

## 8.1 Polarization of Dielectrics.

While describing the polarization of dielectrics (without conduction) in an external field, the dependence of a polarization vector and the macroscopic field strength is under investigation. Besides the local electric field is being found. The dipole moment of an atom depends on that field strength.

### 8.1.1 Polarization

The dipole moment of a unit volume is called the **polarization**:

$$\mathbf{P} = N\mathbf{p}_i, \quad (8.1)$$

$N$  – the number of molecules per unit volume,  $\mathbf{p}_i$  – the dipole moment of a molecule.

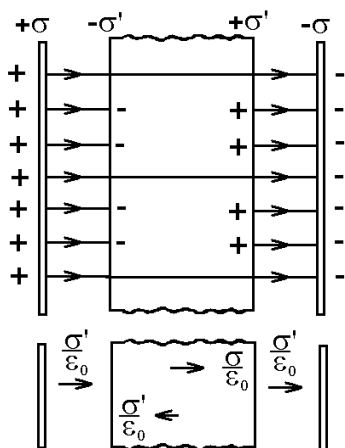


Fig.8.1. A capacitor with a dielectric

A constant voltage is applied through a capacitor (Fig.8.1). The quantity  $\sigma$  is the surface-bound density of free charges. A dielectric installed inside the capacitor would be polarized. Upon the external surfaces of dielectric, the bound charges would be generated (the surface density  $\sigma'$ ).

The macroscopic field consists of two components: the external field  $\mathbf{E}_0$  and the polarization field  $\mathbf{E}_1$ :

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1. \quad (8.2)$$

A glance at Fig.8.1 shows that the polarization field is opposite to the external one.

In an isotropy dielectric, the polarization is proportional to the effective field:

$$\mathbf{P} = \chi\epsilon_0\mathbf{E}, \quad (8.3)$$

The quantity  $\chi$  is called the **dielectric susceptibility**,  $\epsilon_0$  is called the **electric constant**.

Because of polarization, the electric field inside the dielectric differs from the external field. The bound charge density  $\sigma'$  and normal component of polarization  $P_n$  are related by:  $\sigma' = P_n$ , i.e.

$$\sigma' = \chi\epsilon_0 E, \quad (8.4)$$

$E$  is the normal component.

The electric field strength in dielectric:

$$E = \frac{(\sigma - \sigma')}{\epsilon_0} = \frac{(\sigma - \chi\epsilon_0 E)}{\epsilon_0}, \quad (8.5)$$

The Gauss theorem:

$$\oint \epsilon_0 \mathbf{E} ds = (q + q'), \quad (8.6)$$

The quantities  $q$  and  $q'$  – the free and bound charges. Having in mind that  $\oint \mathbf{P} ds = -q'$

$$\oint (\epsilon_0 \mathbf{E} + \mathbf{P}) ds = q. \quad (8.7)$$

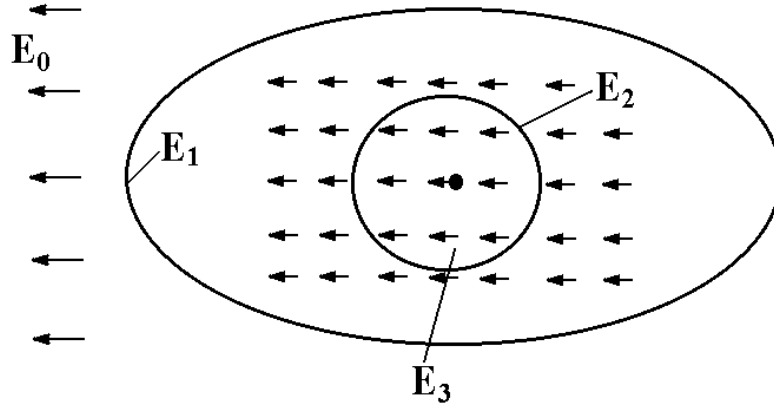
The **displacement** vector:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}. \quad (8.8)$$

The field acting upon a molecule can considerably differ from the macroscopic one (Fig.8.2). Lorenz has shown that the local field strength is represented by:

$$\mathbf{E}_l = \mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 \quad (8.9)$$

$\mathbf{E}_0$  – the field produced by external charges,  $\mathbf{E}_1$  - polarization field produced by the



$\mathbf{E}_0$  - внешнее поле;  $\mathbf{E}_1$  - деполаризирующее поле, связанное с зарядами на внешней поверхности образца;  $\mathbf{E}_2$  - поле, создаваемое зарядами на внутренней поверхности сферы;  $\mathbf{E}_3$  - поле от диполей внутри полости. Для кристаллов кубической симметрии  $\mathbf{E}_3$  равно нулю.

**Fig.8.2. The intrinsic electric field acting upon an atom**

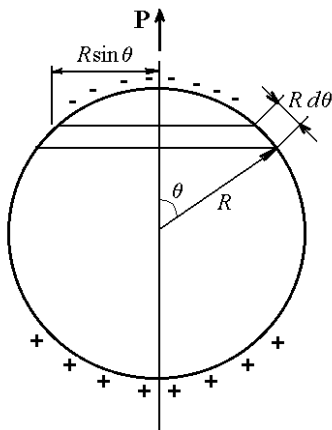
bound charges.  $\mathbf{E}_2$  – the Lorenz field (in a cavity),  $\mathbf{E}_3$  – an atomic field inside the cavity.

The sum of external and polarization field is an effective macroscopic field in dielectric:  $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$ . The field  $\mathbf{E}_3$  is the field produced by the molecules, which are located inside the sphere in its center. It was shown [2] that usually (because of the spherical symmetry)  $\mathbf{E}_3$  is zero. It takes place in gases, non-polar liquids and crystals of cubic structure.

Let us imagine that there is a spherical cavity inside of a uniform dielectric. There is a molecule inside the sphere. The size of the molecule is much less than that of the sphere. Upon the sphere surface, a bound charge is induced. Its density:

$$\sigma_s = P_n = P \cos \theta. \quad (8.10)$$

The induced charge generates the field  $\mathbf{E}_2$  (Lorenz, 1878).



