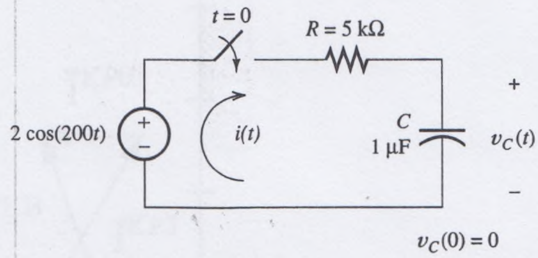


Figure 4.18 The circuit for Exercise 4.7.

EXERCISE 4.7 Repeat Example 4.4 if the source voltage is changed to $2\cos(200t)$ and the initial voltage on the capacitor is $v_C(0) = 0$. The circuit with these changes is shown in Figure 4.18.
Ans. $i(t) = -200\sin(200t) + 200\cos(200t) + 200e^{-t/RC}$ μA, in which $\tau = RC = 5$ ms.

EXERCISE 4.8 Solve for the current in the circuit shown in Figure 4.19 after the switch closes. [*Hint:* Try a particular solution of the form $i_p(t) = Ae^{-t}$.]
Ans. $i(t) = 20e^{-t} - 15e^{-t/2}$ μA.

4.5 SECOND-ORDER CIRCUITS

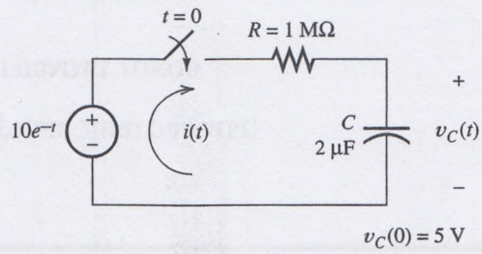
In this section we consider circuits that contain two energy-storage elements. In particular, we look at circuits that have an inductance and a capacitance either in series or in parallel.

4.5.1 Differential Equation

To derive the general form of the equations that we encounter in circuits with two energy-storage elements, consider the series circuit shown in Figure 4.20a. Writing a KVL equation, we have

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)dt + v_C(0) = v_s(t) \tag{4.57}$$

Figure 4.19 The circuit for Exercise 4.8.



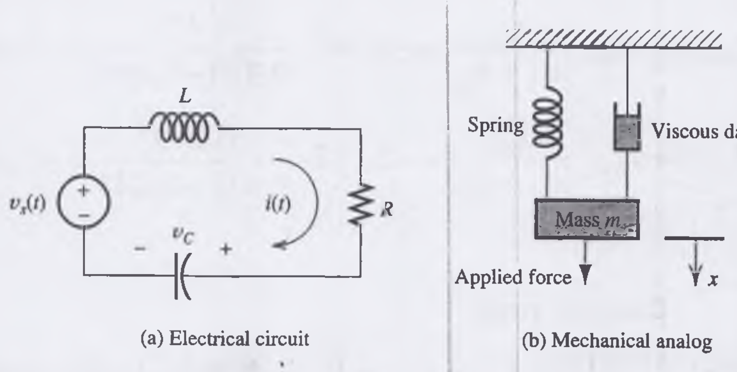


Figure 4.20 The series RLC circuit and its mechanical analog.

Taking the derivative with respect to time, we have

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv_s(t)}{dt} \tag{4.58}$$

Dividing through by L, we obtain

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dv_s(t)}{dt} \tag{4.59}$$

Now we define the **damping coefficient** as

$$\alpha = \frac{R}{2L} \tag{4.60}$$

and the **undamped resonant frequency** as

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{4.61}$$

The **forcing function** is

$$f(t) = \frac{1}{L}\frac{dv_s(t)}{dt} \tag{4.62}$$

Using these definitions, Equation 4.59 can be written as

$$\frac{d^2i(t)}{dt^2} + 2\alpha\frac{di(t)}{dt} + \omega_0^2 i(t) = f(t) \tag{4.63}$$

This is a linear second-order differential equation with constant coefficients. Thus we refer to circuits having two energy-storage elements as second-order circuits. (An exception occurs if we can combine the energy-storage elements in series or parallel. For example, if we have two capacitors in parallel, we can combine them into a single equivalent capacitance, and then we would have a first-order circuit.)

