Tomsk Polytechnic University

PHYSICS I

Textbook

Mechanics. Mechanical Oscillations and Waves.
Molecular Physics and Thermodynamics.
Statistical Distributions

Tomsk 2000-2015

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Mechanics. Mechanical Oscillations and Waves
Molecular Physics and Thermodynamics
Statistical Distributions

This textbook is a brief version of second part of the general course in physics for technical universities. The main theoretical concepts are formulated in logical and continuous fashion. There are many examples and practice problems to be solved. This textbook has been approved by the Department of Theoretical and Experimental Physics and the Department of General Physics of TPU.

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PREFACE

Physics, the most fundamental science, is concerned with the basic principles of the Universe. It provides the basis for other sciences.

This textbook is a course in introductory physics for students mastering science or engineering. The main objectives of this introductory physics textbook are twofold: to provide the student with a clear and logical presentation of the basic concepts of physics, and to strengthen an understanding of these concepts through applications to the real world. So we have attempted to motivate readers through practical examples. The mathematical background of the student taking this course should include one semester of calculus. A large number of examples of varying difficulty are presented as an aid in understanding concepts. In many cases, these examples serve as models for solving another problems.

In general, all the physical phenomena are parts of one or more of the following areas of physics:

1. Classical mechanics, which is concerned with the motion of objects moving at speeds that are low compared to the speed of light
2. Relativity, which describes objects moving at speeds approaching the speed of light
3. Thermodynamics, which deals with heat, work, temperature, and statistical behavior of a large number of particles
4. Electromagnetism, which involves the theory of electricity, magnetism, and electromagnetic fields
5. Quantum mechanics, a theory dealing with the behavior of particles on the submicroscopic level

The first part of this textbook deals with classical or Newtonian mechanics. Mechanics is of vital importance for students from all disciplines. The laws introduced in mechanics retain their importance in fundamental theories that follow, including theories of modern physics. The second part gives an introduction to the special theory of relativity, with emphasis on some of its consequences. The third part deals with mechanical oscillations and wave motion. In fourth part, we turn to the study of thermodynamics, which is concerned with the concepts of heat and temperature. Thermodynamics is very successful in explaining properties of matter and correlation between these properties and mechanics of atoms and molecules.

And finally, there is no simple answer to the question: “How to study physics and prepare for examination?” But there are some recommendations based on our experience in learning and teaching over the years. The first and the main recommendation is: maintain a positive attitude toward the subjects’ matter, keep in mind that physics is most fundamental of all natural sciences. Other science courses that follow will use the same physical principles, so it is important that you understand and be able to apply various concepts and theories discussed in the text.

Contents

Part 1. Mechanics

Mechanics studies the effect of forces on bodies and their mechanical motion.

Spatiotemporal changes in relative positions of bodies or different parts of one body with respect to other bodies are called ***the mechanical motion.*** Every motion is relative. Classical (Newton) mechanics studies the motion of bodies whose speed v is small compared to that of light, i.e., $v << c$. The motion with $v \cong c$ is a subject of relativistic mechanics.

To describe the mechanical motion, the following abstract concepts are conventionally used:

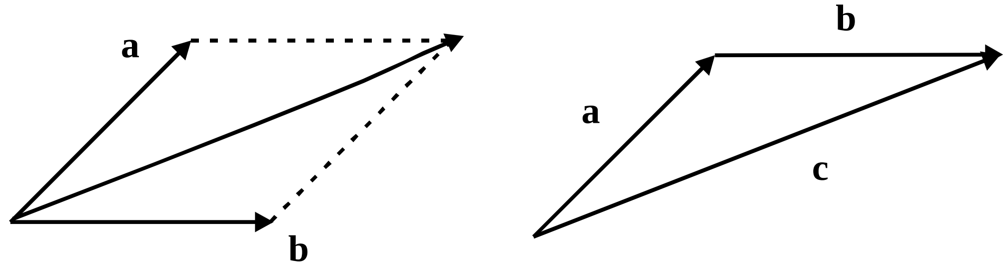
1. ***Mass (material) point*** is a body whose dimensions are neglected when solving a specific problem. Sometimes the term ***point particle*** is used.
2. A body whose deformations may be disregarded for a given problem is called ***perfectly rigid***, or simply ***rigid body.***

Motion occurs both in space and time. Consequently, to describe the motion, ***a reference frame*** is needed. A set of stationary bodies with respect to which the motion is being considered, a coordinate system attached to the reference body, and a timepiece indicating the time form ***a reference frame***. In classical mechanics, the features of space are described by the Euclidean geometry and the time is assumed to be the same in all reference frames.

1. Kinematics

1.1. Vector Quantities

Scalar, vector, and tensor quantities are widely used in physics. These mathematical concepts can be applied to solve a lot of physical problems. Vectors are defined as quantities characterized by magnitude and direction. They are added by the triangle or parallelogram method (see Figure 1.1).



Parallelogram method Triangle method

Figure 1.1. **c = a + b** (addition of vectors).

Usually a Cartesian (rectangular) coordinate system is chosen in a lot of physical applications and an arbitrary vector **a** can be expressed as follows:

$$\mathbf{a} = \mathbf{i} \cdot a_x + \mathbf{j} \cdot a_y + \mathbf{k} \cdot a_z . \quad (1.1)$$

Here $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the *x*, *y*, and *z* axes, respectively; a_x, a_y, and a_z are the projections of the vector onto the coordinate axes. So if you have to add or subtract two vectors (**a** and **b**), this can be done in such a way: $\mathbf{c} = \mathbf{a} \pm \mathbf{b}$,

$$\mathbf{c} = \mathbf{i} \cdot c_x + \mathbf{j} \cdot c_y + \mathbf{k} \cdot c_z = \mathbf{i} \cdot (a_x \pm b_x) + \mathbf{j} \cdot (a_y \pm b_y) + \mathbf{k} \cdot (a_z \pm b_z). \quad (1.2)$$

The magnitude of the vector can be expressed as follows:

$$|\mathbf{a}| = a = \sqrt{a_x^2 + a_y^2 + a_z^2}\,. \quad (1.3)$$

The vector division cannot be defined in general case, but it is possible to multiply vectors. Two vectors **a** and **b** can be multiplied in two ways. One of them results in a scalar quantity, and the other in a certain new vector.

Scalar product is defined as follows:

$$c = (\mathbf{ab}) = ab\cos\alpha, \quad (1.4)$$

where α is the angle between vectors **a** and **b.** It can be expressed in terms of projections

$$c = (\mathrm{ab}) = a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z. \quad (1.5)$$

Vector product is defined as the vector **c** determined by the equation

$$\mathbf{c} = [\mathbf{ab}] = (ab\sin\alpha)\mathbf{n} \quad (1.6)$$

(see Figure 1.2).

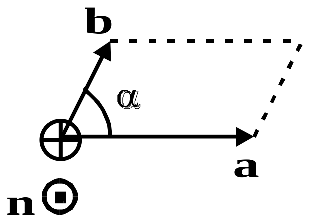


Figure 1.2.

Figure 1.2 shows that the magnitude of the vector product has a simple geometrical meaning. The expression $ab\sin\alpha$ numerically equals the area of the parallelogram constructed on the vectors being multiplied. We determine the direction of the vector **c** by relating it to the direction of rotation from the first multiplier to the second one according to the right-screw rule. Using the projections, **c** can be expressed as follows:

$$\mathrm{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} =$$

$$= \mathbf{i} \cdot (a_y \cdot b_z - a_z \cdot b_y) + \mathbf{j} \cdot (a_z \cdot b_x - a_x \cdot b_z) + \mathbf{k} \cdot (a_x \cdot b_y - a_y \cdot b_x). \quad (1.7)$$

Sometimes, a vector triple product is used

$$\mathbf{d} = [\mathbf{c}[\mathbf{ab}]]. \quad (1.8)$$

A scalar triple product is also used

$$\mathrm{d} = (\mathrm{c}[\mathbf{ab}]). \quad (1.9)$$

Vectors can be polar (true vectors) and axial (pseudovectors). Vectors whose direction is related to that of rotation are called ***pseudovectors***.

A position vector or ***radius vector*** **r** of a point is defined as the vector drawn from the origin of coordinates to the given point (See Figure 1.3).

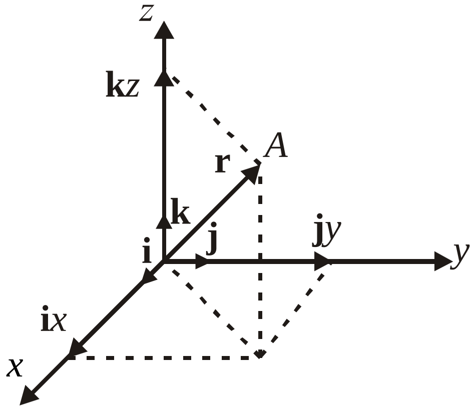


Figure 1.3.

Its projections upon the Cartesian coordinate system are *x*, *y*, *z*; so

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \quad (1.10)$$

and in accordance with Eq. (1.3), we have

$$r = \sqrt{x^2 + y^2 + z^2}\ . \quad (1.11)$$

1.2. Linear Kinematic Characteristics of Motion. General Case

A material point moves along a certain line. The latter is called ***a trajectory***. The length of the trajectory is called ***the distance*** traveled by the particle. The vector $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ (see Figure 1.4) is called ***a displacement vector*** (or just ***displacement***).

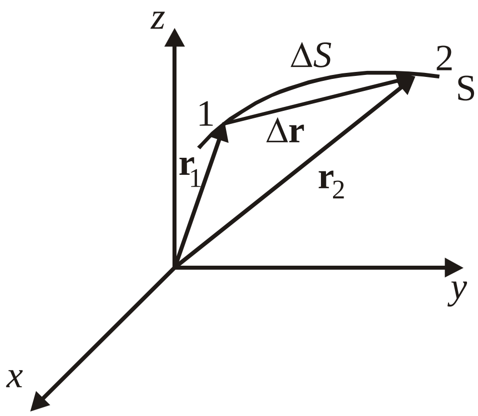


Figure 1.4.

The derivative of the displacement vector with respect to time is called ***the velocity***

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad (1.12)$$

Obviously, the velocity is a vector quantity. It is easy to understand that the direction of the velocity coincides with the tangent to the curve. Equation (1.12) can be written in terms of projections:

$$\mathbf{v} = \mathbf{i} \cdot \frac{dx}{dt} + \mathbf{j} \cdot \frac{dy}{dt} + \mathbf{k} \cdot \frac{dz}{dt}, \tag{1.13}$$

where

$$\frac{dx}{dt} = v_x, \; \frac{dy}{dt} = v_y, \; \frac{dz}{dt} = v_z$$

are components of the velocity vector. The modulus of the velocity vector is given by

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{1.14}$$

(Dots above symbols indicate the derivative of this quantity with respect to time).

If the motion is uniform and linear, then **v**=const and **a**=0. The derivative of **v** with respect to time is called ***the acceleration***, i.e.,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{i} \cdot \dot{x} + \mathbf{j} \cdot \dot{y} + \mathbf{k} \cdot \dot{z} \tag{1.15}$$

or

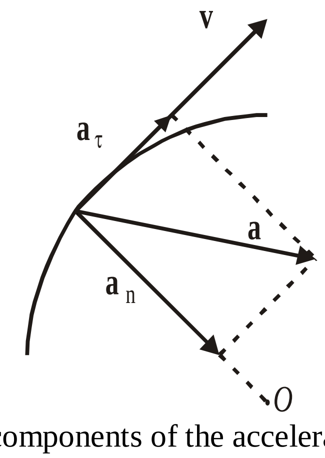
$$\mathbf{a} = \mathbf{i} \cdot \ddot{x} + \mathbf{j} \cdot \ddot{y} + \mathbf{k} \cdot \ddot{z}. \tag{1.16}$$

Here $\ddot{x} = \dot{v}_x = a_x$ and so on. Obviously,

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{1.17}$$

For the curvilinear motion it is convenient to represent the vector **a** as a sum of two components (see Figure 1.5)

$$\mathrm{a} = \mathrm{a}_\tau + \mathrm{a}_n. \tag{1.18}$$



Normal, $\mathbf{a}_n$, and tangential, $\mathbf{a}_\tau$, components of the acceleration vector. *O* is the center of curvature.
Figure 1.5.

The first, $\mathbf{a}_\tau$, has the direction coinciding with that of the tangent to the trajectory and therefore is called ***the tangential acceleration***. Its magnitude is

$$a_\tau = \frac{dv}{dt}. \qquad (1.19)$$

The second component $\mathbf{a}_n$ is directed along the normal to the trajectory toward the center of curvature. Its magnitude is

$$a_n = \frac{v^2}{R}, \qquad (1.20)$$

where R is the radius of curvature. It is clear that $\mathbf{a}_\tau$ and $\mathbf{a}_n$ are perpendicular.

1.3. Angular Kinematic Characteristics of Motion. Rotation

If a solid body rotates about a fixed axis, the trajectories of all its points are circular. The angular speeds of points are the same, but the linear velocities are different.

Let us assume that a particle moves along a circular trajectory in the XY plane (see Figure 1.6). The Z axis is said to be the axis of rotation. The rotation of the particle through an angle $d\varphi$ can be represented by the straight line whose length is $d\varphi$ and whose direction coincides with the axis of rotation. The direction of rotation about the given axis can be reckoned by ***the right-hand screw rule*** (Figure 1.7): it is the same as the direction of advance of a right-hand screw if rotated clockwise.

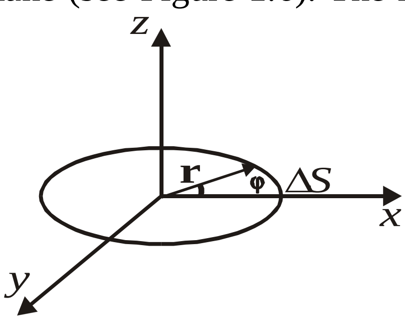


Figure 1.6.

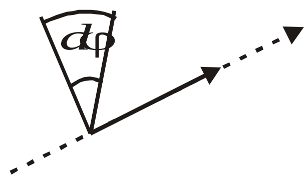


Figure 1.7.

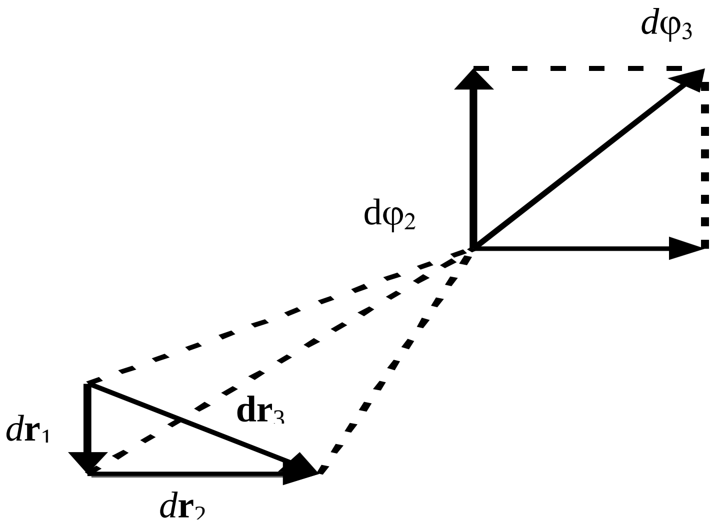


Figure 1.8.

The distance traveled by any point of the body (displacement) when rotated through a small angle ($d\varphi$) can be assumed the straight line (Figure 1.8). Consequently, two small circular motions $d\boldsymbol{\varphi}_1$ and $d\boldsymbol{\varphi}_2$ perfomed sequentially, as can be seen, result in the same displacement $d\mathbf{r}_3 = d\mathbf{r}_1 + d\mathbf{r}_2$ of any point of the body as the circular motion $d\boldsymbol{\varphi}_3$ obtained from $d\boldsymbol{\varphi}_1$ and $d\boldsymbol{\varphi}_2$ by the parallelogram method. Hence it follows that elementary angular displacements can be considered as vectors (we denote these vectors by $d\boldsymbol{\varphi}$). So we can write

$$d\boldsymbol{\varphi}_3=d\boldsymbol{\varphi}_1+d\boldsymbol{\varphi}_2+... \tag{1.21}$$

Owing to the fact that the direction of the vector $d\boldsymbol{\varphi}$ is associated with the direction of rotation of the body, $d\boldsymbol{\varphi}$ is a pseudovector rather than a true vector. It should be noted that rotations through finite angles cannot be added by the parallelogram method and are therefore not vectors. The vector quantity

$$\omega = \frac{d\varphi}{dt} \tag{1.22}$$

is called ***the angular velocity***. The angular velocity $\boldsymbol{\omega}$ is directed along the axis about which the body rotates in accordance with the right-hand screw rule (Figure 1.9).

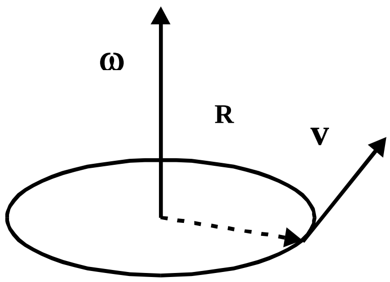


Figure 1.9

The magnitude of the angular velocity is usually called the angular speed. If $\boldsymbol{\omega}$ is constant, the motion is called ***angular uniform***. For a uniform circular motion, we can write

$$\omega = \frac{\varphi}{t}, \tag{1.23}$$

where φ is the finite angle of rotation during the time t (compare with $v = \frac{s}{t}$). The uniform circular motion can be characterized by ***the period of revolution*** T. This quantity is defined as the time over which a body completes one revolution, i.e., rotates through an angle of 360 degrees or 2 radians. Since the time interval $\Delta t=T$ corresponds to the angle of rotation $\Delta\varphi=2\pi$,

$$\omega = \frac{2\pi}{T} \tag{1.24}$$

and

$$T = \frac{2\pi}{\omega}. \tag{1.25}$$

The number of revolutions per unit time (frequency) is equal to

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \tag{1.26}$$

or

$$\omega = 2\pi\nu . \tag{1.27}$$

If angular motion is nonuniform, it can be characterized by ***the angular acceleration*** $\boldsymbol{\varepsilon}$. This quantity is defined as

$$\boldsymbol{\varepsilon} = \frac{d\boldsymbol{\omega}}{dt} \tag{1.28}$$

or

$$\boldsymbol{\varepsilon} = \frac{d^2\varphi}{dt^2} \tag{1.29}$$

(compare with $\mathbf{a} = \frac{d\mathbf{v}}{dt}$ and $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$). In general, the directions of ω and ε do not coincide. But if the direction of the rotation axis does not change, we can write $\boldsymbol{\varepsilon} \uparrow\uparrow \boldsymbol{\omega}$ if ε > 0 and $\boldsymbol{\varepsilon} \uparrow\downarrow \boldsymbol{\omega}$ if **ε** < 0.

There are simple relations between the linear and angular characteristics of motion. Let us find them using Figure 1.10. Examination of the figure shows that the direction of the vector product [**ω**·**r**] coincides with vector **v**, and its magnitude is $\omega r \sin\alpha = \omega R$.

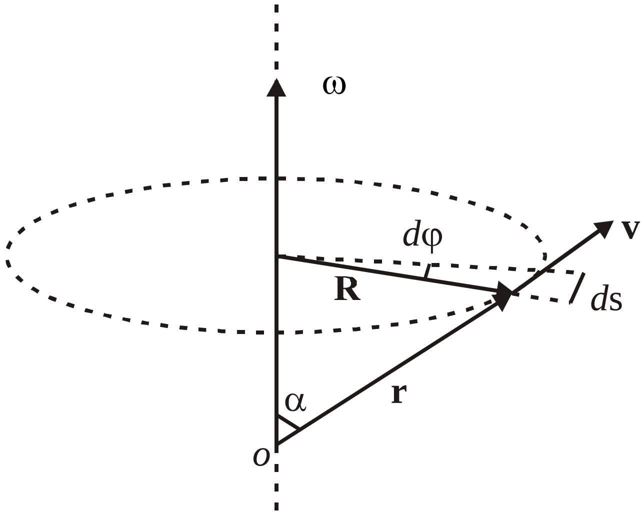


Figure 1.10.

But it is quite clear that

$$v = \frac{ds}{dt} = R \cdot \frac{d\varphi}{dt} = R \cdot \omega. \tag{1.30}$$

So we can write

$$\mathbf{v} = [\boldsymbol{\omega}\mathbf{r}]. \tag{1.31}$$

Using the conventional expression

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

we have

$$\mathbf{a} = \frac{d}{dt}[\boldsymbol{\omega}\mathbf{r}] = [\dot{\boldsymbol{\omega}}\mathbf{r}] + [\boldsymbol{\omega}\dot{\mathbf{r}}] = [\boldsymbol{\varepsilon}\mathbf{r}] + [\boldsymbol{\omega}\mathbf{v}]. \tag{1.32}$$

The first term in the above equation is called ***a tangential acceleration***

$$\mathbf{a}_\tau = [\boldsymbol{\varepsilon}\mathrm{r}]. \tag{1.33}$$

The second term is called ***a normal acceleration***

$$\mathbf{a}_n = [\boldsymbol{\omega}\mathbf{v}]. \tag{1.34}$$

If we introduce the vector **R** drawn from the axis of rotation to the given point of the body at the right angle to the axis, then we can write (using the expression $v = R\cdot\omega$)

$$a_n = \omega^2 \cdot R, \tag{1.35}$$

$$a_\tau = \varepsilon \cdot R, \tag{1.36}$$

$$a_\tau = -\omega^2 \cdot R \tag{1.37}$$

So, the normal and tangential accelerations linearly increase with distance from the axis of rotation to the point.

2. Dynamics of a Point Particle

2.1. Newton's Laws of Motion

Kinematics describes the motion of bodies without consideration of the cause why a body moves exactly in a given way and not in a different one. Dynamics studies the motion of bodies with reference to the cause of the motion (the interactions between bodies). The so-called classical or Newtonian mechanics is based on three laws of motion that were formulated by Isaac Newton in 1687. Newton's laws (like all other laws of physics) were the result of generalizing many experimental facts.

Newton's first law is formulated as follows: a body at rest remains at rest, and a body in motion remains in uniform motion in a straight line unless acted on by an external unbalanced force. Both states are characterized by the acceleration equal to zero. Therefore, the first law can also be formulated as follows: the velocity of a body remains constant (in particular, zero) until the action of other bodies causes it to change. So

$$\mathbf{a} = 0, \text{ if } \mathbf{F} = 0. \qquad (2.1)$$

Newton's second law is formulated as follows: the rate of change of the momentum of a body is equal to the force **F** acting on the body:

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}. \qquad (2.2)$$

This equation is called the ***equation of motion of a body***.

Substituting $m\mathbf{v}$ for **P** and taking into account that in Newtonian mechanics the mass is assumed constant, we have

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt}(m\mathbf{v}) = m\mathbf{a}, \text{ i.e., } \mathbf{F} = m\mathbf{a}. \qquad (2.3)$$

Newton's third law is formulated as follows: the forces exerted by interacting bodies on each other are equal in magnitude and opposite in direction, i.e.,

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \qquad (2.4)$$

2.2. Inertial Reference Frames. Galilean Principle of Relativity

The reference frames where Newton's first law holds are called inertial. Any reference frame which moves at a constant velocity relative to the given inertial frame is also ***inertial***. A reference frame which moves with an acceleration relative to an inertial reference frame is called ***noninertial***. It is obvious that there are an infinite number of reference frames of both types.

It has been established experimentally that the reference frame whose center coincides with the Sun and whose axes are directed toward some stars is an inertial one. This system is called ***the heliocentric reference frame***.

The Earth moves relative to the Sun and stars along a curvilinear trajectory.

The Earth also rotates about its axis. Because of this, a reference frame affixed to the Earth moves with an acceleration relative to the heliocentric one and is not inertial in the strict sense. However, the acceleration of the frame is small and this frame may be considered inertial in many cases. Accelerations caused by the rotation of the Earth about its axis and the Sun are very small (in comparison with the free fall acceleration g = 9.8 m/s^2), and are equal to $3.4 \cdot 10^{-2}$ and $6 \cdot 10^{-3}$ m/s^2, respectively.

Transition from one inertial frame to another can be performed with the help of the Galilean transformation. If the inertial frame $K'(x', y', z')$ moves with the constant velocity $\mathbf{v}_0$ relative to the inertial frame $K(x, y, z)$ (see Figure 1.11), we can write

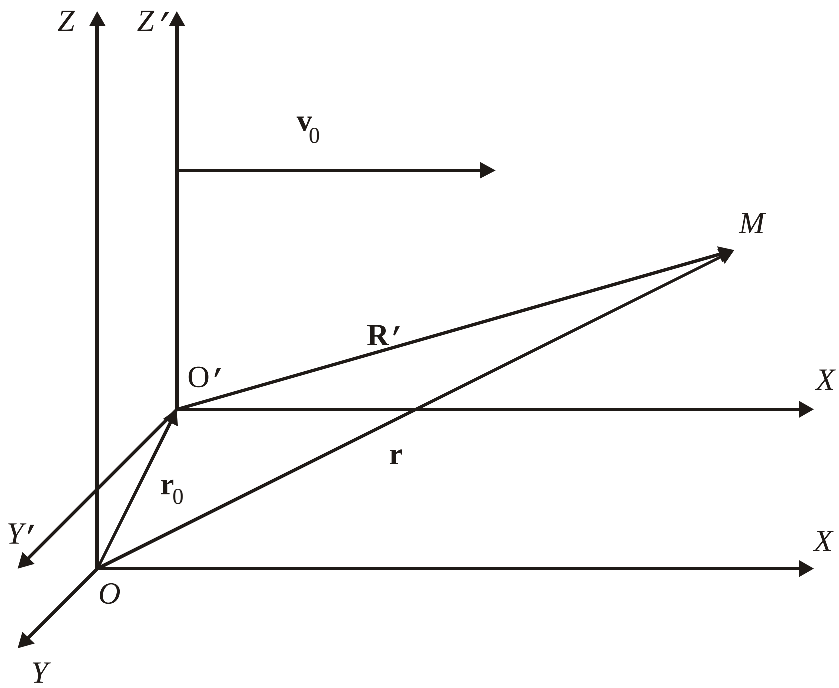


Figure 1.11.

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{r}'. \qquad (2.5)$$

Here $\mathbf{r}$ and $\mathbf{r}'$ specify positions of a body in coordinate systems x, y, z and x', y', z'; $\mathbf{r}_0$ is the radius-vector of the origin of the coordinate system x', y', z' in the coordinate system x, y, z.