**Fig.8.3. A Lorenz field**

### 8.1.2. The Lorenz Field

In the spherical coordinate system, the elementary area:

$$dS = R^2 \sin \theta d\theta d\varphi. \quad (8.11)$$

In accordance with the Coulomb law, taking into account (8.10), the electric field strength in the sphere center:

$$dE = \frac{\sigma_s dS}{4\pi\epsilon_0 R^2} = \frac{P \cos \theta \sin \theta d\theta d\varphi}{4\pi\epsilon_0}. \quad (8.12)$$

Obviously:

$$dE_2 = dE \cos \theta = \frac{P \cos^2 \theta \sin \theta d\theta d\varphi}{4\pi\epsilon_0}. \quad (8.13)$$

Integration of (8.13) leads to:

$$4\pi\epsilon_0 E_2 = P \int_0^\pi \cos^2 \theta \sin \theta d\theta \int_0^{2\pi} d\varphi = 2\pi P \left( -\frac{\cos^3 \theta}{3} \right) \Big|_0^\pi. \quad (8.14)$$

Thus:

$$\mathbf{E}_2 = \frac{\mathbf{P}}{3\epsilon_0} \quad (8.15)$$

The local field strength (SI-units):

$$\mathbf{E}_L = \mathbf{E} + \frac{\mathbf{P}}{3\epsilon_0} \quad (8.16)$$

The ratio between the local and macroscopic field is as follows:

$$\mathbf{E}_L = \frac{\epsilon + 2}{3} \mathbf{E} \quad (8.17)$$

When  $\epsilon \geq 1$ ,  $\mathbf{E}_L \approx \mathbf{E}$ . When  $\epsilon \gg 1$ ,  $\mathbf{E}_L > \mathbf{E}$ .

Thus the field acting upon a molecule differs from the macroscopic field. The local field strength consists of three components: the macroscopic field, Lorenz field, and molecular field.

## 8.2 A Klausius – Mocotty Equation

The equation connects the permittivity and polarizability of a dielectric. Polarization of dielectric is proportional the field strength:

$$\mathbf{P} = \epsilon_0 n \alpha \mathbf{E}_L. \quad (8.18)$$

The quantity  $n$  is the concentration of molecules,  $\alpha$  -polarizability.

The quantity  $\alpha$  is a sum of an electron ( $\alpha_{el}$ ) and dipole ( $\alpha_{dip}$ ) polarizability. For non-polar dielectrics:  $\alpha = \alpha_{el}$ ; for polar dielectrics:  $\alpha = \alpha_{el} + \alpha_{dip}$ .

Substituting (8.17) into (8,18) we get:

$$\mathbf{P} = \epsilon_0 n \alpha \frac{\epsilon + 2}{3} \mathbf{E}. \quad (8.19)$$

Having in mind that:

$$\mathbf{P} = \epsilon_0 (\epsilon - 1) \mathbf{E}. \quad (8.20)$$

We arrive at The Klausius – Mocotty equation:

$$\frac{\epsilon - 1}{\epsilon + 2} = \frac{n\alpha}{3}. \quad (8.21)$$

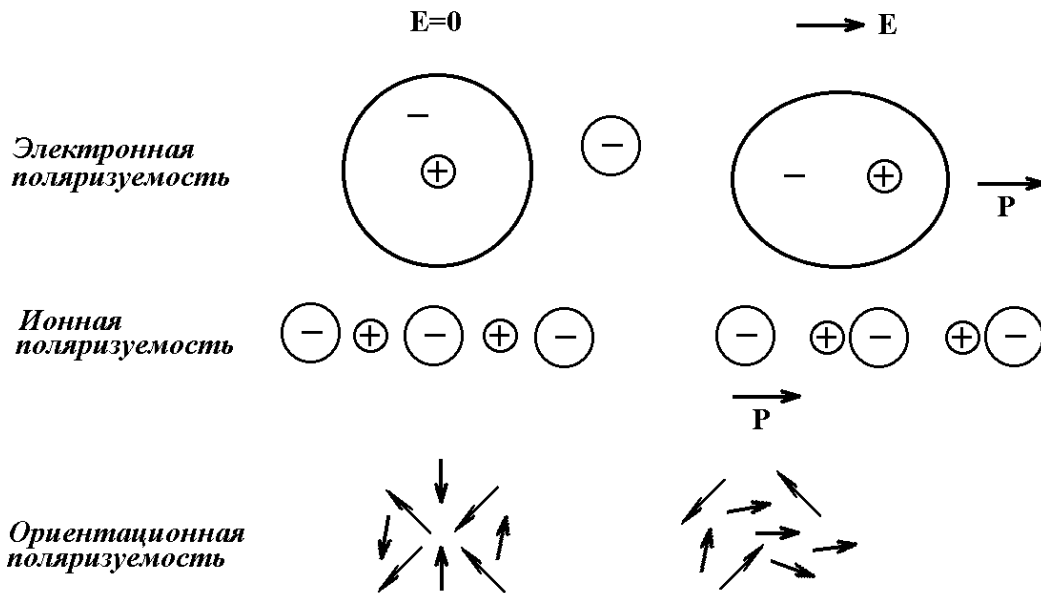
For a mixture of molecules:

$$\frac{\epsilon - 1}{\epsilon + 2} = \sum_i \frac{n_i \alpha_i}{3}. \quad (8.22)$$

### 8.3 Polarizability

There are three components of polarizability: electron, ion, and dipole. The first one is due to the electron shell displacement, the second the displacement of ions. The third one is caused by the orientation of polar molecules in an electric field (see Fig.8.4).

Fig.8.4. The basic components of polarization



At high frequency, the electron polarizability is of importance (because of inertia of atoms and molecules). In the optic frequency region, the equation (8.22) transforms into:

$$\frac{n^2 - 1}{n^2 + 2} = \sum_i \frac{n_i \alpha_i^{el}}{3}. \quad (8.23)$$

Experimental data confirm that formula.

#### 8.3.1 An Electron Polarizability

The frequency of resonance absorption of electromagnetic waves by atoms:

$$\omega_0 = (\beta/m)^{1/2}, \quad (8.24)$$

The quantity  $\beta$  is called the **force constant**. The electron displacement produced by the field can be found from:

$$-eE_{loc} = \beta x = m\omega_0^2 x. \quad (8.25)$$

For static electron polarizability:

$$\alpha^{el} = \frac{p}{E_{loc}} = -\frac{ex}{E_{loc}} = \frac{e^2}{m\omega_0^2}. \quad (8.26)$$

An atomic electron may be considered as a harmonic oscillator, which oscillates in a periodical electric field  $E_{loc} \sin \omega t$ :

$$m \frac{d^2 x}{dt^2} + m\omega_0^2 x = -E_{loc} \sin \omega t. \quad (8.27)$$

The solution is chosen as:  $x = x_0 \sin \omega t$ . Thus:

$$m(-\omega^2 + \omega_0^2)x_0 = -eE_{loc}. \quad (8.28)$$

The dipole amplitude:

$$p_0 = -ex_0 = \frac{e^2 E_{loc}}{m(\omega_0^2 - \omega^2)}, \quad (8.29)$$

Hence:

$$\alpha^{el} = \frac{e^2/m}{\omega_0^2 - \omega^2}. \quad (8.30)$$

In an optical frequency region, the dispersion is rather weak for a great number of transparent substances.

### 8.3.2 An Orientation (Dipole) Polarizability

We remind our readers that quantity  $p = ql$  is called the **dipole**. In an electric field, the dipole tends to orient along the field direction. The heat motion violates the orientation influence of the electric field.