4.5.2 Mechanical Analog

The mechanical analog of the circuit is shown in Figure 4.20b. The displacement x of the mass is analogous to electrical charge, the velocity dx/dt is analogous to current, and force is analogous to voltage. The mass plays the role of the inductance, the spring plays the role of the capacitance, and the damper plays the role of the resistance. The equation of motion for the mechanical system can be put into the form of Equation 4.63.

Based on an intuitive consideration of Figure 4.20, we can anticipate that the sudden application of a constant force (dc voltage) can result in a displacement (current) that either approaches steady-state conditions asymptotically or oscillates before settling to the steady-state value. The type of behavior depends on the relative values of the mass, spring constant, and damping coefficient.

4.5.3 Solution of the Second-Order Equation

We will see that the circuit equations for currents and voltages in circuits having two energy-storage elements can always be put into the form of Equation 4.63. Thus let us consider the solution of

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t) \tag{4.64}$$

where we have used $x(t)$ for the variable, which could represent either a current or a voltage.

Here again, the general solution $x(t)$ to this equation consists of two parts: a particular solution $x_p(t)$ plus the complementary solution $x_c(t)$.

$$x(t) = x_p(t) + x_c(t) \tag{4.65}$$

Particular Solution. The particular solution is any expression $x_p(t)$ that satisfies the differential equation

$$\frac{d^2x_p(t)}{dt^2} + 2\alpha\frac{dx_p(t)}{dt} + \omega_0^2 x_p(t) = f(t) \tag{4.66}$$

The particular solution is also called the **forced response**.

We will be concerned primarily with either constant (dc) or sinusoidal (ac) forcing functions. For dc sources we can find the particular solution directly from the circuit by replacing the inductances by short circuits, replacing the capacitances by open circuits, and solving. This technique was discussed in Section 4.2. In Chapter 5 we will learn efficient methods for finding the forced response due to sinusoidal sources.

Complementary Solution. The complementary solution $x_c(t)$ is found by solving the homogeneous equation, which is obtained by substituting 0 for the forcing function $f(t)$. Thus the homogeneous equation is

$$\frac{d^2x_c(t)}{dt^2} + 2\alpha\frac{dx_c(t)}{dt} + \omega_0^2 x_c(t) = 0 \tag{4.67}$$

In finding the solution to the homogeneous equation, we start by substituting the trial solution $x_c(t) = Ke^{st}$. This yields

$$s^2Ke^{st} + 2\alpha sKe^{st} + \omega_0^2 Ke^{st} = 0 \tag{4.68}$$

Factoring, we obtain

$$(s^2 + 2\alpha s + \omega_0^2)Ke^{st} = 0 \tag{4.69}$$

Since we want to find a solution Ke^{st} that is nonzero, we must have

$$s^2 + 2\alpha s + \omega_0^2 = 0 \tag{4.70}$$

This is called the **characteristic equation**.

The **damping ratio** is defined as

$$\zeta = \frac{\alpha}{\omega_0} \tag{4.71}$$

The form of the complementary solution depends on the value of the damping ratio. The roots of the characteristic equation are given by

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \tag{4.72}$$

and

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \tag{4.73}$$

We have three cases depending on the value of the damping ratio ζ compared to unity.

1. *Overdamped case* ($\zeta > 1$). If $\zeta > 1$ (or equivalently, if $\alpha > \omega_0$), the roots of the characteristic equation are real and distinct. Then the complementary solution is

$$x_c(t) = K_1e^{s_1t} + K_2e^{s_2t} \tag{4.74}$$

In this case we say that the circuit is **overdamped**.

2. *Critically damped case* ($\zeta = 1$). If $\zeta = 1$ (or equivalently, if $\alpha = \omega_0$), the roots are real and equal. Then the complementary solution is

$$x_c(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t} \tag{4.75}$$

In this case we say that the circuit is **critically damped**.

3. *Underdamped case* ($\zeta < 1$). Finally, if $\zeta < 1$ (or equivalently, if $\alpha < \omega_0$), the roots are complex. (By the term *complex*, we mean that the roots involve the imaginary number $\sqrt{-1}$.) In other words, the roots are of the form

$$s_1 = -\alpha + j\omega_n \quad \text{and} \quad s_2 = -\alpha - j\omega_n$$

in which $j = \sqrt{-1}$ and the **natural frequency** is given by

$$\omega_n = \sqrt{\omega_0^2 - \alpha^2} \tag{4.76}$$

(In electrical engineering, we use j rather than i to stand for the imaginary number $\sqrt{-1}$, because we use i for current.)

