and VCO. The output of the phase detector goes to a low-pass filter to remove higher-frequency components. The output of the low-pass filter, which is the output of the system, changes the frequency of the VCO such that the difference between the input and the VCO is reduced. This process continues until the input and the VCO have the same frequency. At this point, we say that the PLL is **locked** or is in **phase lock**. There is a difference between the phases of the input and VCO that produces an error voltage that keeps the PLL locked.

If the input to the PLL shown in Fig. 5.36 is an FM signal, then the output is the demodulated signal. In addition to FM demodulation, the PLL has numerous other practical applications. For some of these, the output of the VCO is the output of the system. Under this circumstance, the connection between the VCO and the phase detector is the feedback element.

Although the concept of the PLL was originated by British scientists in 1932, its utilization for most applications was economically unfeasible until its appearance as an integrated-circuit (IC) package in the 1970s.

5.5 The Laplace Transform

In Section 5.4 we saw that the transfer function $\mathbf{H}(s)$ for a linear circuit or system is a ratio of polynomials in the complex-frequency variable s [see Equations (5.9) and (5.10)]. Such a circuit or system characterization was obtained by considering forced responses to damped sinusoids. Although this may seem like a very restrictive characterization, we will now see how we can generalize it so that complete responses to arbitrary inputs can be obtained without the need for writing and solving differential equations.

In order to accomplish this, the transformation of circuits from the time domain to the frequency domain will not be done by using phasors (for the sinusoidal case or the damped-sinusoidal case), but instead will be done with the use of a more sophisticated mathematical transformation—the Laplace transform.

Definition of the Laplace Transform

Given a function of time $f(t)$, we define its **Laplace transform**[4]—designated either $\mathcal{L}[f(t)]$ or $\mathbf{F}(s)$—to be

$$\boxed{\mathcal{L}[f(t)] = \mathbf{F}(s) = \int_0^\infty f(t)e^{-st}\,dt} \qquad (5.12)$$

[4]Named for the French mathematician Marquis Pierre Simon de Laplace (1749–1827).

where $s = \sigma + j\omega$. Because the lower limit of the integral in the definition of the Laplace transform is zero, the Laplace transform treats a function $f(t)$ as if $f(t) = 0$ for $t < 0$ s. Consequently, we will consider only such functions.

Example 5.11

Recall the unit step function $u(t)$ defined by

$$u(t) = \begin{cases} 0 \text{ for} & t < 0 \text{ s} \\ 1 \text{ for} & t \geq 0 \text{ s} \end{cases}$$

Then the Laplace transform of a unit step function is

$$\mathcal{L}[u(t)] = \int_0^\infty u(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = -\frac{1}{s}(0-1) = \frac{1}{s}$$

For the decaying exponential $e^{-at}u(t)$, where $a > 0$, we have that

$$\mathcal{L}[e^{-at}u(t)] = \int_0^\infty e^{-at}u(t)e^{-st}\,dt = \int_0^\infty e^{-(s+a)t}\,dt$$

$$= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^\infty = -\frac{1}{s+a}(0-1) = \frac{1}{s+a}$$

Drill Exercise 5.11

Determine the Laplace transform of the function $f(t) = (1 - e^{-at})u(t)$.

ANSWER $a/(s^2 + as)$

Properties of the Laplace Transform

Although the Laplace transform of a function $f(t)$ may be obtained by using the defining integral Eq. 5.12, sometimes it is more convenient to use some of the properties of this transform. We will now derive some of the more useful properties.

If $f(t) = f_1(t) + f_2(t)$, then

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty [f_1(t) + f_2(t)]e^{-st}\,dt$$

$$= \int_0^\infty [f_1(t)e^{-st} + f_2(t)e^{-st}]\,dt = \int_0^\infty f_1(t)e^{-st}\,dt + \int_0^\infty f_2(t)e^{-st}\,dt$$

$$= \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)] \tag{5.13}$$

In other words, the Laplace transform of a sum of functions is equal to the sum of the transforms of the individual functions.

If K is a constant, then

$$\mathcal{L}[Kf(t)] = \int_0^\infty Kf(t)e^{-st}\,dt = K\int_0^\infty f(t)e^{-st}\,dt = K\mathcal{L}[f(t)] \tag{5.14}$$

In other words, if a function is scaled by a constant, then the Laplace transform of the function is scaled by the same constant.

The properties of the Laplace transform given by Eq. 5.13 and Eq. 5.14 collectively are referred to as the **linearity property** of the Laplace transform. We also say that the Laplace transform is a **linear transformation**.

Example 5.12

Let us find the Laplace transform of $f(t) = 3(1 - e^{-2t})u(t)$.

Since we can express this function in the form

$$f(t) = (3 - 3e^{-2t})u(t) = 3u(t) - 3e^{-2t}u(t) = f_1(t) + f_2(t)$$

where $f_1(t) = 3u(t)$ and $f_2(t) = -3e^{-2t}u(t)$. Then by the linearity property of the Laplace transform

$$\mathcal{L}[f(t)] = \mathcal{L}[3u(t)] + \mathcal{L}[-3e^{-2t}u(t)] = 3\mathcal{L}[u(t)] - 3\mathcal{L}[e^{-2t}u(t)]$$

$$= 3\left(\frac{1}{s}\right) - 3\left(\frac{1}{s+2}\right) = \frac{3(s+2) - 3s}{s(s+2)} = \frac{6}{s(s+2)}$$

Drill Exercise 5.12

Find the Laplace transform of $e^{-a(t-1)}u(t)$.

ANSWER $e^{a}/(s + a)$

Now recall Euler's formula (see p. 193)

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{5.15}$$

Replacing θ by $-\theta$, we get

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta)$$

Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, then

$$e^{-j\theta} = \cos\theta - j\sin\theta \tag{5.16}$$

By adding Eq. 5.15 and Eq. 5.16, we get

$$\cos\theta = (e^{j\theta} + e^{-j\theta})/2 \tag{5.17}$$

and by subtracting Eq. 5.16 from Eq. 5.15, we obtain

$$\sin\theta = (e^{j\theta} - e^{-j\theta})/j2 \tag{5.18}$$

As we will now see, Eq. 5.17 and Eq. 5.18 can be used find the Laplace transforms of sinusoids.