The torque:

$$M = pE \sin \theta, \quad (8.31)$$

$\theta$  is the angle between  $\mathbf{p}$  and  $\mathbf{E}$ . An elementary work performed by the field while rotating the dipole through the angle  $d\theta$ :

$$dA = Md\theta = pE \sin \theta d\theta, \quad (8.32)$$

The dipole potential energy:

$$U = \int Md\theta = -pE \cos \theta, \quad (8.33)$$

In accordance with the classical distribution, the probability of orientation inside the space angle  $d\Omega$ :

$$d\varpi = ae^{-\frac{U}{kT}} d\Omega = ae^{-\frac{pE \cos \theta}{kT}} d\Omega, \quad (8.34)$$

The number of electrons with dipole moment oriented inside the angle  $d\Omega$ :

$$dn = nae^{-\frac{pE \cos \theta}{kT}} d\Omega. \quad (8.35)$$

Polarization of the substance:

$$dP = p \cos \theta dn, \quad (8.36)$$

$$P = \int p \cos \theta dn = \int_0^\pi pnae^{-\frac{pE \cos \theta}{kT}} \cos \theta \sin \theta d\theta \int_0^{2\pi} d\varphi. \quad (8.37)$$

Designate by  $\bar{p}$  the molecule dipole (along the field direction). (8.38)

Introduce the coordinates:

$$y = \cos \theta, \quad x = pE/kT. \quad (8.39)$$

Equation (8.38) takes the form:

$$\bar{p} = p \frac{\int_{-1}^{+1} e^{yx} y dy}{\int_{-1}^{+1} e^{yx} dy}. \quad (8.40)$$

The denominator integral is elementary. The standard procedure is used to calculate the numerator integral:

$$\int_{-1}^{+1} e^{yx} dy = \frac{1}{x} (e^x - e^{-x}) \quad (8.41)$$

$$\int_{-1}^{+1} e^{yx} y dy = \frac{1}{x} (e^x - e^{-x}) - \frac{1}{x^2} (e^x - e^{-x}) \quad (8.42)$$

Hence:

$$\frac{\bar{p}}{p} = \frac{e^x + e^{-x}}{e^x - e^{-x}} - \frac{1}{x} = \operatorname{cthx} - \frac{1}{x} = L(x). \quad (8.43)$$

The function  $L(x)$  is called the **Lanjeven function**. When  $x < 1$  (weak field, or high temperature) the function can be decompose in series:

$$L(x) = \frac{x}{3} - \frac{x^3}{45} + \dots \quad (8.44)$$

We can write:

$$\frac{\bar{p}}{p} = \frac{pE}{3kT} \left( 1 - \frac{p^2 E^2}{15k^2 T^2} + \dots \right). \quad (8.45)$$

In zero approximation:

$$\bar{p} = \frac{p^2 E}{3kT}. \quad (8.46)$$

If  $E = E_{cr} = 3kT/p$ ,  $\bar{p} = p$ , polarization saturates.

Having in mind that

$$\bar{p} = \varepsilon_0 \alpha_{dip} E. \quad (8.47)$$

and comparing (8.46) and (8.47) we get:

$$\alpha_{dip} = \frac{p^2}{3\varepsilon_0 kT}. \quad (8.48)$$

The Klausius-Mocotti equation:

$$\frac{\varepsilon - 1}{\varepsilon + 2} = \frac{n}{3} \left( \alpha^{el} + \frac{p^2}{3\varepsilon_0 kT} \right). \quad (8.49)$$

Assuming  $\varepsilon + 2 \approx 3$ :

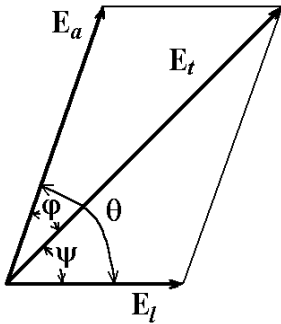
$$\varepsilon - 1 = n \left( \alpha^{el} + \frac{p^2}{3\varepsilon_0 kT} \right). \quad (8.50)$$

At room temperature, dipole polarizability  $\alpha_{dip}$  is of the same order as electron one  $\alpha^{el}$ . The dependence of the quantity  $(\varepsilon - 1)$  on the inverse  $1/T$  is lineary (Fig.8.6). At high temperature,  $\varepsilon - 1 \approx n\alpha^{el}$  (if  $T \rightarrow \infty$ ,  $\alpha_{dip} \rightarrow 0$ ). The inclination angle is defined by:

$$\text{tg}\beta = \frac{np^2}{3\varepsilon_0 k}.$$

Oxides, vapors of water and organic substances are polar gases.

### Polarizability of polar liquids



**Fig.8.5. An electric field of polar liquids**

The water, alcohol, cellulose are polar liquids. The dipole moments of water and alcohol are about  $6 \cdot 10^{30}$  Col·m  $\approx$  2debye. The intrinsic electric field is about  $10^9$  V/m. That field handicaps the dipoles to orient in an external field.

Let us assume that besides the local field  $\mathbf{E}_l$  (8.16), the additional field  $E_a \gg E_l$  acts upon a polar molecule. Orientation of  $\mathbf{E}_a$  is arbitrary. Because  $E_a \gg E_l$  we assume that  $\theta \approx \psi$  (Fig.8.5). The equilibrium condition:

$$E_l \sin \psi = E_a \sin \varphi, \quad (8.51)$$

that can be written ( $\theta \approx \psi$ ) as follows:

$$\sin \varphi = \frac{E_l}{E_a} \sin \theta, \quad (8.52)$$

The electric field strength:

$$E_t = \sqrt{E_a^2 + E_l^2 + 2E_a E_l \cos \theta}. \quad (8.53)$$

Having in mind that  $E_a \gg E_l$  and decomposing the radical:

$$E_t = E_a \left( 1 + \frac{E_l}{E_a} \cos \theta \right). \quad (8.54)$$

The average projection  $\bar{p}$  of the molecule dipole moment  $\mathbf{p}$  upon the field direction  $\mathbf{E}_t$  can be described [see (8.43)] as follows:

$$\frac{\bar{p}}{p} = L \left( \frac{pE_t}{kT} \right) = L(y), \quad (8.55)$$

$$y = \frac{pE_t}{kT} = \frac{pE_a}{kT} \left( 1 + \frac{E_l}{E_a} \cos \theta \right). \quad (8.56)$$

Introducing  $x = \frac{pE_a}{kT}$  and decomposing the Lanjeven function in series by the powers of  $E_l/E_a$  we get (in the first approximation):

$$L(y) = L \left[ x \left( 1 + \frac{E_l}{E_a} \cos \theta \right) \right] = L(x) + \frac{dL(x)}{dx} x \frac{E_l}{E_a} \cos \theta. \quad (8.57)$$

The dipole moment projection  $\bar{p}_t$  (onto the macroscopic field  $\mathbf{E}$ ):

$$\bar{p}_t = p \left[ L(x) + \frac{dL(x)}{dx} x \frac{E_l}{E_a} \cos \theta \right]. \quad (8.58)$$