For complex roots, the complementary solution is of the form

$$x_c(t) = K_1 e^{-\alpha t}\cos(\omega_n t) + K_2 e^{-\alpha t}\sin(\omega_n t) \tag{4.77}$$

In this case we say that the circuit is **underdamped**.

Example 4.5

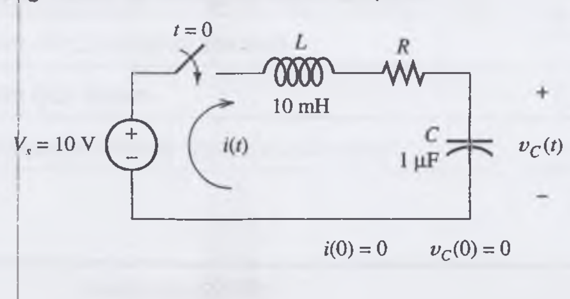
A dc source is connected to a series RLC circuit by a switch that closes at $t = 0$ as shown in Figure 4.21. The initial conditions are $i(0) = 0$ and $v_C(0) = 0$. Write the differential equation for $v_C(t)$. Solve for $v_C(t)$ if $R = 300$, 200, and 100 Ω.

Solution

First we can write an expression for the current in terms of the voltage across the capacitance.

$$i(t) = C\frac{dv_C(t)}{dt} \tag{4.78}$$

Figure 4.21 The circuit for Example 4.5.



Then we write a KVL equation for the circuit:

$$L\frac{di(t)}{dt} + Ri(t) + v_C(t) = V_s \tag{4.79}$$

Using Equation 4.78 to substitute for $i(t)$, we have

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = V_s \tag{4.80}$$

Dividing through by LC, we have

$$\frac{d^2v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{V_s}{LC} \tag{4.81}$$

As expected, the differential equation for $v_C(t)$ has the same form as Equation 4.64.

Next, we find the particular solution. Since we have a dc source, we can find this part of the solution by replacing the inductance by a short circuit and the capacitance by an open circuit. This is shown in Figure 4.22. Then the current is zero, the drop across the resistance is zero, and the voltage across the capacitance (open circuit) is equal to the dc source voltage. Thus the particular solution is

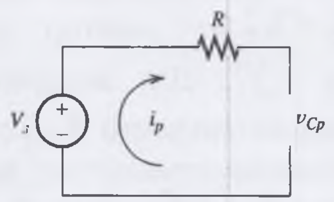
$$v_{Cp}(t) = V_s = 10 \text{ V} \tag{4.82}$$

(It can be verified that this is a particular solution by substituting it into Equation 4.81.) Notice that in this circuit the particular solution for $v_C(t)$ is the same for all three values of resistance.

Next we find the homogeneous solution and general solution for each value of R. For all three cases we have

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10^4 \tag{4.83}$$

Figure 4.22 The equivalent circuit for Figure 4.21 under steady-state conditions. The inductor has been replaced by a short circuit and the capacitor by an open circuit.



Case I ($R = 300\ \Omega$)

In this case we have

$$\alpha = \frac{R}{2L} = 1.5 \times 10^4 \tag{4.84}$$

The damping ratio is $\zeta = \alpha/\omega_0 = 1.5$. Because we have $\zeta > 1$, this is the overdamped case. The roots of the characteristic equation are given by Equations 4.72 and 4.73. Substituting values we have

$$\begin{aligned} s_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ &= -1.5 \times 10^4 - \sqrt{(1.5 \times 10^4)^2 - (10^4)^2} \\ &= -2.618 \times 10^4 \end{aligned}$$

and

$$\begin{aligned} s_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} \\ &= -0.3820 \times 10^4 \end{aligned}$$

The homogeneous solution has the form of Equation 4.74. Adding the particular solution given by Equation 4.82 to the homogeneous solution, we obtain the general solution

$$v_C(t) = 10 + K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{4.85}$$

Now we must find values of K_1 and K_2 so the solution matches the known initial conditions in the circuit. It was given that the initial voltage on the capacitance is zero.