The relation between the coordinates of systems K and K' can be easily found. If we count time from the moment when the x coordinates of two frames coincide $x' = x = 0$, then it is easy to see that

$$x = x' + v_0 \cdot t,\ y = y' + a,\ z = z' + b, \qquad (2.6)$$

where a and b are constants. These relations are called ***the Galilean transformation***.

In classical mechanics, the time is the same in both reference frames. So, for the velocities we have (using Eq. (2.5))

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 \qquad (2.7)$$

or

$$v_x = v_x' + v_0,\ v_y = v_y',\ v_z = v_z',\ \mathbf{a} = \frac{d}{dt}(\mathbf{v}_0 + \mathbf{v}_1)$$

or

$$\mathbf{a} = \mathbf{a}'. \qquad (2.8)$$

The acceleration of a body does not depend on the relative velocity of an inertial frame and is the same in each inertial frame. The body's mass (v << c) in each inertial reference frame is also the same. Just because of this, it is impossible to say whether an inertial frame moves or does not move (using any physical devices which are inside the inertial frame). In every inertial reference frame, all mechanical laws are the same. The equation of motion does not change in going from one reference frame to another.

2.3. Force. Mass

Four interaction types are distinguished in modern physics: (1) gravitational, (2) electromagnetic, (3) strong, and (4) weak.

1. ***Gravitational*** interaction is caused by the universal gravitation. The gravitational attraction acting between two point masses is given by Newton's universal law of gravitation

   $$F = G \cdot \frac{m_1 \cdot m_2}{r^2}, \qquad (2.9)$$

   where m_1, m_2 are the masses of the bodies, r is the distance between them, and G is the gravitational constant.
2. ***Electromagnetic*** interaction is due to an electromagnetic field. The electric component of this field acting between two point electric charges q_1 and q_2 is defined by Coulomb's law

   $$F = k \cdot \frac{q_1 \cdot q_2}{r^2}, \qquad (2.10)$$

   where k is the coefficient depending on a chosen system of units, and r is the distance between the point electric charges. If the charges are moving, then magnetic forces act on them in addition to the forces defined by Eq. (2.10)

   $$\mathbf{F} = k' q[\mathbf{vB}]. \qquad (2.11)$$

   Here **B** is the magnetic induction, and k' is the proportionality factor.
3. ***Strong*** interaction holds the particles in the atomic nucleus together.
4. ***Weak*** interaction is involved in the radioactive decay of some nuclei when so-called leptons (electrons, positrons, muons, and neutrinos) are emitted.

In classical mechanics, we deal with gravitational and electromagnetic forces and also with elastic and friction forces. The last two forces have an electromagnetic origin and are determined by the nature of the interaction between molecules of a substance.

Gravitational and electromagnetic forces are fundamental, they cannot be reduced to any other simpler forces. Elastic and friction forces, on the other hand, are not fundamental.

Mass is a scalar physical quantity. On the one hand, it is the measure of inertia, and on the other hand, it determines the gravitational properties of a body. The concept of mass was introduced by I. Newton when he defined the momentum $\mathbf{p} = m\mathbf{v}$ and the force $\mathbf{F} = m\mathbf{a}$. The mass acts as a source of the gravitational field. In the theory of gravitation (using Eq. (2.9)), the free fall acceleration can be written in the form:

$$g = G \cdot \frac{m_e}{R_e^2}. \tag{2.12}$$

Here m_e and R_e are the Earth's mass and radius, respectively. The weight of a body on the Earth's surface may be expressed as

$$p = m \cdot g. \tag{2.13}$$

Numerous experimental facts indicate that the inertial and gravitational masses of bodies are directly proportional to each other. This indicates that these masses became identical if the units of measurements are properly selected. This statement expresses ***the equivalence principle***.

2.4. Space and Time

The space and time are the basic concepts of physics describing the order of all events in the Universe. In accordance with Newton, the space, the time, and the matter are independent; the space is uniform and isotropic, and the time is absolute.

Later (in the early twentieth century) it was shown that the properties of space and time are not absolute and can vary when a body moves with a high velocity ($v \cong c$) or a strong gravitational field is present. These concepts are the main principles of relativity. The expression "space-time continuum" is widely used in the special theory of relativity some aspects of which will be discussed in the next sections.

3. Work and Mechanical Energy

3.1. Energy. Work of a Force

Energy is a scalar physical quantity that is the measure of different forms of motion of matter and the interactions associated with this motion. For different forms of motion, different types of energy are introduced (for example, mechanical, internal, electromagnetic, chemical, nuclear, heat energy, and so on).

Classical mechanics deals with the mechanical energy which is the measure of mechanical motion of a body or a system of bodies.

The mechanical energy of a body changes if external forces act upon the body. In order to describe this process, the physical concept of mechanical work done by some forces is introduced.

The elementary ***work*** *dA* produced by a force **F** exerted on an object and the distance *ds* the object moves in the direction of the force is defined by the scalar product of **F** and *d***r** (*d***r** is the displacement vector)

$$dA = (\mathbf{F} \cdot d\mathbf{r}) = F \cdot dr \cdot \cos\alpha = F_t \cdot dr. \qquad (3.1)$$

Here α is the angle between **F** and *d***r**, F_t is the projection of **F** upon the vector *d***r**. Total work can be expressed by a definite integral

$$A_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r}. \qquad (3.2)$$

If the function $F_t(r)$ is known (Fig.1.12), Eq. (3.2) can be written in the form

$$A_{12} = \int_{r_1}^{r_2} F_t(r) \cdot \mathrm{d}r. \qquad (3.3)$$

So numerically it is equal to the area of irregular tetragonal r_1-1-2-r_2.

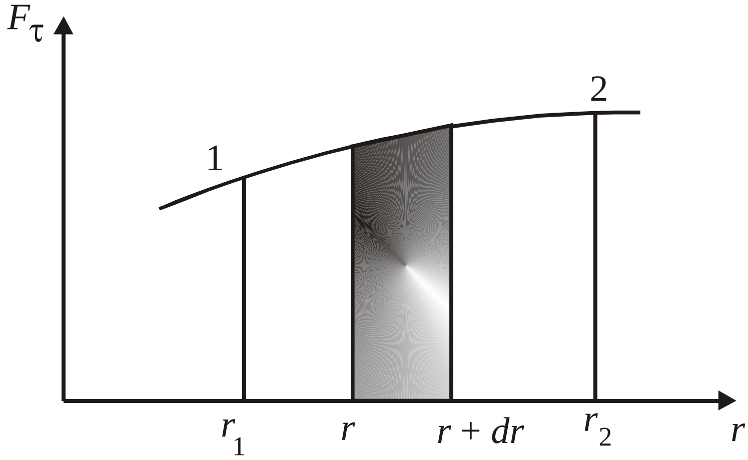


Figure 1.12.

In terms of projections, Eq. (3.2) can be represented as the sum of three definite integrals

$$A_{12} = \int_{x_1}^{x_2} F_x \cdot dx + \int_{y_1}^{y_2} F_y \cdot dy + \int_{z_1}^{z_2} F_z \cdot dz. \qquad (3.4)$$

Sometimes it is very convenient to calculate the amount of work done using an expression

$$A_{12} = \int_{t_1}^{t_2} Fv dt. \qquad (3.5)$$

Conservative forces can be defined in two ways:

1. as forces whose work does not depend on the path along which a particle moves from one point to another,
2. as forces whose work along any closed path is equal to zero.

Forces acting in electrostatic and stationary gravitational fields are just the forces of this type.

3.2. Kinetic and Potential Energy

Let us consider the simplest system consisting of a single point particle. The equation of motion of the particle is

$$m\dot{\mathbf{v}} = \mathbf{F}. \tag{3.6}$$

Here **F** is the ***resultant*** force acting on the particle. Multiplying Eq. (3.6) by the displacement of the particle $d\mathbf{r}=\mathbf{v}\cdot dt$, we obtain

$$m\mathbf{v}\dot{\mathbf{v}}dt = \mathbf{F}d\mathbf{r}. \tag{3.7}$$

Using Eq. (3.7) we obtain

$$d\left(\frac{m\cdot\mathbf{v}^2}{2}\right) = \mathbf{F}d\mathbf{r}. \tag{3.8}$$

If the system is closed, i.e. **F**=0, then the quantity

$$E_k = \frac{mv^2}{2} \tag{3.9}$$

remains constant. This quantity is called ***the kinetic energy*** of the particle. For an isolated particle the kinetic energy is ***an integral of motion***. If there are several particles in a mechanical system, the total kinetic energy is the sum of their kinetic energies, i.e.,

$$E_k = \sum_{(i)} E_{ki}. \tag{3.10}$$

When a particle moves from point 1 to point 2, then using Eq. (3.10) we obtain

$$E_2 - E_1 = \int_1^2 \mathbf{F}d\mathbf{r} \tag{3.11}$$

or

$$\Delta E = A. \tag{3.12}$$

In other words, the work done by the external forces acting upon a particle equals the increment of the kinetic energy of the particle. It is easy to see that

$$E_k = \frac{p^2}{2m}. \tag{3.13}$$

For many physical applications, it is possible to represent the force in the form

$$\mathbf{F} = -\mathrm{grad}U, \tag{3.14}$$

$$\mathrm{grad}U = \mathbf{i}\frac{\partial U}{\partial x} + \mathbf{j}\frac{\partial U}{\partial y} + \mathbf{k}\frac{\partial U}{\partial z} \tag{3.15}$$

or using the Laplacian operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{3.16}$$

we have

$$\mathbf{F} = -\nabla U. \tag{3.17}$$

The scalar quantity U is called ***the potential energy***. Fields whose force can be expressed by Eq. (3.17) are called the potential fields.

In general physics it is proved that the potential energy numerically equals to the work done by the external forces when a particle moves from the given point **r** to the infinity, i.e.,

$$U(r) = \int_r^\infty \mathbf{F} d\mathbf{r} . \qquad (3.18)$$

Using Eqs. (3.17)-(3.18), it is easy to derive formulas of the potential energy for:

a spring, $U = -\frac{k \cdot x^2}{2}$,

two interacting electric charges, $U = k \cdot \frac{q_1 \cdot q_2}{r}$,

two mass points, $U = -c \cdot \frac{m_1 \cdot m_2}{r}$,

a mass point in a uniform gravitational field, $U=mgh$,

so-called centrifugal potential, $U = \frac{m \cdot w^2 \cdot r^2}{2}$, and so on.

4. Mechanics of a Rigid Body

4.1. Center of Mass of a Body

In some applications, a body cannot be represented by a point mass. However, by dividing the body into elementary masses m_i, we can represent it as a system of point particles whose arrangement remains unchanged. Any elementary mass is acted upon by inertial forces due to its interaction with other elementary masses of the body being considered and by external forces. For each elementary mass we have

$$m\mathbf{a}_i = \mathbf{f}_i + \mathbf{F}_i , \qquad (4.1)$$

where $\mathbf{f}_i$ and $\mathbf{F}_i$ are the resultant internal ($\mathbf{f}_i$) and external ($\mathbf{F}_i$) forces exerted to the given elementary mass. Summation of Eqs. (4.1) over the elementary masses yields

$$\sum m\mathbf{a}_i = \sum \mathbf{f}_i + \sum \mathbf{F}_i . \qquad (4.2)$$

The sum of internal forces acting in the system equals zero (in accordance with Newton's third law). So we have

$$\sum m_i \mathbf{a}_i = \sum \mathbf{F}_i . \qquad (4.3)$$

Here $\mathbf{F} = \sum \mathbf{F}_i$ is the resultant external force acting on the body. To simplify Eq. (4.3), the concept of ***the center of mass*** ($\mathbf{r}_c$) of the body can be introduced

$$\mathbf{r}_c = \frac{1}{m} \cdot \sum m_i \cdot \mathbf{r}_i \qquad (4.4)$$

Here

$$m = \sum m_i \qquad \textbf{(}4.5)$$

is the total mass of the body. Differentiating Eq. (4.4) twice with respect to time and taking into account that $\ddot{\mathbf{r}}_i = \mathbf{a}_i$ and $\ddot{\mathbf{r}}_c = \mathbf{a}_c$, we obtain

$$\sum m_i \cdot \mathbf{a}_i = m \cdot \mathbf{a}_c . \qquad (4.6)$$

Comparing Eqs. (4.3) and (4.6), we can write

$$m \cdot \mathbf{a}_c = \sum \mathbf{F}_i . \qquad (4.7)$$

Thus, the center of mass of a rigid body moves in the same way as a point particle whose mass is equal to that of the body.

4.2. Torque. Angular Momentum. Rotary Inertia

These physical quantities are introduced to describe the rotation of a rigid body about an axis.

Torque is defined by the formula

$$\mathbf{M} = [\mathbf{rF}] . \qquad (4.8)$$

Angular momentum is defined by the formula

$$\mathbf{L} = [\mathbf{rP}] . \qquad (4.9)$$

Here **r** is the radius-vector, **F** is the force, and **P** is the momentum. For the rotary motion, the fundamental equation of angular motion holds

$$\frac{d\mathbf{L}}{dt} = \mathbf{M} . \qquad (4.10)$$

This equation holds for any rotational motion, but it can be considerably simplified if the particle moves along a circular trajectory. In this case, the origin of coordinates can be placed on the axis of rotation. So $\mathbf{r} \perp \mathbf{v}$, $|\mathbf{r}| = \mathrm{r} = \mathrm{const}$, and we can write

$$\mathbf{L} = mr^2 \boldsymbol{\omega} , \qquad (4.11)$$

$$\mathbf{M} = mr^2 \frac{d\boldsymbol{\omega}}{dt} . \qquad (4.12)$$

The physical quantity $m \cdot r^2$ is called ***the rotary inertia*** or the moment of inertia (I) of the mass point

$$I = mr^2 . \qquad (4.13)$$

Using this concept, we can express Eqs. (4.11) and (4.12) in the form

$$\mathbf{L} = I\boldsymbol{\omega} , \qquad (4.14)$$

$$\mathbf{M} = I \frac{d\boldsymbol{\omega}}{dt} . \qquad (4.15)$$

For the given example of circular motion of the mass point, vectors **L** and **M** are collinear with the axis of rotation. But for arbitrary rotational motion the situation is not so simple.

From the definition of the rotary inertia (Eq. (4.13)) it is clear that it is an additive quantity. This means that the rotary inertia of a body is equal to the sum of the moments of inertia of its parts

$$I = \sum \Delta m_i r_i^2 . \qquad (4.16)$$

Here, instead of m_i we use the symbol Δm_i to stress the fact that we tentatively subdivided the continuous rigid body into elementary parts.

Having in mind that $\Delta m = \rho dV$ (ρ is the density of the body), the rotary inertia can be expressed by the formula

$$I = \int_{(V)} \rho r^2 dV . \quad (4.17)$$

Here r is the distance between the given volume element dV and the axis of rotation. It should be noted that every body has a certain rotary inertia about an arbitrary axis whether it rotates or stays at rest.

4.3. Steiner Theorem

The Steiner theorem on parallel axes is formulated as follows: the rotary inertia I about an arbitrary axis equals the moment of inertia I_c about an axis parallel to the given one and passing through the body center of mass plus the product of the body mass m and the square distance (a) between the axes

$$I = I_c + ma^2 . \quad (4.18)$$

The Steiner theorem is widely used in many physical applications. The value of I_c is usually tabulated (for regular bodies), and the only thing you must do is just to apply this theorem. For example (see Fig. 1.13), the rotary inertia of a disk (mass m) about point a can be expressed, according to the Steiner theorem, as follows:

$$I_a = I_o + mR^2 = \frac{1}{2}mR^2 + mR^2 = \frac{3}{2}mR^2 . \quad (4.19)$$

The value $I_o = \frac{1}{2}mR^2$ is tabulated.

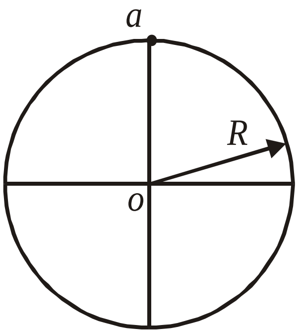


Figure 1.13.

4.4. Rotational Kinetic Energy. Rotational Work

A rigid body can be represented as a system of mass points. The kinetic energy of every mass point is $E_i = \frac{1}{2} \cdot m_i \cdot v_i^2$. Obviously, $v_i = r_i \omega$ (ω is the angular speed and r_i is the distance between the given mass point and the axis of rotation). Taking into account that kinetic energy is an additive quantity, we derive an expression for the total kinetic energy of the rotating body

$$E = \sum E_i = \frac{\omega^2}{2} \sum m_i r_i^2 = \frac{I\omega^2}{2} . \quad (4.20)$$

In general, the work done by the torque acting upon a body can be expressed by the formula

$$\delta A = \mathbf{M} \cdot d\boldsymbol{\varphi} . \quad (4.21)$$

It is easy to see that for a body in circular motion Eq. (4.21) can be written in the form

$$\delta A = M \cdot d\varphi. \qquad (4.22)$$

Indeed, $\delta A = F \cdot ds = F \cdot r \cdot d\varphi = M \cdot d\varphi$.

Table 1.4 compares the formulas of mechanics of rotation with the similar formulas of mechanics of translation. This comparison shows that in all cases of rotation, the part of mass is played by the rotary inertia, the part of force by the torque, the part of momentum by the angular momentum, the part of radius vector by the angle of rotation, and so on.

Table 1.4. Comparison of translational and rotational motion characteristics.

Translation	Rotation
R = radius vector	**φ** = angle of rotation
v = linear velocity	**ω =** angular velocity
a= $\dot{\mathbf{v}}$ **=** linear acceleration	**ε =** $\dot{\omega}$ = angular acceleration
m = mass	*I* = rotary inertia
p = m·**v** = momentum	**L** = *I*·**ω** = angular momentum
F = force	**M** = torque
$\dot{\mathbf{p}}$ = **F**	$\dot{\mathbf{L}}$ = **M**
m·**a** = **F**	$I \cdot \varepsilon_z = M_z$ (*z* is the axis of rotation)
$E_k = \frac{1}{2} m \cdot v^2$	$E_k = \frac{1}{2} I \cdot \omega^2$ (for the axis of rotation)
$\delta A = M \cdot dr$ = work	$\delta A = M_z \cdot d\varphi$ = work (*z* is the axis of rotation)

It should be noted that in general case *I* is a tensor (the quantity characterized by nine components) rather than a scalar and Eqs. (4.20) and (4.14) assume the form:

$$E_k = \frac{1}{2} \sum I_{ik} \cdot \omega_i \cdot \omega_k, \qquad (4.23)$$

$$i, k = x, y, z,$$

$$L = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \cdot \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}. \qquad (4.24)$$

EXAMPLE 1

A boat sails due north and crosses a wide river with a speed of 10.0 km/h relative to water. The river flows due east with a uniform speed of 5.00 km /h relative to the Earth. Determine the velocity of the boat relative to a ground observer.

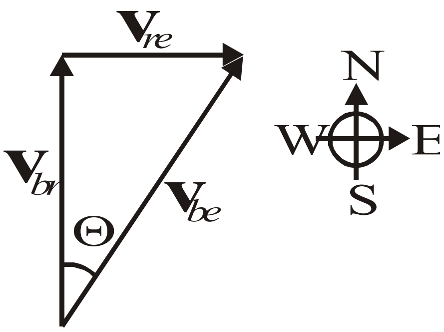
Solution. Given:

$\mathbf{v}_{br}$ = the velocity of the boat, *b*, relative to the river, *r*

$\mathbf{v}_{re}$ = the velocity of the river, *r*, relative to the Earth, *e*.

Required: $\mathbf{V}_{be}$, the velocity of the boat relative to the Earth. The relationship between these three quantities is

$$\mathbf{V}_{be} = \mathbf{V}_{br} + \mathbf{V}_{re}.$$



The terms in the equation are vector quantities; the vectors are shown in Figure 1. The quantity $\mathbf{V}_{br}$ is due north, $\mathbf{V}_{re}$ is due east, and the vector sum of the two, $\mathbf{V}_{be}$, is at angle Θ, as shown in Figure 1. Thus, the speed of the boat relative to the Earth can be found from the Pythagoras theorem:

Figure 1.

$$\upsilon_{be} = \sqrt{\upsilon_{br}{}^2 + \upsilon_{re}{}^2} = \sqrt{10.0^2 + 5.0^2} = 11.2 \text{km/h}.$$

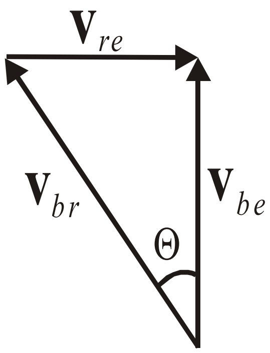
The direction of $\mathbf{V}_{be}$ is

$$\Theta = \tan^{-1}\left(\frac{\upsilon_{re}}{\upsilon_{br}}\right) = \tan^{-1}\left(\frac{5.0}{10.0}\right) = 26.6^\circ.$$

Therefore, the boat will sail at a speed of 11.2 km/h relative to the Earth in the northeast direction at an angle of 26.6° to the north direction.

EXAMPLE 2

If the boat of the preceding example sails due north with the same speed (10.0 km/h) relative to water, as shown in Figure 2, what will be its direction?



Solution. As in the previous example, we know V_{br} and V_{re} and we want to find V_{be}. The relationship between these three quantities, $\mathbf{V}_{be} = \mathbf{V}_{br} + \mathbf{V}_{re}$, is shown in Figure 2. That is, the boat must sail upstream in order to cross the river due north. The speed V_{be} can be found from the Pythagoras theorem

$$\text{V}_{be} = \sqrt{\text{V}_{br}{}^2 - \text{V}_{re}{}^2} = \sqrt{10.0^2 - 5.0^2} = 8.66 \text{ km/h}.$$

Figure 2.

The direction of $\mathbf{V}_{be}$ is

$$\Theta = \tan^{-1}\left(\frac{\text{V}_{re}}{\text{V}_{br}}\right) = \tan^{-1}\left(\frac{5.0}{8.66}\right) = 30.0^\circ.$$

So, the boat must sail northwest, at an angle of 30.0° to the north direction.

EXAMPLE 3

A particle starts from the origin at t = 0 with an initial velocity having an *x* component of 20 m/s and an *y* component of -15 m/s. The particle moves in the *xy* plane with the *x* component of acceleration only, $a_x = 4.0 \text{ m/s}^2$.

(a) Determine the velocity components as functions of time and the total velocity vector at time *t*.

Solution. With $\upsilon_{xo} = 20\,\text{m/s}$ and $a_x = 4.0\,\text{m/s}^2$, the kinematic equations give

$$\upsilon_x = \upsilon_{xo} + a_x t = (20 + 4t)\ \text{m/s}.$$

Also, with $\upsilon_{yo} = -15\,\text{m/s}$ and $a_y = 0$,

$$\upsilon_y = \upsilon_{yo} = -15\,\text{m/s}.$$

Therefore, using these results and noting that the velocity vector **V** has two components, we obtain

$$\mathbf{V} = \upsilon_x \mathbf{i} + \upsilon_y \mathbf{j} = (20 + 4.0t)\mathbf{i} - 15\mathbf{j}\ \text{m/s}.$$

(b) Calculate the velocity and speed of the particle at $t = 5.0\,\text{s}$.

Solution. At $t = 5.0\,\text{s}$, from (a) we obtain

$$\mathbf{V} = [20 + 4(5.0)t]\mathbf{i} - 15\mathbf{j}\ \text{m/s} = (40\mathbf{i} - 15\mathbf{j})\ \text{m/s}.$$

That is, at $t = 5.0\,\text{s}$, $\upsilon_x = 40\,\text{m/s}$ and $\upsilon_y = -15\,\text{m/s}$. Knowing these two components, we know the velocity vector. To determine the angle (that **V** makes with the *x* axis), we take advantage of the fact that $\tan\theta = \upsilon_x / \upsilon_y$, or

$$\theta = \tan^{-1}\left(\frac{\upsilon_y}{\upsilon_x}\right) = \tan^{-1}\left(\frac{-15.0}{40}\right) = -21^\circ.$$

The speed is the magnitude of $\mathbf{V}$:

$$\upsilon = |\mathbf{V}| = \sqrt{\upsilon_x{}^2 + \upsilon_y{}^2} = \sqrt{40^2 + (-15)^2} = 43\,\text{m/s}.$$

(Note: If you calculate υ_o from the *x* and *y* components of υ, you will find that $\upsilon > \upsilon_o$. Why?)

(c) Determine the *x* and *y* coordinates of the particle at time *t* and the displacement vector at this time.

Solution. Since at $t = 0$, $x_0 = y_0 = 0$, the kinematic equation gives

$$x = \upsilon_{x0} t + a_x t^2 / 2 = 20t + 2.0t^2\ \text{m}.$$

Therefore, the displacement vector at time *t* is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = \left[\left(20t + 2.0t^2\right)\mathbf{i} - 15t\mathbf{j}\right].$$

Thus, for example, at *t* = 5.0 s, *x* = 150 m and *y* = 75 m, $\mathbf{r} = (150\mathbf{i} - 7\mathbf{j})$. Hence it follows that the distance of the particle from the origin to this point is the magnitude of the displacement:

$$|\mathbf{r}| = r = \sqrt{150^2 + (-75)^2} = 170\,\text{m}.$$

Note that this is ***not*** the distance the particle travels in this time! Can you determine this distance from the available data?

EXAMPLE 4

A mass m_1 on a rough, horizontal surface is connected to a second mass m_2 by a weightless cord over a weightless, frictionless pulley, as shown in Figure 4a. A force of magnitude F at an angle Θ with the horizontal is applied to m_1. The coefficient of kinetic friction between m_1 and the surface is μ. Determine the magnitude of the acceleration of the masses and the tension in the cord.

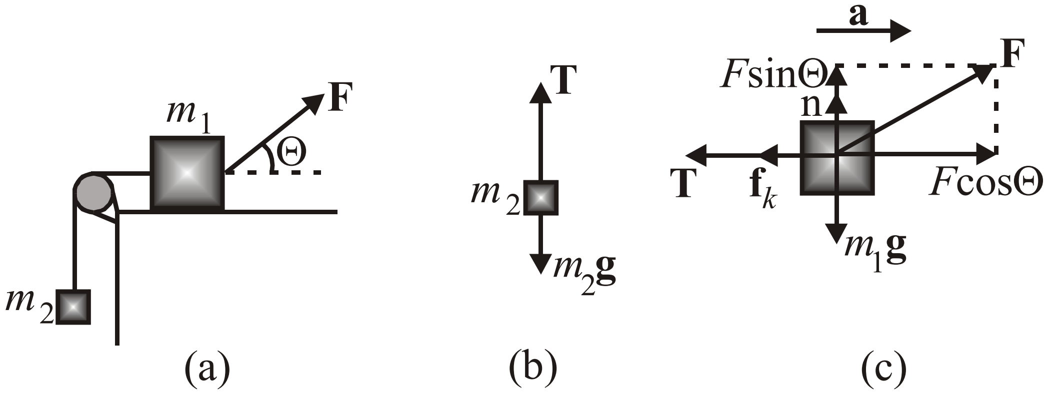


Figure 4.

