a_1}s + \overline{a_2}s^2 + \overline{a_3}s^3 + \underline{a_4}s^4 + \underline{a_5}s^5 + \overline{a_6}s^6 + \dots.$$

$$V_8(s) = \underline{a_0} + \overline{a_1}s + \underline{a_2}s^2 + \overline{a_3}s^3 + \underline{a_4}s^4 + \overline{a_5}s^5 + \underline{a_6}s^6 + \dots.$$

Each interval polynomial is defined to correspond to a certain vertex V_m, $m = \overline{1,8}$ of the interval coefficients polyhedron.

Proposition 4. The maximum estimate (after) η^* of the robust stability degree of an interval polynomial (7) is determined as $\eta^* = \min\limits_{V_m} \eta^{V_m}$, when the following conditions are fulfilled:

$$\begin{cases} \lambda_{iz}(\eta) = \dfrac{\overline{a_{i-1}a_{i+2}}}{\left(\underline{a_i} - \underline{\overline{a_{i+1}}}\,(n-i-1)\,\eta\right)\left(\underline{\overline{a_{i+1}}} - \overline{a_{i+2}}(n-i-2)\eta\right)} = 0.465, \\ \qquad i = \overline{1,\ n-2},\ z{=}0 \text{ at } \underline{a_{i+1}};\ z = 1 \text{ at } \overline{a_{i+1}}; \\ \lambda_{jz}(\eta) = \dfrac{\overline{a_{j-1}a_{j+2}}}{\left(\underline{a_j} - \underline{\overline{a_{j+1}}}\,(n-i-1)\,\eta\right)\left(\underline{\overline{a_{j+1}}} - \overline{a_{j+2}}(n-i-2)\eta\right)} < 0.465, \\ \qquad j = \overline{1, n-2},\ j \neq i,\ z = 0 \text{ at } \underline{a_{j+1}};\ z = 1 \text{ at } \overline{a_{j+1}}; \\ \underline{a_l} - \overline{a_{l+1}}(n-l-1)\eta \geq 0, \qquad l = \overline{1,\ n-1}; \\ \underline{a_0} - \overline{a_1}\eta + \dfrac{2\underline{a_2}\eta^2}{3} \geq 0. \end{cases} \tag{12}$$

Proof: For each system (12) with the number m, we increase η and determine the maximum values of η^{V_m} (taking m-vertex V_m as an index) for which the conditions (12) are met. Choosing the minimum of the values η^{V_m} gives the maximum estimate η^* of the robust stability degree for an interval polynomial, i.e., $\eta^* = \min\limits_{V_m} \eta^{V_m}$. ■

If a control system has some interval parameters, a design engineer is interested not only in providing its robust stability. The engineer also is interested in obtaining a specified robustness quality, e.g., the quasi-maximal robust stability degree that best moves the root localization area away from the imaginary axis. To synthesize a robust controller, it is possible to apply the interval approach to the conditions (6).

Proposition 5. The robust controller with the setting $\overline{k}^*$ provides the quasi-maximal degree of robust stability $\eta^*_{\max}$ in interval system with a characteristic polynomial (7) if the following conditions are fulfilled

$$\begin{cases} \dfrac{\overline{a_{i-1}(\overline{k})a_{i+2}(\overline{k})}}{\left(\underline{a_i(\overline{k})} - \overline{a_{i+1}(\overline{k})}(n-i-1)\eta\right)\left(\underline{a_{i+1}(\overline{k})} - \overline{a_{i+2}(\overline{k})}(n-i-2)\eta\right)} = \lambda^*, \\ \qquad i = \overline{1,\, n-2},\;\; z = 0 \text{ at } \underline{a_{i+1}};\, z = 1 \text{ at } \overline{a_{i+1}}; \\ \dfrac{\overline{a_{j-1}(\overline{k})a_{j+2}(\overline{k})}}{\left(\underline{a_j(\overline{k})} - \overline{a_{j+1}(\overline{k})}(n-j-1)\eta\right)\left(\underline{a_{j+1}(\overline{k})} - \overline{a_{j+2}(\overline{k})}(n-j-2)\eta\right)} < \lambda^*, \\ \qquad j = \overline{1,\, n-2},\;\; j \neq i,\;\; z{=}0 \text{ at } \underline{a_{j+1}};\, z = 1 \text{ at } \overline{a_{j+1}}; \\ \underline{a_l(\overline{k})} - \overline{a_{l+1}(\overline{k})}(n-l-1)\eta \geq 0, \qquad l = \overline{1,\; n-1}; \\ \underline{a_0(\overline{k})} - \overline{a_1(\overline{k})}\eta + \dfrac{2\underline{a_2(\overline{k})}\eta^2}{3} \geq 0. \end{cases} \tag{13}$$

Proof: It is similar to the proof of Proposition 1. We should only take into account that it is necessary to consider the larger system of equations determined by the number of possible combinations of coefficients extremes $a_{i+1}(\overline{k})$ and $a_{j+1}(\overline{k})$. ■

By doing so, the robust controller parametric synthesis that provides the quasi-maximal degree of the interval system robust stability anticipates the consideration of the eight vertices of a system parametric polyhedron, in contrast to the four vertices from Kharitonov's Theorem that are necessary only for robust stability analysis.