Example 5.13

For the case that $f(t) = \cos\beta t\, u(t)$, then

$$\mathscr{L}[\cos\beta t\, u(t)] = \mathscr{L}\left[\frac{1}{2}(e^{j\beta t} + e^{-j\beta t})u(t)\right] = \frac{1}{2}\mathscr{L}[e^{j\beta t}u(t) + e^{-j\beta t}u(t)]$$

$$= \frac{1}{2}\mathscr{L}[e^{j\beta t}u(t)] + \frac{1}{2}\mathscr{L}[e^{-j\beta t}u(t)]$$

Just as $\mathscr{L}[e^{-at}u(t)] = 1/(s + a)$, so too $\mathscr{L}[e^{-j\alpha t}u(t)] = 1/(s + ja)$. Therefore,

$$\mathscr{L}[\cos \beta t\, u(t)] = \frac{1}{2}\left(\frac{1}{s - j\beta}\right) + \frac{1}{2}\left(\frac{1}{s + j\beta}\right) = \frac{1}{2}\,\frac{s + j\beta + s - j\beta}{(s - j\beta)(s + j\beta)}$$

$$= \frac{s}{s^2 + \beta^2}$$

Drill Exercise 5.13

Find the Laplace transform of $f(t) = \sin \beta t\, u(t)$.

ANSWER $\beta/(s^2 + \beta^2)$

Differentiation

Another property of the Laplace transform involves the derivative of a function. Specifically,

$$\mathscr{L}\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt} e^{-st}\, dt = \int_0^\infty e^{-st} \frac{df(t)}{dt}\, dt$$

We may employ the formula for integration by parts:

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

By selecting $u = e^{-st}$ and $dv = [df(t)/dt]dt = df(t)$, we have that

$$du = -se^{-st}\, dt \qquad \text{and} \qquad v = f(t)$$

Thus,

$$\mathscr{L}\left[\frac{df(t)}{dt}\right] = e^{-at} f(t)\Big|_0^\infty - \int_0^\infty f(t)[-se^{-st}]\, dt = 0 - f(0) + s\int_0^\infty f(t)e^{-st}\, dt$$

$$= -f(0) + s\mathscr{L}[f(t)] = -f(0) + s\mathbf{F}(s)$$

This result is known as the **differentiation property** of the Laplace transform.

Example 5.14

Let us find the Laplace transform of sin $\beta t\, u(t)$ by using the differentiation property. Since

$$\frac{d[\sin \beta t\, u(t)]}{dt} = \beta \cos \beta t\, u(t)$$

then taking the Laplace transform of both sides of this expression, we obtain

$$\mathscr{L}\left(\frac{d[\sin \beta t\, u(t)]}{dt}\right) = \beta\mathscr{L}[\cos \beta t\, u(t)]$$

Therefore,

$$-\sin 0\, u(0) + s\mathscr{L}[\sin \beta t\, u(t)] = \beta \frac{s}{s^2 + \beta^2}$$

and hence

$$\mathscr{L}[\sin \beta t\, u(t)] = \frac{\beta}{s^2 + \beta^2}$$

Drill Exercise 5.14

Define the unit ramp function $r(t)$ by $r(t) = t\,u(t)$. Use the fact that $dr(t)/dt = u(t)$ to determine the Laplace transform of $r(t)$.

ANSWER $1/s^2$

In order to find $\mathscr{L}[d^2f(t)/dt^2]$, we can apply the differentiation property twice to obtain

$$\mathscr{L}\left[\frac{d^2f(t)}{dt^2}\right] = -\frac{df(0)}{dt} - sf(0) + s^2\mathscr{L}[f(t)]$$

where $df(0)/dt$ is the derivative of $f(t)$ evaluated at $t = 0$ s.

Formulas for the Laplace transforms of higher-order derivatives can be obtained by repeated applications of the differentiation property.

Complex Translation

Another important property of the Laplace transform is obtained as follows: Suppose that $\mathbf{F}(s) = \mathcal{L}[f(t)]$. Then

$$\mathcal{L}[e^{-at}f(t)] = \int_0^\infty e^{-at}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s+a)t}\,dt = \mathbf{F}(s+a)$$

In other words, $\mathcal{L}[e^{-at}f(t)]$ can be obtained from $\mathcal{L}[f(t)]$—simply replace each s in $\mathcal{L}[f(t)]$ by $s + a$. This result is referred to as the **complex-translation property** of the Laplace transform.

Example 5.15

Since $\mathcal{L}[u(t)] = \mathbf{F}(s) = 1/s$, then by the complex-translation property

$$\mathcal{L}[e^{-at}u(t)] = \mathbf{F}(s+a) = \frac{1}{s+a}$$

Furthermore,

$$\mathcal{L}[\cos\beta t\, u(t)] = \frac{s}{s^2+\beta^2} \quad\Rightarrow\quad \mathcal{L}[e^{-\alpha t}\cos\beta t\, u(t)] = \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$$

Also,

$$\mathcal{L}[\sin\beta t\, u(t)] = \frac{\beta}{s^2+\beta^2} \quad\Rightarrow\quad \mathcal{L}[e^{-\alpha t}\sin\beta t\, u(t)] = \frac{\beta}{(s+\alpha)^2+\beta^2}$$

Drill Exercise 5.15

Use the complex-translation property and the result from Drill Exercise 5.14 to determine the Laplace transform of $te^{-at}u(t)$.

ANSWER $1/(s+a)^2$

Complex Differentiation

Given that $\mathcal{L}[f(t)] = \mathbf{F}(s)$, then

$$\frac{d\mathbf{F}(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty f(t)\frac{d[e^{-st}]}{ds}\,dt$$

$$= \int_0^\infty f(t)[-te^{-st}]\,dt = -\int_0^\infty tf(t)e^{-st}\,dt = -\mathcal{L}[tf(t)]$$

Thus,

$$\mathcal{L}[tf(t)] = -\frac{d\mathbf{F}(s)}{ds} = -\frac{d}{ds}\{\mathcal{L}[f(t)]\}$$

and this is known as the **complex-differentiation property** of the Laplace transform.