The projection onto the field  $\mathbf{E}_l$

$$p_l = \bar{p}_t \cos \psi. \quad (8.59)$$

Taking into account (8.52) and that the angle is small:

$$\cos \psi = \cos(\theta - \varphi) = \cos \theta \cos \varphi + \sin \theta \sin \varphi = \cos \theta + \frac{E_l}{E_a} \sin^2 \varphi, \quad (8.60)$$

$$\begin{aligned} p_l &= \bar{p}_t \left( \cos \theta + \frac{E_l}{E_a} \sin^2 \varphi \right) = \\ &= p \left[ L(x) + \frac{dL(x)}{dx} x \frac{E_l}{E_a} \cos \theta \right] \left( \cos \theta + \frac{E_l}{E_a} \sin^2 \varphi \right), \end{aligned} \quad (8.61)$$

Finding the average quantities and having in mind that  $\overline{L(x)} = L(x)$ :

$$\overline{\cos \theta} = 0; \quad (8.62)$$

$$\overline{\sin^2 \theta} = \frac{1}{4\pi} \int_0^{4\pi} \sin^2 \theta d\Omega = \frac{1}{4\pi} \int_0^\pi \sin^2 \theta 2\pi \sin \theta d\theta = \frac{2}{3}; \quad (8.63)$$

$$\overline{\cos^2 \theta} = \frac{1}{4\pi} \int_0^{4\pi} \cos^2 \theta d\Omega = \frac{1}{4\pi} \int_0^\pi \cos^2 \theta 2\pi \sin \theta d\theta = \frac{1}{3}. \quad (8.64)$$

In the first approximation:

$$\begin{aligned} p_l &= p \left[ L(x) \overline{\cos \theta} + L(x) \frac{E_l}{E_a} \overline{\sin^2 \theta} + \frac{dL(x)}{dx} x \frac{E_l}{E_a} \overline{\cos^2 \theta} \right] = \\ &= p \frac{E_l}{3E_a} \left[ 2L(x) + x \frac{dL(x)}{dx} \right]. \end{aligned} \quad (8.65)$$

Differentiating the Lanjeven function with respect to  $x$ :

$$p_l = \frac{p^2 E_l}{3kT} \left[ 1 - L^2(x) \right]. \quad (8.66)$$

$$p_l = \varepsilon_0 \alpha_{dip} E_l. \quad (8.67)$$

It follows from (8.66) and (8.67):

$$\alpha_{dip} = \frac{p^2}{3\varepsilon_0 kT} \left[ 1 - L^2(x) \right] = \frac{p^2}{3\varepsilon_0 kT} R(x). \quad (8.68)$$

The quantity  $R(x) = 1 - L^2(x)$  is called the **reduction factor**. It is a ratio of polarizability of the liquid and gas. There is an intrinsic friction inside the liquid. Hence that quantity is less than unit. The reduction factor of water is 0.2.



Thus the Klausius-Mocotti equation for polar liquids differs from (8.49) only by the reduction factor:

$$\frac{\varepsilon - 1}{\varepsilon + 1} = \frac{n}{3} \left( \alpha^{el} + \frac{p^2}{3\varepsilon_0 kT} R \right). \quad (8.69)$$

In order to find polarizability, the solution of polar dielectrics in non-polar solvent is used. At low concentration the distance between soluble molecules is great. The gas approximation is valid [see (8.49)]. Having in mind that  $\alpha_2 = \alpha_2^{el} + \alpha_{dep}$ :

$$\frac{\varepsilon - 1}{\varepsilon + 1} = \frac{n_1 \alpha_1^{el}}{3} + \frac{n_2 \alpha_2^{el}}{3} + \frac{n_2 p^2}{9\varepsilon_0 kT}. \quad (8.70)$$

### 8.3.3 Polarization of Ion Dielectric Solids and Heterogeneous Compounds

In any dielectric there are two types of polarization: electronic and ionic. The relaxation time of electron polarization is  $10^{-14}$  c. The relaxation time of ion polarization is  $10^{-13}$  c. The relaxation time depends on the free-running frequency of dipole oscillation. For a binary crystal, the effective mass must be used:

$$M = \frac{M_1 M_2}{M_1 + M_2}, \quad (8.71)$$

The frequency of free oscillation:

$$\omega = \sqrt{k/M}, \quad (8.72)$$

The quantity  $k$  is the elasticity coefficient. Hence:  $\omega^2 M = \sqrt{k/M}$ .

Under action of the force  $qE$ , ions displace from an equilibrium position at a distance  $\Delta x = qE/k$ . A dipole moment is produced:  $p = q^2 E/k$ . Taking into account that  $p = \varepsilon_0 \alpha_i E$ :

$$\alpha_i = \frac{q^2}{\varepsilon_0 \omega^2} \cdot \frac{M_1 + M_2}{M_1 M_2}. \quad (8.73)$$

Multiplying by the square of the Avogadro number:

$$\alpha_i = \frac{q^2 N_A}{\varepsilon_0 \omega^2} \cdot \frac{\mu_1 + \mu_2}{\mu_1 \mu_2}, \quad (8.74)$$

$\mu_1$  and  $\mu_2$  is a molar mass. Introducing the ion concentration  $N$  we arrive at:

$$\mathbf{p}_i = \varepsilon_0 N \frac{q^2 N_A}{\varepsilon_0 \omega^2} \cdot \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \mathbf{E}. \quad (8.75)$$

Taking into account the electron polarization:

$$\mathbf{p}_i = \varepsilon_0 N \left( \alpha^{el} + \alpha_i \right) \mathbf{E} = \varepsilon_0 N \left( \alpha^{el} + \frac{q^2 N_A}{\varepsilon_0 \omega^2} \cdot \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \right) \mathbf{E}. \quad (8.76)$$

The quantity  $\alpha^{el} = \alpha_1^{el} + \alpha_2^{el}$  is the sum of electron polarizability of components. Having in mind that:

$$\mathbf{p}_i = \varepsilon_0 (\varepsilon - 1) \mathbf{E}, \quad (8.77)$$

we arrive at last to:

$$N\alpha^{el} + N \frac{q^2 N_A}{\epsilon_0 \omega^2} \cdot \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} = \epsilon - 1 \quad (8.78)$$

That equation is known as the Born formula. The quantity  $\epsilon$  includes the electron and ion polarization. Sometime it is convenient to use the **optic permittivity**  $\epsilon_\infty$ :

$$\frac{\epsilon - 1}{\epsilon + 2} = \frac{N}{3} \alpha^{el} \quad (8.79)$$

$$\epsilon + 2 \approx 3$$

$$\epsilon_\infty = 1 + \alpha^{el}. \quad (8.80)$$

We remind our readers that  $n = \sqrt{\epsilon_\infty \mu}$ . For non-magnetic substance:  $\epsilon_\infty = n^2$ . Thus, we get:

$$\epsilon = n^2 + N \frac{q^2 N_A}{\epsilon_0 \omega^2} \cdot \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} \quad (8.81)$$

It should be noted that the theoretical quantities are always less the experimental data. It is due to the fact that the macroscopic field was used instead the local one. For crystals with great permittivity, the formula (8.81) is not true.