Reasoning. First we draw the free-body diagrams of m_1 and m_2, as in Figures 4b and 4c. Then, we apply Newton's second law to each block and take advantage of the fact that the magnitude of the force of kinetic friction is proportional to the normal force $f_k = \mu n$. Finally, we solve for the acceleration in terms of the parameters given.

Solution. The applied force **F** has the components $F_x = F\cos\theta$ and $F_y = F\sin\theta$.

Applying Newton's second law to both masses and assuming the motion of m_1 is to the right, we obtain

Motion of m_1:

$$\sum F_x = F\cos\theta = f_k - T = m_1 a,$$

$$\sum F_y = n + F\,sin\,\theta = m_1 g = 0. \qquad (1)$$

Motion of m_2:

$$\sum F_x = 0,$$

$$\sum F_y = T - m_3 g = m_2 a \qquad (2)$$

But $f_k = \mu n$, and from Eq. (1), $n = m_1 g - F\sin\theta$ (note that in this case n is not equal to $m_1 g$; therefore,

$$f_k = \mu(m_1 g - F\,sin\,\boldsymbol{\Theta}). \qquad (3)$$

That is, the frictional force is reduced because of the positive y component of F. Substituting Eq. (3) and the value of **T** from Eq. (2) into Eq. (1), we obtain

$$F\cos\theta - \mu(m_1 g - F\sin\theta) - m_2(a+g) = m_1 a.$$

Solving for a, we obtain

$$a = \frac{F(\cos\theta + \mu\sin\theta) - g(m_2 + \mu m_1)}{m_1 + m_2}. \qquad (4)$$

We can find T by substituting this value of a into Eq. (2). Note that the acceleration for m_1 can be directed either to the right or to the left, depending on the sign of the numerator in Eq. (4). If the motion of m_1 is to the left, we must reverse the sign of f_k, because the frictional force must oppose the motion. In this case, the value of a is the same as in Eq. (4) with μ replaced by $-\mu$.

EXAMPLE 5

Find the moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop passing through its center (Figure 5).

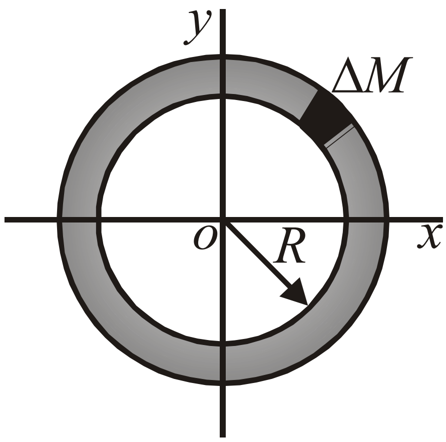


Figure 5.

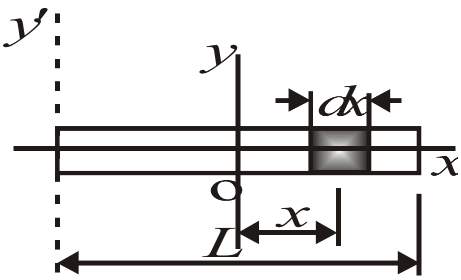
Solution. All mass elements are at the same distance $r = R$ from the axis, so, applying the equation for rotary inertia, for the moment of inertia about the z axis passing through O we obtain

$$I_z = \int r^2 dm = R^2 \int dm = MR^2 .$$

The mass elements of the uniform hoop are all at the same distance from O.

EXAMPLE 6

Calculate the moment of inertia of a uniform rigid rod of length L and mass M (Figure 6) about an axis perpendicular to the rod (the y axis) and passing through its center of mass.



Uniform rigid rod of length L. The moment of inertia about the y axis is less than that about the y' axis.

Figure 6.

Solution. The shaded element of length dx has the mass dm equal to the mass per unit length multiplied by dx:

$$dm = \frac{M}{L} dx .$$

Substituting this expression for dm into equation for rotary inertia ($r = x$), we obtain

$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx = \frac{M}{L} \int_{-L/2}^{L/2} x^2 dx = \frac{M}{L} \left[x^3 / 3 \right]_{-L/2}^{L/2} = ML^2 / 12 .$$

Exercise. Calculate the moment of inertia of a uniform rigid rod about an axis perpendicular to the rod and passing through its end (the *y*' axis). Note that the calculation requires that the integration limits be from *x* = 0 to *x* = *L*.
Answer: *ML*/3.

EXAMPLE 7
Suppose a rod is nonuniform such that its mass per unit length varies linearly with *x* according to the expression $\lambda = \alpha x$, where α is a constant. Find the *x* coordinate of the center of mass as a fraction of *L*.

Solution. In this case, we replace *dm* by λdx, where λ is not constant. Therefore,

$$x_{CM} = \frac{1}{M}\int_0^L x dm = \frac{1}{M}\int_0^L x\lambda dx = \frac{\alpha}{M}\int_0^L x^2 dx = \frac{\alpha L^3}{3M}.$$

We can eliminate α considering that the total mass of the rod is related to α by the expression

$$M = \int dm = \int_0^L \lambda dx = \int_0^L \alpha x dx = \alpha L^2/2.$$

Substituting this into the expression for *x*, we obtain

$$x_{CM} = \frac{\alpha L^3}{3\alpha L^2/2} = \frac{2}{3}L.$$

EXAMPLE 8
The turntable of a record player rotates initially at a rate of 33 rev/min. It stops in 20.0 s. (a) What is the angular acceleration of the turntable, assuming it is uniform?

Solution. Recall that 1 rev = 2π rad. We see that the initial angular speed is

$$\omega_0 = \frac{33.02\pi}{60} = 3.46 \text{ rad/s}.$$

Using $\omega = \omega_0 + \alpha t$ and the fact that $\omega = 0$ at $t = 20.0\,\text{s}$,we obtain

$$\alpha = -\frac{\omega_0}{t} = -\frac{3.46}{20.0} = -1.73\,\text{rad/s}^2,$$

where the negative sign indicates that ω decreases.
(b) How many revolutions does the turntable conduct before it stops?

Solution. Using rotational kinematic equations for angular motion, we find that the angular displacement in 20.0 s is

$$\Delta\theta = \theta - \theta_0 = \omega_0 t + \alpha t^2/2 = 3.46\cdot 20.0 + \frac{1}{2}(-0.173)(20.0)^2 = 34.6 \text{ rad}.$$

This correspond to 34.6/ 2π rev or 5.50 rev.

(c) If the radius of the turntable is 14.0 cm, what are the magnitudes of the radial and tangential components of the linear acceleration of a point on the rim at $t = 0$?

Solution. We can use $a_t = r\alpha$ and $a_r = r\omega^2$, which give

$$a_t = r\alpha = 14.0(-0.173) = -2.42 \text{ cm/s}^2, \quad a_r = r\omega^2 = 14.0(3.46)^2 = 168 \text{ cm/s}^2.$$

Exercise. What is the initial linear speed of a point on the rim of the turntable?
Answer : 48.4 cm/s.

EXAMPLE 9

A ski jumper skis down and leaves the ski-jump with a horizontal speed of 25.0 m/s, as in Figure 9. The landing is inclined at an angle of 35.0°. (a) Where does she land?

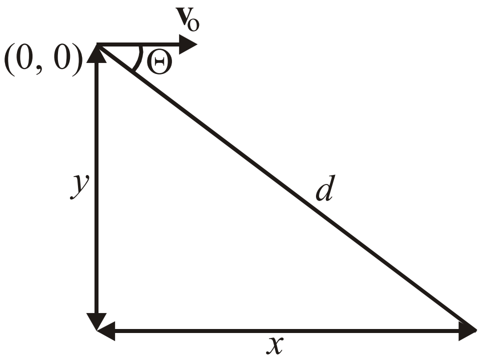


Figure 9.

Solution. It is convenient to chose the origin $(x = y = 0)$ at the beginning of her jump. Since $v_{x0} = 25\text{m/s}$, and $\upsilon_{y0} = 0$ in this case, Eqs. (4.12) and (4.13) yield

$$x = \upsilon_{x0} t = 25.0t, \qquad (1)$$

$$y = \upsilon_{y0} t - \frac{1}{2} g t^2 = -\frac{1}{2} 9.8 t^2. \qquad (2)$$

Taking d to be the distance before landing, from Figure 9 we see that x and y coordinates of the point of landing are $x = d\cos 35.0^o$. Substituting them into Eqs. (1) and (2), we obtain

$$d\cos 35.0^\circ = 25.0t, \qquad (3)$$

$$-d\sin 35.0^\circ = -\frac{1}{2} 9.8 t^2. \qquad (4)$$

Canceling t from these equations, we obtain d = 109 m. Hence, the x and y coordinates of the point at which she lands are

$$x = d\cos 35.0^o = 109\cos 35.0^o = 89.3 \text{ m},$$

$$y = -d\sin 35.0^\circ = -109\sin 35.0^\circ = -62.5 \text{ m}.$$

Exercise. Determine how long the ski jumper is into the air and the vertical component of her velocity just before she lands.
Answer: $3.57 \text{ s}, \upsilon_y = -35.0$ m/s.

4.5. Mechanical Deformation of a Body. Hooke's Law

Any real body becomes deformed, i.e., changes its dimensions and shape, under the action of forces applied to it. If the body returns to its initial dimensions and shape when the forces are removed, the deformation or strain is called ***elastic***. If the dimensions and shape are kept when the forces are removed, the deformation is called ***plastic***. There are a lot of different kinds of deformations. The most simple are ***the elongation*** and ***shear***. Any kind of mechanical deformation can be caused by different factors: mechanical forces, electric and magnetic fields, heating, and so on.

Experiments show that the elastic force and deformation are directly proportional. This statement is called ***Hooke's law***. For elongation (Figure 1.14*a*) and shear (Figure 1.14*b*) it is expressed by the formulas

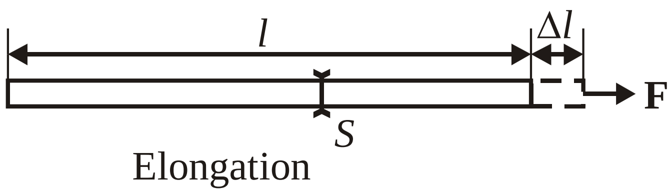


Figure 1.14*a*.

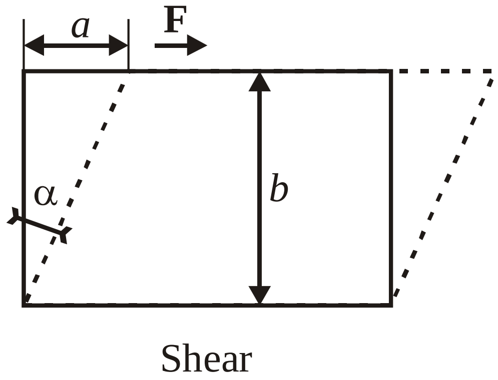


Figure 1.14*b*.

Elongation

$$\Delta l = \frac{F \cdot l}{E \cdot S}. \quad (4.25)$$

Here F is the force, l is the length of a sample, S is the cross sectional area, and E is the ***elastic*** (***Young's***) ***modulus***.

Shear

$$\tan\alpha = \frac{F}{G \cdot S}. \quad (4.26)$$

Here S is the cross sectional area of a sample and G is the shear modulus.

Analogous formulas can be given for all kinds of elastic deformations including ***bending***, ***rotation***, and ***restraint***.

The quantity

$$\sigma = \frac{F}{S} \quad (4.27)$$

is called ***the stress***. The maximum stress a body can experience without becoming permanently deformed is called ***the elastic limit*** or the elastic stress.

5. Mechanical Conservation Laws

A mechanical system is called ***closed*** if the resultant external force acting upon the system is equal to zero. In this case, three mechanical conservation laws can be formulated.

1. ***Energy conservation law***

In any closed mechanical system, the total energy does not change. This law is based on ***the uniformity of time***, i.e., on the equivalence of all moments of time. The equivalence should be understood in the sense that the substitution of the moment of time t_2 for the moment t_1 without changing the coordinates and velocities of the particles does not change the mechanical properties of the system. This means that after such a substitution, the coordinates and velocities of the particles at any moment of time $t_2 + t$ will be the same as at the moment $t_1 + t$.

2. ***Momentum conservation law***

In any closed mechanical system, the total momentum does not change. The conservation of momentum is based on ***the uniformity of space***, i.e., on the identical properties of space at all points. It should be understood in the sense that a translation of a closed system from one position in space to another $(\mathbf{r} \to \mathbf{r} + \mathbf{a}, \mathbf{a} = \text{const})$ without changing the mutual arrangement and velocities of the particles does not change the mechanical properties of the system.

3. ***Angular momentum conservation law***

In any closed mechanical system, the total angular momentum does not change. The conservation of angular momentum is based on ***the isotropy of space***, i.e., on the identical properties of space in all directions. It should be understood that the rotation of a closed system as a whole does not change its mechanical properties.

The laws of conservation are very powerful means for solving many physical problems. They often allow a problem to be solved without accurate consideration of motion equations which sometimes are very complicated.

The laws of conservation are more general than Newton's laws. They are strictly obeyed even when Newton's laws (particularly, the third one) are violated. It should be noted that the laws of energy, momentum, and angular momentum conservation are exact laws that are also strictly obeyed in relativistic situations.

5.1. Collision of Two Bodies

Two extreme types of collisions are distinguished: elastic and inelastic ones.

1. ***Completely elastic collision*** is a collision at which the kinetic energy of the bodies after collision is equal to the kinetic energy before collision.

2. ***Completely inelastic collision*** is a collision between the bodies when after collision the system of bodies moves as a whole. Obviously, in this case the final kinetic energy is less then the initial one.

Let us consider a system of two colliding homogeneous spheres having masses m_1 and m_2. The collision is assumed to be ***central***, i.e., the centers of masses of the bodies lie on the collision line.

1. Completely elastic collision is illustrated by Figure 1.15.

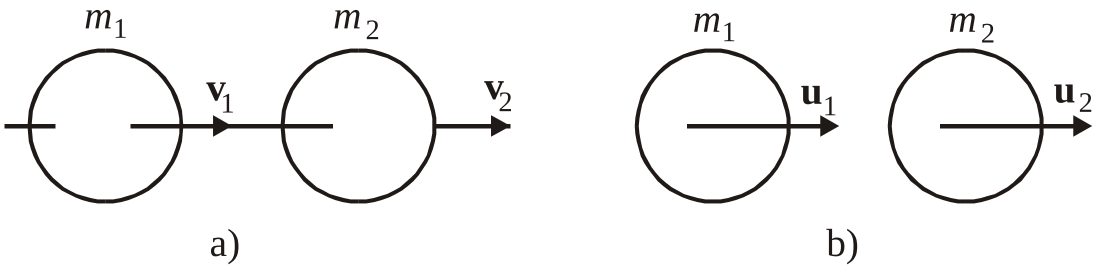


Figure 1.15

Here $\mathbf{v}_1$ and $\mathbf{v}_2$ are the initial velocities, and $\mathbf{u}_1$ and $\mathbf{u}_2$ are the final velocities. Using the momentum and energy conservation laws, we obtain

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{u}_1 + m_2\mathbf{u}_2, \quad (5.1)$$

$$\frac{m_1 \cdot v_1^2}{2} + \frac{m_2 \cdot v_2^2}{2} = \frac{m_1 \cdot u_1^2}{2} + \frac{m_2 \cdot u_2^2}{2}. \quad (5.2)$$

Having solved this system of equations, we obtain two symmetric relations:

$$\mathbf{u}_1 = \frac{(m_1 - m_2)\cdot \mathbf{v}_1 + 2\cdot m_2 \cdot \mathbf{u}_2}{m_1 + m_2}, \quad (5.3)$$

$$\mathbf{u}_2 = \frac{(m_2 - m_1)\cdot \mathbf{v}_2 + 2\cdot m_1 \cdot \mathbf{u}_1}{m_2 + m_1}. \quad (5.4)$$

Let us consider two special cases:

(A): $m_1\mathbf{v}_1 = - m_2\mathbf{v}_2$

Then $\mathbf{u}_1 = -\mathbf{v}_1$ and $\mathbf{u}_2 = -\mathbf{v}_2$, i.e., the momentum of the body changes its sign.

(B): $\mathbf{v}_2 = 0$

Then $\mathbf{u}_1 = \frac{(m_1 - m_2)\cdot \mathbf{v}_1}{m_1 + m_2}$ and $\mathbf{u}_2 = \frac{2\cdot m_1 \cdot \mathbf{u}_1}{m_1 + m_2}$.

If $m_1 > m_2$, the first body moves (after collision) in the same direction, but its final speed is less than the initial one. The speed of the second body is higher compared to the first one.

If $m_1 = m_2$, $\mathbf{u}_1 = 0$ and $\mathbf{u}_2 = \mathbf{v}_1$, i.e., the first ball stops and the second ball moves at a speed equal to the initial speed of the first ball.

If $m_1 < m_2$, $\mathbf{u}_1 = -\frac{(m_2 - m_1)\cdot \mathbf{v}_1}{m_1 + m_2}$ and $\mathbf{u}_2 = \frac{2\cdot m_1 \cdot \mathbf{u}_1}{m_1 + m_2}$, i.e., the first ball moves in the opposite direction, and the second moves in the same direction at a speed $u_2 < v_1$.

2. Completely inelastic collision

In accordance with the momentum conservation law,

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{v} \quad (5.5)$$

($\mathbf{v}$ is the identical velocity of both particles after collision).

Hence,

$$\mathbf{v} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}.$$

(Obviously, for the situation shown in Figure 1.15, $\mathbf{v}_1$ is higher than $\mathbf{v}_2$.)

5.2. Space Velocities

The first space or ***orbital velocity*** is the velocity needed for a body to become an artificial satellite of a planet (for example, the Earth). Let us designate by m the mass of a body, by M the mass of the planet, and by R the radius of the planet. In order to move along a circular orbit, the force of attraction should be equal to $\frac{m \cdot v^2}{R}$. So we have

$$G \cdot \frac{m \cdot M}{R^2} = \frac{m \cdot v^2}{R}. \quad (5.6)$$

Hence,

$$v = \sqrt{\frac{G \cdot M}{R}}. \quad (5.7)$$

Talking into account that

$$g = \frac{G \cdot M}{R^2}, \quad (5.8)$$

we can write the last equation in the form

$$v = \sqrt{g \cdot R}. \quad (5.9)$$

For the Earth, the first orbital velocity is v_1 = 7900 m/s.

The second space or ***escape velocity*** is the velocity needed for a body to leave a planet (in other words, to be able to overcome the gravitational attractive forces and to move to infinity). When $r \to \infty$, E_p = 0, and the body velocity for $r \to \infty$ can be assumed zero (we want the body just to leave the planet). So we can write

$$\frac{m \cdot v_2^2}{R} - \frac{G \cdot m \cdot M}{R} = 0. \quad (5.10)$$

Hence,

$$v_2 = \sqrt{\frac{2 \cdot G \cdot M}{R}} \quad (5.11)$$

or, using Eq. (5.8), we obtain

$$v_2 = \sqrt{2 \cdot g \cdot R}. \quad (5.12)$$

For the Earth, the escape velocity is v_2 = 12200 m/s.

5.3. Gyroscopes

A ***gyroscope*** (or a ***top***) is a massive symmetrical body spinning rapidly about a symmetry axis. We call this axis ***the spin axis***. Physically, it is a metallic spindle. To have the two main features of the gyroscope, two conditions should be met:

1. The mechanical construction of the gyroscope should be such that the spindle has three degrees of freedom.
2. The angular velocity of gyroscope spinning about its axis should be much higher compared to that when the direction of the spindle is being changed.

A continuously driven gyroscope holds the direction of its spin axis under any perturbations caused by external forces. This property of the gyroscope is due to the angular momentum conservation law.

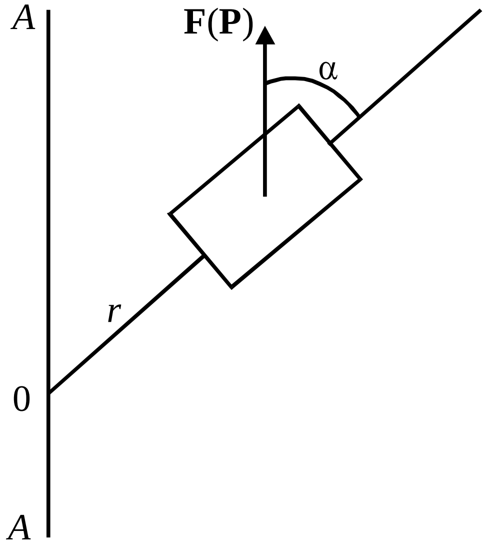


Figure 1.16.

If an external force **F** acts upon the gyroscope spindle (see Figure 1.16), the spin axis does not turn in the direction of this force, but in a perpendicular direction. As a result, the gyroscope begins to rotate with angular velocity equal to

$$\omega = \frac{M}{I \cdot \Omega}. \quad (5.13)$$

Here, M is the external torque, Ω is the angular velocity of the gyroscope about its axis, and I is its rotary inertia.

The gyroscope properties are widely used in technology, for example, in satellite or nautical navigation in order to stabilize the direction of ship motion; they are also used in geodesy and topography.

5.4. Laws of Planetary Motion

The laws of planetary motion were established by a German astronomer Johannes Kepler (1571–1630) in the early 17th century.

Kepler's first law. All the planets of the Solar system circulate around the Sun in elliptical orbits with the Sun at one focus. Kepler found that planets move fastest when closest to the Sun, slowest when farthest away.
Kepler's second law. The radius vector from the Sun to the planet sweeps out equal areas in equal times (Figure 1.17).

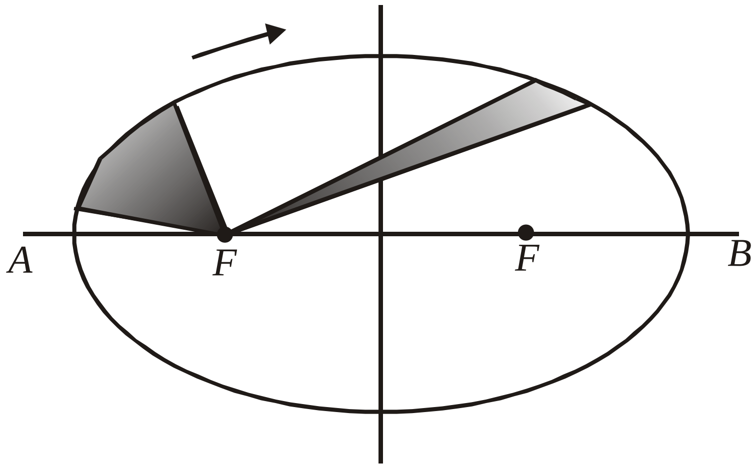


Figure 1.17.

So, the velocity at point A (perihelion) is higher than that at point B (aphelion).
Kepler's third law. The ratio of the square of the revolution period of a planet T to the third power of the larger semi-axis of the ellipse a, $\frac{T^2}{a^3}$, is the same for all the planets of the Solar system.

These three laws were established as a result of very hard work done by Kepler. Later, Newton analyzed the Kepler's laws and had come to his famous law of the universe gravitation.

6. Relativistic Mechanics

6.1. Basic Postulates of Relativity

In the early twentieth century it appeared that Newtonian mechanics held only for bodies moving at speeds that are much lower than the speed of light in vacuum (this speed is denoted by the symbol c). In order to describe motion at speeds comparable with c, the special theory of relativity was introduced. Many prominent physicists (Poincaré, Larmor, Lorentz, Einstein, Fitzgerald, Minkowski, and Voigt) took part in developing this theory. The basic principles of relativity are historically called Einstein's postulate and the postulate on constancy of speed of light.

Einstein's postulate of relativity is an extension of Galileo's mechanical principle to all physical phenomena without any exception. According to this postulate, all laws of nature are the same for all inertial reference frames. This postulate can be formulated as follows: equations expressing laws of nature do not change with respect to transformation of coordinates and time from one inertial reference frame to another.

The postulate on constancy of speed of light states that the speed of light in vacuum is the same for all inertial reference frames and does not depend on the motion of sources and receivers of light.

6.2. Lorentz Transformations

It is shown in the special theory of relativity (for inertial reference frames) that to have the laws of nature independent of the relative speed of inertial reference frames, it is necessary to transform space-time coordinates in accordance with the so-called Lorentz transformations (See Figure 1.18):

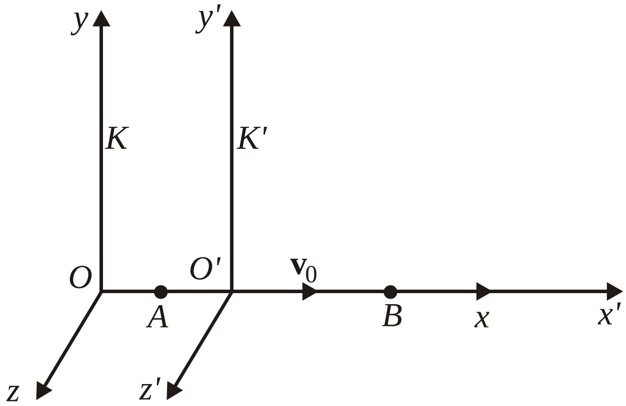


Figure 1.18.

$$x' = \frac{x - v \cdot t}{\sqrt{1 - \frac{v^2}{c^2}}},$$

$$y' = y, \qquad (6.1)$$

$$z' = z,$$

$$t' = \frac{t - \frac{v \cdot x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Inverse transformations can be easily obtained by changing the sign of v (the relative velocity of reference frames K and K'):

$$x = \frac{x' - v \cdot t'}{\sqrt{1 - \frac{v^2}{c^2}}},$$

$$y = y', \qquad (6.2)$$

$$z = z',$$
$$t = \frac{t' - \frac{v \cdot x'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

6.3. Interval

Let (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) be two ***world points*** in reference frame *K*. Let us designate

$$\begin{aligned} t_2 - t_1 &= \Delta t, \\ x_2 - x_1 &= \Delta x, \\ y_2 - y_1 &= \Delta y, \\ z_2 - z_1 &= \Delta z. \end{aligned} \tag{6.3}$$

The quantity

$$\Delta S = \sqrt{c^2 \cdot \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2} \tag{6.4}$$

is called ***the interval***. It is very easy to show (using formulas of the Lorentz transformation) that in any reference frame the value of the interval is the same:

$$\Delta S = \Delta S'. \tag{6.5}$$

So, the interval is ***an invariant***.
The quantity

$$\Delta\tau = \frac{\Delta S}{c} \tag{6.6}$$

is called the ***proper time***. Thus, the proper time is also an invariant.