The suggested method of maximizing the system robust stability degree in its parametric polyhedron vertices is tested when choosing the settings of linear P, PI, or PID - controllers of interval control systems.

5 A Practical Example

We are given the transfer function of an open-loop system with the unity feedback

$$W_{OL}(s) = \frac{k_1 s + k_0}{s} \cdot \frac{K}{c_3 s^3 + c_2 s^2 + c_1 s + c_0},$$

where K is a fixed transmission factor of the object under control, k_0 and k_1 are tunable controller parameters, and $\boldsymbol{c}_0$, $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, and $\boldsymbol{c}_3$ are such interval parameters of controlled object that $\boldsymbol{c}_0 \in [0.1, 0.7]$, $\boldsymbol{c}_1 \in [1.1, 1.5]$, $\boldsymbol{c}_2 \in [0.45, 0.49]$, and $\boldsymbol{c}_3 \in [0.07, 0.09]$. We wish to determine PI-controller parameters that provide the quasi-maximal robust stability degree for the system, assuming that its accuracy is fixed.

First, we write the characteristic equation of the system in the form

$$\boldsymbol{a}_4 \cdot s^4 + \boldsymbol{a}_3 \cdot s^3 + \boldsymbol{a}_2 \cdot s^2 + \boldsymbol{a}_1 \cdot s + \boldsymbol{a}_0 = 0,$$

where $\boldsymbol{a}_0(k_0) = k_0 \cdot K$, $\boldsymbol{a}_1(k_1) = \boldsymbol{c}_0 + k_1 \cdot K$, $\boldsymbol{a}_2 = \boldsymbol{c}_1$, $\boldsymbol{a}_3 = \boldsymbol{c}_2$, and $\boldsymbol{a}_4 = \boldsymbol{c}_4$.

We suggest determining the controller coefficient k_0 on the basis of requirements to the system accuracy. According to the error classification in [46], k_0 can be determined uniquely from the coefficients of the interval characteristic polynomial of the system and a required gain-bandwidth [44]. We set the value of the gain-bandwidth as $D = 3$. Then

$$D = \frac{k_0 K}{\underline{c_0}} \Rightarrow k_0 = \frac{D \underline{c_0}}{K} = \frac{3 \cdot 0.1}{1} = 0.3.$$

Next, we have to form expressions for the stability indices $\lambda_i(\overline{k}, \eta)$ according to (13):

$$\lambda_{10}(k_1, \eta) = \frac{0.3 \cdot 0.49}{\big((0.1 + k_1) - 1.1 \cdot (4 - 1 - 1)\eta\big)\big(1.1 - 0.49 \cdot (4 - 1 - 2)\eta\big)} = \frac{0.147}{\big((0.1 + k_1) - 2.2\eta\big)\big(1.1 - 0.49\eta\big)},$$

$$\lambda_{20}(k_1, \eta) = \frac{0.49 \cdot (k_1 + 0.7)}{\big(1.1 - 0.45 \cdot (4 - 2 - 1)\eta\big)\big(0.45 - (4 - 2 - 2)\eta\big)} = \frac{0.49 \cdot (k_1 + 1)}{0.45 \cdot (1.1 - 0.45\eta)}.$$

Yet, according to Proposition 3, it is necessary to consider a pair of equations, because values of the coefficients $\overline{\underline{a_{i+1}}}$ and $\overline{\underline{a_{j+1}}}$ in the denominator can take both maximum and minimum values from their intervals.

$$\lambda_{11}(k_1, \eta) = \frac{0.3 \cdot 0.49}{\big((0.1 + k_1) - 1.5 \cdot (4 - 1 - 1)\eta\big)\big(1.5 - 0.49 \cdot (4 - 1 - 2)\eta\big)} = \frac{0.147}{\big((0.1 + k_1) - 3\eta\big)\big(1.5 - 0.49\eta\big)},$$

$$\lambda_{21}(k_1, \eta) = \frac{0.09 \cdot (k_1 + 0.7)}{\big(1.1 - 0.49 \cdot (4 - 2 - 1)\eta\big)\big(0.49 - (4 - 2 - 2)\,\eta\big)} = \frac{0.09 \cdot (k_1 + 1)}{0.49 \cdot (1.1 - 0.49\eta)}.$$

According to Proposition 5, it is necessary to construct systems of inequalities and find a simultaneous solution for two systems provided that $\lambda_{10}(k_1, \eta) = \lambda^*$. Then the same should be done for two systems subject to $\lambda_{20}(k_1, \eta) = \lambda^*$.