Example 5.16

Since $\mathcal{L}[e^{-at}u(t)] = 1/(s + a)$, then

$$\mathcal{L}[te^{-at}u(t)] = -\frac{d}{ds}\left(\frac{1}{s + a}\right) = \frac{1}{(s + a)^2}$$

Setting $a = 0$, we get

$$\mathcal{L}[t\,u(t)] = 1/s^2$$

Drill Exercise 5.16

Use the complex-differentiation property to determine the Laplace transform of $t^2e^{-at}u(t)$.

ANSWER $2/(s + a)^3$

A summary of some of the properties of the Laplace transform, as well as the transforms of some important functions, is given in Table 5.1.

Table 5.1 Table of Laplace Transforms

$f(t)$	Property	$\mathbf{F}(s)$
$f(t)$	Definition	$\int_0^\infty f(t)e^{-st}\,dt$
$f_1(t) + f_2(t)$	Linearity	$\mathbf{F}_1(s) + \mathbf{F}_2(s)$
$Kf(t)$	Linearity	$K\mathbf{F}(s)$
$\dfrac{df(t)}{dt}$	Differentiation	$s\mathbf{F}(s) - f(0)$
$\dfrac{d^2f(t)}{dt^2}$	Differentiation	$s^2\mathbf{F}(s) - sf(0) - \dfrac{df(0)}{dt}$
$\int_0^t f(t)\,dt$	Integration	$\dfrac{1}{s}\mathbf{F}(s)$
$tf(t)$	Complex differentiation	$-\dfrac{d\mathbf{F}(s)}{ds}$
$e^{-at}f(t)$	Complex translation	$\mathbf{F}(s + a)$
$f(t - a)\,u(t - a)$	Real translation	$e^{-as}\mathbf{F}(s)$
$u(t)$		$\dfrac{1}{s}$
$e^{-at}u(t)$		$\dfrac{1}{s + a}$
$\cos \beta t\, u(t)$		$\dfrac{s}{s^2 + \beta^2}$
$\sin \beta t\, u(t)$		$\dfrac{\beta}{s^2 + \beta^2}$
$e^{-\alpha t} \cos \beta t\, u(t)$		$\dfrac{s + \alpha}{(s + \alpha)^2 + \beta^2}$
$e^{-\alpha t} \sin \beta t\, u(t)$		$\dfrac{\beta}{(s + \alpha)^2 + \beta^2}$
$t\,u(t)$		$\dfrac{1}{s^2}$
$te^{-at}u(t)$		$\dfrac{1}{(s + a)^2}$

5.6 Inverse Laplace Transforms

Given a function $\mathbf{F}(s)$, in order to determine the function $f(t)$ such that $\mathcal{L}[f(t)] = \mathbf{F}(s)$, we must take the **inverse Laplace transform** of $\mathbf{F}(s)$—which is denoted as $\mathcal{L}^{-1}[\mathbf{F}(s)] = f(t)$. Although this can be done with the use of a mathematical formula, such an approach requires the use of advanced mathematics. Therefore, we will take inverse Laplace transforms instead by inspecting the table of Laplace transforms (see Table 5.1) to see what function $f(t)$ has the Laplace transform $\mathbf{F}(s)$. If $\mathbf{F}(s)$ is not in the table, we will decompose it into functions that are in the table or are readily obtainable by using the properties of Laplace transforms.

Just as the Laplace transform is a linear transformation, so too the inverse Laplace transform is a linear transformation. Specifically, if $\mathbf{F}(s) = \mathbf{F}_1(s) + \mathbf{F}_2(s)$, then

$$\mathcal{L}^{-1}[\mathbf{F}(s)] = \mathcal{L}^{-1}[\mathbf{F}_1(s) + \mathbf{F}_2(s)] = \mathcal{L}^{-1}[\mathbf{F}_1(s)] + \mathcal{L}^{-1}[\mathbf{F}_2(s)]$$

Furthermore,

$$\mathcal{L}^{-1}[K\mathbf{F}(s)] = K\,\mathcal{L}^{-1}[\mathbf{F}(s)]$$

Example 5.17

Let us determine $f_1(t)$ given that its Laplace transform is

$$\mathcal{L}[f_1(t)] = \mathbf{F}_1(s) = \frac{14s + 23}{s^2 + 4s + 5}$$

By completing the square for the denominator, this function can be written in the form

$$\mathbf{F}_1(s) = \frac{14s + 23}{(s + 2)^2 + 1^2} = \frac{14(s + 2) - 5}{(s + 2)^2 + 1^2}$$

$$= \frac{14(s + 2)}{(s + 2)^2 + 1^2} - \frac{5(1)}{(s + 2)^2 + 1^2}$$

Since the inverse Laplace transform of a sum is equal to the sum of the individual inverse Laplace transforms, by using Table 5.1, we get

$$f_1(t) = 14e^{-2t}\cos t\, u(t) - 5e^{-2t}\sin t\, u(t) = e^{-2t}(14\cos t - 5\sin t)u(t)$$

Next let us find $f_2(t)$ given that its Laplace transform is

$$\mathscr{L}[f_2(t)] = \mathbf{F}_2(s) = \frac{14s + 23}{s^2 + 5s + 4}$$

In this case the denominator cannot be put into the form $(s + \alpha)^2 + \beta^2$, where α and β are real numbers. However, it is true that—and we will see why shortly—

$$\mathbf{F}_2(s) = \frac{14s + 23}{s^2 + 5s + 4} = \frac{3}{s + 1} + \frac{11}{s + 4} \tag{5.19}$$

Therefore, from Table 5.1, the inverse Laplace transform of $\mathbf{F}_2(s)$ is

$$f_2(t) = 3e^{-t}u(t) + 11e^{-4t}u(t) = (3e^{-t} + 11e^{-4t})u(t)$$

Drill Exercise 5.17

Determine the inverse Laplace transform of

$$\frac{2s + 26}{s^2 + 6s + 25}$$

ANSWER $e^{-3t}(2 \cos 4t + 5 \sin 4t)u(t)$

Partial-Fraction Expansions

In Example 5.17, Eq. 5.19 indicates that $\mathbf{F}_2(s)$ can be expressed as the sum of two functions, each of which is in the form of a function in Table 5.1. There is a systematic method for decomposing a function into a sum of simpler functions—such a decomposition is called a **partial-fraction expansion**, and we now describe a procedure for obtaining it.