According to Eq. (6.4), the interval can be ***real*** (if $c\Delta t > \Delta r^2$), ***imaginary*** (if $c\Delta t < \Delta r^2$), or equal to zero (if $c\Delta t = \Delta r^2$). The last case occurs for events of light signal emission from the point (x_1, y_1, z_1) at the moment t_1 and the arrival of this signal at the point (x_2, y_2, z_2) at the moment t_2.

For a real interval, we have

$$c^2 \Delta t^2 - \Delta r^2 = c^2 \Delta t'^2 - \Delta r'^2 < 0. \tag{6.7}$$

It can be seen from this relation that it is possible to find a frame *K*′ in which $\Delta r' = 0$, i.e., both events coincide in space. No reference frame exists, however, in which $\Delta t' = 0$ (the interval would become imaginary for this value of $\Delta t'$). Thus, events separated by a real interval cannot become simultaneous in any reference frame. For this reason, real intervals are called ***time-like***.

For an imaginary interval, we have

$$c^2 \cdot \Delta t^2 - \Delta r^2 = c^2 \cdot \Delta t'^2 - \Delta r'^2 < 0. \tag{6.8}$$

Hence, it is possible to find a frame *K*′ in which $\Delta t' = 0$, i.e., both events occur at the same moment *t*′. No reference frame exists, however, in which we would have $\Delta r' = 0$ (the interval would be real for this value of $\Delta r'$). Thus, events separated by an imaginary interval cannot coincide in space in any reference frame. For this reason, imaginary intervals are called ***space-like***.

The distance Δr between the points at which the events separated by a space-like interval occur exceeds $c\cdot\Delta t$. Therefore, these events cannot affect each other in any way, i.e., cannot be ***causally related*** to each other. Causally related events can be separated only by a time-like or zero interval.

6.4. Corollaries of the Lorentz Transformations

a) ***Lorentz (or Fitzgerald) contraction***

Let us consider a rod arranged along the x' axis and at rest relative to the reference frame K' (Figure 1.19).

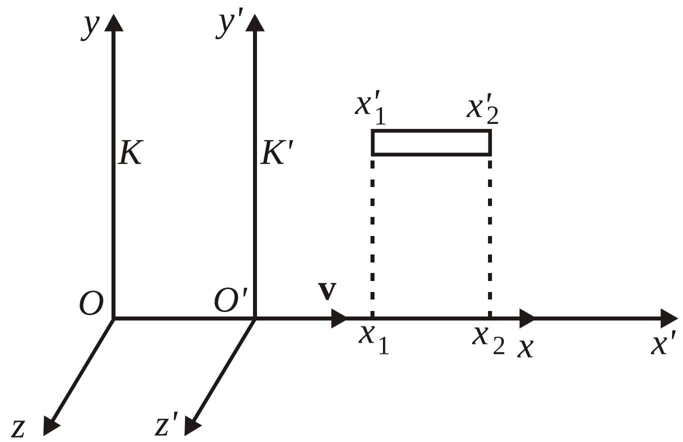


Figure 1.19.

The length of the rod in K' $l_0 = x_2' - x_1'$. The coordinates x'_2 and x'_1 can be measured at different time moments, but in the reference frame K they should be measured simultaneously. Using the Lorentz transformation formulas

$$x_1' = \frac{x_1 - v\cdot t}{\sqrt{1-\frac{v^2}{c^2}}}, \quad x_2' = \frac{x_2 - v\cdot t}{\sqrt{1-\frac{v^2}{c^2}}}$$

and designating $x_2 - x_1 = l$ (the length of the rod relative to the reference frame K), we obtain

$$x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1-\frac{v^2}{c^2}}}, \text{ or}$$

$$l = l_0 \cdot \sqrt{1-\frac{v^2}{c^2}} \tag{6.9}$$

(l_0 is called ***the proper length***).

Thus, for moving bodies their dimensions contract in the direction of their motion the greater, the higher is the velocity. This phenomenon is called the

Lorentz (or Fitzgerald) contraction. It should be noted that the dimensions of the rod are identical in all the reference frames in directions of the y and z axes.

b) ***Time dilation***

Let us consider a clock at rest in the reference frame K' (its position is x'). Using the Lorentz transformation formulas, we obtain

$$t_1 = \frac{t_1' + \frac{v}{c^2} \cdot x'}{\sqrt{1 - \frac{v^2}{c^2}}}, \; t_2 = \frac{t_2' + \frac{v}{c^2} \cdot x'}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Hence, $t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1 - \frac{v^2}{c^2}}}$, $t'_2 - t'_1 = \Delta t_0$ is the time interval registered by the clock at rest (Δt_0 is called the ***proper time***); t_2 - $t_1 = \Delta t$ is the corresponding time interval registered in the reference frame K relative to which the clock moves. Thus;

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{6.10}$$

Obviously, $\Delta t > \Delta t_0$, so in the reference frame K', moving at a speed v relative to the reference frame K, the time slows down (from the K – observer's point of view). This slowing of time is referred to as time dilation.

c) ***Addition of velocities***

For an arbitrary direction of the velocity v' in the reference frame K', the formulas for the velocity in the reference frame K are rather complicated. But if a body is moving parallel to the x axis, its velocity v relative to the frame K coincides with v_x, and its velocity v' relative to the frame K' coincides with v'_x. In this case, the law of velocity addition has the form:

$$v = \frac{v' + v_0}{1 + v_0 v' / c^2}. \tag{6.11}$$

Here, v_0 designates the relative velocity of the systems K' and K. It should be noted that according to Eq. (6.11), velocities comparable with that of light are not added in accordance with the parallelogram rule; the resultant speed v cannot exceed the speed of light.

6.5. Dynamics of Relativity

The basic equation of motion has the form:

$$\frac{d}{dt}\left(\frac{m_0\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\mathbf{F}, \tag{6.12}$$

where m_0 designates the "conventional" Newtonian mass. The quantity

$$\mathbf{P}=\frac{m_0\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{6.13}$$

is called the ***relativistic momentum***. Thus, formally Eq. (6.12) coincides with Eq. (2.3) of classical mechanics. But in general case, $\mathbf{F} = m\cdot\mathbf{a}$ does not hold. Using Eq. (6.12), it is rather easy to show that only in two cases the force **F** and acceleration **a** are collinear:

(a) Transverse force ($\mathbf{F}\perp\mathbf{v}$)

$$\frac{m_0\mathbf{a}}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{6.14}$$

(b) Longitudinal force ($\mathbf{F}\uparrow\uparrow\mathbf{v}$, or $\mathbf{F}\uparrow\downarrow\mathbf{v}$)

$$\frac{m_0\mathbf{a}}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{\frac{3}{2}}}=\mathbf{F}. \tag{6.15}$$

In general, the directions of **a** and **F** do not coincide. Sometimes, the quantity

$$m_\perp=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \tag{6.16}$$

is called ***the transverse mass***, and the quantity

$$m_{||}=\frac{m_0}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^{\frac{3}{2}}} \tag{6.17}$$

is called ***the longitudinal mass***. The quantity

$$E=\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{6.18}$$

is called ***the total energy*** of the particle, and the quantity

$$E_0 = m_0 \cdot c^2 \tag{6.19}$$

is called ***the rest energy***. The rest energy of mass particles can be liberated when particles annihilate, i.e., convert into field quanta (for example, $e^- + e^+ \to 2\gamma$). The kinetic energy of the particle is

$$E_k = E - E_0 \tag{6.20}$$

or

$$E_k = m_0 \cdot c^2 \cdot \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right). \tag{6.21}$$

When $\frac{v}{c} << 1$, the last equation yields $E_k = \frac{m_0 \cdot c^2}{2}$, i.e., the classical expression of the kinetic energy.

EXAMPLE 10

Playing billiards, a player wishes to sink the target ball in the corner pocket, as shown in Figure 10. If the angle with respect to the corner pocket is 35°, at what angle Θ will the cue ball be deflected? Assume that friction and rotational motion can be neglected and the collision is elastic.

Solution. Since the target ball is initially at rest, v_{2i} = 0, and the law of conservation of energy gives

$$\frac{1}{2} m_1 {v_{1i}}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 .$$

However, $m_1 = m_2$, so that

$${v_{1i}}^2 = v_{1f}^2 + v_{2f}^2 . \tag{1}$$

Applying the law of conservation of momentum to the collision between two balls, we obtain

$$v_{1i} = v_{1f} + v_{2f} . \tag{2}$$

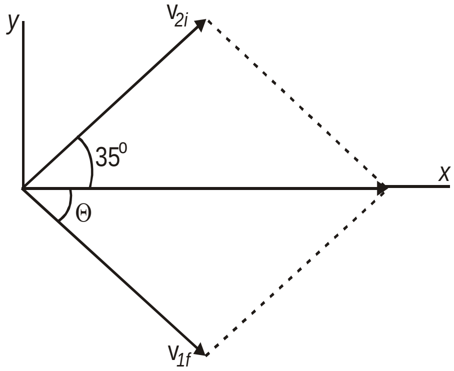


Figure 10.

If we square both sides of this equation, we obtain

$$v_{1i}{}^2 = (v_{1f} + v_{2f})\cdot(v_{1f} + v_{2f}) = \upsilon_{1f}{}^2 + \upsilon_{2f}{}^2 + 2v_{2f}v_{2f}.$$

However, $v_{1f}v_{2f} = v_{1f}v_{2f}\cos(\Theta + 35^{o})$, and

$$v_{1i}{}^2 = v_{1f}{}^2 + v_{2f}{}^2 2v_{1f}v_{2f}\cos(\Theta + 35^{o}). \qquad (3)$$

Subtracting (1) from (3) gives

$2v_{1f}v_{2f}\cos(\Theta + 35^{o}) = 0$, $\cos(\Theta + 35^{o}) = 0$, $\Theta + 35^{o} = 90^{o}$ or $\Theta = 55^{o}$.

Again, this result shows that whenever two equal masses undergo a glancing elastic collision and one of them is initially at rest, after the collision they will move at right angle with respect to each other.

EXAMPLE 11

A small sphere of mass 2.00 g drops from rest to a large vessel filled with oil. The sphere reaches a terminal speed of 5.00 cm/s. Determine the time constant τ and the time it takes the sphere to reach 90% of its terminal velocity. Assume that the resistive force is proportional to the sphere velocity **R** = b**v**.

Solution. Applying Newtons second law to the vertical motion, choosing the downward direction to be positive, and noting that $\Sigma F_y = mg - bv$ and $a = \frac{dv}{dt}$, we obtain

$$mg - bv = m\frac{dv}{dt}, \qquad (1)$$

where the acceleration is downward. Simplification of this expression gives

$$\frac{dv}{dt} = g - \frac{b}{m}v. \qquad (2)$$

Since the terminal speed is given by $v_t = mg/b$, the coefficient b is $b = mg/v_t = 392\ g/s$ Eq. (2) is a differential one. Solving this equation for v, we obtain

$$\int_0^v \frac{dv}{g - \frac{b}{m}v} = \int_0^t dt,$$

$$\ln\left(1 - \frac{b}{m}v\right) = -\frac{b}{m}t,$$

$$v = \frac{mg}{b}\left(1 - e^{-bt/m}\right) = v_t\left(1 - e^{-t/\tau}\right). \qquad (3)$$

As t increases, the resistive force increases and the acceleration decreases. Eventually, the acceleration becomes zero when the resistive force equals the weight. At this point, the sphere reaches its terminal speed v_t and continues to move with zero acceleration. The terminal speed can be obtained from last equation by setting $t=\infty$. This gives

$$v_t = \frac{mg}{b}.$$

The coefficient b is

$$b = \frac{mg}{v_t} = \frac{2.0\cdot 9.8}{0.05} = 0.392\ \text{kg/s}.$$

Therefore, the time constant τ is given by

$$\tau = \frac{m}{b} = \frac{2.0}{0.392} = 5.1\cdot 10^{-3}\text{s}.$$

The speed of the sphere as a function of time is given by Eq. (3). To find the time t it takes the sphere to reach a speed of $0.9v_t$, we substitute $v=0.9v_t$ into Eq. (3) and solve it for t:

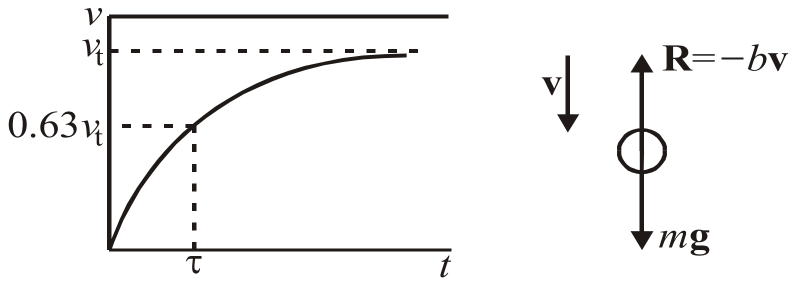
$$0.9v_t = v_t\left(1-e^{-t/\tau}\right),$$

$$1-e^{-t/\tau} = 0.9,$$

$$e^{-t/\tau} = 0.1,$$

$$-t/\tau = \ln 0.1 = -2.3,$$

$$t = 2.3\tau = 2.3\cdot(5.1\cdot 10^{-3})\text{s} = 11.7\ \text{ms}.$$



(b) (a)

Figure 11.

Exercise. What will be the sphere speed in oil in 11.7 ms? Compare this value with the speed the sphere would have had if it had fallen in vacuum and hence had been influenced only by gravity.

Answer. 4.50 cm/s in oil versus 11.5 cm/s in free fall.

EXAMPLE 12

Consider a car of mass m accelerating up a hill, as in Figure 12. Assume that the magnitude of the resistive force is

$$|\mathbf{f}| = \left(218 + 0.7v^2\right)N,$$

where v is the speed, in meters per second. Calculate the power the engine must deliver to the wheels.

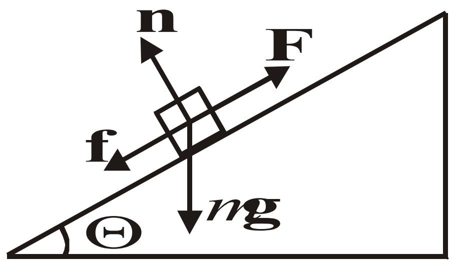


Figure 12.

Solution. The forces acting on the car are shown in Figure 12, where **F** is the force of static friction that propels the car, and the remaining forces have their usual meaning. Newton's second law applied to the motion along the road surface gives

$$\sum F_x = F - |\mathbf{f}| - mg\sin\Theta = ma,$$

$$F = ma + mg\sin\Theta + |\mathbf{f}| = ma + mg\sin\Theta + \left(218 + 0.7v^2\right).$$

Therefore, the power required for propulsion is

$$P = Fv = mva + mvg\sin\Theta + 218v + 0.70v^2,$$

where mva represents the power the engine must deliver to accelerate the car. If the car moves at constant speed, this term is zero, and the power requirement is reduced. The term $mvg\sin\Theta$ is the power required to overcome the force of gravity when the car moves up the incline. This term would be zero for motion along a horizontal surface. The term $218v$ is the power required to counterbalance the rolling friction. Finally, the term $0.70v^2$ is the power needed to overcome the air drag. If we take m = 1450 kg, v = 27 m/s, a = 1.0 m/s^2, and Θ = 10°, the terms entering into P are calculated to be

$$mva = 39\text{ kW},$$

$$mvg\sin\Theta = 1450\text{ kg}\cdot 27\text{ m/s}\cdot 9.8\text{ m/s}^2\cdot\sin 10^\circ = 67\text{ kW},$$

$$0.7v^3 = 14\text{ kW}.$$

Hence, the total power required is 126 kW. Note that the power requirements for moving at constant speed along a horizontal surface are only 20 kW (the sum of the last two terms). Furthermore, if the mass is halved (as in compact cars), the power required is also reduced by almost the same factor.

EXAMPLE 13

Two blocks are connected by a massless cord that passes over a frictionless pulley and a frictionless peg as in Figure 14. One end of the cord is attached to mass m_1 = 3.00 kg that is at distance R = 1.20 m from the peg. The other end of the cord is connected to a block of mass m_2 = 6.00 kg resting on a table. From what angle Θ (measured from the vertical) must the 3.00-kg mass be released in order to begin to lift the 6.00-kg block from the table?

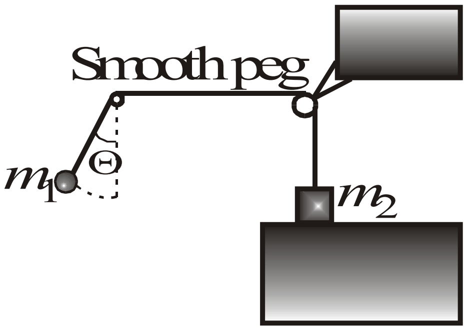


Figure 13.

Reasoning. It is necessary to use several concepts to solve this problem. First, we use the law of conservation of energy to find the speed of mass m_1 at its lowest position on a vertical circle as a function of Θ and the radius of the circle. Next, we apply Newton's second law to the 3.00-kg mass at its lowest position to find the tension as a function of the given parameters. Finally, we note that the 6.00-kg block lifts from the table when the upward force exerted on it by the cord exceeds the force of gravity acting on the block. This procedure enables us to find the required angle.

Solution. Applying the law of conservation of energy to the 3.0-kg mass, we obtain

$$K_i + U_i = K_f + U_f\,,$$

$$0 + m_1 g y_i = \frac{1}{2} m_1 v^2 + 0\,, \qquad (1)$$

where v is the speed of the 3.00-kg mass at its lowest position. (Note that $K_i = 0$ since the 3.00-kg mass starts from rest, and $U_f = 0$ because at the lowest position the potential energy vanishes.) From the geometry in Figure 13 we see that $y_i = R - R\cos\Theta = R(1-\cos\Theta)$. Substituting this relation into Eq. (1), we obtain

$$v^2 = 2gR(1-\cos\Theta)\,. \qquad (2)$$

Now we apply Newton's second law to the 3.00-kg mass when it is at its lowest position:

$$T - m_1 g = m_1 \frac{v^2}{R}\,,$$

$$T = m_1 g + m_1 \frac{v^2}{R}\,. \qquad (3)$$

This same force acts on the 6.00-kg block, and to lift it from the table, the resultant normal force must be zero, that is, $T = m_2 g$. Using this condition and Eqs. (2) and (3), we derive

$$m_2 g = m_1 g + m_1 \frac{2gR(1-\cos\Theta)}{R}\,.$$

Solving this equation for Θ and substituting the given parameters, we obtain

$$\cos\Theta = \frac{3m_1 - m_2}{2m_1} = \frac{33.0 - 6.0}{} 23.0 = \frac{1}{2},$$

$\Theta = 60.0^{o}$.

Exercise. If the initial angle is Θ = 40.0°, find the speed of the 3.00-kg mass and the tension in the cord when the 3.00-kg mass is at its lowest position.

Answer. 2.35 m/s, 43.2 N.

EXAMPLE 14

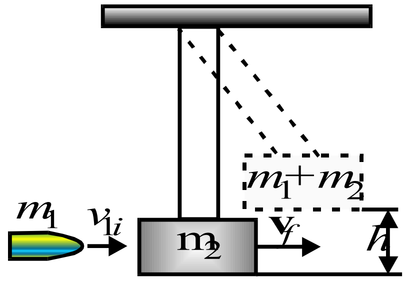


Figure 14.

A ballistic pendulum (see Figure 14) is a system used to measure the speed of rapidly moving projectile, such as a bullet. The bullet is fired into a large block of wood suspended from light wires. The bullet is stopped by the block, and the entire system swings at a height *h*. Because the collision is perfectly inelastic, momentum is conserved. Equation for the momentum conservation gives the speed of the system after collision in the impulse approximation. The kinetic energy after collision is

$$K = \frac{1}{2}(m_1 + m_2){v_f}^2. \qquad (1)$$

With $v_{2i} = 0$, equation for momentum conservation becomes

$$v_f = \frac{m_1 v_{1i}}{m_1 + m_2}. \qquad (2)$$

Substituting this value of v_{1i} into Eq. (1), we obtain

$$K = \frac{{m_1}^2 {v_{1i}}^2}{2(m_1 + m_2)},$$

where v_{1i} is the initial speed of the bullet. Note that this kinetic energy is less than the initial kinetic energy of the bullet. After collision, however, the total energy remains constant; the kinetic energy at the lowest position is transformed into the potential energy at the height *h*:

$$\frac{{m_1}^2 {v_{1i}}^2}{2(m_1 + m_2)} = (m_1 + m_2)gh,$$

$$v_{1i} = \left(\frac{m_1 + m_2}{m_1}\right)\sqrt{gh}.$$

Hence, it is possible to obtain the initial speed of the bullet by measuring *h* and the two masses. Why would it be incorrect to equate the initial kinetic energy of the bullet to the final gravitational energy of the bullet block combination?

Exercise. In a pendulum experiment, suppose that h = 5.0 cm, m_1 = 5.0 g, and m_2 = 1.0 kg. Find (*a*) the initial speed of the projectile, and (*b*) the energy loss due to the collision.
Answer. 199 m/s, 98.5 J.

EXAMPLE 15
Two masses m_1 and m_2 are connected by a light cord that passes over a pulley of radius R having the moment of inertia I about its axle, as shown in Figure 15. The mass m_2 slides on a frictionless horizontal surface. Determine the acceleration of the two masses using the concepts of angular momentum and torque.

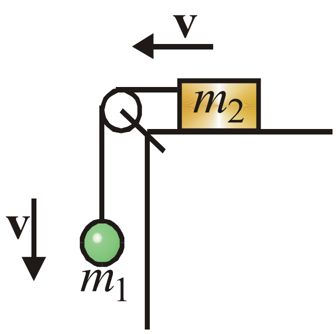


Figure 15.

Solution. First, we calculate the angular momentum of the system, which consists of the two masses plus the pulley. Then we calculate the torque about the axis passing along the axle of the pulley through 0. At the instant m_1 and m_2 have a speed v, the angular momentum of m_1 is m_1vR, and that of m_2 is m_2vR. At the same instant, the angular momentum of the pulley is $I\omega = Iv/R$. Therefore, the total angular momentum of the pulley is

$$L = m_1vR + m_2vR + Iv/R. \qquad (1)$$

Now let us evaluate the total external torque on the system about the axle. Because it has zero moment arm, the force exerted by the axle on the pulley does not contribute to the torque. Furthermore, the normal force acting on m_2 is balanced by its weight m_2g, and so these forces do not contribute to the torque. The external force m_1g produces a torque about the axle equal in magnitude to m_1gR, where R is the moment arm of the force about the axle. This is the total external torque about 0. Using this result together with Eq. (1) and Eq. (4.10), we obtain

$$\tau_{ext} = \frac{dL}{dt},$$

$$m_1gR = \frac{d}{dt}\left[(m_1 + m_2)Rv + I\frac{v}{R}\right],$$

$$m_1gR = (m_1 + m_2)R\frac{dv}{dt} + \frac{I}{R}\frac{dv}{dt}.$$

Because $dv/dt = a$, we can solve the above equation for a to obtain

$$a = \frac{m_1g}{(m_1 + m_2) + I/R^2}.$$

You may wonder why we neglected the forces that the cord exerts on the objects in evaluating the net torque about the axle. The reason is that these forces are internal to the system under consideration. Only the external torque contribute to the change in the angular momentum.

EXAMPLE 16

A projectile of mass m and velocity $\mathbf{v}_o$ is fired at a solid cylinder of mass M and radius R (Fig. 16). The cylinder is initially at rest and is mounted on a fixed horizontal axle that passes through the center of mass. The trajectory of motion of the projectile is perpendicular to the axle and ends at a distance $d < R$ from the center. Find the angular speed of the system after the projectile strikes and adheres to the surface of the cylinder.

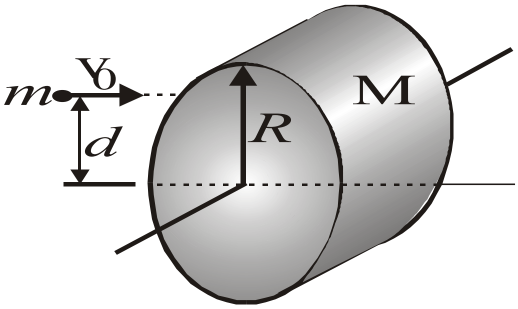


Figure 16.

Reasoning. Let us evaluate the angular momentum of the system (projectile + cylinder) about the axle of the cylinder. The net external torque on the system about this axle is zero. Hence, the angular momentum of the system is the same before and after collision.

Solution. Before collision, only the projectile has the angular momentum with respect to a point on the axle. The magnitude of this angular momentum is $mv_o d$, and it is directed along the axle away from us. After collision, the total angular momentum of the system is $I\omega$, were I is the total moment of inertia about the axle (projectile + cylinder). Since the total angular momentum is constant, we obtain

$$mv_o d = I\omega = (\frac{1}{2}MR^2 + mR^2)\omega,$$

$$\omega = \frac{mv_o d}{\frac{1}{2}MR^2 + mR^2}.$$

This suggests another technique for measuring the speed of a bullet.

Exercise. In this example, the mechanical energy is not conserved since the collision is inelastic. Show that $\frac{1}{2}I\omega^2 < \frac{1}{2}mv_o^2$. What are the factors responsible for the energy loss?

Part 2. Mechanical Oscillations and Waves.

7.0 Oscillations. General Information

Periodically repeated processes or motions are called ***oscillations***. For example, swings of a clock pendulum, vibrations of a string or a tuning fork, and the voltage across the plates of a capacitor in a radio receiver circuit have this property of repetition.

Depending on the physical nature of repeating process, we distinguish mechanical, sound, electromagnetic, and other oscillations. Oscillations (***vibrations***) are widespread in nature and engineering.

Oscillations are called ***periodic*** if the value of an oscillating physical quantity (or better to say, the state of a system) is the same in identical time intervals.

Depending on the nature of the action on an oscillating system, we distinguish ***free*** (***or natural***) ***oscillations***, ***forced oscillations***, ***self-oscillations***, and ***parametric oscillations***.

Free or natural oscillations occur in a system left alone after an impetus was imparted to it or it was brought out of the equilibrium position.

The ***forced oscillations*** occur when the oscillating system is acted upon by a small external periodically changing force; the frequency of the force differs from the natural frequency of the system.