For a two-phase heterogeneous compound, a **Lichteneker formula** is valid.

$$\epsilon^m = x_1 \epsilon_1^m + (1 - x_1) \epsilon_2^m \quad (8.81)$$

The quantity  $x_1$  and  $(1 - x_1)$  is the volume concentration,  $\epsilon_1$  and  $\epsilon_2$  is permittivity,  $m$  depends of the system geometry. The formula is widely used when the permittivity of slide dielectric is under investigation.

## 8.4 Polarization of Dielectrics in an Alternating Field and the Dielectric Losses

In previous section, the polarization of dielectrics in a constant electric field has been discussed. In the following section, the transition processes are being investigated.

### 8.4.1 Transition Process when a Field is Switched in and Switched off

Assume that an external uniform electric field  $\mathbf{E}$  is switched in instantaneously. The polarization  $\mathbf{P}$  retards from the field, and its maximal magnitude  $P_c$  is installed only after a certain time interval. That interval can be subdivided into two parts.

In time interval of order of  $10^{-14} - 10^{-13}$ , the “non-inertial” polarization ( $P_\infty$ ) is installed. In the following period, the relaxation polarization is installed.

For the non-inertial polarization:

$$P_\infty = \epsilon_0 (\epsilon_\infty - 1) E, \quad (8.82)$$

The installed magnitude:

$$P_c = \epsilon_0 (\epsilon_c - 1) E. \quad (8.83)$$

The relaxation polarization:

$$P_{pc} = P_c - P_\infty = \varepsilon_0(\varepsilon_c - \varepsilon_\infty)E. \quad (8.84)$$

The time dependence:

$$P_p = P_{pc} \left( 1 - e^{-\frac{t}{\tau}} \right). \quad (8.85)$$

If in the dielectric there are no volume charges and its resistance is constant, the displacement current and relaxation polarization current density can be easily found.

When an external field is switched in, the polarization instantaneously is diminished by  $P_\infty$ . In the following time interval:

$$P_p = P_{pc} e^{-\frac{t}{\tau}}. \quad (8.86)$$

There is a relation between the relaxation time ( $\tau$ ) and the lifetime of relaxators ( $\tau_{lc}$ ).

$$\tau = \frac{\tau_{lc}}{1 - \frac{\varepsilon_\infty + 2}{9} \alpha_d N}, \quad (8.87)$$

$N$  is the molecule concentration.

Thus the speed of polarization is dominated by the dipole component.

Obviously:

$$\frac{\varepsilon_c - 1}{\varepsilon_c + 2} = \frac{n}{3} \alpha^{el} + \frac{n}{3} \alpha_d, \quad (8.88)$$

$$\frac{\varepsilon_c - 1}{\varepsilon_c + 2} = \frac{\varepsilon_\infty - 1}{\varepsilon_\infty + 2} + \frac{n}{3} \alpha_d. \quad (8.88a)$$

Hence:

$$\tau = \frac{\varepsilon_c + 2}{\varepsilon_\infty + 2} \tau_{lc}. \quad (8.89)$$

The formula (8.89) is a relation between the relaxation time and lifetime of a molecule. Because  $\varepsilon_c > \varepsilon_\infty$ ,  $\tau > \tau_{lc}$ . Inequality  $\tau > \tau_{lc}$  is due to the fact that the local field acting upon a relaxator increases up to its stationary magnitude.

For an arbitrary alternating field:

$$P_p(t) = \varepsilon_0(\varepsilon_c - \varepsilon_\infty) \int_{-\infty}^t E(t_i) e^{-\frac{t-t_i}{\tau}} \frac{dt_i}{\tau}. \quad (8.90)$$

The entire polarization:

$$P(t) = P_\infty + P_p = \varepsilon_0(\varepsilon_\infty - 1)E(t) + \varepsilon_0(\varepsilon_c - \varepsilon_\infty) \int_{-\infty}^t E(t_i) e^{-\frac{t-t_i}{\tau}} \frac{dt_i}{\tau}. \quad (8.91)$$

## 8.4.2 Polarization in an Alternating Field

Assume that the polarization field follows the law:

$$E(t) = E_m \sin \omega t. \quad (8.92)$$

In accordance with (8.91):

$$P(t) = \varepsilon_0(\varepsilon_\infty - 1)E_m \sin \omega t + \varepsilon_0(\varepsilon_c - \varepsilon_\infty) \int_{-\infty}^t E_m \sin(\omega t) \cdot e^{-\frac{t-t_i}{\tau}} \frac{dt_i}{\tau}. \quad (8.93)$$

The integral:

$$\int_{-\infty}^t E_m \sin(\omega t) \cdot e^{-\frac{t-t_i}{\tau}} dt_i = \frac{e^{-\frac{t}{\tau}} \tau}{1 + (\omega\tau)^2} (\sin \omega t - \omega\tau \cos \omega t).$$

Hence:

$$P(t) = \varepsilon_0(\varepsilon_\infty - 1)E_m \sin \omega t + \varepsilon_0(\varepsilon_c - \varepsilon_\infty) \frac{E_m}{1 + (\omega\tau)^2} (\sin \omega t - \omega\tau \cos \omega t). \quad (8.93a)$$

At low frequency and  $\tau$ , when  $\omega\tau \ll 1$ :

$$P(t) = \varepsilon_0(\varepsilon_\infty - 1)E_m \sin \omega t + \varepsilon_0(\varepsilon_c - \varepsilon_\infty)E_m \sin \omega t = \varepsilon_0(\varepsilon_c - 1)E_m \sin \omega t. \quad (8.94)$$

The field and polarization are in phase.

When the condition  $\omega\tau \ll 1$  is not valid, the formula (8.93) is true. Designate:

$$P_\infty(t) = \varepsilon_0(\varepsilon_\infty - 1)E(t) \quad (8.95)$$

$$P_p(t) = \varepsilon_0(\varepsilon_c - \varepsilon_\infty) \frac{E_m}{1 + (\omega\tau)^2} (\sin \omega t - \omega\tau \cos \omega t). \quad (8.96)$$

Then:

$$P(t) = P_\infty(t) + P_p(t). \quad (8.97)$$

### An «active» and «reactive» polarization in a sine field

Retarding of polarization from the polarizing field leads to the energy dissipation, i.e. to energy loss.

The component of relaxation polarization associated with liberation of power is called the **active component**  $P_{pa}$ . The component coinciding in phase with the external field is called the **reactive component**  $P_{pr}$ .