Suppose we are given a function $\mathbf{F}(s) = \mathbf{N}(s)/\mathbf{D}(s)$, where $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are polynomials in s with real coefficients. If the roots of $\mathbf{D}(s)$ are $s_1, s_2, s_3, \ldots s_n$, we can write $\mathbf{F}(s)$ in the form

$$\mathbf{F}(s) = \frac{\mathbf{N}(s)}{\mathbf{D}(s)} = \frac{\mathbf{N}(s)}{(s - s_1)(s - s_2)(s - s_3) \cdots (s - s_n)} \tag{5.20}$$

If the degree of $\mathbf{D}(s)$ is greater than the degree of $\mathbf{N}(s)$, and if the roots of $\mathbf{D}(s)$—that is, the poles of $\mathbf{F}(s)$—are distinct, then we may write

$$\mathbf{F}(s) = \frac{K_1}{s - s_1} + \frac{K_2}{s - s_2} + \frac{K_3}{s - s_3} + \cdots + \frac{K_n}{s - s_n} \tag{5.21}$$

To find K_1, first multiply both sides of Eq. 5.21 by $s - s_1$. This yields

$$(s - s_1)\mathbf{F}(s) = K_1 + \frac{K_2(s - s_1)}{s - s_2} + \frac{K_3(s - s_1)}{s - s_3} + \cdots + \frac{K_n(s - s_1)}{s - s_n}$$

If we set $s = s_1$, then this equation becomes

$$(s - s_1)\mathbf{F}(s)\Big|_{s=s_1} = K_1$$

Similarly,

$$(s - s_2)\mathbf{F}(s)\Big|_{s=s_2} = K_2$$

et cetera.

Example 5.18

Let us take the partial-fraction expansion of

$$\mathbf{F}(s) = \frac{2s^2 + 11s + 19}{(s + 1)(s + 2)(s + 3)} = \frac{K_1}{s + 1} + \frac{K_2}{s + 2} + \frac{K_3}{s + 3}$$

Multiplying this expression by $s + 1$ and then setting $s = -1$, we get

$$K_1 = \frac{2s^2 + 11s + 19}{(s + 2)(s + 3)}\Bigg|_{s=-1} = \frac{2(-1)^2 + 11(-1) + 19}{(-1 + 2)(-1 + 3)} = 5$$

Multiplying $\mathbf{F}(s)$ by $s + 2$ and then setting $s = -2$, we obtain

$$K_2 = \left.\frac{2s^2 + 11s + 19}{(s + 1)(s + 3)}\right|_{s=-2} = \frac{2(-2)^2 + 11(-2) + 19}{(-2 + 1)(-2 + 3)} = -5$$

Finally,

$$K_3 = \left.\frac{2s^2 + 11s + 19}{(s + 1)(s + 2)}\right|_{s=-3} = \frac{2(-3)^2 + 11(-3) + 19}{(-3 + 1)(-3 + 2)} = 2$$

Hence

$$\mathbf{F}(s) = \frac{5}{s + 1} - \frac{5}{s + 2} + \frac{2}{s + 3}$$

and from Table 5.1, the inverse Laplace transform of $\mathbf{F}(s)$ is

$$f(t) = 5e^{-t}u(t) - 5e^{-2t}u(t) + 2e^{-3t}u(t) = (5e^{-t} - 5e^{-2t} + 2e^{-3t})u(t)$$

Drill Exercise 5.18

Determine the inverse Laplace transform of

$$\frac{600}{s^3 + 40s^2 + 300s}$$

ANSWER $(2 - 3e^{-10t} + e^{-30t})u(t)$

Multiple and Complex Poles

In Example 5.18, the function $\mathbf{F}(s)$ has poles that are both distinct and real. In general, however, the poles of $\mathbf{F}(s)$ can be nondistinct or complex—occurring in conjugate pairs. Although there are formal procedures for taking partial-fraction expansions for these cases as well, we will study only circuits or systems whose complexity is such that we may deal with these cases by utilizing the technique for distinct, real poles.

and from Table 5.1, we get that

$$f_2(t) = -3u(t) + 8e^{-t}\cos 2t\, u(t) + 14e^{-t}\sin 2t\, u(t)$$

Drill Exercise 5.20

Determine the inverse Laplace transform of

$$\frac{14s - 50}{s^3 + 6s^2 + 25s}$$

ANSWER $-2u(t) + 2e^{-3t}\cos 4t\, u(t) + 5e^{-3t}\sin 4t\, u(t)$

5.7 Application of the Laplace Transform

One very important application of the Laplace transform is to the solution of differential equations.

Example 5.21

Suppose that we wish to solve the linear, second-order differential equation

$$\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t) = 4e^{-3t}u(t) \tag{5.22}$$

subject to the initial conditions $x(0) = 2$ and $dx(0)/dt = -1$. Taking the Laplace transform of this differential equation, if we let $\mathbf{X}(s) = \mathcal{L}[x(t)]$, then

$$\left[s^2\mathbf{X}(s) - sx(0) - \frac{dx(0)}{dt}\right] + 3[s\mathbf{X}(s) - x(0)] + 2\mathbf{X}(s) = \frac{4}{s+3}$$

from which

$$s^2\mathbf{X}(s) - 2s + 1 + 3s\mathbf{X}(s) - 6 + 2\mathbf{X}(s) = \frac{4}{s+3}$$

Therefore,

$$(s^2 + 3s + 2)\mathbf{X}(s) = \frac{4}{s+3} + 2s + 5 = \frac{2s^2 + 11s + 19}{s+3}$$

and hence

$$\mathbf{X}(s) = \frac{2s^2 + 11s + 19}{(s+1)(s+2)(s+3)}$$

By Example 5.18, the solution to Eq. 5.22 subject to the given initial conditions is

$$x(t) = (5e^{-t} - 5e^{-2t} + 2e^{-3t})u(t)$$

Drill Exercise 5.21

Find the solution of the differential equation

$$\frac{d^2x(t)}{dt^2} + 8\frac{dx(t)}{dt} + 12x(t) = 0$$

subject to the initial conditions $x(0) = 0$ and $dx(0)/dt = -12$.