Self-oscillations, like forced ones, are excited by the action of external forces on the oscillating system, but the moments of time when these actions are exerted are set by the oscillating system itself – it controls the external action.

In ***parametric oscillations***, an external action causes periodic changes in a parameter of the system, for example, in the length of a thread on which an oscillating body is suspended.

The time T needed for a physical system to repeat its state is called the ***period of oscillations***. The reciprocal quantity, $\nu = \dfrac{1}{T}$, is called ***the frequency***. The frequency in SI units is herz or cycle (one oscillation per second). The quantity $\omega = 2\pi\nu$ is called ***the cyclic frequency***.

Harmonic oscillations are the simplest ones. There are oscillations when the oscillating quantity changes with time according to a sine or cosine law. This kind of oscillations is especially important for the following reasons: first, oscillations in nature and engineering are often close to harmonic ones in their character, and second, periodic processes having different time dependences can be represented as the superposition of several harmonic oscillations.

7.1. Free (Natural) Harmonic Oscillations

Let us consider a mechanical system whose position can be defined by a single quantity x. The system is said to have one degree of freedom. In this case, the oscillating motion equation has the form (we chose a cosine law):

$$x = A \cdot \cos(\omega_0 \cdot t + \varphi). \quad (7.1)$$

Here A is the amplitude of oscillations, ω_0 is the natural cyclic frequency, and φ is the initial phase. Note: Eq. (7.1) can be written in a complex form

$$x = A^{i \cdot (\omega_0 \cdot t + \varphi)}. \quad (7.2)$$

Differentiation of Eq. (7.1) with respect to time yields:

$$\dot{x} = -A \cdot \omega_0 \cdot \sin(\omega_0 \cdot t + \varphi) = A \cdot \omega_0 \cdot \cos(\omega_0 \cdot t + \varphi + \frac{\pi}{2}), \quad (7.3)$$

$$\ddot{x} = -A \cdot \omega_0^2 \cdot \cos(\omega_0 \cdot t + \varphi) = A \cdot \omega_0^2 \cdot \cos(\omega_0 \cdot t + \varphi + \pi). \quad (7.4)$$

Thus, the velocity and acceleration have the same time dependences, but are displaced in time by additional phase shifts $\frac{\pi}{2}$ and π, respectively. Obviously,

$$v_{\max} = A \cdot \omega_0, \quad (7.5)$$

$$a_{\max} = A \cdot \omega_0^2. \quad (7.6)$$

7.2. Vector Diagram

One of the basic features of vector quantities is that they are summed by the parallelogram rule, i.e., their projections are algebraically added. Harmonic oscillations are widely known to be represented with the help of so-called vector diagrams (see Figure 2.1).

Here, $\mathbf{A}_1$ and $\mathbf{A}_2$ are arbitrary vectors and $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$ is their sum. Let us imagine that all three vectors are rotating (about point O in counterclockwise direction with angular speed ω_0). The time is counted from the moment when these vectors are at angles φ_1, φ_2, and φ to the x axis. Obviously, the time dependence of their projections can be expressed in the form:

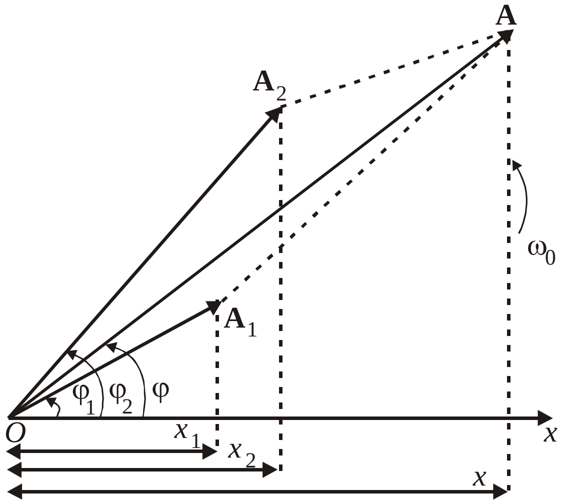


Figure 2.1.

$$\left. \begin{aligned} x_1 &= A_1 \cdot \cos(\omega_0 \cdot t + \varphi_1) \\ x_2 &= A_2 \cdot \cos(\omega_0 \cdot t + \varphi_2) \\ x_1 + x_2 &= x = A \cdot \cos(\omega_0 \cdot t + \varphi) \end{aligned} \right\}. \quad (7.7)$$

The quantities on the right side of Eq. (7.7) represent harmonic oscillations. Thus, the vector **A** represents the resultant oscillation. It can be seen from the figure that

$$A^2 = A_1^2 + A_2^2 + 2A_1 \cdot A_2 \cdot \cos(\varphi_2 - \varphi_1), \quad (7.8)$$

$$\tan\varphi = \frac{A_1 \cdot \sin\varphi_1 + A_2 \cdot \sin\varphi_2}{A_1 \cdot \cos\varphi_1 + A_2 \cdot \cos\varphi_2}. \qquad (7.9)$$

Thus, the representation of harmonic oscillations by means of vectors makes it possible to reduce the addition of several oscillations (in the same directions) to the operation of vector addition. This procedure is especially useful in optics and electrical engineering. Of course, Eqs. (7.8) and (7.9) can be derived by trigonometric transformations. But the vector diagram method is much simpler and very effective.

If the frequencies of oscillations x_1 and x_2 are not the same, the vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ will rotate with different angular velocities. In this case, the resultant vector **A** varies in magnitude and rotates with a varying velocity. Consequently, in this case the resultant motion will be a complex oscillating process. It should be noted that any periodic (but not necessary harmonic) oscillation can be represented in the form of the Fourier transform in terms of sine (or cosine) functions representing the harmonic oscillations with frequencies ω, 2·ω, 3·ω, and so on (ω is called ***the basic frequency***)

$$x(t) = \sum_{(n)}^{\infty} A_n \cdot \cos(n \cdot \omega \cdot t + \varphi_n). \qquad (7.10)$$

7.3. Beats

Of special interest is the case when two harmonic oscillations of the same direction being added differ only slightly in frequency. To avoid unnecessary complications, let us assume that the amplitudes of both oscillations are identical and their initial phases are zero. Thus, $x_1 = A \cdot \cos(\omega \cdot t)$, $x_2 = A\cos((\omega + \Delta\omega)t)$, and $\Delta\omega << \omega$. By summing these expressions and using the trigonometric formula for the sum of cosines, we have

$$x = x_1 + x_2 = (2A\cos(\frac{\Delta\omega}{2}t))\cos(\omega t) \qquad (7.11)$$

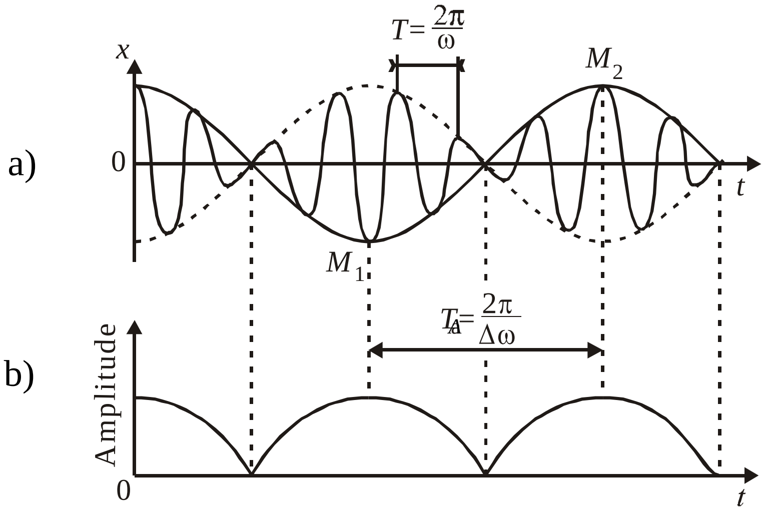


Figure 2.2.

(in the second multiplier we disregard the term $\frac{\Delta\omega}{2}$ in comparison with ω). The plot of function (7.11) is shown in Figure 2.2 ($\frac{\Delta\omega}{\omega} = \frac{1}{10}$).

Owing to the condition $\Delta\omega << \omega$, the multiplier in parentheses does not virtually change during the time the multiplier cos(ω*t*) completes several oscillations. So, oscillation (7.11) can be considered as a harmonic oscillation of frequency ω whose amplitude changes by a periodic law. Obviously, the analytic expression for the amplitude has the form

$$amplitude = \left|2A\cos(\frac{\Delta\omega}{2}t)\right| \text{ (Figure 2.2}b\text{)}. \qquad (7.12)$$

The quantity
$$T_A = \frac{2\pi}{\Delta\omega} \qquad (7.13)$$
is called ***the period of beats***.

It should be noted that the multiplier $2A\cos(\frac{\Delta\omega}{2}t)$ affects the oscillation phase (see points M_1 and M_2 in Figure 2.2*a*).

7.4. Addition of Mutually Perpendicular Oscillations

Let us consider a point particle which can oscillate both along the *x* and *y* axes. If both oscillations are induced, the particle will move along a curved trajectory (in general case) whose shape depends on the phase shift between the two oscillations. Assume also that the oscillation frequencies are identical. Thus,
$$x = A\cdot\cos(\omega\cdot t + \varphi_1),\ y = B\cdot\cos(\omega\cdot t + \varphi_2). \qquad (7.14)$$
Equations (7.14) describing the trajectory are given in the parametric form. To obtain an equation of the trajectory in the conventional form, the parameter *t* should be excluded. After simple transformations, we obtain
$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - 2\cdot\frac{x\cdot y}{A\cdot B}\cdot\cos(\varphi_2 - \varphi_1) = \sin^2(\varphi_2 - \varphi_1). \qquad (7.15)$$
Equation (7.15) is well known to be the equation of an ellipse whose axes are oriented arbitrary relative to the *x* and *y* coordinate axes (see Figure 2.3). The orientation of the ellipse and the lengths of its semi-axes depend in a rather complicated way on the amplitudes *A* and *B* and the phase difference ($\varphi_2 - \varphi_1$). It should be noted, however, that the ellipse is always inscribed in a rectangle with sides equal to 2*A* and 2*B*. The oscillations described by Eq. (7.15) are called the

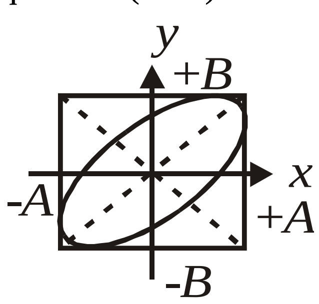


Figure 2.3.

elliptic oscillations. Let us discuss some particular cases (Figure 2.4).

1. The phase difference is $\varphi_2 - \varphi_1 = \pm 2\pi n$ (n=0, 1, 2, …). Then we have (Figure 2.4*a*)

$$y = \frac{B}{A} x . \tag{7.16}$$

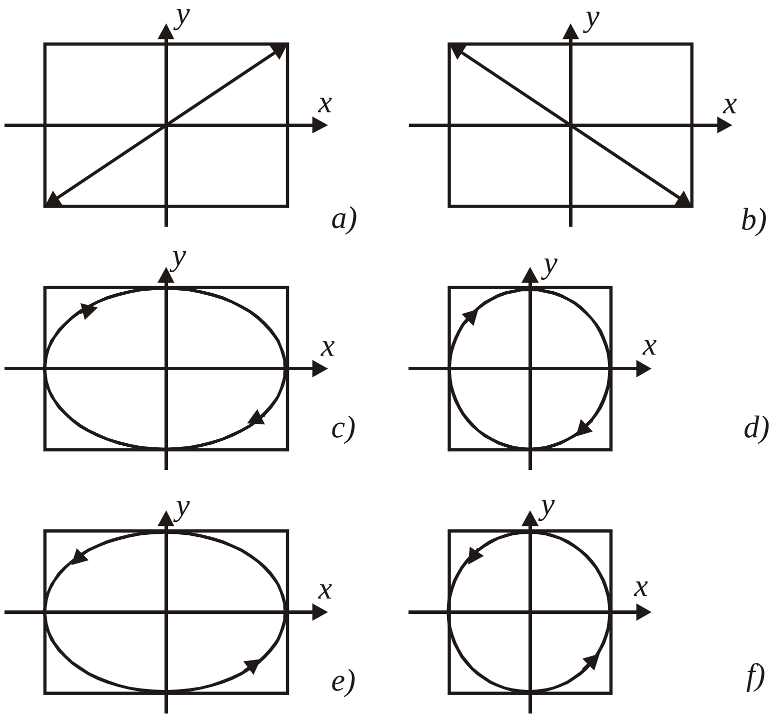


Figure 2.4.

2. The phase difference is $\varphi_2 - \varphi_1 = \pm(2n+1)\pi$. Then we have (Figure 2.4*b*)

$$y = -\frac{B}{A} x . \tag{7.17}$$

The oscillations of these two types are called linearly polarized oscillations.

3. The phase shift is $\varphi_2 - \varphi_1 = \frac{\pi}{2} \pm 2\pi n$. Then we have an ellipse whose axes are parallel to the x and y axes, and the point particle moves clockwise (Figure 2.4*c*). If $A = B$, the ellipse degenerates into a circle (Figure 2.4*d*).

4. The phase shift is $\varphi_2 - \varphi_1 = -\frac{\pi}{2} \pm 2\pi n$. Then again we have an ellipse whose axes are parallel to the x and y axes, but the point particle moves counterclockwise (Figure 2.4*e*). If $A = B$, the ellipse degenerates into a circle (Figure 2.4*f*).

The oscillations of the last two types are called circularly polarized oscillations. In general, when two perpendicular oscillations are added, we have a resultant motion of a very complicated form, and the trajectory of motion can be open. But when n and m are integers, we can write

$$x = A\cos(m\omega t + \varphi_m) , \tag{7.18}$$

$$y = B\cos(n\omega t + \varphi_n) , \tag{7.19}$$

and the trajectory of motion is closed. These rather intricate curves are called ***Lissajou's figures***. Figure 2.5 shows a simple trajectory obtained at a ratio of frequencies of 1:2 and a phase difference of $\frac{\pi}{2}$.

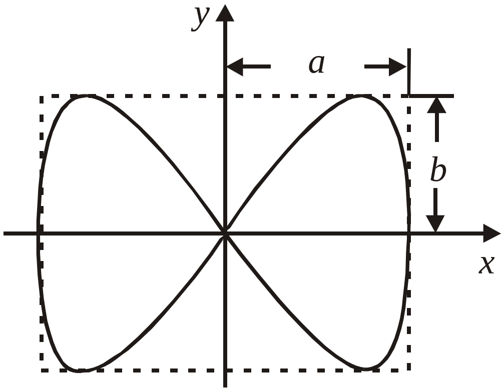


Figure 2.5.

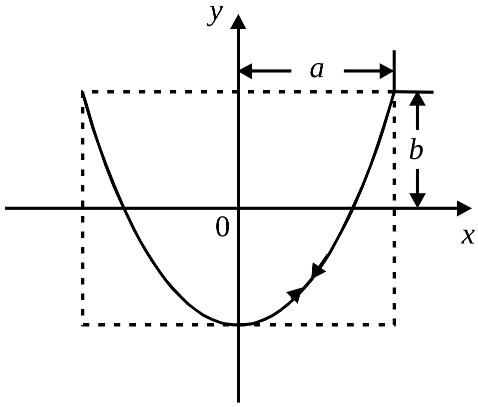


Figure 2.6.

The equations of oscillations have the form

$$x = A\cos(\omega t),$$
$$y = B\cos(2\omega t + \frac{\pi}{2}). \quad (7.20)$$

If the ratio of frequencies is 1:2 and the phase shift is zero, this trajectory degenerates into ***an open curve*** (Figure 2.6) along which the particle moves back and forth.
The equations of oscillations have the form

$$x = A\cos(\omega t),$$
$$y = B\cos(2\omega t). \quad (7.21)$$

7.5. Free Small-Amplitude Oscillations

An oscillating system (with a single degree of freedom x) whose equation of motion has the form

$$\ddot{x} + \omega_0^2 x = 0 \quad (7.22)$$

is called ***the linear harmonic oscillator***. It should be noted that this equation is true for any oscillating system under conditions that the amplitude of oscillations is sufficiently small. The solution of Eq. (7.22) are Eqs. (7.1) and (7.2).
The kinetic and potential energies of the linear oscillator can be written in the form:

$$E_k = \frac{1}{2} mA^2\omega_0^2 \sin^2(\omega_0 t + \varphi), \quad (7.23)$$

$$E_p = \frac{1}{2} mA^2\omega_0^2 \cos^2(\omega_0 \cdot t + \varphi). \quad (7.24)$$

Obviously, the total energy of an oscillating material point (for example, a spring pendulum) is

$$E = \frac{1}{2} m A^2 \omega_0^2. \tag{7.25}$$

Using Eqs. (7.23) and (7.24), we easily obtain

$$\frac{x^2}{A^2} + \frac{p^2}{m^2 A^2 \omega_0^2} = 1, \tag{7.26}$$

i.e., an ellipse (in the *x*-*p* coordinate plane). The area of the ellipse is

$$S = \pi m A^2 \omega_0 . \tag{7.27}$$

Thus, the total energy of oscillations can be represented as follows:

$$E = \frac{S\omega_0}{2\pi}. \tag{7.28}$$

It must be taken into consideration that for a spring pendulum

$$\omega_0 = \sqrt{\frac{k}{m}}, \tag{7.29}$$

where k is the spring rigidity. In other situations ω_0 can be expressed in some other way.

A mathematical or simple pendulum is defined as an ideal system consisting of a weightless and unstretchable string on which a mass concentrated at one point is suspended. A sufficiently close approximation to a simple pendulum is a small heavy sphere suspended on a long thin thread (Figure 2.7).

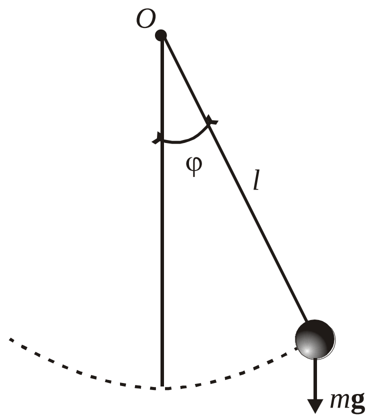


Figure 2.7.

The system has a single degree of freedom (the angle φ). Using the basic equation of rotational motion $J\boldsymbol{\varepsilon} = \mathbf{M}$, we obtain

$$ml^2\ddot{\varphi} = -mg\sin\varphi \text{ or } \ddot{\varphi} + \frac{g}{l}\sin\varphi = 0. \tag{7.30}$$

Let us consider only small-amplitude oscillations ($\sin\varphi \cong \varphi$). Introducing the notation

$$\frac{g}{l} = \omega_0^2, \tag{7.31}$$

we arrive at the equation

$$\ddot{\varphi} + \omega_0^2\varphi = 0. \tag{7.32}$$

By analogy with Eq. (7.22), its solution has the form

$$\ddot{\varphi} + \omega_0^2\varphi = 0. \tag{7.33}$$

Here A is the angular amplitude, and α is the initial angular phase.

Consequently, for small-amplitude oscillations the angular displacement of a simple pendulum changes with time according to a harmonic law. It is easy to see that the period of oscillations is

$$T = 2\pi\sqrt{\frac{l}{g}}\,. \tag{7.34}$$

If an oscillating body cannot be treated as a point particle, the pendulum is called ***a physical*** or ***compound pendulum*** (Fig. 2.8).

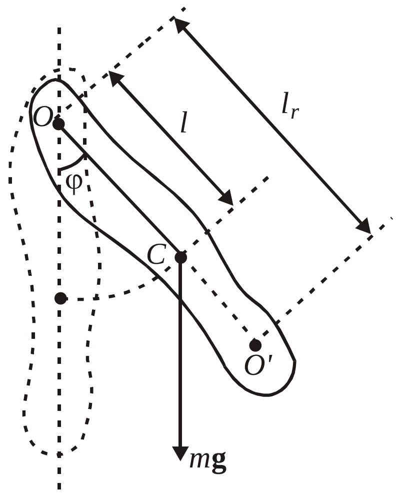


Figure 2.8.

Denoting the rotary inertia of the pendulum about the axis passing through the suspension point O by the symbol I, we obtain

$$I\ddot{\varphi} = -mgl\sin\varphi\,. \tag{7.35}$$

For small amplitude oscillations, Eq. (7.35) transforms into Eq. (7.32):

$$\ddot{\varphi} + \omega_0^2\varphi = 0\,.$$

Here ω_0 stands for the following quantity:

$$\omega_0^2 = \frac{mgl}{I}\,.$$

Hence,

$$T = 2\pi\sqrt{\frac{I}{mgl}}\,. \tag{7.36}$$

A comparison of these equations shows that a simple pendulum of length

$$l_r = \frac{I}{ml} \tag{7.37}$$

has the same period of oscillations as the given compound pendulum. The quantity l_r is called ***the reduced length*** of the physical pendulum.

The point on the straight line joining the point of suspension (O) and the center of mass (C) at a distance of the reduced length from the rotation axis is called ***the center of oscillations*** of the physical pendulum (O'). It can be shown (we recommend you to do this) that when the pendulum is suspended by its center of oscillations O', its reduced length and, consequently, its oscillation period will be the same. Hence, the point of suspension and the center of oscillations are interchangeable.

7.6. Damped Oscillations

If a point particle oscillates in a medium, the friction force acts upon the particle. The force of friction (for sufficiently small velocities) is supposed to be proportional to the value of the velocity, i.e.,

$$F_{fr} = -r\dot{x}.$$

The quantity r is called ***the resistance coefficient***. The equation of Newton's second law has the form

$$m\ddot{x} = -kx - r\dot{x}. \quad (7.38)$$

Introducing the notation $2\beta = \frac{r}{m}$ and $\omega_0^2 = \frac{k}{m}$, we can rewrite Eq. (7.38) as follows:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0. \quad (7.39)$$

This differential equation describes damping oscillations of the system. Its solution is

$$x = A_0 e^{-\beta t}\cos(\omega t + \alpha). \quad (7.40)$$

Here A_0 and α are arbitrary constants and $\omega = \sqrt{\omega_0^2 - \beta^2}$.

The motion of a system described by Eq. (7.40) can be considered a harmonic oscillation of frequency ω with an amplitude varying by the law (see Figure 2.9)

$$A(t) = A_0 \cdot e^{-\beta t}. \quad (7.41)$$

The upper dashed curve in Figure 2.9 depicts the function $A(t)$, $x_0 = A_0\cos\alpha$.

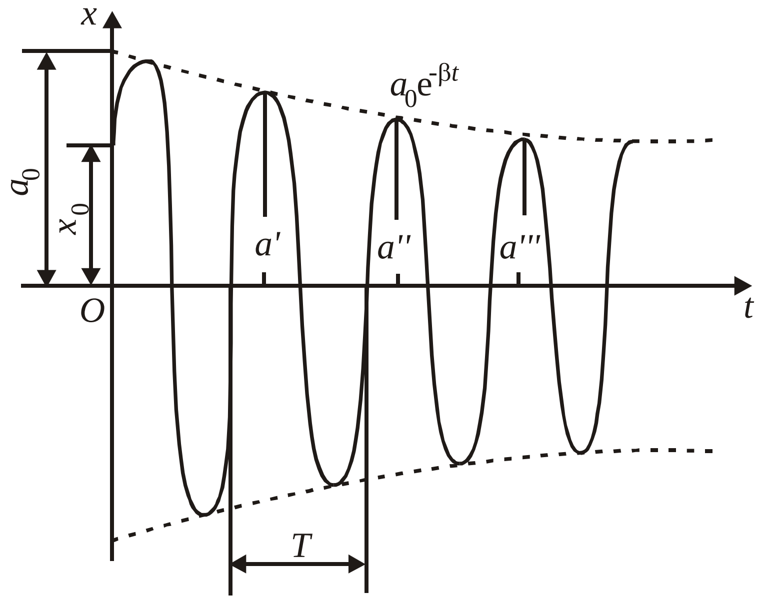


Figure 2.9.

The rate of damping is determined by the quantity $\beta = \frac{r}{2m}$ defined as ***the damping factor***. It is easy to see that the damping factor is the reciprocal of the time interval needed for the amplitude to be diminished *e* times. Taking into account Eq. (7.40), the period of damped oscillations can be represented as

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}}. \tag{7.41a}$$

When the resistance of the medium is insignificant, $T = \frac{2\pi}{\omega_0}$. The period of oscillations grows with increasing damping factor. If $\omega_0 \le \beta$, the motion is called ***aperiodic***.

The ratio of the amplitudes corresponding to moments of time that differ by the period

$$\frac{A(t)}{A(t+T)} = e^{\beta T} \tag{7.42}$$

is called ***the damping decrement***, and its logarithm is called ***the logarithmic decrement***:

$$\lambda = \ln\left(\frac{A(t)}{A(t+T)}\right) = \beta T. \tag{7.43}$$

The logarithmic decrement is the reciprocal of the number of complete cycles of motion (N_e) after which the amplitude decreases *e* times.

The oscillatory system is also characterized by the quantity

$$Q = \frac{\pi}{\lambda} = \pi \cdot N_e. \tag{7.44}$$

The total energy of oscillating system is proportional to the square of the amplitude, i.e.,

$$E = E_0 e^{-2\beta t}. \tag{7.45}$$

Time differentiation of this equation gives

$$\frac{dE}{dt} = -2\beta E. \tag{7.46}$$

For small-amplitude oscillations under condition $\omega_0 >> \beta$, the reduction of the energy for the period can be found by multiplying Eq. (7.45) by *T*:

$$\Delta E = -2\beta TE \tag{7.47}$$

or

$$|\Delta E| = 2\beta TE. \tag{7.48}$$

Hence,

$$\frac{E}{|\Delta E|} = \frac{1}{2\beta T} = \frac{1}{2\lambda} = \frac{Q}{2\pi}. \tag{7.49}$$

7.7. Forced Oscillations

When the driving force varies by a harmonic law (this situation is often observable), the oscillations are described by the equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t) . \qquad (7.50)$$

Here β is the damping factor, ω_0 is the natural frequency of the system, $f_0 = \frac{F_0}{m}$ (F_0 is the amplitude of the driving force), and ω is the frequency of the force.

Equation (7.50) is nonhomogeneous. The general solution of a nonhomogeneous equation is the sum of the general solution of the corresponding homogeneous equation (the right side is zero) and a partial solution of the nonhomogeneous equation. We already know the general solution of homogeneous equation (see Eq. (7.4)). Let us write it in the form

$$x = A_0 e^{-\beta t} \cos(\omega' t + \alpha) , \qquad (7.51)$$

where $\omega' = \sqrt{\omega_0^2 - \beta^2}$, and A_0 and α are arbitrary constants. The symbol ω without prime stands for the frequency of the driving force. After a certain period of time, the damped oscillations vanish, and only a partial solution ought to be taken into consideration.