In accordance with (8.96) the active component is shifted by the angle  $\pi/2$  relative to the field.

$$P_{pa} = \varepsilon_0(\varepsilon_c - \varepsilon_\infty) \frac{\omega\tau}{1 + (\omega\tau)^2} E_m \cos \omega t = P_{pam} \cos \omega t, \quad (8.98)$$

The reactive component [see (8.96)]:

$$P_{pr} = \varepsilon_0(\varepsilon_c - \varepsilon_\infty) \frac{E_m}{1 + (\omega\tau)^2} \sin \omega t = P_{prm} \sin \omega t. \quad (8.99)$$

The reactive component is in phase with the external field.

We should remember that the non-inertial polarization  $P_\infty$  coincides in phase with the external field. Thus the resulting polarization, which coincides in phase with the field is:

$$P_\phi(t) = P_\infty(t) + P_{pr} = \left[ \varepsilon_0(\varepsilon_\infty - 1) + \frac{\varepsilon_0(\varepsilon_c - \varepsilon_\infty)}{1 + (\omega\tau)^2} \right] E_m \sin \omega t. \quad (8.100)$$

## 8.5 The Energy Loss in Relaxation Polarization and Through Conductivity Dielectrics

In an alternating field, the energy loss in dielectric is greater than in a constant field. In alternating field there are the polarization and through currents, in constant field there is only the thorough current.

The electric current in dielectric consists of displacement and conduction components.

$$j = j_{dis} + j_{con} , \quad (8.101)$$

$$j_{con} = \sigma E = \sigma E_m \sin \omega t = \frac{\sigma V_m}{d} \sin \omega t , \quad (8.102)$$

$$j_{dis} = \frac{dD}{dt} = \frac{d(\epsilon_0 E + P)}{dt} , \quad (8.103)$$

The quantity  $\sigma$  is the specific conductivity in a constant field,  $d$  is the thickness of dielectric,  $V_m = E_m d$  is an external voltage. The dielectric losses of conduction do not depend on frequency and depend on the current strength and voltage.

Having in mind (8.97), equation (8.103) can be written as follows:

$$j_{dis} = \epsilon_0 \omega \frac{V_m}{d} \cos \omega t + \epsilon_0 (\epsilon_\infty - 1) \omega \frac{V_m}{d} \cos \omega t + \epsilon_0 (\epsilon_c - \epsilon_\infty) \frac{V_m}{d} \frac{(\omega \cos \omega t - \omega^2 \tau \sin \omega t)}{1 + (\omega \tau)^2} . \quad (8.104)$$

The current density:

$$j = j_{am} \sin \omega t + j_{rm} \cos \omega t . \quad (8.105)$$

The amplitude of active and reactive current density:

$$j_{am} = \sigma \frac{V_m}{d} + \epsilon_0 (\epsilon_c - \epsilon_\infty) \frac{\omega^2 \tau}{1 + (\omega \tau)^2} \cdot \frac{V_m}{d} ; \quad (8.106)$$

$$j_{rm} = \epsilon_0 \epsilon_\infty \omega \frac{V_m}{d} + \epsilon_0 (\epsilon_c - \epsilon_\infty) \frac{\omega}{1 + (\omega \tau)^2} \cdot \frac{V_m}{d} . \quad (8.107)$$

The current component that is in phase with the field is called the active one. The current component shifted at angle  $\pi/2$  with respect to the field is called the reactive current. The active component causes the energy loss. At high frequencies, the active current amplitude reaches its maximum:

$$j_{am} = \sigma \frac{V_m}{d} + \epsilon_0 (\epsilon_c - \epsilon_\infty) \frac{V_m}{\tau d} . \quad (8.108)$$

The dielectric loss:

$$W = \frac{1}{T} \int_0^T j S V dt , \quad (8.109)$$

$T$  is the period,  $S$  – the cross-section. площадь сечения. Having in mind (8.105):

$$W = \frac{I_{am} V_m}{T} \int_0^T \sin^2 \omega t dt + \frac{I_{rm} V_m}{T} \int_0^T \sin \omega t \cos \omega t dt . \quad (8.110)$$

The second addend in (8.110) is zero because  $\sin \omega t \cos \omega t = \frac{1}{2} \sin 2\omega t$ .

$$\int_0^T \sin^2 \omega t dt = \int_0^T \frac{1 - \cos 2\omega t}{2} dt = \frac{T}{2}. \text{ Thus:}$$

$$W = \frac{I_{am} V_m}{2} = \sigma S \frac{V_m^2}{2d} + \epsilon_0 (\epsilon_c - \epsilon_\infty) \frac{S \omega^2 \tau}{1 + (\omega \tau)^2} \frac{V_m^2}{2d}. \quad (8.111)$$

The effective voltage:

$$V = V_m / \sqrt{2}, \quad (8.112)$$

$$W = \sigma S \frac{V^2}{d} + \epsilon_0 (\epsilon_c - \epsilon_\infty) \frac{S \omega^2 \tau}{1 + (\omega \tau)^2} \frac{V^2}{d}. \quad (8.113)$$

The first addend is the energy loss of conductivity:

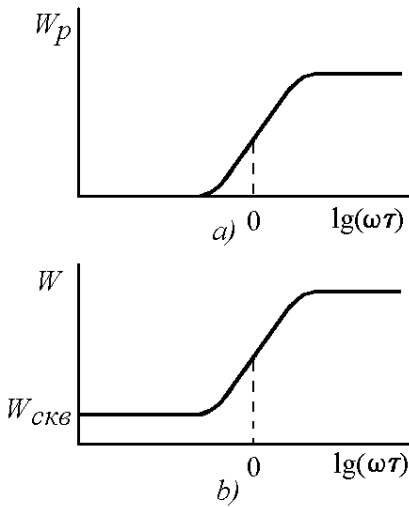
$$W = \sigma S \frac{V^2}{d} = \frac{V^2}{R}. \quad (8.114)$$

$R = d / (\sigma S)$  is the electric resistance of dielectric at zero frequency..

The second addend in (8.113) is the relaxation energy loss:

$$W_p = \epsilon_0 (\epsilon_c - \epsilon_\infty) \frac{S \omega^2 \tau}{1 + (\omega \tau)^2} \frac{V^2}{d}. \quad (8.113)$$

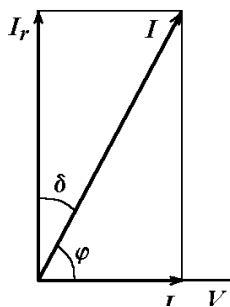
The relaxation loss is proportional to the active relaxation polarization amplitude and frequency. The energy loss frequency dependence is shown in Fig.8.6.



**Fig.8.6. The relaxation (a), and total (b) energy loss as function of frequency**

At low frequency ( $\omega \tau \ll 1$ ), the dielectric loss depends only on the through conductivity. When  $\omega \tau \sim 1$ , the function curvature decreases and the saturation is reached...