ANSWER $(3e^{-6t} - 3e^{-2t})u(t)$ (See Drill Exercise 3.12 on p. 155.)

Application to Circuit Analysis

Of course, we can write the differential equation or equations that characterize a circuit and then use Laplace transforms to solve such equations. However, we can avoid writing differential equations if we employ Laplace-transform (frequency-domain) concepts directly. Let's see how this is done.

For a resistor having a value of R ohms, we know that

$$v(t) = Ri(t)$$

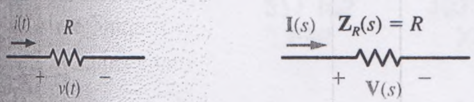
Taking the Laplace transform of both sides of this equation results in

$$\mathbf{V}(s) = R\mathbf{I}(s)$$

where $\mathbf{V}(s) = \mathscr{L}[v(t)]$ and $\mathbf{I}(s) = \mathscr{L}[i(t)]$. Defining the impedance $\mathbf{Z}_R(s)$ of the resistor to be the ratio of voltage transform to current transform, we get

$$\boxed{\mathbf{Z}_R(s) = \frac{\mathbf{V}(s)}{\mathbf{I}(s)} = R}$$

The circuit symbols of a resistor in the time domain and in the frequency (transform) domain are shown in Fig. 5.37*a* and *b*, respectively.



(a) Time domain (b) Frequency domain

Fig. 5.37 Resistor circuit symbols.

For an inductor having a value of L henries,

$$v(t) = L\frac{di(t)}{dt}$$

Taking the Laplace transform, we get

$$\mathbf{V}(s) = L[s\mathbf{I}(s) - i(0)] = Ls\mathbf{I}(s) - Li(0) \tag{5.23}$$

from which

$$\mathbf{I}(s) = \frac{1}{Ls}\mathbf{V}(s) + \frac{i(0)}{s} \tag{5.24}$$

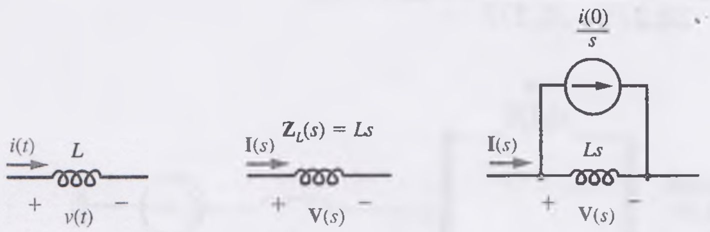
For the case of zero initial conditions, $i(0) = 0$ A, and thus

$$\mathbf{V}(s) = Ls\mathbf{I}(s)$$

We then define the impedance $\mathbf{Z}_L(s)$ of the inductor to be the ratio of voltage transform to current transform when the initial current is zero. Thus

$$\boxed{\mathbf{Z}_L(s) = \frac{\mathbf{V}(s)}{\mathbf{I}(s)} = Ls}$$

The time-domain circuit symbol for an inductor is shown in Fig. 5.38*a*, whereas Fig. 5.38*b* shows the circuit symbol in the frequency domain when the initial current is zero. For the case that the initial current is not necessarily zero, the parallel connection shown in Fig. 5.38*c* models the frequency-domain description given by Eq. 5.24. Note that if $i(0) = 0$ A, then the model in Fig. 5.38*c* is equivalent to the one in Fig. 5.38*b*.



(a) Time domain

(b) Frequency domain—zero initial current

(c) Frequency domain—nonzero initial current

Fig. 5.38 Inductor circuit symbols.

For a capacitor having a value of C farads,

$$i(t) = C\frac{dv(t)}{dt}$$

Taking the Laplace transform, we get

$$\mathbf{I}(s) = C[s\mathbf{V}(s) - v(0)] = Cs\mathbf{V}(s) - Cv(0) \tag{5.25}$$

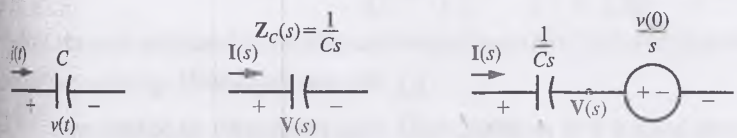
from which

$$\mathbf{V}(s) = \frac{1}{Cs}\mathbf{I}(s) + \frac{v(0)}{s} \tag{5.26}$$

The impedance $\mathbf{Z}_C(s)$ of the capacitor (for zero initial voltage) is

$$\boxed{\mathbf{Z}_C(s) = \frac{\mathbf{V}(s)}{\mathbf{I}(s)} = \frac{1}{Cs}}$$

The circuit symbols for a capacitor in the time domain and the frequency domain, with zero and nonzero initial conditions, are shown in Fig. 5.39.



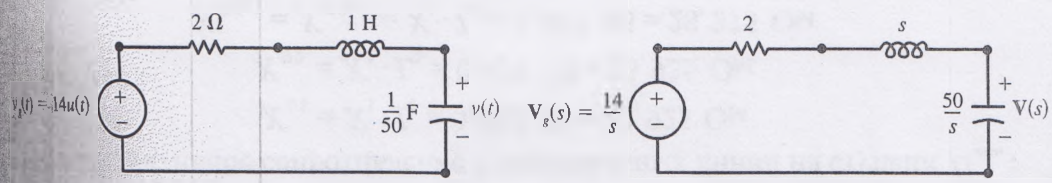
(a) Time domain (b) Frequency domain—zero initial voltage (c) Frequency domain—nonzero initial voltage

Fig. 5.39 Capacitor circuit symbols.

Since we have the same expressions for the impedances of resistors, inductors, and capacitors as we had for the damped-sinusoidal case, we can use the same circuit analysis techniques—the difference being that we use the Laplace transforms of time functions rather than their phasor representations.

Example 5.22

Suppose that the series *RLC* circuit shown in Fig. 5.40*a* has zero initial conditions. Let us find the step response $v(t)$.



(a) Time domain (b) Frequency domain

Fig. 5.40 A series *RLC* circuit (*a*) in the time domain, and (*b*) in the frequency domain.

