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BASICS OF FINANCIAL MATHEMATICS

A study guide

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BASICS OF FINANCIAL MATHEMATICS

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The study guide describes the basic notions of the quantitative analysis of financial transactions and methods of evaluating the yield of commercial contracts, investment projects, risk-free securities and optimal portfolio of risk-laden securities.
The study guide is designed for students with the major 231300 *Applied Mathematics*, 230700 *Application Informatics*, and master's program students with the major 140400 *Power Engineering and Electrical Engineering*.

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Part 1. Lecture Course

Introduction

The main goal of the science of finances consists in studying *how the financial agents* (persons and institutions) distribute the resources limited *in time.* The accent exactly on the *time*, but not other distribution types studied in economics (in regions, industries, enterprises), is a distinguishing feature of the financial science. The solutions made by the persons with regard to the time distribution of resources are *financial decisions.* From the point of view of the person(s) taking the such decisions, the resources distributed refer to either *expenses* (expenditures) or *earnings* (inflows). The financial decisions are based on commensuration of the values of expenses and profit streams. In the term *payment* the temporal character of resource distribution is reflected. The problems concerning the time distribution of resources (in the most general sense), are *financial problems.*

Since the solution of financial problems implies *the commensuration of values of expenses* (expenditures) and *the results* (earnings), the existence of some common measure to evaluate the cost (value) of the distributed resources is supposed. In practice, the cost of the resources (assets) is measured in these or those *currency units.* However, it is only one aspect of the problem. The other one concerns the consideration of *time* factor*.* If the *problem of time distribution* of resources is an identifying characteristic of financial problems, then the financial theory must give means for commensuration of values *referring to different* time moments. This aspect of the problem has an aphoristic expression *time is money*. The ruble, dollar, etc. have different values today and tomorrow.

Besides, there is one more crucially important aspect. In *all the real* financial problems which one must face in practice, there is an *uncertainty* referring to both the value of the *future* expenses and income, and the time points which they refer to. This very fact that the financial problems are connected with time stipulates the uncertainty characteristic of them. Talking of uncertainty, we imply, of course, the uncertainty of the *future,* but not past. The uncertainty of the past is usually connected (at least in financial problems) with the lack of information and in this sense, in principle, it is removable along with the accumulation and refinement of the data; whereas, the uncertainty of the future is not removable *in principle*. This uncertainty is that is characteristic of financial problems leads to the risk situation at their solution. Due to uncertainty, *any solution* on the financial problems *may* lead to the results different from the *expected ones*, however thorough and thoughtful the solution may be.

The financial theory develops the concepts and methods for financial problem solution. As any other theory, it builds the *models* of real *financial processes.* Since such basic elements as time, value, risk, and criteria for choosing the desired distribution of resources obtain a quantitative expression, these models bear the character of *mathematical models,* if necessary. The majority of the models studied in the modern financial theory, have a strongly marked mathematical character. Along with that, the mathematical means used to build and analyze the financial models, vary from the elementary algebra to the fairly complicated divisions of random processes, optimal management, etc.

Although, as it was mentioned, the uncertainty and risk are inseparable characteristics of financial problems, in a number of cases it is possible to neglect them either due to the *stability* of conditions in which the decision is made, or in idealized situations, when the model considered ignores the existence of these or those risk types due to its specificity. Financial models of this type are called the models with *total information*, deterministic models, etc. The study of such models is important because of two factors.

First, in a number of cases, these models are fairly applicable for a direct use. This refers to, tor instance, the majority of models of the classical and financial mathematics devoted to models of the simplest financial transactions, such as bank deposit, deal on the promissory note, etc.

Second, one of the ways for studying the models in the uncertainty conditions is *modeling*, i.e. the analysis of possible future situations or scenarios. Each scenario corresponds to a certain, fairly determined, future *course of events*. The analysis of this scenario is made, naturally, within the deterministic model. Then, on the basis of the carried out analysis of different variants of event development, a *common solution* is made.

Chapter 1

Accumulation and Discounting

1.1. Time Factor in Quantitative Analysis of Financial Transactions

The basic elements of financial models are *time* and *money.* In essence, *financial models* reflect to one extent or another the *quantitative relations* between *sums of money* referring to *various time points.* The fact that with time the *cost* or, better to say, the *value* of money *changes* now due to constant inflation, is obvious to everyone. The ruble today and the ruble tomorrow, in a week, month or year – are different things. Perhaps, it is less obvious, at least not for an economist, that even without inflation, the time factor nevertheless influences the value of money.