### 8.5.1. The Dielectric Loss Angle



**Fig.8.7. A vector current diagram of a capacitor**

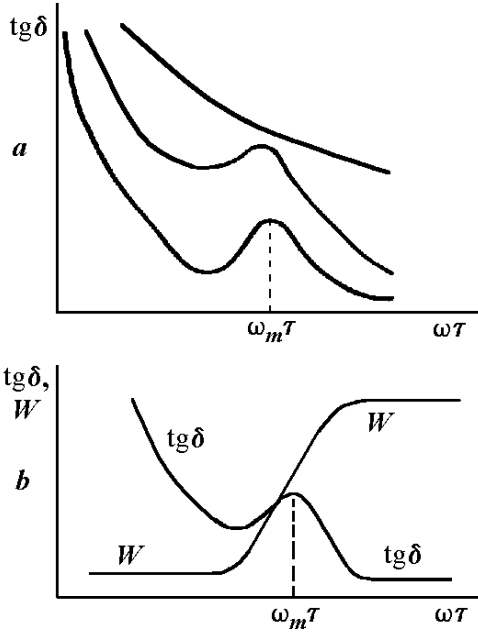
The dielectric loss angle characterizes the process of the energy liberation in a dielectric (see Fig.8.7).

The electric power liberated in a dielectric:

$$W = VI \cos \varphi = VI_a = VI_r \operatorname{tg} \delta. \quad (8.114)$$

The quantities  $V$  and  $I$  is an effective voltage and current:  $V = V_m / \sqrt{2}$ ,  $I = I_m / \sqrt{2}$ . The ratio of the effective current  $I_a$  to the reactive current  $I_r$  is the tangent of the energy loss angle:

$$\operatorname{tg} \delta = I_a / I_r = I_{am} / I_{rm} = j_{am} / j_{rm}. \quad (8.115)$$



**Fig.8.8. (a)  $\text{tg}\delta$  as function of frequency and temperature at different  $\sigma$ ; (b) dependence of the energy loss  $W$  and  $\text{tg}\delta$  on frequency for relaxation polarization**

Introducing (8.106) and (8.107) into (8.115):

$$\text{tg}\delta = \frac{\sigma + \epsilon_0(\epsilon_c - \epsilon_\infty) \frac{\omega^2 \tau}{1 + (\omega\tau)^2}}{\epsilon_0 \epsilon_\infty \omega + \epsilon_0(\epsilon_c - \epsilon_\infty) \frac{\omega}{1 + (\omega\tau)^2}}. \quad (8.116)$$

When  $\omega \rightarrow 0$ :

$$\text{tg}\delta = \frac{\sigma}{\epsilon_0 \epsilon_\infty \omega}. \quad (8.117)$$

When the electric conductivity is small:

$$\text{tg}\delta = \frac{(\epsilon_c - \epsilon_\infty) \frac{\omega\tau}{1 + (\omega\tau)^2}}{\epsilon_\infty + \frac{\epsilon_c - \epsilon_\infty}{1 + (\omega\tau)^2}}. \quad (8.118)$$

At high frequency ( $\omega\tau \gg 1$ ):

$$\text{tg}\delta = \frac{(\epsilon_c - \epsilon_\infty)\omega\tau}{\epsilon_c + \epsilon_\infty(\omega\tau)^2}. \quad (8.119)$$

Let us find the maximum of (8.119):

$$\frac{d\text{tg}\delta}{d(\omega\tau)} = \frac{(\epsilon_c - \epsilon_\infty) [\epsilon_c + \epsilon_\infty(\omega\tau)^2] - 2\epsilon_\infty(\omega\tau)^2(\epsilon_c - \epsilon_\infty)}{[\epsilon_c + \epsilon_\infty(\omega\tau)^2]^2}. \quad (8.120)$$

The denominator is not a zero. Hence:

$$(\epsilon_c - \epsilon_\infty) [\epsilon_c - \epsilon_\infty(\omega\tau)^2] = 0. \quad (8.121)$$

Having in mind that  $\epsilon_c \gg \epsilon_\infty$ :

$$\omega_m \tau = \sqrt{\epsilon_c / \epsilon_\infty}. \quad (8.122)$$

Thus:

$$(\text{tg}\delta)_m = \frac{(\epsilon_c - \epsilon_\infty)}{2\epsilon_c} \sqrt{\frac{\epsilon_c}{\epsilon_\infty}}. \quad (8.123)$$

The function  $\text{tg}\delta = f(\omega\tau)$  is shown in Fig.8.8a. When the conductivity is great the maximum of  $(\text{tg}\delta)_m$  is not expressed distinctly (Fig.8.8a). For comparing, the power  $W$  and  $\text{tg}\delta$  as function of frequency for relaxation polarization is shown (Fig.8.8b). For good dielectrics with electron polarization,  $\text{tg}\delta$  already at the frequency 50Hz is small (of the order of  $10^{-4}$ ). For dielectrics with relaxation polarization, at the same frequency  $\text{tg}\delta \approx 10^{-2}$ . The magnitude of the relaxation maximum depends on the type of conductivity. For ferrites at 50Hz,  $\text{tg}\delta \gg 1$ .

## 8.6. The Complex Dielectric Permittivity. The Cole-Cole Diagram

The dielectric permittivity can be represented as a complex quantity:

$$\hat{\varepsilon} = \varepsilon' - i\varepsilon'' \quad (8.124)$$

Analogously, the current density:

$$j = j_a - ij_r \quad (8.125)$$

From the other hand:

$$j = d\dot{Q}/dt, \quad (8.126)$$

$\dot{Q}$  - the density of the surface complex charge:

$$\dot{Q} = \varepsilon C_0 \dot{V} = \varepsilon C_0 V_m e^{i\omega t}; \quad (8.127)$$

$C_0 = \varepsilon_0/d$  — the capacitance of a plate vacuum capacitor of unit square.  $\dot{V} = V_m e^{i\omega t}$  is a complex voltage.

The current density:

$$j = \varepsilon'' \omega C_0 \dot{V} + \varepsilon' \omega C_0 \dot{V}. \quad (8.128)$$

Comparing (8.128) and (8.125):

$$j_a = \varepsilon'' \omega C_0 \dot{V} = \varepsilon'' \omega C_0 V_m e^{i\omega t} \quad (8.129)$$

$$j_r = \varepsilon' \omega C_0 \dot{V} = \varepsilon' \omega C_0 V_m e^{i\omega t}. \quad (8.130)$$

Thus:

$$\operatorname{tg}\delta = j_{am}/j_{rm} = \varepsilon''/\varepsilon', \quad (8.131)$$

$$\varepsilon'' = \varepsilon' \operatorname{tg}\delta. \quad (8.132)$$

We see that  $\varepsilon''$  is defined by the energy losses. The quantity  $\varepsilon'$  is the dielectric permittivity  $\varepsilon$ . Hence, the real part of (8.124) is usual dielectric permittivity. Comparing (8.106) and (8.107) with (8.129) and (8.130) we arrive at the Debye formula:

$$\varepsilon'' = \frac{\sigma}{\varepsilon_0 \omega} + \frac{(\varepsilon_c - \varepsilon_\infty) \omega \tau}{1 + (\omega \tau)^2}, \quad (8.133)$$

$$\varepsilon' = \varepsilon_\infty + \frac{\varepsilon_c - \varepsilon_\infty}{1 + (\omega \tau)^2}. \quad (8.134)$$