Figure 5.40*b* shows the circuit in the frequency domain. Of course this circuit can be analyzed by using either mesh analysis or nodal analysis. However, even more simply, by voltage division we can write

$$\mathbf{V}(s) = \frac{50/s}{50/s + s + 2}\mathbf{V}_g(s) = \frac{50}{s^2 + 2s + 50}\left(\frac{14}{s}\right)$$

or

7. If there are no cancellations of common poles and zeros, the poles of a network function indicate the form of the natural response.

8. A linear system can be simulated with integrators, adders, and scalers (i.e., with an analog computer).

9. Systems are often represented by block diagrams.

10. Feedback can improve system performance and can be used for purposes of control.

11. The Laplace transform is a linear transformation that can be used to solve linear differential equations or analyze linear circuits.

12. The inverse Laplace transform can be found by using a table of transforms and various transform properties, as well as partial-fraction expansions.

13. The impedance of an R-ohm resistor is R, of an L-henry inductor is Ls, and of a C-farad capacitor is $1/Cs$.

14. An inductor (or a capacitor) with a nonzero initial condition can be modeled by an independent source and an inductor (or capacitor) with a zero initial condition.

15. Circuit analysis using Laplace transforms results in complete (both forced and natural) responses.

Problems

5.1 Sketch the phase response $\text{ang}(\mathbf{V}_2/\mathbf{V}_1)$ versus ω for the high-pass filter given in Fig. 5.5 on p. 269.

5.2 For the circuit given in Fig. 5.5 on p. 269, replace the capacitor C with an inductor L, and sketch the phase response $\text{ang}(\mathbf{V}_2/\mathbf{V}_1)$ versus ω for the resulting low-pass filter.

5.3 Sketch the amplitude response of $\mathbf{V}_2/\mathbf{V}_1$ for the op-amp circuit shown in Fig. P5.3. Determine the half-power frequency. What type of filter is this circuit?

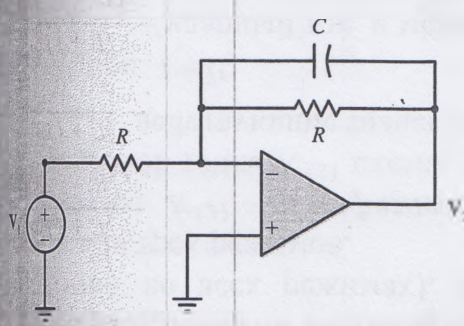


Fig. P5.3

5.4 Show that for the circuit given in Fig. P5.4 the voltage transfer function is

$$\mathbf{H}(j\omega) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_2(1 + j\omega R_1 C_1)}{(R_1 + R_2) + j\omega R_1 R_2 (C_1 + C_2)}$$

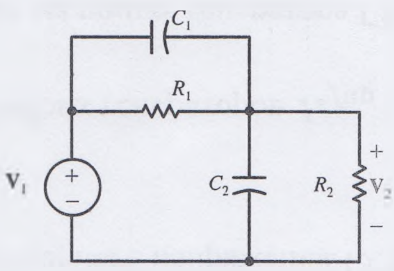


Fig. P5.4

5.5 For the circuit shown in Fig. P5.4, suppose that $R_1 = R_2 = R$ and $C_1 = C_2 = C$. Sketch the amplitude response and the phase response of $\mathbf{V}_2/\mathbf{V}_1$.

5.6 For the circuit shown in Fig. P5.4, suppose that $R_1 = R_2 = R$, $C_1 = C$ and $C_2 = 0$ F. Sketch the amplitude response of $\mathbf{V}_2/\mathbf{V}_1$. What is the half-power frequency?

5.7 For the circuit shown in Fig. P5.4, suppose that $R_1 = R_2 = R$, $C_1 = 0$ F and $C_2 = C$. Sketch the amplitude response of $\mathbf{V}_2/\mathbf{V}_1$. What is the half-power frequency?

5.8 For the op-amp circuit shown in Fig. P5.8, sketch the amplitude response of $\mathbf{V}_2/\mathbf{V}_1$, indicating the half-power frequency. What type of filter is this circuit?

If only the forced response of an ac circuit is of interest,[5] Laplace transform techniques can be avoided by taking the simpler phasor-analysis approach discussed in Chapter 4. In particular, for this circuit see Example 4.4 on p. 204.

Drill Exercise 5.24

For the circuit given in Fig. 5.42, replace the $\frac{1}{2}$-F capacitor with a $\frac{1}{2}$-H inductor. Use Laplace transform techniques to determine $i(t)$ and $v_o(t)$.

ANSWER $[3e^{-2t} - 3\cos 2t + 3\sin 2t]u(t)$ A,
$[-3e^{-2t} + 3\cos 2t + 3\sin 2t]u(t)$ V

Application to Linear Systems

Suppose that the input to a linear system has a Laplace transform of $\mathbf{X}(s)$, and suppose that the Laplace transform of the output, given that all the initial conditions are zero, is $\mathbf{Y}(s)$. Then the **transfer function** $\mathbf{H}(s)$ of the system is defined to be

$$\boxed{\mathbf{H}(s) = \frac{\mathbf{Y}(s)}{\mathbf{X}(s)}}$$

If the transfer function of a linear system is known, then when the input is specified, the output transform can be determined from the equation

$$\boxed{\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{X}(s)}$$

Taking the inverse Laplace transform of $\mathbf{Y}(s)$ yields the corresponding output $y(t)$ in the time domain.

Example 5.25

Consider a low-pass filter with a voltage transfer function of

$$\mathbf{H}(s) = \frac{\mathbf{V}_2(s)}{\mathbf{V}_1(s)} = \frac{3}{s + 3}$$

[5] The natural response is negligible after just a few time constants—in this example the time constant is $\frac{1}{3}$ s.

Taking the inverse Laplace transform of **I** gives us

$$i(t) = -3e^{-2t}u(t) + 3\cos 2t\, u(t) + 3\sin 2t\, u(t)\ \text{A}$$

But note that this response is a complete response—the natural response is $i_n(t) = -3e^{-2t}u(t)$ A and the forced response is $i_f(t) = 3\cos 2t\, u(t) + 3\sin 2t\, u(t) = 3\sqrt{2}\cos(2t - 45°)\, u(t)$ A.