Rather simple but cumbersome calculations give the following partial solutions:

$$x = A\cos(\omega t + \varphi) . \qquad (7.52)$$

Here

$$A = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta\omega^2}} , \qquad (7.53)$$

$$\tan\varphi = \frac{2\beta\omega}{\omega_0^2 - \omega^2} . \qquad (7.54)$$

The dependence of the amplitude of forced oscillations on the frequency of the driving force is shown in Figure 2.10.

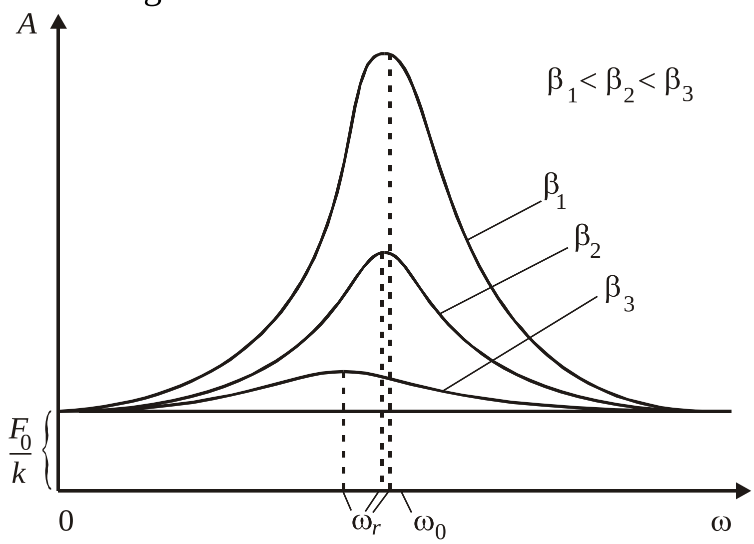


Figure 2.10.

The curves correspond to the indicated values of parameter β. These curves are called ***resonance curves***. Using Eq. (7.53) and the condition

$$\frac{dA}{d\omega} = 0, \tag{7.55}$$

it is not too difficult (we invite our readers to do this as an exercise) to obtain

$$A_{\max} = \frac{F_0}{2\beta m\sqrt{\omega_0^2 - \beta^2}}, \tag{7.56}$$

$$\omega_r = \sqrt{\omega_0^2 - 2\beta}. \tag{7.57}$$

Here $A_{\max}$ is the amplitude in resonance, ω_r is ***the resonance frequency***, i.e., the frequency at which the amplitude is maximum. It is clear that the larger is the value of β, the lower is the resonance curve.

7.8. Parametric Resonance

In the previous section we dealt with an external force which causes the direct displacement of an oscillating system. There is another kind of external action by means of which great oscillations can be imparted to a system. This kind of action consists in periodical change of a parameter of the system in step with its oscillations. This phenomenon is called ***the parametric resonance*** and is described by the differential equation

$$\ddot{x} + f(t)x = 0, \tag{7.58}$$

where $f(t)$ is a periodic function of time.

Let us take as an example a simple pendulum whose length is being periodically changed (Figure 2.11); it increased when the pendulum is at its extreme position and decreased when the pendulum is at its middle position.

This being done, the pendulum starts swinging violently. This can be explained as follows: the negative work of the external force in stretching of the pendulum is smaller in magnitude than the positive work done in compressing the pendulum thread. As a result, the total work done by the external force during the period is positive.

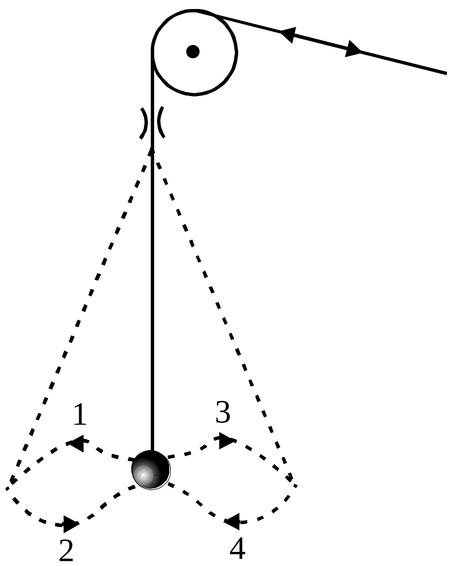


Figure 2.11.

7.9. Waves. General Information

If particles of an elastic (solid, fluid, or gas) medium are made to oscillate, then due to interactions of the particles, these oscillations will propagate through the medium from particle to particle with a certain velocity v. The process of oscillation propagation in the elastic medium is called ***the elastic wave***. The region of the medium where the particles oscillate is called ***the wave field***. The interface between the oscillating particles and the particles at rest, which do not oscillate, is called ***the wave front***. The locus of the points oscillating in the same phase is known as ***the wave surface***. Wave surfaces remain stationary, while the wave front is in continuous motion. Wave surfaces can have arbitrary shapes. In the simplest cases, they are planes, cylinders, or spheres. The wave in these cases is called ***plane***, ***cylindrical***, or ***spherical***. In ***the plane wave***, the wave surfaces are a multitude of parallel planes, in ***the cylindrical wave*** – of coaxial cylinders, and in ***the spherical wave*** – of concentric spheres.

The particles of the medium in which the wave propagates are not made to perform the translation motion, they only oscillate about their equilibrium positions. If the particles of the medium oscillate parallel to the direction of the wave motion, the wave is called ***longitudinal***. If oscillations are perpendicular to the direction of wave motion, the wave is called ***transverse***. Elastic transverse waves can appear only in media having a shear resistance. Therefore, only longitudinal waves can appear in gases and fluids. Both longitudinal and transverse waves can propagate in solids.

7.10. Wave Equation

The wave propagation in a medium is described by the differential equation

$$\nabla^2\xi - \frac{1}{v^2}\cdot\frac{\partial^2\xi}{\partial t^2} = 0. \qquad (7.59)$$

In Cartesian reference frame it has the form

$$\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2} = \frac{1}{v^2}\cdot\frac{\partial^2\xi}{\partial t^2}. \qquad (7.60)$$

Here ∇^2 is called ***the Laplacian operator***, v is ***the phase velocity*** of the wave (for ***monochromatic waves*** this quantity coincides with the velocity of wave propagation), ξ is the displacement of the particles of the medium. In general, a solution of Eq. (7.59) is very complicated, depending on ***the initial*** and ***boundary conditions***.

It can be rather easily solved for three basic types of space symmetry: plane, axial, and central. As a result, we obtain

$$\text{plane waves: } \xi = A\sin(\omega t - kx), \qquad (7.61)$$

$$\text{cylindrical waves: } \xi = \frac{A}{\sqrt{r}}\sin(\omega t - kr), \qquad (7.62)$$

$$\text{spherical waves: } \xi = \frac{A}{r}\sin(\omega t - kr). \qquad (7.63)$$

Here A is the wave amplitude, ω is the wave frequency, k is the wave number,

$$\frac{k}{\omega} = v, \qquad (7.64)$$

r is the distance from the axis of symmetry (Eq. (7.62)) or from the center of symmetry (Eq. (7.63)).

The plane wave (Eq. (7.61)) propagates from left to right (i.e., along the *x* axis). Indeed, the quantity in parenthesis at a distance Δx from a given point *x* after the moment *t* can be repeated at the moment $t+\Delta t$ if

$$\omega(t+\Delta t)-k(x+\Delta x)=\omega t-kx.$$

So we have $\omega\Delta t=k\Delta x$, or

$$\frac{\Delta x}{\Delta t}=\frac{k}{\omega}=v, \tag{7.65}$$

i.e., the phase velocity of wave.

If the sign in parenthesis is changed, the wave propagates from right to left. The distance between two neighboring points having the same phase is called the wavelength λ. It follows from Eq. (7.61) that

$$kx=k(x+\lambda),\text{ i.e., } k\lambda=2\pi.$$

So

$$\lambda=\frac{2\pi}{k},\ k=\frac{2\pi}{\lambda}. \tag{7.66}$$

Figure 2.12 shows the wave form of the transverse wave as a function of time.

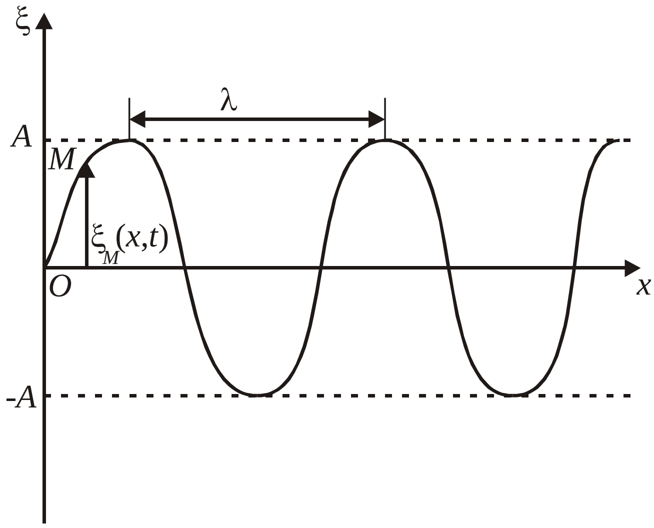


Figure 2.12.

To make the particles of the medium to oscillate, the wave must give them a certain energy. Thus, the wave propagating in a medium transfers the energy. The energy *dw* carried through the area $dS_\perp$ perpendicular to the direction of wave propagation during time *dt*,

$$j=\frac{\Delta w}{\Delta S_\perp\cdot\Delta t}, \tag{7.67}$$

is called ***the energy flux density***. This quantity is a vector and can be represented in the form

$$\mathbf{j}=\omega\mathbf{v}, \tag{7.68}$$

where ω is ***the energy density*** at a given point. The energy flux density may be different at different points of space. At a given point, it varies with time by a sine square law. Its average value is

$$\langle\mathbf{j}\rangle=\langle\omega\rangle\mathbf{v}=\frac{1}{2}\rho A^2\omega^2\mathbf{v}. \tag{7.69}$$

Equation (7.69) holds for an arbitrary wave. It should be noted that when we consider the intensity of wave at a given point, we have in mind the time-average value of the energy flux density transferred by the wave. The concept of the energy flux density was introduced into physics by the Russian physicist N. Umov at the end of the nineteenth century, and the quantity **j** is usually called Umov's vector.

Only longitudinal waves can propagate in gaseous media. The speed of propagation is given by the relation

$$v = \sqrt{\gamma \frac{p}{\rho}}. \tag{7.70}$$

Here p is the pressure, ρ is the gas density, and γ is the ratio of the gas heat capacities at constant pressure and constant volume.

At atmospheric pressure and standard temperature, most gases are close to an ideal gas in their properties. Therefore, we can assume that the ratio $\frac{p}{\rho}$ equals $\frac{RT}{M}$ and rewrite the last equation in the from

$$v = \sqrt{\frac{\gamma RT}{M}}. \tag{7.71}$$

The foregoing shows that longitudinal and transverse waves can propagate in solids. The phase velocity of longitudinal elastic waves is

$$v = \sqrt{\frac{E}{\rho}}, \tag{7.72}$$

where E is Young's modulus, and ρ is the density of the medium.

The phase velocity of transverse elastic waves is

$$v = \sqrt{\frac{\sigma}{\rho}}. \tag{7.73}$$

Here, σ is the shear modulus.

7.11. Standing Waves

If several waves propagate in an elastic medium simultaneously, the oscillations of the particles are geometrically added. This statement follows from the experiments and is called ***the superposition principle***.

If the oscillations due to separate waves at each point of the medium have a constant phase shift, the waves are called ***coherent***. The interaction of coherent waves when they are superposed is called ***interference***.

A very important case of interference is observed in case of superposition of two oncoming plane waves having the same amplitudes. The resulting oscillatory process is called ***a standing wave***. Standing waves are produced when the waves are reflected from obstacles. The wave striking an obstacle and the oncoming reflected wave produce a standing wave as a result of superposition.

The equations of two plane waves propagating along the x axis in opposite directions are

$$\xi_1 = A\cos(\omega t - kx + \alpha_1),$$
$$\xi_2 = A\cos(\omega t + kx + \alpha_2). \qquad (7.74)$$

Adding these two equations, we obtain

$$\xi = \xi_1 + \xi_2 = 2A\cos(kx + \frac{\alpha_2 - \alpha_1}{2})\cos(\omega t + \frac{\alpha_1 + \alpha_2}{2}). \qquad (7.75)$$

To simplify it, let us choose the origin of x and t so that the terms (α_2 - α_1) and ($\alpha_1 + \alpha_2$) vanish. Then Eq. (7.75) can be written in the form

$$\xi = (2A\cos(2\pi\frac{x}{\lambda}))\cos\omega t. \qquad (7.76)$$

Thus, at every point of standing wave the oscillations have the same frequency as the opposite waves and the amplitude depending on x. At the points whose coordinates meet the condition

$$2\pi\frac{x}{\lambda} = \pm\pi n \; (n = 0, 1, 2, ...) \qquad (7.77)$$

the amplitude of the oscillations reaches its maximum. These points are called ***the antinodes*** of the standing wave

$$X_{anti} = \pm n\frac{\lambda}{2} \; (n = 0, 1, 2, ...). \qquad (7.78)$$

It should be emphasized that an antinode is not a single point, but a plane whose points are specified by Eq. (7.78). At the points whose coordinates meet the condition

$$2\pi\frac{x}{\lambda} = \pm(n + \frac{1}{2})\pi \;\; (n = 0, 1, 2, ...) \qquad (7.79)$$

the amplitude of the standing waves vanishes. These points are called ***the nodes*** of the standing waves

$$X_{node} = \pm(n + \frac{1}{2})\frac{\lambda}{2}, \;\; (n = 0, 1, 2, ...). \qquad (7.80)$$

A node, like an antinode, is not a single point, but a plane whose points are specified by Eq. (7.80). Figure 2.12 shows that the distance between the adjacent antinodes is equal to that between the ajacent nodes and is $\frac{\lambda}{2}$.

The antinodes and nodes are displaced by a quarter of wavelength. The arrows show the velocities of the particles.

7.12. Doppler Effect

If the source and the receiver of waves are stationary relative to the medium in which the waves are propagating, the frequency of oscillations recorded by the receiver will be equal to the frequency ν_0 of oscillations of the source. If the source or the receiver (or both) are moving relative to the medium, then the frequency ν

recorded by the receiver may differ from ν_0. This phenomenon is called ***the Doppler effect***

$$\nu = \nu_0 \frac{v \pm v_r}{v \mp v_s}. \tag{7.81}$$

Here v_r is the velocity of the receiver, v_s is the velocity of the source, and v is the velocity of waves relative to the medium. The upper sign in Eq. (7.81) is chosen when the distance between the source and the receiver decreases, and *vice versa* when the distance between them increases.

If the directions of the velocities $\mathbf{v}_s$ and $\mathbf{v}_r$ do not coincide with the straight line passing through the source and the receiver, the projections of the vectors $\mathbf{v}_s$ and $\mathbf{v}_r$ onto this straight line must be substituted for v_s and v_r in Eq. (7.81).

EXAMPLE 18

A body is in simple harmonic motion along the x axis. Its displacement varies with time according to the equation

$$x = 4.0\cos(\pi t + \frac{\pi}{4}),$$

where t is in seconds and the angles in the parentheses are in radians.

(a) Determine the amplitude, frequency, and period of the motion.

Solution. Comparing this equation with the general equation for simple harmonic motion**,** $x = A\cos(\omega t + \Phi)$, we see that A = 4.00 m and $\omega = \pi$ rad/s; therefore, we find $f = \omega / 2\pi = \pi / 2\pi = 0.5\,\text{s}^{-1}$and $T = 1/f$ = 2.00 s.

(b) Calculate the velocity and acceleration of the body at time t.

Solution

$$v = \frac{dx}{dt} = -4.0\sin(\pi t + \frac{\pi}{4})\frac{d}{dt}(\pi t) = -4.0\pi\sin(\pi t + \frac{\pi}{4}),$$

$$a = \frac{dv}{dt} = -4.0\pi\cos(\pi t + \frac{\pi}{4})\frac{d}{dt}(\pi t) = -4.0\pi^2\cos(\pi t + \frac{\pi}{4}).$$

(c) Using the results of part (b), determine the position, velocity, and acceleration of the body at t = 1.00 s.

Solution. Noting that the angles in the trigonometric functions are in radians, we obtain for t = 1.00 s

$$x = 4.0\cos(\pi t + \frac{\pi}{4}) = 4.0\cos\left(\frac{5\pi}{4}\right) = -2.83\,\text{m},$$

$$v = -4.0\pi\sin\left(\frac{5\pi}{4}\right) = 8.89\,\text{m/s},$$

$$a = -4.0\pi^2 \cos\left(\frac{5\pi}{4}\right) = 27.9\ \text{m/s}^2 .$$

(*d*) Determine the maximum speed and the maximum acceleration of the body.

Solution. From general expressions for *v* and *a* found in part (*b*), we see that the maximum values of the sine and cosine functions are unity. Therefore, *v* varies between ±4.0π m/s, and *a* varies between $\pm 4.0\pi^2$ m/s^2. Thus, $v_{max} = 4.0\pi$ m/s, and $a_{max} = 4.0\pi^2$ m/s^2. The same results can be obtained using $v_{max} = \omega A$ and $a_{max} = \omega^2 A$, where $A = 4.0$ m and $\omega=\pi$ rad/s.

(*e*) Find the displacement of the body between $t = 0$ and $t = 1.0$ s.

Solution. The *x* coordinate at $t = 0$ is

$$x_0 = 4.0\cos\left(0 + \frac{\pi}{4}\right) = 2.83\,\text{m}.$$

In part (*c*) we have already found that the coordinate at $t = 1.0$ s is $x = -2.83$ m; therefore, the displacement between $t = 0$ and $t = 1.0$ s is

$$\Delta x = x - x_o = -2.83 - 2.83 = -5.66\ \text{m}$$

Because the particle velocity changes its sign during the first second, the magnitude of Δx is not the same as the distance traveled for the first second.

Exercise. What is the phase of the motion at $t = 2.0$ s?

Answer. $9\pi/4$ rad.

EXAMPLE 19

A sinusoidal wave traveling in the positive *x* direction has an amplitude of 15.0 cm, a wavelength of 40.0 cm, and a frequency of 8.0 Hz. The vertical displacement of the medium at $t = 0$ and $x = 0$ is also 15.0 cm, as shown in Figure 19. Find the angular wave number, period, angular frequency, and speed of the wave.

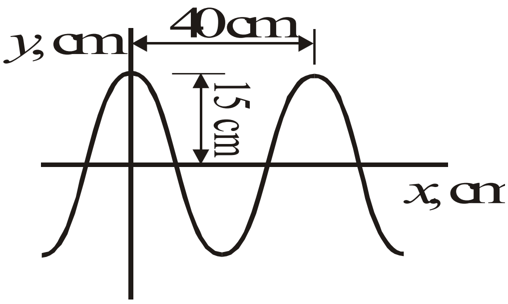


Figure 19.

Solution. Using expressions for the angular wave number, period, angular frequency, and speed of the wave, we obtain

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{40.0} = 0.157,$$
$$T = \frac{1}{f} = \frac{1}{8.0} = 0.125,$$
$$w = 2\pi f = 2\pi 8.0 = 50.3,$$
$$v = f\lambda = 8.040.0 = 320.$$

(*b*) Determine the phase ϕ, and write a general expression for the wave function.

Solution Since *A* = 15.0 cm and *y* = 15.0 cm at *x* = 0 and *t* = 0, their substitution into Eq. (7.1) yields

$$15 = 15\sin(-\phi) \text{ or } \sin(-\phi) = 1.$$

Since $\sin(-\phi) = -\sin\phi$, we see that $\phi = -\pi/2\,\text{rad}$ (or −90°). Hence, the wave function is of the form

$$y = A\sin(kx - \omega t + \frac{\pi}{2}) = A\cos(kx - \omega t).$$

That the wave function must have this form, can be seen by inspection noting that the cosine argument is displaced by 90° from the sine function. Substituting the values for *A, k*, and ω into this expression, we obtain

$$y = 15.0\cos(0.157x - 50.3t).$$

EXAMPLE 20

An object moves in a smooth, straight tunnel between two points on the Earth's surface (Fig. 20). Show that the object moves in a simple harmonic motion and find the period of its motion. Assume that the Earth's density is uniform throughout its volume.

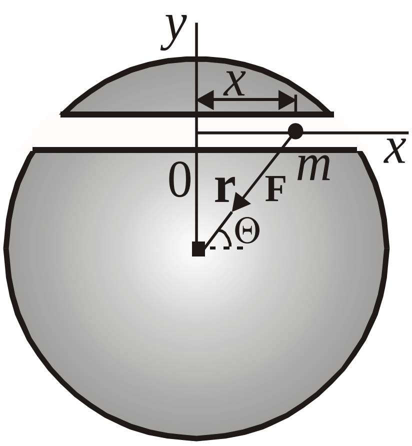


Figure 20.

The object moves along the tunnel through the Earth. The component of the gravitational force $\mathbf{F}_g$ along the *x* axis is the driving force for the motion. Note that this component always acts toward the origin of coordinates 0.

Solution. When the object is in the tunnel, the gravitational force exerted to the object acts toward the Earth's center and is given by the formula

$$F_g = -\frac{GmM_E}{R_E^{\ 3}} r.$$

The *y* component of this force is balanced by the normal force exerted by the tunnel wall, and the *x* component is

$$F_x = -\frac{GmM_E}{R_E{}^3} r\cos\Theta .$$

Since the *x* coordinate of the object is $x = r\cos\Theta$, we can write

$$F_x = -\frac{GmM_E}{R_E{}^3} x .$$

Applying Newton's second law to the motion along *x*, we obtain

$$F_x = -\frac{GmM_E}{R_E{}^3} x = ma ,$$

$$a = -\frac{GM_E}{R_E{}^3} x = -w^2 x .$$

But this is the equation of simple harmonic motion with angular speed ω, where

$$\omega = \sqrt{\frac{GM_E}{R_E{}^3}} .$$

The period is

$$T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{R_E{}^3}{GM_E}} = 2\pi\sqrt{\frac{(6.3710^6)^3}{6.6710^{-11}5.9810^{24}}} = 5.0610^3 \text{ s} = 84.3 \text{ min}$$

This period is the same as that of a satellite orbiting just above the Earth's surface. Note that the result is independent of the length of the tunnel.

EXAMPLE 21

Calculate the escape speed from the Earth for a 5000-kg spacecraft, and determine the kinetic energy it must have at the Earth's surface in order to escape from the Earth's gravitational field.

Solution. Using the equation for the escape speed with $M_E = 5.9810^{24}$ kg and $R_E = 6.37110^6$ m, we obtain

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{\frac{26.6710^{-11}5.9810^{24}}{6.3710^6}} = 1.1210^4 \text{ m/s}.$$

The kinetic energy of the spacecraft is

$$K = \frac{1}{2} m v_{esc}{}^2 = 3.1410^{11} \text{ J} .$$

Finally, you should note that Eq. (5.10) can be extended to any planet, that is, in general the escape speed from the surface of any planet of mass *M* and radius *R* is

$$v_{esc} = \sqrt{2\frac{GM}{R}} .$$

EXAMPLE 22

Calculate the work required for an Earth's satellite of mass m to pass from a circular orbit of radius $2R_E$ to the orbit of radius $3R_E$.

Solution. Using the equation for the total energy , $E=-\frac{GMm}{2r}$, we obtain for the total initial and final energies

$$E_i=-\frac{GM_E m}{4R_E},\ E_f=-\frac{GM_E m}{6R_E}.$$

Therefore, the work required to increase the energy of the system is

$$W=E_f-E_i=-\frac{GM_E m}{6R_E}-\ \frac{GM_E m}{4R_E}=\frac{GM_E m}{12R_E}.$$

For example, if we take m = l000 kg, we find that the work required is W=5.2·10^9 J, which is the energy equivalent of 150 liters of gasoline.

If we wish to determine how the energy is distributed after the work on the system has been done, we find that the change in the kinetic energy is

$$\Delta K=-\frac{GM_E m}{12R_E}\quad \text{(it decreases)},$$

while the corresponding change in the potential energy is

$$\Delta U=\frac{GM_E m}{6R}\quad \text{(it increases)}.$$

Thus, the work done by the system is $W=\Delta K+\Delta U=\frac{GM_E m}{12R_E}$, as we calculated above. In other words, part of the work done goes to increasing the potential energy and part goes to decreasing the kinetic energy.

Part 3. Molecular Physics and Thermodynamics

8.1. General Information

Molecular physics is a branch of physics studying the structure and properties of substances on the basis of the so-called molecular-kinetic concepts. According to these concepts, any body (solid, liquid, or gaseous) consists of an enormous number of very small particles – molecules (atoms can be considered as monatomic molecules). The molecules of a substance are in disordered chaotic motion (at least in gases and liquids) having no preferred direction. Its intensity depends on the temperature of the substance. For example, there are $2.7 \cdot 10^{19}$ molecules per 1 cm^3 in the air under standard conditions. It is quite obvious that to solve Newtonian equations for such a number of molecules is a useless task, and some other methods of consideration are to be applied. Two mutually complementary methods (thermodynamic and statistical) are known to be widely used for this purpose.

The objective of statistical physics based on the molecular-kinetic theory and mathematical concepts of the theory of probability is to interpret directly observed properties of bodies as the net result of action of molecules. Statistical physics does not deal with individual molecules. Only average quantities characterizing the motion of an enormous number of particles are under consideration.

Thermodynamics also studies various properties of bodies and changes in the state of a substance. However, thermodynamics studies macroscopic properties of bodies and natural phenomena without being interested in their microscopic structure. Thermodynamics allows one to arrive at conclusions how a process goes on without taking molecules and atoms into consideration and without treating the process from a microscopic standpoint.

8.2. Basic Results of Thermodynamics

Thermodynamics is a macroscopic phenomenological theory of heat. A physical system is described by a few parameters which can be controlled and measured. Relationships between these parameters and general laws by which the parameters are being changed are deduced from axioms regarded as experimental facts. Systems under consideration are ***equilibrium systems***, i.e., systems whose macroscopic parameters (average microscopic values) do not change too rapidly. It would be better to call this theory ***the thermostatics***, but the above-mentioned definition (thermodynamics) is conventional. Classical thermodynamics was formulated in the 19th century. It is a criterion for the validity of any statistical theory developed for "usual" macroscopic systems.