In the first case, when $\lambda_{10}\,(k_1, \eta) = \lambda^*$, we propose to express the desired controller parameter $k_1(\eta)$ from the equality $\lambda_{10}\,(k_1, \eta) = \lambda^*$.

$$k_{10}(\eta) = -\frac{1.078\eta^2 - 2.469\eta - 0.206}{1.1 - 0.49\eta}$$

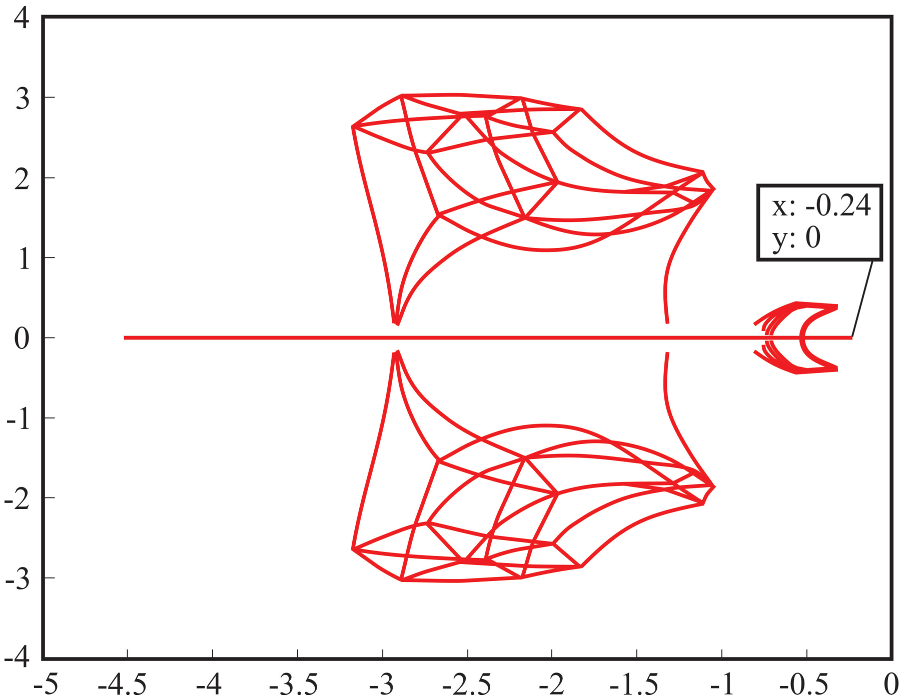

Figure 2: Localization area of system poles

when $\lambda_{11}(k_1, \eta) = \lambda^*$ we have

$$k_{11}(\eta) = -\frac{1.470\eta^2 - 4.549\eta - 0.166}{1.5 - 0.49\eta}$$

As a result of our transformations, the only unknown variable in the inequality system becomes η. It is worth noting that $\eta = \eta^*_{\max}$.

We determine $\eta^*_{\max} = 0.22$ by solving the simultaneously obtained inequality systems. This solution corresponds to the condition $\lambda_{11}(k_1, \eta^*_{\max}) = \lambda^*$. Therefore, the value $\eta^*_{\max}$ should be put into the expression that corresponds to the condition on the controller parameter k_1, yielding $k_1 = 0.787$.

Next, using the criterion $\lambda_{20}(k_1, \eta) = \lambda^*$, we examine the corresponding inequality systems, and they turn out to have no solutions.

To check the accuracy of the settings of the proportional plus reset controller, we write the interval characteristic polynomial of the system taking into account values of the controller coefficients:

$$[0.07, 0.09] \cdot s^4 + [0.45, 0.49] \cdot s^3 + [1.1, 1.5] \cdot s^2 + [0.887, 1.487] \cdot s + 0.3 = 0.$$

Figure 2 displays a pole localization area for the system having the above interval characteristic polynomial. Also, Figure 2 shows that the actual robust stability degree $\alpha_r = 0.24$ for the system is even higher than the quasi-maximal robust stability degree $\eta^*_{\max} = 0.22$ obtained during the parametric synthesis of the robust controller.

6 Conclusion

The results presented in the article are based on representation of an interval control system as a multimode system (operation modes are determined by the parametric polyhedron vertices of an automatic control system). One more basis of the work is the application of the stability degree of stationary systems to interval automatic control systems. More precisely, the quality index of the interval automatic control system is the robust stability degree that determines the maximum transient period of the automatic control system in its worst mode.

The main result of our work is a new approach to the linear controller parametric synthesis for multimode systems that maximizes the robust stability degree and decreases of the worst transient time. Another important result is the determination of the parametric polyhedron vertex set for interval automatic control systems, where it is worth synthesizing the robust controller according to the maximum stability degree criterion or analyzing the robust stability degree.

Acknowledgements

The authors wish to express sincere thanks to Prof. S.P. Shary and reviewers of the paper for their attention to the work and revealing its defects.

The work has been supported by the Ministry of Education and Science of the Russian Federation under the Federal Program "Scientific, Research and Educational Personnel of Innovative Russia for 2009–2013".

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