To find $v_o(t)$, we first calculate $\mathbf{V}_o$ from

$$\mathbf{V}_o = \mathbf{Z}_C\mathbf{I} = \frac{2}{s}\,\frac{12s}{(s+2)(s^2+4)} = \frac{24}{(s+2)(s^2+4)}$$

$$= \frac{K_2}{s+2} + \mathbf{F}_2(s) \tag{5.30}$$

where

$$K_2 = \left.\frac{24}{s^2+4}\right|_{s=-2} = 3$$

and

$$\mathbf{F}_2(s) = \mathbf{V}_o - \frac{K_2}{s+2} = \frac{24}{(s+2)(s^2+4)} - \frac{3}{s+2)}$$

$$= \frac{-3(s^2-4)}{(s+2)(s^2+4)} = \frac{-3(s-2)}{s^2+4}$$

Substituting the above values of K_2 and $\mathbf{F}_2(s)$ into Eq. 5.30 yields

$$\mathbf{V}_o = \frac{3}{s+2} + \frac{-3s}{s^2+4} + \frac{6}{s^2+4} = \frac{3}{s+2} - \frac{3s}{s^2+2^2} + \frac{3(2)}{s^2+2^2}$$

Taking the inverse Laplace transform of $\mathbf{V}_o$ gives us

$$v_o(t) = 3e^{-2t}u(t) - 3\cos 2t\, u(t) + 3\sin 2t\, u(t)\ \text{V}$$

This response again is a complete response—the natural response is $v_n(t) = 3e^{-2t}u(t)$ V and the forced response is $v_f(t) = -3\cos 2t\, u(t) + 3\sin 2t\, u(t) = 3\sqrt{2}\cos(2t - 135°)\, u(t)$ V.

Example 5.24

Let us determine the responses $i(t)$ and $v_o(t)$ to a sinusoidal excitation (which begins at time $t = 0$ s) for the circuit shown in Fig. 5.42a. The frequency-domain representation of this circuit is shown in Fig. 5.42b.

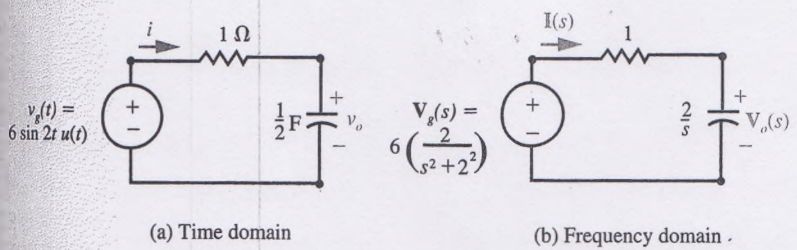


Fig. 5.42 A series *RC* circuit (a) in the time domain, and (b) in the frequency domain.

By KVL,

$$\mathbf{V}_g = 1\mathbf{I} + \frac{2}{s}\mathbf{I} = \frac{s+2}{s}\mathbf{I} \quad \Rightarrow \quad \mathbf{I} = \frac{s}{s+2}\mathbf{V}_g = \frac{12s}{(s+2)(s^2+4)}$$

However, we can express $\mathbf{I}$ in the form

$$\mathbf{I} = \frac{12s}{(s+2)(s^2+4)} = \frac{K_1}{s+2} + \mathbf{F}_1(s) \tag{5.29}$$

where

$$K_1 = \left.\frac{12s}{s^2+4}\right|_{s=-2} = -3$$

From Eq. 5.29,

$$\mathbf{F}_1(s) = \mathbf{I} - \frac{K_1}{s+2} = \frac{12s}{(s+2)(s^2+4)} - \frac{-3}{s+2} = \frac{3s+6}{s^2+4}$$

Substituting the above values of K_1 and $\mathbf{F}_1(s)$ into Eq. 5.29 yields

$$\mathbf{I} = \frac{-3}{s+2} + \frac{3s}{s^2+4} + \frac{6}{s^2+4} = \frac{-3}{s+2} + \frac{3s}{s^2+2^2} + \frac{3(2)}{s^2+2^2}$$

Let us find the output voltage $v_2(t)$ for the case that the input voltage is $v_1(t) = 2e^{-3t}u(t)$ V.

Since

$$\mathbf{V}_1(s) = \mathcal{L}[2e^{-3t}u(t)] = \frac{2}{s+3}$$

then

$$\mathbf{V}_2(s) = \mathbf{H}(s)\mathbf{V}_1(s) = \left(\frac{3}{s+3}\right)\left(\frac{2}{s+3}\right) = \frac{6}{(s+3)^2},$$

and

$$v_2(t) = 6te^{-3t}u(t) \text{ V}$$

In this case, the forced and natural responses combine into the single term $6te^{-3t}u(t)$. This is a consequence of exciting the system at its pole; that is, the pole of $\mathbf{V}_1(s)$ is the same as the pole of $\mathbf{H}(s)$.

Drill Exercise 5.25

A high-pass filter has the voltage transfer function $\mathbf{H}(s) = 3s/(s+3)$. Find the output voltage $v_2(t)$ when the input voltage is $v_1(t) = 2e^{-3t}u(t)$ V.

ANSWER $(6e^{-3t} - 18te^{-3t})u(t)$ V

SUMMARY

1. The frequency response of a circuit consists of the amplitude response and the phase response.

2. The frequencies at which the amplitude response drops to $1/\sqrt{2}$ of its maximum value are the half-power frequencies.

3. The frequencies at which an impedance (or admittance) is purely real are the resonance frequencies of the impedance (or admittance).

4. The bandwidth and the quality factor are measures of the sharpness of an amplitude response.

5. An impedance can be expressed as a function of the complex frequency s, as can other parameters like the voltage transfer function.