When  $\sigma$  is small and  $\omega \tau \gg 1$ :

$$\operatorname{tg}\delta = \frac{\varepsilon''}{\varepsilon'} = \frac{(\varepsilon_c - \varepsilon_\infty) \omega \tau}{\varepsilon_c + \varepsilon_\infty (\omega \tau)^2}. \quad (8.135)$$

Equation (8.134) coincides with (8.119). The position of maximum of the relaxation component  $\varepsilon''$  is a little shifted from that one of  $\operatorname{tg}\delta$ . At low temperature:

$$\varepsilon'' = \frac{(\varepsilon_c - \varepsilon_\infty) \omega \tau}{1 + (\omega \tau)^2}, \quad (8.136)$$

Finding the extreme of  $\varepsilon''$  we get:

$$\omega_m \tau = 1. \quad (8.137)$$



That condition leads to a smaller resonance frequency than that one for  $\text{tg}\delta$ . In accordance with (8.135) and (8.133), the magnitude of  $\varepsilon''$  and  $\varepsilon'$  corresponding to maximum of dielectric permittivity:

$$\varepsilon''_m = \frac{\varepsilon_c - \varepsilon_\infty}{2}, \quad (8.138)$$

$$\varepsilon'_m = \frac{\varepsilon_c + \varepsilon_\infty}{2}. \quad (8.139)$$

The limit quantity of  $\varepsilon'$  at low and high frequency (assuming  $\omega\tau \ll 1$  and  $\omega\tau \gg 1$ ) in accordance with (8.133):

$$\varepsilon' = \varepsilon_c \text{ и } \varepsilon' = \varepsilon_\infty. \quad (8.140)$$

It is convenient to represent the quantities  $\varepsilon''$  and  $\varepsilon'$  by the **Cole-Cole diagram**. Excluding the quantity  $\omega\tau$  from (8.133) and (8.135) let us find the analytic dependence  $\varepsilon'' = f(\varepsilon')$ .

$$(\varepsilon' - \varepsilon'_y)^2 + (\varepsilon'')^2 = R, \quad (8.141)$$

$$R = \frac{\varepsilon_c - \varepsilon_\infty}{2}, \quad (8.142)$$

$$\varepsilon'_y = \varepsilon_\infty + \frac{\varepsilon_c - \varepsilon_\infty}{2} = \frac{\varepsilon_c + \varepsilon_\infty}{2}. \quad (8.143)$$

The function (8.141) is shown in Fig.8.9. It follows from (8.134) and (8.136) that at a glance on the figure shows that when  $\omega\tau = 0$ ,  $\varepsilon' = \varepsilon_c$  and  $\varepsilon'' = 0$ ; when  $\omega\tau = \infty$ ,  $\varepsilon' = \varepsilon_\infty$  and  $\varepsilon'' = 0$ .

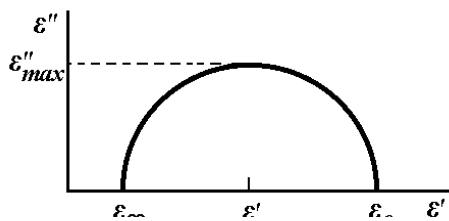


Fig.8.9. A Cole-Cole diagram

The dielectric dispersion is shown in Fig.8.10.

The specific effective conductivity:

$$\sigma_a = \varepsilon'' \varepsilon_0 \omega = \sigma + \varepsilon_0 (\varepsilon_c - \varepsilon_\infty) \frac{\omega^2 \tau}{1 + (\omega\tau)^2}. \quad (8.144)$$

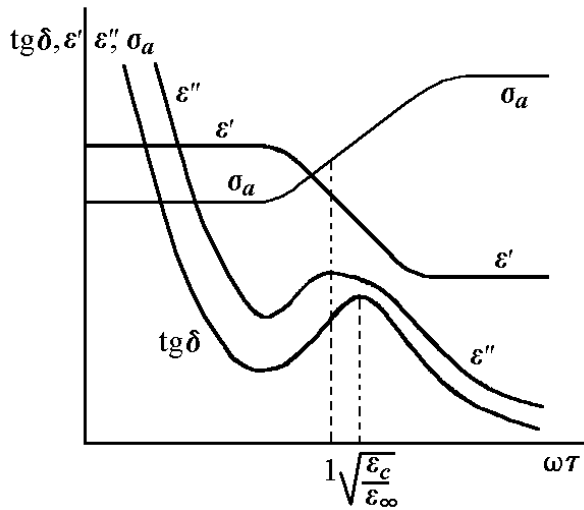
Hence:

$$j_a = \sigma_a E. \quad (8.145)$$

The quantity  $\varepsilon'$  varies from  $\varepsilon_c$  up to  $\varepsilon_\infty$ . At  $\omega\tau = 1$  there is a curve bend. The maximum of the function  $\text{tg}\delta = f(\omega)$  is a little right from that one of the function  $\varepsilon'' = f(\omega)$ . The function  $\varepsilon'' = f(\omega)$  is upward of function  $\text{tg}\delta = f(\omega)$  because  $\varepsilon'' = \varepsilon' \text{tg}\delta$  and  $\varepsilon' > 1$ . The specific effective electric conductivity  $\sigma_a$  at low frequency and that one in a stationary field are identical. When  $\omega\tau \gg 1$ :

$$\sigma_a = \sigma + \varepsilon_0 (\varepsilon_c - \varepsilon_\infty) \frac{1}{\tau}, \quad (8.146)$$

Thus, the quantity  $\sigma_a$  reaches its maximum (Fig.8.10).



**Fig.8.10. The dependence of  $\text{tg}\delta$ ,  $\epsilon'$ ,  $\epsilon''$ , and  $\sigma_a$  on the parameter  $\omega T$**

The relaxation time depends on temperature. For dipole relaxation polarization, the relaxation time:

$$\tau = \tau_0 e^{\frac{\Delta E}{kT}} \quad (8.147)$$

The maximum of the function  $\epsilon'' = f(\omega)$  is displaced to the higher frequency with temperature.

The behavior of the function  $\text{tg}\delta = f(\omega)$  maximum is analogous but it is rather difficult to use the criterion (8.122) because the permittivity also depends on temperature. When the temperature increases the energy loss becomes greater because of the increasing of the through conductivity.

If inside a dielectric there are cavities filled with a gas, the strike ionization can be observed in high electric fields. It is due to the fact that the electric stability of gases is considerably less than that one of liquids and solids. It leads to increasing of  $\text{tg}\delta$  and energy losses, which in that situation are called the **ionization losses**.