$$v_C(0) = 0$$

Evaluating Equation 4.85 at $t = 0$, we obtain

$$10 + K_1 + K_2 = 0 \tag{4.86}$$

Furthermore, the initial current was given as $i(0) = 0$. Since the current through the capacitance is given by

$$i(t) = C \frac{dv_C(t)}{dt}$$

we conclude that

$$\frac{dv_C(0)}{dt} = 0$$

Taking the derivative of Equation 4.85 and evaluating at $t = 0$, we have

$$-s_1 K_1 - s_2 K_2 = 0 \tag{4.87}$$

Now we can solve Equations 4.86 and 4.87 for the values of K_1 and K_2. The results are $K_1 = 1.708$ and $K_2 = -11.708$. Substituting these values into Equation 4.85, we have the solution.

$$v_C(t) = 10 + 1.708e^{s_1 t} - 11.708e^{s_2 t}$$

Plots of each of the terms of this equation and the complete solution are shown in Figure 4.23.

Case II ($R = 200\ \Omega$)

In this case we have

$$\alpha = \frac{R}{2L} = 10^4 \tag{4.88}$$

Because $\zeta = \alpha/\omega_0 = 1$, this is the critically damped case. The roots of the characteristic equation are given by Equations 4.72 and 4.73. Substituting values, we have

$$s_1 = s_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha = -10^4$$

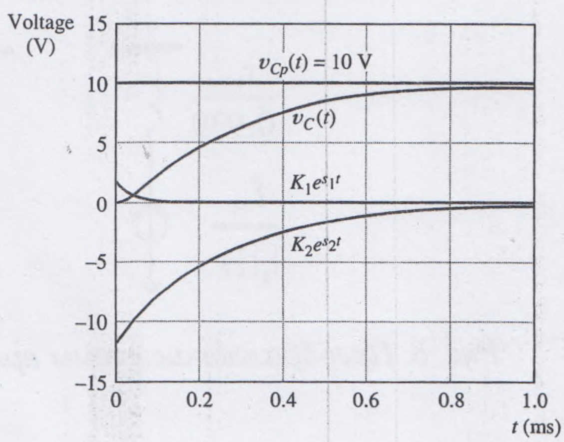
The homogeneous solution has the form of Equation 4.75. Adding the particular solution (Equation 4.82) to the homogeneous solution, we have

$$v_C(t) = 10 + K_1 e^{s_1 t} + K_2 t e^{s_1 t} \tag{4.89}$$

As in case I, the initial conditions require $v_C(0) = 0$ and $dv_C(0)/dt = 0$. Thus substituting $t = 0$ into Equation 4.89, we have

$$10 + K_1 = 0 \tag{4.90}$$

Figure 4.23 Solution for $R = 300\ \Omega$.



Differentiating Equation 4.89 and substituting $t = 0$ yields

$$s_1 K_1 + K_2 = 0 \tag{4.91}$$

Solving Equations 4.90 and 4.91 yields $K_1 = -10$ and $K_2 = -10^5$. Thus the solution is

$$v_C(t) = 10 - 10e^{s_1 t} - 10^5 t e^{s_1 t} \tag{4.92}$$

Plots of each of the terms of this equation and the complete solution are shown in Figure 4.24.

Case III ($R = 100\ \Omega$)

For this value of resistance we have

$$\alpha = \frac{R}{2L} = 5000 \tag{4.93}$$

Because $\zeta = \alpha/\omega_0 = 0.5$, this is the underdamped case. Using Equation 4.76, we compute the natural frequency.

$$\omega_n = \sqrt{\omega_0^2 - \alpha^2} = 8660 \tag{4.94}$$

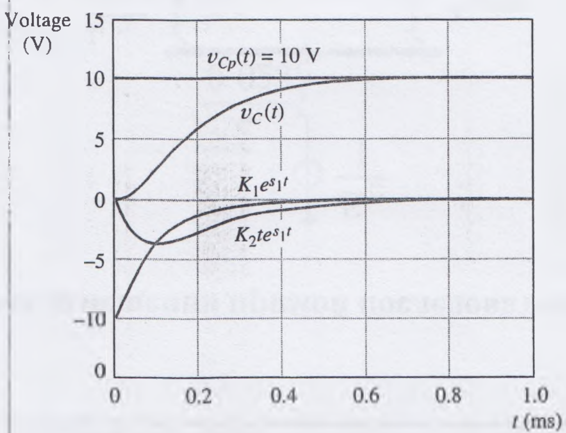
The homogeneous solution has the form of Equation 4.77. Adding the particular solution found earlier to the homogeneous solution, we obtain the general solution.