6. The poles and zeros of a ratio of polynomials in s can be depicted in the s plane with a pole-zero plot.

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{s/2} + \frac{\mathbf{V}_2 - 2/s}{8/s} = \frac{1}{s} \quad \Rightarrow$$

$$-16\mathbf{V}_1 - (s^2 + 16)\mathbf{V}_2 = 2s + 8 \tag{5.28}$$

Using Cramer's rule to solve simultaneous Eq. 5.27 and Eq. 5.28 results in

$$\mathbf{V}^2 = \frac{\begin{vmatrix} s + 10 & -5 \\ -16 & 2s + 8 \end{vmatrix}}{\begin{vmatrix} s + 10 & -5 \\ -16 & s^2 + 16 \end{vmatrix}}$$

$$= \frac{(s + 10)(2s + 8) - 80}{(s + 10)(s^2 + 16) - 160} = \frac{4}{s + 2} + \frac{-2}{s + 8}$$

Since $\mathbf{V} = \mathbf{V}_2$, then

$$v(t) = 4e^{-t}u(t) - 2e^{-8t}u(t) = (4e^{-2t} - 2e^{-8t})u(t) \text{ V}$$

and this is an example of an overdamped natural response (see Example 3.12 on p. 153).

Drill Exercise 5.23

Determine $v(t)$ for the series RLC circuit shown in Fig. 5.41 when $R = 4\ \Omega$, $L = 1$ H, and $C = \frac{1}{4}$ F subject to the initial conditions $v(0) = 2$ V and $i(0) = 1$ A. (See Example 3.15 on p. 163.)

ANSWER $(2e^{-2t} + 8te^{-2t})u(t)$ V

Circuits with Sinusoidal Sources

In Chapter 4 we saw how to find forced sinusoidal responses by using phasors. We will now see an example of how to determine the complete response to a sinusoidal source (which is zero for $t < 0$ s) by using Laplace-transform techniques.

Nonzero Initial Conditions

Having analyzed a circuit with zero initial conditions, let us now consider the case of a circuit with nonzero initial conditions. In the following examples, we will simplify the notation by replacing $\mathbf{V}_1(s)$, $\mathbf{V}_2(s)$, and $\mathbf{V}(s)$ with $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}$, respectively.

Example 5.23

Suppose that we wish to find $v(t)$ for the series *RLC* circuit shown in Fig. 5.41*a* subject to the initial conditions $v(0) = 2$ V and $i(0) = 1$ A.

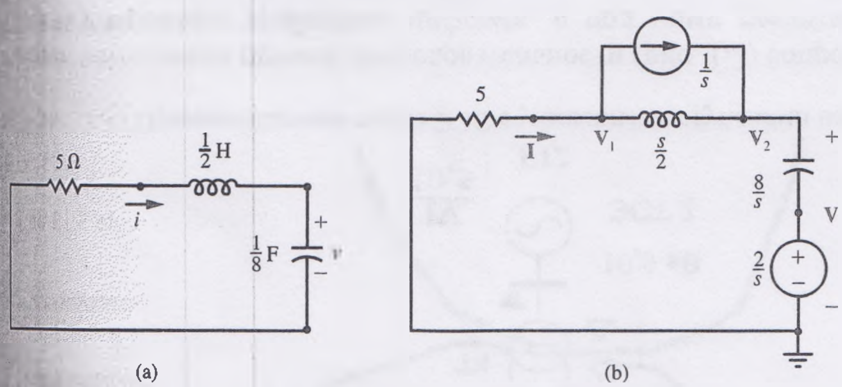


Fig. 5.41 A series *RLC* circuit (*a*) in the time domain, and (*b*) in the frequency domain.

In the frequency domain, the circuit is as shown in Fig. 5.41*b*. Here, the initial voltage across the capacitor is modeled with an independent voltage source, and the initial current through the inductor is modeled with an independent current source. Note that the voltage (transform) $\mathbf{V}$ that is to be determined is the voltage across the series combination of the $\frac{1}{8}$-F capacitor and a voltage source having a value of $v(0)/s = 2/s$. Although two node voltages $\mathbf{V}_1$ and $\mathbf{V}_2$ are indicated in Fig. 5.41*b*, note that $\mathbf{V}_2 = \mathbf{V}$.

Summing the currents directed out of the node labeled $\mathbf{V}_1$, by KCL, we get

$$\frac{\mathbf{V}_1}{5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{s/2} + \frac{1}{s} = 0 \quad \Rightarrow \quad (s + 10)\mathbf{V}_1 - 10\mathbf{V}_2 = -5 \qquad (5.27)$$

By KCL at the node labeled $\mathbf{V}_2$, we obtain

$$\mathbf{V}(s) = \frac{700}{s(s^2 + 2s + 50)} = \frac{K_0}{s} + \mathbf{F}(s)$$

where

$$K_0 = s\mathbf{V}(s)\Big|_{s=0} = \frac{700}{s^2 + 2s + 50}\Big|_{s=0} = 14$$

Thus

$$\mathbf{F}(s) = \mathbf{V}(s) - \frac{K_0}{s} = \frac{700}{s(s^2 + 2s + 50)} - \frac{14}{s} = \frac{-14s - 28}{s^2 + 2s + 50}$$

Hence

$$\mathbf{V}(s) = \frac{14}{s} - \frac{14s + 28}{s^2 + 2s + 50} = \frac{14}{s} - \frac{14s + 28}{(s + 1)^2 + 7^2}$$

$$= \frac{14}{s} - \frac{14(s + 1)}{(s + 1)^2 + 7^2} - \frac{(2)(7)}{(s + 1)^2 + 7^2}$$

Therefore, from Table 5.1, the inverse Laplace transform of $\mathbf{V}(s)$ is

$$v(t) = 14u(t) - 14e^{-t}\cos 7t\, u(t) - 2e^{-t}\sin 7t\, u(t)$$

$$= [14 - e^{-t}(14\cos 7t + 2\sin 4t)]u(t) \text{ V}$$

and this is the complete response—i.e., forced and natural responses—for an underdamped series *RLC* circuit. (See Example 3.17 on p. 170.)

Drill Exercise 5.22

For the series *RLC* circuit shown in Fig. 5.40, change the value of the resistor to 16 Ω, change the value of the inductor to 2 H, and find the step response $v(t)$ when $v_g(t) = 6u(t)$ V. (See Drill Exercise 3.17 on p. 172.)

ANSWER $[6 - e^{-4t}(6\cos 3t + 8\sin 3t)]u(t)$ V