Thermodynamic system can be characterized by its properties (total mass, chemical structure, and so on) and how it is separated from the outer space (walls of a vessel, boundary surfaces, external fields, and so on). Using the concept of “thermodynamic system” we have in mind any system of bodies. An example of such a system is a liquid and vapor in equilibrium.

A parameter does not always have a definite value. For example, if the temperature at different points of a body is not the same, then a definite value of the parameter T cannot be ascribed to the body. In this case, the body is said to be in ***a nonequilibrium state***. If such a body is isolated from other bodies and left alone, its temperature will level out and will take the same value for all points. Thus, the body will pass over to an equilibrium state. The same may also occur with other parameters. The process of transition of a system from a nonequilibrium state to an equilibrium one is called ***a relaxation process*** or just ***relaxation***. The time needed for such a transition is called ***the relaxation time***. Thus, in an equilibrium state all the macroscopic parameters remain constant until the external conditions are changed.

A process, i.e., a transition of a system from one state to another, consisting of a continuous sequence of equilibrium states is called ***an equilibrium*** or ***quasistatic***. An equilibrium process can be conducted in the reverse direction. The system will pass through the same states as in the forward process, but in the opposite order. This is why equilibrium processes are also called ***reversible***. A process by which a system after a number of changes returns to its initial state is called ***a cyclic process*** or ***a cycle***.

All the macroscopic parameters can be subdivided into two groups. ***External parameters*** are defined by external bodies or fields; ***internal parameters*** are defined by the system itself when the external parameters are fixed. For example, in Figure 3.1*a*, V is an external parameter, and p is an internal parameter; in Figure 3.1*b* the situation is *vice versa*.

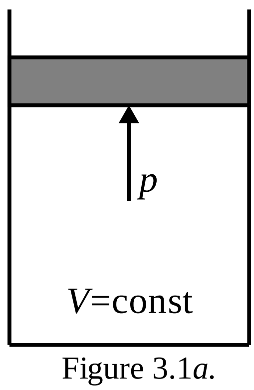


Figure 3.1*a*.

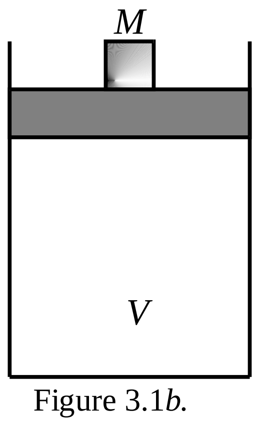


Figure 3.1*b*.

$$p = \frac{mg}{S} = \text{const}$$

Basic axioms of thermodynamics are given below.

Equilibrium axiom. Every thermodynamic system under invariable external conditions has an equilibrium state in which all its parameters do not change with time and the system cannot leave this state spontaneously.

Additivity postulate. Energy of a thermodynamic system equals the sum of energies of its parts (***subsystems***).

Temperature postulate. Adiabatically isolated equilibrium systems when brought into contact, form an equilibrium thermodynamic system only when the initial temperatures of subsystems are the same. This statement is known to be called ***the zero law of thermodynamics***. In accordance with the additivity and temperature postulates, the temperature is introduced in thermodynamics as all parameters defining equilibrium systems (or subsystems). All inter equilibrium parameters are functions of external parameters and temperature.

Energy conservation law. This law is the extrapolation of the same law of classical Newtonian mechanics to thermodynamic systems and expresses ***the conservation law of motion*** in its general form. It allows one to introduce ***the heat quantity*** as the energy imported to the system in contact of the given system with external bodies rather then by mechanical work. Let us use the following notations: dE is the energy increment, ∂A is the elementary work done by the system on external bodies, and ∂Q is the amount of heat imported to the system. Then

$$\partial Q = dE + \partial A. \tag{8.1}$$

Here, the designations ∂Q and ∂A are used to emphasize the fact that these quantities are partial differentials compared to dE which is the perfect one. Eq. (8.1) is usually called ***the first law of thermodynamics***. It can be formulated as follows: the amount of heat imported to a system is spent to increase the internal energy of the system and to produce the work done by the system on external bodies.

The second law of thermodynamics. The postulates formulated by Clausius and Kelvin are known to be the essence of the second law of thermodynamics. ***Clausius' statement***: processes are impossible whose only final result would be the heat flow from a colder body to a warmer one. ***Kelvin's statement***: processes are impossible whose only final result would be the removal of a definite amount of heat from a body and the complete conversion of this heat into work. Later it was shown that the second statement could be obtained from the first one and *vice versa*. From both postulates, it follows that a perpetual motion machine of the second kind, i.e., a periodically operating engine that receives heat from a single reservoir and completely converts this heat into work, is impossible.

The most important result of the second law is the possibility to introduce the new function of state – ***the entropy*** S in accordance with the formula

$$\partial Q = Tds \text{ (}ds\text{ is the perfect differential).} \tag{8.2}$$

So, substituting Eq. (8.2) into Eq. (8.1) and having in mind that

$$dA = pdV\,, \tag{8.3}$$

we arrive at ***the basic differential equation***

$$Tds = dE + pdV\,. \tag{8.4}$$

For nonequilibrium processes, Eq. (8.4) transforms into

$$Tds \geq dE + pdV\,. \tag{8.5}$$

It is also proved that the increment ΔS cannot be negative for adiabatic processes, i.e.,

$$\Delta S \geq 0\,. \tag{8.6}$$

The above relation can be formulated as ***the law of increasing of entropy***

$$\frac{dS}{dt} \geq 0 \text{ (}t\text{ is time).} \quad (8.7)$$

Nernst's theorem

$$S \to 0, T \to 0. \quad (8.8)$$

This theorem is known as ***the third law of thermodynamics***. In other words, it can be formulated as follows: when the temperature of a body tends to absolute zero, its entropy tends to zero, or it is impossible to reach the absolute zero temperature.

Problems solved by classical thermodynamics

Classical thermodynamics sets a close relation between thermal and caloric equations of state. The equations of state including T, P, and V are called ***thermic*** (***thermal***), for example, the Clapeyron equation of state for an ideal gas

$$PV_m = RT \quad (8.9)$$

and the van der Waals equation of state for a real gas

$$(P + \frac{a}{V_m^2})(V_m - b) = RT. \quad (8.10)$$

Here P is the pressure exerted on the gas from the outside (equal to the pressure of the gas on the walls of the vessel it occupies), a and b are the van der Waals constants, R is the molar gas constant equal to $8.31 \frac{\text{J}}{\text{mol} \cdot \text{K}}$, T is the temperature, and V_m is the volume occupied by one mole of the gas at the given P and T (thus, $V_m = \frac{V\mu}{m}$, where m is the mass of the gas, and μ is the molar mass). The equation of state that relates the energy E to the external parameters and temperature, is called the caloric equation. For example, the equation for the energy of an ideal gas has the form

$$E = \frac{3}{2} RT. \quad (8.11)$$

For the molar specific heat, we have

$$C_V = \frac{3}{2} R. \quad (8.12)$$

Work of an ideal gas in different processes

The work done by a body on external bodies when it passes from state 1 to state 2 is

$$A_{12} = \int_{V_1}^{V_2} PdV. \quad (8.13)$$

Graphically, it is expressed by the hatched area (see Figure 3.2).

In a process, a gas, in addition to the equation of state, meets some other conditions determined by the nature of process. For example, the condition P = const is met in ***the isobaric process***. The condition V = const holds in ***an isochoric process***. The condition T = const describes ***an isothermal process***. A process going on without heat exchange with surroundings is called ***adiabatic***. Processes in which the heat capacity of a body remains constant are defined as ***polytropic*** ones.

The equations of these processes and the expressions for the work done are given below.

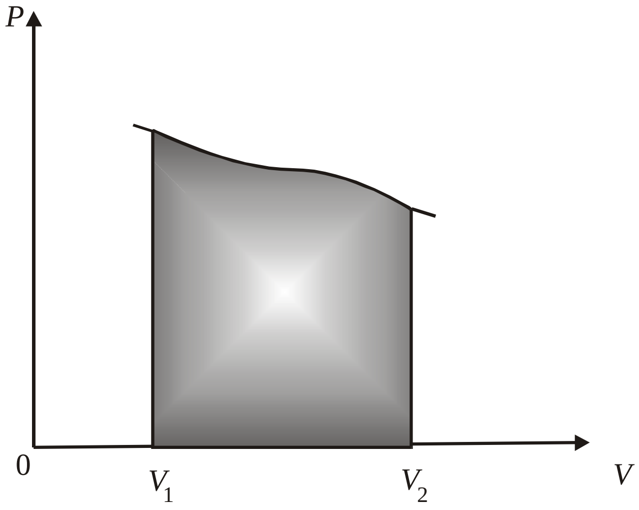
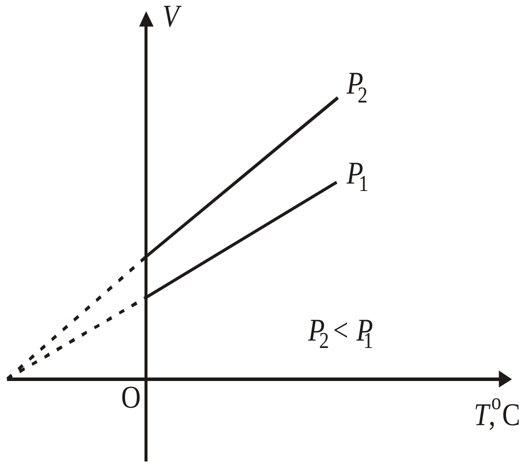
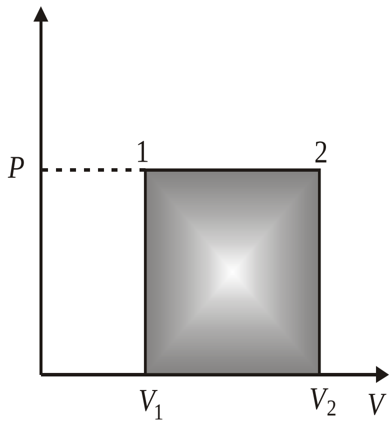


Figure 3.2.

1. ***Isobaric process***. P = const, $\frac{V}{T} = \text{const}$, $V = V_o(1 + \alpha t)$ (Gay-Lussac's law).

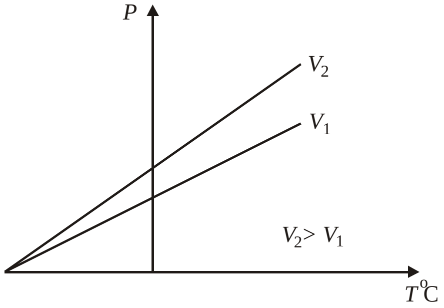
Here V_o is the gas volume at $t = 0^oC$ and $\alpha = \frac{1}{273.15}K^{-1}$. (See Figures below.)

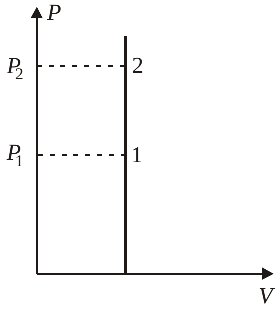




Obviously, $A = P(V_2 - V_1)$.

2. ***Isochoric process***. V = const, $\frac{P}{T} = \text{const}$, and $P = P_o(1 + \alpha t)$.

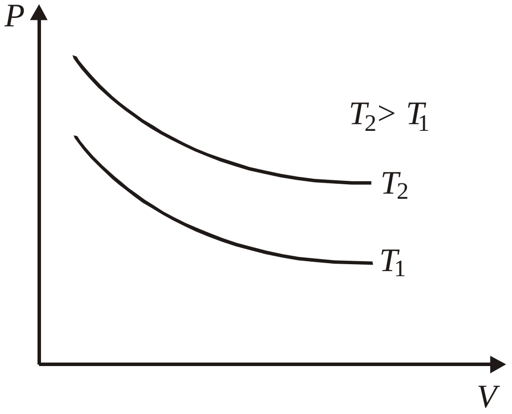


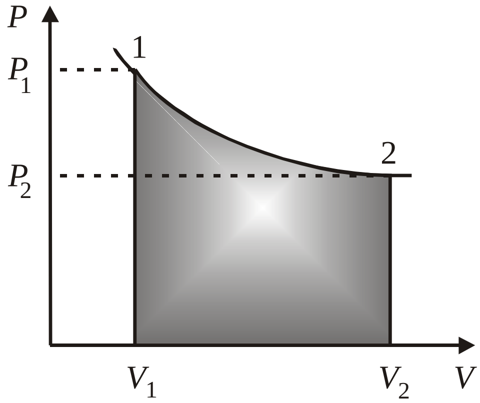


Here P_o is the gas pressure at $t = 0^oC$ and $\alpha = \frac{1}{273.15}K^{-1}$. (See Figures.)

Obviously, $A = 0$.

3. ***Isothermal process***. T = const, PV = const. (See Figures below.)





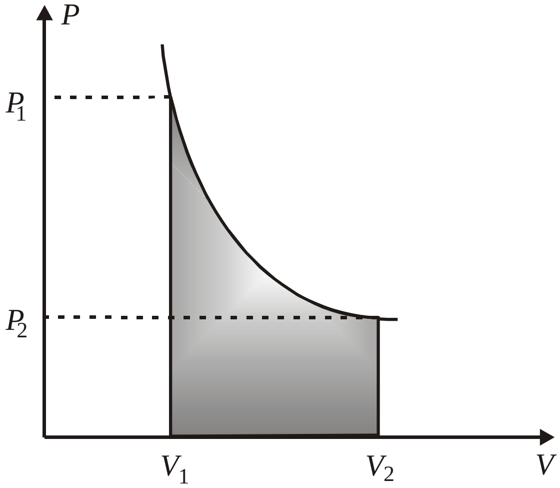
$$A = \frac{m}{\mu} RT \ln\left(\frac{V_2}{V_1}\right) = \frac{m}{\mu} RT \ln\left(\frac{P_1}{P_2}\right).$$

4. ***Adiabatic process***. $dQ = 0$.

Equation of an adiabat of an ideal gas (Poisson equation) is

$$PV^{\gamma} = \text{const.}$$

Here $\gamma = C_P / C_V$, C_V is the heat capacity at constant volume, and C_P is the heat capacity at constant pressure (see the Figure below).



The work done is

$$A = \frac{m}{\mu} \frac{RT}{(\gamma - 1)} \left[1 - \left(\frac{V_1}{V_2}\right)^{\gamma - 1}\right].$$

For one mole of an ideal gas

$$C_P = C_V + R. \tag{8.14}$$

The quantity

$$\gamma = C_P / C_V \tag{8.15}$$

depends on the number of degrees of freedom. For monatomic gases, γ=1.67; for diatomic gases, γ=1.40; for triatomic gases γ=1.33. The value of γ is determined by the number and nature of the degrees of freedom of the molecule.

Using the equation of an isotherm, we have

$$\frac{dP}{dV} = -\frac{P}{V}, \tag{8.16}$$

and for adiabatic processes

$$\frac{dP}{dV} = -\gamma \frac{P}{V}. \tag{8.17}$$

Thus, the slope of the adiabat is γ times greater compared to the isotherm (see Figure 3.3).

8.2 Polytropic processes

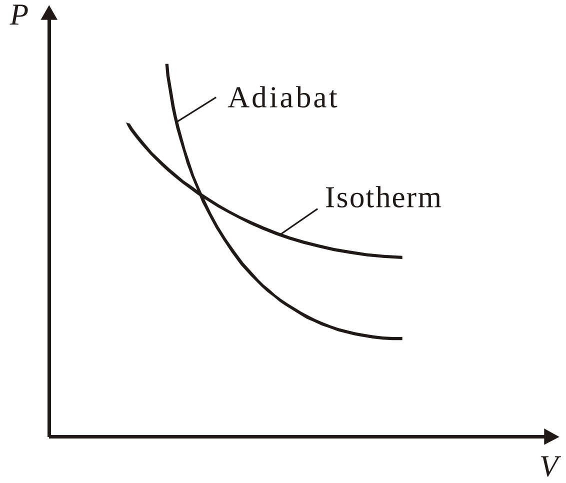


Figure 3.3.

c = *const*. The equation of a polytrope for an ideal gas is

$$pV^n = \text{const}, \qquad (8.18)$$

where

$$n = \frac{c - c_p}{c - c_v}. \qquad (8.19)$$

The quantity n determined by Eq. (8.18) is called ***the polytropic exponent*** or ***index***.

$$A = \frac{m}{\mu} \cdot \frac{RT_1}{n-1} \cdot \left[1 - \left(\frac{V_1}{V_2}\right)^{n-1}\right]. \qquad (8.20)$$

The four ***isoprocesses*** treated above belong to the category of polytropic processes (see the Table).

Process	n
Isobaric	0
Isochoric	∞
Isothermal	1
Adiabatic	γ

8.3. Heat machines. The Carnot cycle

A heat engine (machine) is a cyclic device made for heat-mechanic energy transformation. The main parts of the heat engine are: ***a heater***, ***a cooler***, and ***a working body*** (usually a gas). A gas absorbs the heat Q_1 from the heater, expands, does some work, deposits the heat Q_2 to the cooler, and returns to its initial state. The work done is

$$A = Q_1 - Q_2. \qquad (8.21)$$

Here, Q_1 is the heat transferred to the gas from the heater, and Q_2 is the heat transferred from the gas to the cooler. ***The efficiency*** of the heat engine is

$$\eta = \frac{Q_1 - Q_2}{Q_1}. \qquad (8.22)$$

The maximum theoretical efficiency is

$$\eta_{max} = \frac{T_1 - T_2}{T_1}. \qquad (8.23)$$

Here, T_1 is the temperature of the heater, and T_2 is the temperature of the cooler. Obviously,

$\eta < \eta_{max} < 1$. The value of η_{max} given by Eq. (8.23) can only be achieved in an ideal cyclic process which is called ***the Carnot cycle***. The Carnot cycle (see Figure 3.4) consists of two isotherms and two adiabats.

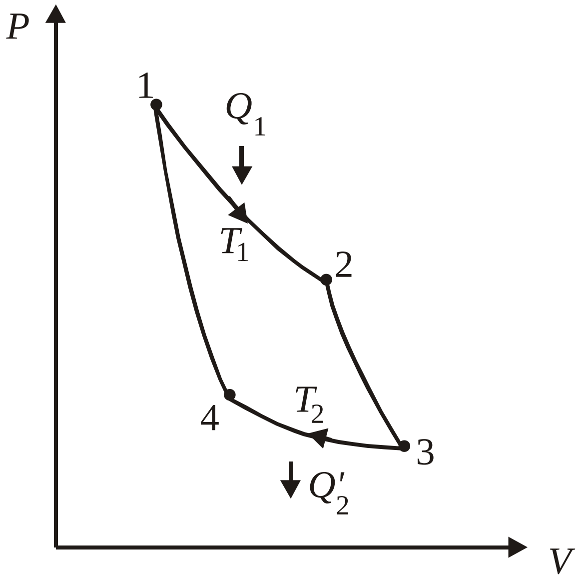


Figure 3.4.

Path 1-2 corresponds to the isothermal expansion, and path 2-3 represents the adiabatic expansion of the gas. And *vice versa*, path 3-4 represents the isothermal compression, and path 4-1 corresponds to the adiabatic compression of the gas.

The Carnot cycle can be represented in a simple form when using the *T-S* diagram (see Figure 3.5).

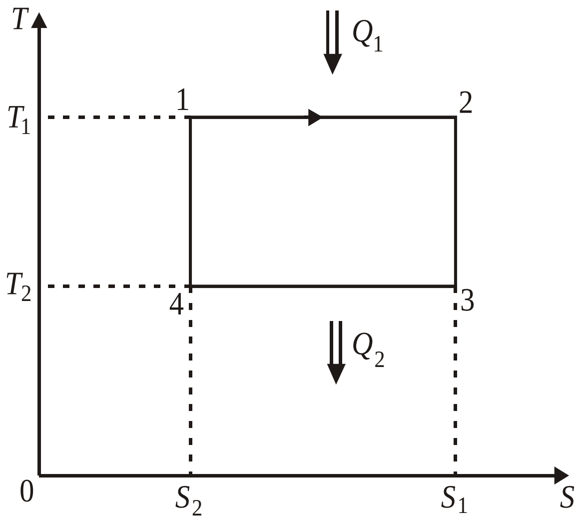


Figure 3.5.

In accordance with equation *dQ* = *T·dS*, we have

$$\Delta S = \int \frac{dQ}{T}.$$

(8.24)

Thus,

$$\Delta S_{12} = \frac{Q_1}{T_1}, \qquad (8.25)$$

$$\Delta S_{34} = -\frac{Q_2}{T_2}, \qquad (8.26)$$

or

$$\frac{Q_1}{T_1} - \frac{Q_2}{T_2} = 0. \qquad (8.27)$$

The total change of the entropy for a cyclic reversible process is zero.

Example 23

An ideal gas occupies a volume of 100 cm^2 at a temperature of 20^oC and a pressure of 100 Pa. Determine the number of gas moles in the container.

Solution. (*a*) The quantities given are volume, pressure, and temperature: $V = 100\ cm^3 = 1 \cdot 10^{-4}\ m^3$, $p = 100$ Pa, and $T = 20^oC = 293K$. Using equation $PV = \nu RT$, we obtain

$$\nu = \frac{pV}{RT} = \frac{100 \cdot 1 \cdot 10^{-4}}{8.31 \cdot 293} = 4.4 \cdot 10^{-6} \text{mol}.$$

Note that you must express T as an absolute temperature (K) when using the ideal gas law.
(b) Calculate the number of molecules in the container taking advantage of the fact that Avogadro's number is $6.02 \cdot 10^{23}$ mol^{-1}.

$$N = \nu N_A = 4.11 \cdot 10^{-6} \cdot 6.02 \cdot 10^{23} = 2.47 \cdot 10^{18} \text{molecules.}$$

Example 24

Pure helium gas is pumped into a tank containing a movable piston. The initial volume, pressure, and temperature of the gas are $15 \cdot 10^{-3}$ m^3, 200 kPa, and 300 K. The volume is decreased to $12 \cdot 10^{-3}$ m^3 and the pressure is increased to 350 kPa. Find the final temperature of the gas. (Assume that helium behaves like an ideal gas.)

Solution. As no gas escapes from the tank, the number of moles remains constant; therefore, using $PV = \nu RT$ at the initial and final points, we obtain

$$\frac{P_i V_i}{T_i} = \frac{P_f V_f}{T_f},$$

where i and f refer to the initial and final states. Solving this equation for T_f, we obtain

$$T_f = \left(\frac{P_f V_f}{P_i V_i}\right) T_i = \frac{350 \cdot 10^3 \cdot 12 \cdot 10^{-3} \cdot 300}{200 \cdot 10^{-3} \cdot 15 \cdot 10^{-3}} = 420 \text{ K.}$$

Example 25

A sealed glass bottle containing air at atmospheric pressure (101 kPa) and having a volume of 30 cm^3 is at 27°C. It is then tossed into an open fire. When the temperature of the air in the bottle reaches 200°C, what is the pressure inside the bottle? Assume any changes of the bottle volume are negligible.

Solution. We start with the expression

$$\frac{P_i V_i}{T_i} = \frac{P_f V_f}{T_f}.$$

Since the initial and final volumes of the gas are assumed equal, this expression reduces to

$$\frac{P_i}{T_i} = \frac{P_f}{T_f}.$$

This gives

$$P_f = \left(\frac{T_f}{T_i}\right) P_i = \frac{437}{300} 101 \cdot 10^3 = 159 \text{ kPa}.$$

Obviously, the higher the temperature, the higher the pressure exerted by the trapped air.

Exercise. Show that 1.00 mole of any gas at atmospheric pressure (101 kPa) and standard temperature (237 K) occupies a volume of 22.4 L.
Exercise. A bubble of marsh gas rises from the bottom of a fresh water lake located at a depth of 4.2 m and having a temperature of 5^oC to the surface, where the water temperature is 12^oC. What is the ratio of the bubble diameter for two positions? (Assume that the bubble gas is in thermal equilibrium with water at each position). (Answer: 1.13)

Example 26

A student eats a dinner rated at 2000 food Calories. He wish to do an equivalent of work in the gymnasium by lifting a 50-kg mass. How many times must he raise the mass to expend this amount of energy? Assume that he raises it at 2 m each time and regains no energy when it is dropped to the floor.
Solution. Since 1 food calorie = $1{\cdot}10^3$ cal, the work required is $2{\cdot}10^6$ cal. Converting this to J, we have for the work required $W = 2{\cdot}10^6{\cdot}4.186 = 8.37{\cdot}10^6$ J. The work done in lifting the mass at height h is equal to mgh, and the work done in lifting it n times is $n{\cdot}mgh$. We equate it to the total work required:

$$W = n{\cdot}mgh = 8.37{\cdot}10^6 \text{ J. } n = W/mgh = (8.37{\cdot}10^6)/(5{\cdot}9.8{\cdot}2) = 8.54{\cdot}10^3 \text{ times.}$$

If the student is in good shape and lifts the weight once every 5s, it will take him about 12 h to perform this feat. Clearly, it is much easier to lose weight by dieting.

Example 27

Calculate the work done by 1 mole of an ideal gas kept at 0^oC during expansion from 3 to 10 liters.

Solution. The work done by the gas is given by the equation $W = \int_{v_i}^{v_f} PdV$. Since the gas is ideal and the process is quasi-static, we can apply $PV = \nu RT$ for each point of the path. Therefore, we have

$$W = \int_{v_i}^{v_f} PdV = \int_{v_i}^{v_f} \frac{\nu RT}{V} dV = \nu RT \ln \frac{V_f}{V_i}.$$

Substituting these values into equation gives $W = 1{\cdot}8.31{\cdot}273{\cdot}\ln(10/3) = 2.7{\cdot}10^3$ J. The thermal energy that must be supplied to the gas from the reservoir to keep T constant is also $2.7{\cdot}10^3$ J.

Exercise. An ideal gas is enclosed in a cylinder that has a movable piston on the top. The piston has a mass of 8000 g and an area of 5 cm^2 and is free to slide up and down, keeping the pressure of the gas constant. How much work is done by 0.2 mole of the gas if its temperature is raised from 20^o C to 300^oC?

Exercise. One mole of an ideal gas does 3000 J of work on the surroundings as it expands isothermally to a final pressure of 1 atm and a volume of 25 L. Determine (*a*) the initial volume and (*b*) the temperature of the gas.

Exercise. Five moles of an ideal gas expand isothermally at 127^{o}C and increase their initial volume four times. Find (*a*) the work done by the gas and (*b*) the thermal energy transferred to the system, both in joules.