$$v_C(t) = 10 + K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t) \tag{4.95}$$

As in the previous cases, the initial conditions are $v_C(0) = 0$ and $dv_C(0)/dt = 0$. Evaluating Equation 4.95 at $t = 0$, we obtain

$$10 + K_1 = 0 \tag{4.96}$$

Figure 4.24 Solution for $R = 200\ \Omega$.



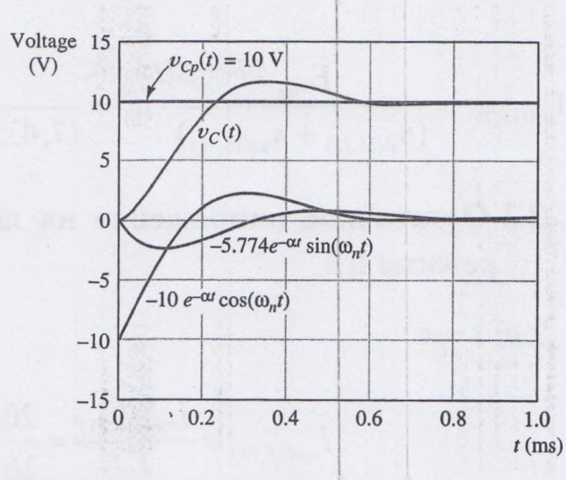


Figure 4.25 Solution for $R = 100\ \Omega$.

Differentiating Equation 4.95 and evaluating at $t = 0$, we have

$$-\alpha K_1 + \omega_n K_2 = 0 \tag{4.97}$$

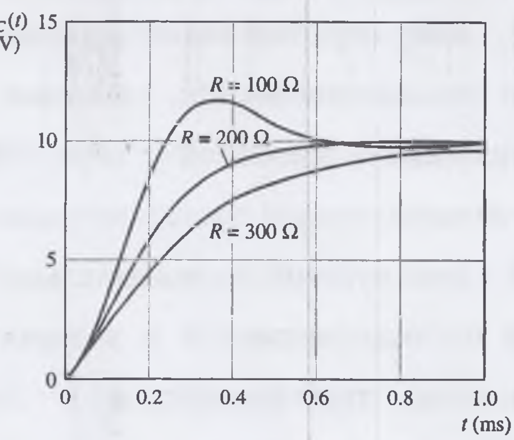
Solving Equations 4.96 and 4.97, we obtain $K_1 = -10$ and $K_2 = -5.774$. Thus the complete solution is

$$v_C(t) = 10 - 10e^{-\alpha t}\cos(\omega_n t) - 5.774e^{-\alpha t}\sin(\omega_n t) \tag{4.98}$$

Plots of each of the terms of this equation and the complete solution are shown in Figure 4.25.

Figure 4.26 shows the complete response for all three values of resistance. ∎

Figure 4.26 Solutions for all three resistances.



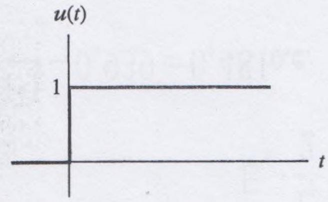


Figure 4.27 A unit step function $u(t)$. For $t < 0$, $u(t) = 0$. For $t \geq 0$, $u(t) = 1$.

4.5.4 Normalized Step Response of Second-Order Systems

When we suddenly apply a constant source to a circuit, we say that the forcing function is a **step function**. A unit step function, denoted by $u(t)$, is shown in Figure 4.27. By definition, we have

$$\begin{aligned} u(t) &= 0 \qquad t < 0 \\ &= 1 \qquad t \geq 0 \end{aligned}$$

For example, if we apply a dc voltage of A volts to a circuit by closing a switch, the applied voltage is a step function.

$$v(t) = Au(t)$$

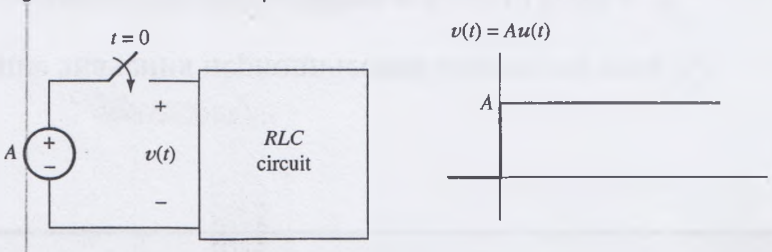
This is illustrated in Figure 4.28.

We often encounter situations, like Example 4.5, in which step forcing functions are applied to second-order systems described by a differential equation of the form

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = Au(t) \tag{4.99}$$

The differential equation is characterized by its undamped resonant frequency ω_0 and damping ratio $\zeta = \alpha/\omega_0$. [Of course, the solution for $x(t)$ also depends on the initial conditions.] Normalized solutions are shown in Figure 4.29 for the initial conditions $x(0) = 0$ and $x'(0) = 0$.

Figure 4.28 Applying a dc voltage by closing a switch results in a forcing function that is a step function.



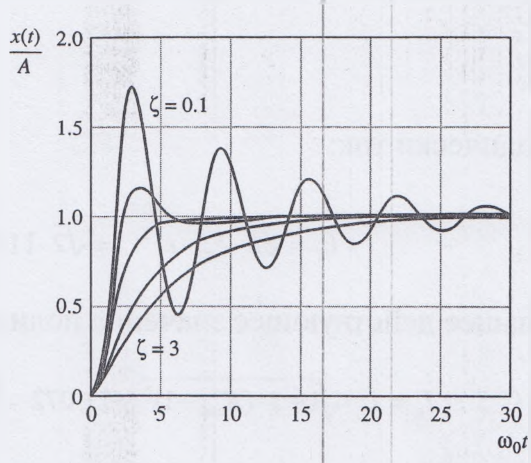


Figure 4.29 Normalized step responses for second-order systems described by Equation 4.99 with damping ratios of $\zeta = 0.1, 0.5, 1, 2,$ and 3. The initial conditions are assumed to be $x(0) = 0$ and $x'(0) = 0$.

The system response for small values of the damping ratio ζ displays **overshoot** and **ringing** before settling to the steady-state value. On the other hand, if the damping ratio is large (compared to unity), the response takes a relatively long time to closely approach the final value.

Sometimes we want to design a second-order system that quickly settles to steady state. Then we try to design for a damping ratio close to unity. For example, the control system for a robot arm could be a second-order system. When a step signal calls for the arm to move, we probably want it to achieve the final position in the minimum time without excessive overshoot and ringing.

4.5.5 Circuits with Parallel L and C

The solution of circuits having an inductance and capacitance in parallel is very similar to the series case. Consider the circuit shown in Figure 4.30a. The circuit inside the box is assumed to consist of sources and resistances. As we saw in Section 2.6, we can find a Norton equivalent circuit for any two-terminal circuit composed of resistances and sources. The equivalent circuit is shown in Figure 4.30b.

We can analyze this circuit by writing a KCL equation at the top node of Figure 4.30b. This results in

$$C\frac{dv(t)}{dt} + \frac{1}{R}v(t) + \frac{1}{L}\int_0^t v(t)\,dt + i_L(0) = i_n(t) \tag{4.100}$$

This can be converted into a pure differential equation by taking the derivative with respect to time.

$$C\frac{d^2v(t)}{dt^2} + \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) = \frac{di_n(t)}{dt} \tag{4.101}$$

Dividing through by the capacitance, we have

$$\frac{d^2v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = \frac{1}{C}\frac{di_n(t)}{dt} \tag{4.102}$$