Exercise. During controlled expansion, the pressure of a gas is $P = 12 \cdot e^{-bV}$ atm, $b = \frac{1}{12\,\text{m}^3}$, where the volume is in m^3 (see the Figure 27). Determine the work performed when the gas expands from 12 m^3 to 36 m^3.

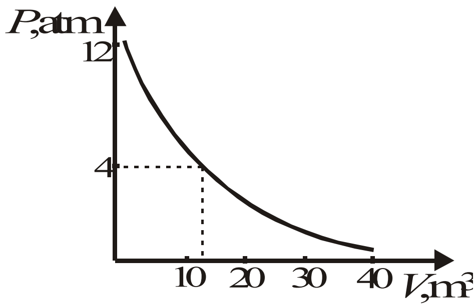


Figure 27.

Exercise. One mole of an ideal gas initially at 300 K is cooled at constant volume so that the final pressure is one-fourth of the initial pressure. Then the gas expands at constant pressure until it reaches the initial temperature. Determine the work done by the gas.

Example 28

A tank with a volume of 0.300 cm^3 contains 2 moles of helium gas at 20^{o}C. Assuming that helium behaves like an ideal gas, (*a*) find the thermal energy of the system. (*b*) What is the average kinetic energy per molecule?

Solution. (*a*) Using equation $E = 2/3\nu RT$, with $\nu = 2$ and $T = 293$ K, we obtain $E = (3/2) \cdot 2 \cdot 8.31 \cdot 293 = 7.3 \cdot 10^3$ J. (*b*) From equation $1/2mv^2 = 3/2kT$, we see that the average kinetic energy per molecule is $1/2mv^2 = 3/2 \cdot 1.38 \cdot 10^{-23} \cdot 293 = 6.07 \cdot 10^{-21}$ J.

Example 29

A cylinder contains 3 moles of helium gas at a temperature of 300 K. (*a*) How much heat must be transferred to the gas to increase its temperature to 500 K if it is heated at constant volume? (*b*) How much thermal energy must be transferred to the gas at constant pressure to raise its temperature to 500 K? (*c*) What is the work done by the gas in this process?

Solution. (*a*) For the constant volume process, the work done is zero. Therefore, from the first law of thermodynamics, we obtain $Q_1 = 3/2\nu R\Delta T = \nu C_v \Delta T$. But $C_v = 12.5$ J/mol·K for He and $\Delta T = 200$ K; therefore, $Q_1 = 3 \cdot 12.5 \cdot 200 = 7.5 \cdot 10^3$ J.

(b) Using the Table, we obtain $Q_2 = nC_p\Delta T = 3 \cdot 208 \cdot 200 = 12.5 \cdot 10^3$ J. (c) $W = Q_2 - Q_1 = 12.5 \cdot 10^3 - 7.5 \cdot 10^3 = 5 \cdot 10^3$ J.

Example 30

Air in the cylinder of a diesel engine at 20^o C is compressed from an initial pressure of 1 atm and volume of 800 cm^3 to a volume of 600 cm^3. Assuming that the air behaves as an ideal gas ($\gamma = 1.4$) and that the compression is adiabatic and reversible, find the final pressure and temperature.

Solution. Using the equation for reversible adiabatic process, $P_i V_i^\gamma = P_f V_f^\gamma$, we find that $P_f = P_i (\frac{V_i}{V_f})^\gamma = 1.00(\frac{800}{60})^{1.4} = 37.6$ atm. Because $PV = \nu RT$ is always valid during the process and no gas escapes from the cylinder, we obtain

$$\frac{P_i V_i}{T_i} = \frac{P_f V_f}{T_f} \quad T_f = T_i \frac{P_f V_f}{P_i V_i} = \frac{37.6 \cdot 60}{1 \cdot 800} 293 = 826 \, \mathrm{K}.$$

9. Statistical Distributions

9.1. General Information

In many physical applications it is very important to know how molecules are distributed in space and how they are distributed by velocity magnitudes. In order to describe the distribution of molecules, two ***probability density functions***, $f(v)$ and $f(\mathbf{r})$, are introduced. Their physical sense is as follows:

$$dp = f(v)dv \tag{9.1}$$

is the probability to find a molecule having a velocity from v to $v + dv$, and

$$dp = f(\mathbf{r})d\mathbf{r} \tag{9.2}$$

is the probability to find a molecule in a position $\mathbf{r}$ within the differential volume $d\mathbf{r}$. (We are sure that our readers possess knowledge of elementary concepts of the theory of probabilities).

Hence, the number of molecules whose velocity components are within the limits from v_x to $v_x + dv_x$, from v_y to $v_y + dv_y$, and from v_z to $v_z + dv_z$ can be written in the form

$$dN_{v_x, v_y, v_z} = Nf(v)dv_x dv_y dv_z. \tag{9.3}$$

Here, N is the total number of molecules. In analogous way, the number of molecules with coordinates from x to $x + dx$, from y to $y + dy$, and from z to $z + dz$ can be represented as follows

$$dN_{x,y,z} = Nf(\mathrm{r})dxdydz. \tag{9.4}$$

The specific form of the probability density functions $f(v)$ and $f(\mathbf{r})$ depends on the classical or quantum approach to the problem and may also be dependent on the type of forces acting between the particles. In this section, we are going to discuss only four types of distributions: the Maxwell, the Boltzmann, the Fermi-Dirac, and the Bose-Einstein distributions. All these distributions can be deduced from the most general distribution known as ***the Gibbs distribution***.

9.2. The Maxwell Distribution

The Maxwell distribution is obeyed for particles interacting by the short-range forces, i.e., for neutral atoms and molecules. However, one must be very careful when using this distribution for a charged plasma or gravitationally interacting particles. The analytical expression of Maxwell's distribution has the form

$$f(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \cdot \exp\left(-\frac{mv^2}{2kT}\right)v^2 . \tag{9.5}$$

The plot of this function is shown in Figure 3.6 (see below).

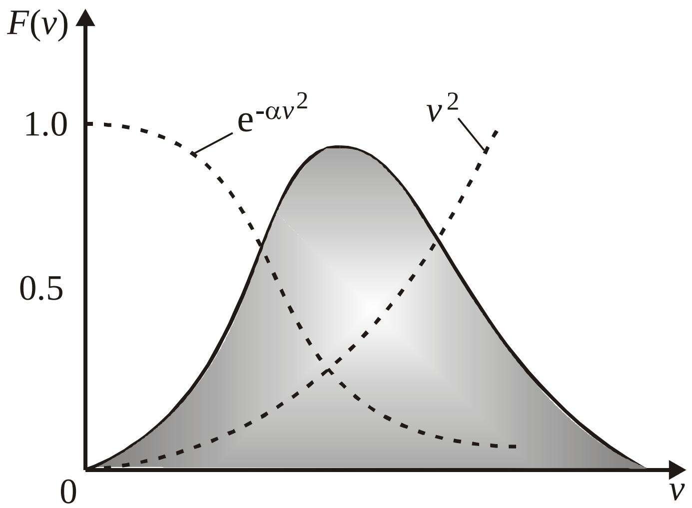


Figure 3.6.

As v increases, the factor $\exp(-\alpha v^2)$ diminishes more rapidly than the factor v^2 grows, and the function, which begins from zero, reaches its peak and then asymptotically tends to zero. The area enveloped by the curve is equal to unity.

The most probable velocity v_{prob} can be easily found by differentiating Eq. (9.5) with respect to v and then by determining the root of the equation $f\,'(v) = 0$. As a result, we obtain

$$v_{prob} = \sqrt{\frac{2kT}{m}} . \tag{9.6}$$

The mean velocity of molecules $<v>$ is given by the relation

$$<v> = \int_0^\infty v f(v) dv . \tag{9.7}$$

After necessary manipulations, we have

$$<v> = \sqrt{\frac{8kT}{\pi \cdot m}} . \tag{9.8}$$

The square root of $<v^2>$ is called ***the mean square velocity***

$$< v^2 > = \int_0^{\infty} v^2 f(v) dv, \qquad (9.9)$$

and

$$v_{m.s.v.} = \sqrt{< v^2 >} = \sqrt{\frac{3kT}{m}}. \qquad (9.10)$$

Thus,

$$v_{prob} : < \mathrm{v} > : v_{m.s.v.} = \sqrt{2} : \sqrt{8/\pi} : \sqrt{3} = 1:1.13:1.22. \qquad (9.11)$$

Using Eqs. (9.5) and (9.6), we can find the maximum value of the function $f(v)$

$$f(v_{prob}) = \frac{4}{e} \cdot \sqrt{\frac{m}{2\pi kT}}. \qquad (9.12)$$

It can be seen from Eqs. (9.6) and (9.8) that when the temperature grows (or the mass of the molecule diminishes), the peak of the curve moves to the right and becomes lower. The area under the curve remains unchanged.

For some applications, it is very convenient to use Maxwell's distribution for the quantity $U=v/v_{m.prob}$

$$f(U) = \frac{4}{\sqrt{\pi}} \cdot e^{-U^2} \cdot U^2. \qquad (9.13)$$

Using the obvious relation

$$f(v)dv = f(E)dE, \qquad (9.14)$$

we obtain the Maxwell energy distribution

$$f(E) = A \cdot \exp(-\frac{E}{kT}) \cdot \sqrt{E}, \qquad (9.15)$$

where

$$A = \frac{2}{\sqrt{\pi}} (kT)^{-3/2}. \qquad (9.16)$$

Calculations show that the velocity of 70% of all the molecules differs from the most probable value by no more than 50%. Only 0.04% of molecules have velocities exceeding $v_{m.prob}$ more then three times. And only one of 12 billions ($12 \cdot 10^9$) molecules, on average, has a velocity exceeding $5 \cdot v_{m.prob}$.

At room temperature the mean velocity of oxygen molecules is about 500 m/s, and that one of hydrogen molecules, having a mass of 1/16 of oxygen molecule, is about 2000 m/s.

9.3. The Boltzmann Distribution

The Boltzmann distribution for the probability density function has the form

$$f(\mathbf{r}) = A \cdot e^{-\frac{U(\mathbf{r})}{kT}} \qquad (9.17)$$

Here, $U(\mathbf{r})$ is the potential energy of particle at a point with the radius-vector $\mathbf{r}$. The constant A can be determined from the normalization condition

$$\int_0^\infty f(\mathbf{r})dV = 1. \qquad (9.18)$$

Here, dV is a volume element. The density of particles at a given point can be represented by the formula

$$n = N \cdot f(\mathbf{r}), \qquad (9.19)$$

where N is the number of particles. For example, in the potential field of the Earth's gravitation, the so-called barometric formula is well known

$$n = n_0 e^{-\frac{mgh}{kT}}, \qquad (9.20)$$

or

$$p = p_0 e^{-\frac{mgh}{kT}}, \qquad (9.21)$$

where n_0, and p_0 are the density of particles and the pressure at see level, respectively, and h is the altitude. The analogous formulas can be written for the particles in a centrifuge

$$n = n_0 e^{\frac{m\omega^2 r^2}{2kT}}, \qquad (9.22)$$

$$p = p_0 e^{\frac{m\omega^2 r^2}{2kT}}. \qquad (9.23)$$

Here, n_0 and p_0 are the density of particles and pressure at the central point of the centrifuge, and r is the distance between the central point and the given one.

The Maxwell and Boltzmann distributions can be combined into the ***Maxwell-Boltzmann's law***, according to which the number of molecules whose velocities are within the limits from $\mathbf{v}$ to $\mathbf{v}+d\mathbf{v}$ and whose coordinates are within the limits from $\mathbf{r}$ to $\mathbf{r}+d\mathbf{r}$ is

$$dN = A \cdot \exp\left(-\frac{U + mv^2/2}{kT}\right) dv d\mathbf{r}. \qquad (9.24)$$

Here A is the normalization factor equal to $n_0 (m/2\pi kT)^{3/2}$.

The potential energy U and the kinetic energy $mv^2/2$, and therefore the total energy E, can take continuous values in distribution (9.24).

If the total energy of particle can take on only discrete values E_1, E_2, …, then the Boltzmann distribution has the form

$$N_i = A \cdot \exp\left(-\frac{E_i}{kT}\right), \qquad (9.25)$$

where N_i is the number of particles in the state with the energy E_i, A is the constant determined by the condition

$$\sum N_i = N, \qquad (9.26)$$

and N is the number of particles.

Example 31

What is the density of air at an altitude of 12 km compared to that at sea level?

Solution. The density of our atmosphere decreases exponentially with altitude by the law

$$n(y) = n_0 e^{-mgy/kT} . \qquad (1)$$

We assume a temperature of 0^oC (T = 273 K) and an average molecular mass of 28.8 a.u. = $4.78 \cdot 10^{-26}$ kg. Taking y = 12 km, the exponent in Eq. (1) is calculated to be $\frac{mgy}{kT} = \frac{4.78 \cdot 10^{-26} \cdot 9.8 \cdot 12 \cdot 10^3}{1.38 \cdot 10^{-23} \cdot 273} = 1.49$. Thus, Eq. (1) gives $n(y) = n_0 e^{-mgy/kT} = n_0 e^{-1.49} = 0.225 n_0$. That is, the air density at an altitude of 12 km is only 22.5% of the air density at sea level.

Example 32

Determine the average height *y* of a molecule in the atmosphere at a temperature of 300 K.

Solution. The exponential function $e^{-mgy/kT}$ that appears in the law for the atmosphere can be interpreted as a probability distribution that gives the relative probability of finding a gas molecule at height *y*. Thus, the probability distribution *p*(*y*) is proportional to the density distribution *n*(*y*). The expression for the average height is

$$y = \frac{\int_0^\infty y n(y) dy}{\int_0^\infty n(y) dy} = \frac{\int_0^\infty y e^{-mgy/kT} dy}{\int_0^\infty e^{-mgy/kT} dy},$$

where the height of the molecule can change from 0 to ∞. The numerator represents the sum of the heights of the particles multiplied by their number, while the denominator is the sum of the numbers of particles. After the integration, we find (taking advantage of the integral $\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^n}$)

$$y = \frac{(kT/mg)^2}{kT/mg} = kT/mg . \qquad (1)$$

Substituting numerical values into Eq. (1), we obtain

$$y = 1.38 \cdot 10^{-23} \cdot 300 / 4.78 \cdot 10^{-26} \cdot 9.8 = 8837 \text{ m}.$$

Example 33

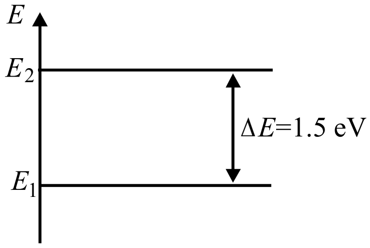


Figure 33.

Consider a gas at a temperature of 2500 K whose atoms can occupy only two energy levels separated by 1.5 eV (see the Figure 33). Determine the ratio of the number of atoms on the higher energy level to their number on the lower energy level.

Solution. The Boltzmann distribution law gives the relative number of atoms on a given energy level. In our case, the atom has two possible energies, E_1 and E_2, where E_1 is the lower energy level. Hence, the ratio of the number of atoms on the higher energy level to the number on the lower level is

$$\frac{n(E_2)}{n(E_1)} = \frac{n_0 e^{-E_2/KT}}{n_0 e^{-E_1/kT}} = e^{-(E_2 - E_1)/kT}.$$

In this problem, $E_2 - E_1 = 1.5$ eV, and since 1 eV = $1.6 \cdot 10^{-19}$ J and $kT = 0.216$ eV. Therefore, the required ratio is

$$\frac{n(E_2)}{n(E_1)} = e^{-1.5/0.216} = e^{-6.94} = 9.64 \cdot 10^{-4}.$$

This result shows that at $T = 2500$ K, only a small fraction of atoms is on the higher energy level. The number of atoms on the higher level increases for higher temperatures, but the distribution law indicates that in equilibrium there are always more atoms on the lower level than on the higher level.

Example 34

Calculate the mean free path and collision frequency for nitrogen molecules at 20°C and 1.00 atm. Assume a molecular diameter of $2 \cdot 10^{-7}$ cm.

Solution. Assuming the gas as ideal, we can use the equation $PV = NkT$ to obtain the number of molecules per unit volume under these conditions:

$$n_V = \frac{N}{V} = \frac{P}{kT} = \frac{1.01 \cdot 10^5}{1.38 \cdot 10^{-23} \cdot 293} = 2.5 \cdot 10^{25} \text{ molecules/m}^3.$$

Hence, the mean free path is

$$l = \frac{1}{\sqrt{2}\pi d^2 n_V} = \frac{1}{\sqrt{2} \cdot \pi \cdot 2 \cdot 10^{-10} \cdot 2.5 \cdot 10^{25}} = 2.25 \cdot 10^{-7} \text{ m}.$$

This value is approximately 10 times greater than the molecular diameter. Since the average speed of the nitrogen molecule at 20°C is about 511 m/s, the collision frequency is

$$f = \frac{v}{e} = \frac{511}{2.25 \cdot 10^{-7}} = 2.27 \cdot 10^9 \text{ s}^{-1}$$

$$(v=\sqrt{\frac{8RT}{\pi M}}=\sqrt{\frac{8\cdot 8.31\cdot 293}{3.14\cdot 28\cdot 10^{-3}}}=511\,\text{m/s}).$$

The molecule collides with other molecules at an average rate of about two billion times each second! The mean path *l* is not the same as the average separation between the particles. In fact, the average separation *d* between the particles is given approximately by $n_V^{-1/3}$. In this example, the average molecular separation is

$$d=\frac{1}{n_V^{1/3}}=\frac{1}{(2.5\cdot 10^{25})^{1/3}}=3.4\cdot 10^{-9}\ \text{m}.$$

Exercise. Find the rms velocity of nitrogen molecules under standard conditions, 0^oC and 1 atm. Recall that 1 mole of any gas occupies a volume of 22.4 liters under standard conditions.

Exercise. A spherical balloon with a volume of 4000 cm^3 contains inside helium at a pressure of 120 kPa. How many moles of helium are in the balloon if each helium atom has an average kinetic energy of $3.6\cdot 10^{-27}$ J? (Answer: 3.32 mol.)

Exercise. A cylinder contains a mixture of helium and argon gas in equilibrium at 150^oC. What is the average kinetic energy of each gas molecule? (Answer: $8.76\cdot 10^{-21}$ J.)

Exercise. (*a*) Determine the temperature at which the rms velocity of He atom equals 500 m/s. (*b*) What is the rms velocity of He on the surface of the Sun, where the temperature is 5800 K? (Answer: 40.1 K, 6.01 km/s.)

Exercise. If the rms velocity of helium atom at room temperature is 1350 m/s, what is the rms velocity of oxygen (O_2) molecule at this temperature? (Answer: 477 m/s.)

Exercise. During a 30-s interval, 5.00 hailstones strike a glass window having an area of 0.60 m^2 at an angle of 45^o to the window surface. Each hailstone has a mass of 5.0 g and a velocity of 8.0 m/s. Given that the collisions are elastic, find the average force and pressure on the window.

Exercise. What is the thermal energy of 100 g of He gas at 77K? How much more energy must be supplied to heat this gas to 24^o C? (Answer: 24.0 kJ, 68.7 kJ.)

Exercise. A container has a mixture of two gases: n_1 moles of gas 1 having molar specific heat C_1 and n_2 moles of gas 2 of molar specific heat C_2. (*a*) Find the molar specific heat of the mixture. (*b*) What is the molar specific heat if the mixture has *m* gases comparing n_1, n_2,…,n_m moles and molar specific heat C_1, C_2,…,C_m, respectively.

Exercise. During the compression stroke of a gasoline engine, the pressure increases from 1 to 20 atm. Given that the process is adiabatic and reversible and that the gas is ideal with γ=1.40, (*a*) by what factor does the volume change and (*b*) by what factor does the temperature change? (Answer: (*a*) 0.118, so the compression ratio $V_t / V_i = 8.50$; (*b*) 2.35.)

Exercise. One mole of an ideal diatomic gas occupies a volume of one liter at a pressure of 0.10 atm. The gas undergoes a process in which the pressure is proportional to volume. At the end of the process, the speed of sound in the gas has doubled from its initial value. Determine the amount of heat transferred to the gas. (Answer: 91.2 J.)

Example 35

Calculate the change in entropy of 2.00 moles of an ideal gas that undergoes free expansion and increases three times its initial volume.

Solution. The free expansion of the gas is clearly neither reversible nor quasi-static process. The work done by the gas against vacuum is zero, and since the walls are insulating, no thermal energy is transferred during expansion. That is, $W = 0$ and $Q = 0$. Using the first law, we see that the change in the internal energy is zero; therefore, $U_i = U_f$. Since the gas is ideal, U depends on the temperature only. So we conclude that $T_i = T_f$. That is, we find an equivalent reversible path that shares the same initial and final states. A simple choice is isothermal reversible expansion during which the gas pushes slowly a piston. Since T is constant in this process, equation for ΔS gives

$$\Delta S = \int \frac{dQ}{T} = \frac{1}{T}\int dQ \ .$$

But $\int dQ$ is simply the work (W) done by the gas during the isothermal expansion from V_i to V_f. Using this result, we find ($W = \int_{V_i}^{V_f} PdV$)

$$\Delta S = \frac{1}{T}\int dQ = \frac{1}{T}\int PdV = \int \frac{\nu RT}{TV} dV = \nu R \ln \frac{V_f}{V_i} \ .$$

Using this equation with $\nu = 2$, and $V_f = 3V_i$, we find that

$$\Delta S = 2 \cdot 8.31 \cdot \ln 3 = 18.3 \text{ J/K}.$$

Example 36

Suppose 1.00 kg of water at 0°C is mixed with an equal mass of water at 100°C. After equilibrium is reached, the mixture has a temperature of 50°C. What is the change in the system entropy?

Solution. The change in entropy can be calculated from the equation

$$\Delta S = m_1 c_1 \ln \frac{T_f}{T_i} + m_2 c_2 \ln \frac{T_1}{T_2} .$$

Using the values $m_1 = m_2 = 1$ kg, $c_1 = c_2 = 4186$ J/kg·K, $T_1 = 0^{o}$C (273 K), $T_2 = 100^{o}$C (373 K), and $T_f = 50^{o}$C (323 K), we obtain

$$\Delta S = 1 \cdot 4186 \cdot \ln \frac{323}{273} + 1 \cdot 4186 \cdot \ln \frac{323}{373} = 704 - 602 = 102 \text{ J/K}.$$

That is, as a result of this irreversible process, the increase in entropy of the cold water is greater than the decrease in entropy of warm water. Consequently, the increase in entropy of the system is 102 J/K.

Exercise. A heat engine absorbs 360 J of thermal energy and performs 25 J of work in each cycle. Find (*a*) the efficiency of the engine and (*b*) the thermal energy expelled in each cycle. (Answer: (*a*) 6.94%, (*b*) 335 J.)

Exercise. The heat absorbed by an engine is three times greater than the work it performs. (*a*) What is its thermal efficiency? (*b*) What fraction of the heat absorbed is expelled to the cold reservoir? (Answer: (*a*) 0.333, (*b*) 0.667.)

Exercise. An ideal gas is compressed to half its initial volume while its temperature is held constant. (*a*) If 1000 J of energy is removed from the gas during compression, how much work is done on the gas? (*b*) What is the change in the internal energy of the gas during compression? (Answer: (*a*)100 kJ, (*b*) 0.)

Exercise. A refrigerator has a coefficient of performance equal to 5. If the refrigerator absorbs 120 J of thermal energy from a cold reservoir in each cycle, find (*a*) the work done in each cycle and (*b*) the thermal energy expelled to the hot reservoir. (Answer: (*a*) 24.0 J, (*b*) 144 J.)

Exercise. How much work is required, using an ideal Carnot refrigerator, to remove 1 J of thermal energy from a helium gas at 4.0 K and to reject this thermal energy to a room-temperature (293 K) environment? (Answer: 72.2 J.)

10. Transport phenomena

The break of equilibrium is accompanied by the flow of molecules, heat, electric charge, and so on. The relevant processes are called ***transport phenomena.***

In this section we briefly discuss three transport phenomena: diffusion, thermal conductivity, and internal friction or viscosity.

Diffusion. It has been established experimentally that the flow of molecules of the *i*th species through a surface S perpendicular to the z axis is determined by the expression

$$N_i = -D \frac{dn_i}{dz} \cdot S, \qquad (10.1)$$

where D is a proportionality factor called ***the diffusion coefficient***, and n_i is the concentration of molecules. Multiplying both sides of Eq. (10.1) by the mass of a molecule of the ith species m_i, we obtain the expression for the mass flow of the ith species

$$M_i = -D\frac{d\rho_i}{dz}\cdot S\,, \qquad (10.2)$$

where $\rho_i = n_i m_i$ is ***the partial density*** of the ith species. Eqs. (10.1) and (10.2) are called ***Fick's law***. The diffusion coefficient D is determined by the formula

$$D = \frac{1}{3}v\lambda\,, \qquad (10.3)$$

where v is the average speed of molecules, and λ is their mean free path.

Thermal conductivity. Experiments show that if there is a temperature gradient along the z axis in a medium, a heat flux is produced whose magnitude is determined by the formula

$$q = -\wp\frac{dT}{dz}\cdot S\,. \qquad (10.4)$$

Here, q is the heat flux through the surface S perpendicular to the z axis, dT/dz is the temperature gradient (more exactly, the projection of the temperature gradient onto the z axis), $\wp$ is the proportionality factor depending on the properties of the medium and called ***the thermal conductivity.*** Eq. (10.4) is known to be ***the Fourier law***. The quantity $\wp$ is given by the formula

$$\wp = \frac{1}{3}v\lambda c_v\rho\,, \qquad (10.5)$$

where c_v is the specific heat capacity, and ρ is the density of the medium.

Internal friction. Experiments show that the force of friction between two layers of a fluid is determined by the formula

$$F = \eta\left|\frac{dU}{dz}\right|\cdot S\,, \qquad (10.6)$$

where η is the viscosity (the viscosity coefficient), dU/dz is the quantity showing how rapidly the velocity of the fluid changes in the direction z perpendicular to the direction of layer motion (the gradient of U), S is the surface area over which the force F acts. According to Newton's second law, the interaction of two layers with the force F can be considered as a process in the course of which the momentum equal to F in magnitude is transmitted from one layer to another for unit time. Therefore, Eq. (10.6) can be written in the form

$$K = -\eta\frac{dU}{dz}\cdot S\,, \qquad (10.7)$$

where K is the momentum transmitted in one second from layer to layer through the surface S (i.e., the momentum flux through S). The viscosity coefficient is determined by the expression

$$\eta = \frac{1}{3}v\lambda\rho\,. \qquad (10.8)$$

Note: the mean free path of molecules is given by

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 n},$$

where d is the molecule diameter, and n is the number of molecules per unit volume.