Now if we define the damping coefficient

$$\alpha = \frac{1}{2RC} \tag{4.103}$$

the undamped resonant frequency

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{4.104}$$

and the forcing function

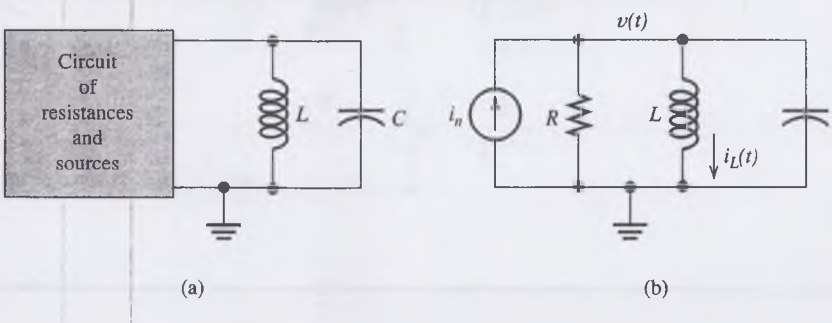
$$f(t) = \frac{1}{C}\frac{di_n(t)}{dt} \tag{4.105}$$

the differential equation can be written as

$$\frac{d^2v(t)}{dt^2} + 2\alpha\frac{dv(t)}{dt} + \omega_0^2 v(t) = f(t) \tag{4.106}$$

This equation has exactly the same form as Equation 4.64. Therefore, transient analysis of circuits with parallel LC elements is very similar to that of series LC circuits. However, notice that the equation for the damping coefficient α is different for the parallel circuit (in which $\alpha = 1/2RC$) than for the series circuit (in which $\alpha = R/2L$).

Figure 4.30 Any circuit consisting of sources, resistances, and a parallel LC combination can be reduced to the equivalent circuit shown in (b).



EXERCISE 4.9 Consider the circuit shown in Figure 4.31 with $R = 25\ \Omega$. **(a)** Compute the undamped resonant frequency, the damping coefficient, and the damping ratio. **(b)** The initial conditions are $v(0-) = 0$ and $i_L(0-) = 0$. Show that this requires that $v'(0+) = 10^6$ V/s. **(c)** Find the particular solution for $v(t)$. **(d)** Find the general solution for $v(t)$, including the numerical values of all parameters.

Ans. **(a)** $\omega_0 = 10^5$, $\alpha = 2 \times 10^5$ and $\zeta = 2$; **(b)** KVL requires that $i_C(0) = 0.1\text{ A} = Cv'(0)$, thus $v'(0) = 10^6$; **(c)** $v_p(t) = 0$; **(d)** $v(t) = 2.89(e^{-0.268\times10^5 t} - e^{-3.73\times10^5 t})$.

EXERCISE 4.10 Repeat Exercise 4.9 for $R = 50\ \Omega$.

Ans. **(a)** $\omega_0 = 10^5$, $\alpha = 10^5$ and $\zeta = 1$; **(b)** KVL requires that $i_C(0) = 0.1\text{ A} = Cv'(0)$, thus $v'(0) = 10^6$; **(c)** $v_p(t) = 0$; **(d)** $v(t) = 10^6 te^{-10^5 t}$.

EXERCISE 4.11 Repeat Exercise 4.9 for $R = 250\ \Omega$.

Ans. **(a)** $\omega_0 = 10^5$, $\alpha = 0.2 \times 10^5$ and $\zeta = 0.2$; **(b)** KVL requires that $i_C(0) = 0.1\text{ A} = Cv'(0)$, thus $v'(0) = 10^6$; **(c)** $v_p(t) = 0$; **(d)** $v(t) = 10.21e^{-2\times10^4 t}\sin(97.98 \times 10^3 t)$.

4.6 TRANSIENT ANALYSIS WITH PSPICE AND PROBE

PSpice and other programs derived from SPICE are capable of performing transient circuit analysis. For example, with transient analysis, we can easily produce plots of the step response of RLC circuits, even those that are too complex for practical manual analysis. PSpice can also readily analyze the transient response of nonlinear electronic circuits, including amplifiers and logic circuits. We consider these applications later in the book.

After running a transient analysis with PSpice, waveforms of currents and voltages can be displayed or plotted using a program called Probe. Output for Probe is requested by including the statement

```
.PROBE
```

in the PSpice program. After the analysis is completed, the Probe program is executed, and the results are observed using menu commands. Free student versions of both PSpice and Probe are available for a variety of computers.

In addition to the dc sources that we learned how to specify in Section 2.9, we can specify several types of time-varying sources, three of which are described next.

Figure 4.31 Circuit for Exercises 4.9, 4.10, and 4.11.