Let us assume that possessing a “free” sum you decide to place it for a time deposit in a bank at a certain interest. In time, the sum on your bank account increases, and at the term end, under favorable conditions, you will get a higher amount of money than you placed initially. Instead of the deposit, you could buy shares or bonds of a company that can also bring you a certain profit after some time. Thus, also in this case, the sum invested initially turns into a larger amount after some time period. Of course, you may choose to not undertake anything and simply keep the money at home or at a bank safe. In this case their sum will not change. The real cost will not change either unless there is inflation. In other case, it will certainly decrease. However, having at least and in principle an opportunity to invest and not doing it is, from the economist’s point of view, irrational and you have *quite* a *real* loss in the *economic sense*. This loss bears the title of *implicit cost or loss of profit.* Therefore, the naïve point of view differs from the economic one. When counting (in absence of inflation) the amount of money that is kept in the safe and does not lose its cost, you, from the economist’s point of view, are mistaken. In this case, the value of money will also change in time.

By all means, the “economic” approach implies the presence of some mechanisms on “managing the cost” of money. In the present society it is realized through the presence of investment, in particular, financial, market. Banks, insurance companies, investment funds, broker’s companies make a wide spectrum of assets whose purchase leads (often but not always) to an increase in the value of the invested capital. Accumulation of the invested capital value starts a “process of transformation” of the value of money in time. Hence, the ruble invested today turns into two rubles in a few years; on the other hand, the future amounts have a lower cost from the point of view of the current (today's) moment at least because in order to acquire them in the future, it is sufficient to invest a smaller amount today.

Summing up, it is possible to formulate the total financial principle determining the influence of time on the value of money:

One and the same sum of money has various costs at various time points. On the other hand, in relation to certain conditions, various sums of money at various time points may be equivalent in the financial and economic context.

The necessity to consider the time factor is expressed in the form of the *principle of money disparities* that refer to various time points. The disparity of two identical money amounts is determined by the fact that any amount of money may be invested and bring profit. The coming profit may be reinvested etc.

The consequence of the *principle of money disparities* is the illegitimacy of summation of money values that belong to different time points at the analysis of financial transactions.

Time factor consideration is based on the fundamental for the financial analysis *principle of payment discounting* and *payment streams*. The notion of discounting is in its turn connected with the notions of interest and interest rates.

1.2. Interest and Interest Rates

Let us use the following symbols:

$t = 0$ - the moment of lending money (the present time point);
T or n - the life of the loan;
P_0 – the sum provided as a loan at the time point $t = 0$;
S_T – the sum of the dischargeable debt at the moment $t = T$;
i – the interest rate (of accumulation);
d – the discount rate;
I – interest and interest money.

The interest money or interest $I = (S_T - P_0)$ are the absolute value of return from providing the money as a loan in its any form, in particular: issue of money loans, sale on credit, placement of money on a savings account, bond purchase etc.

When concluding a financial contract, the parties make an agreement on the amount of the interest rate. In financial mathematics, two types of interest calculation rates are distinguished: interest rate and discount rate.

The interest rate i_T is the relation of the sum of the interest money paid for the fixed period of time to the value of the loan:

$$i_T = \frac{S_T - P_0}{P_0}$$

Here i_T is determined in form of the decimal fraction. In order for the rate to be expressed in per cent, it must be multiplied by 100.

The discount rate d_T is the relation of the sum of interest money paid for the fixed period of time to the amount of the dischargeable debt:

$$d_T = \frac{S_T - P_0}{S_T}$$

Let us consider a simple example. Let the investor place a sum of 5000 rubles in a bank for a year at 8% per annum. This means that at the end of the year the investor will obtain apart, from the invested money, an addition amount I that is called the interest on the deposit that equals

$$I = 5000 \cdot 0.08 = 400 \text{ rubles.}$$

The interval regarding which the interest (discount) rate is determined is called *calculation period.* It may equal a year, six months, a month, a day, etc. Note that the interest is also an *interval* characteristic, i.e. it relates to the period of the transaction.

Example 1.1. Let at the time moment t_0 a credit of $P = 5000$ currency units is issued for a term of T =2 years after which the creditor should obtain $S = 10000$ currency units. Find the interest rate of the deal.

Solution. $i_T = \dfrac{10000 - 5000}{5000} = 1$, i.e. $i_T = 100\%$.

Example 1.2. A credit deal of issuing 2000 rubles for a term of 3 years is considered. Find the amount of repayment, if the interest rate is 60%.

Solution. $S = 2000 \cdot (1 + 0.6) = 3200$.

The rates i_T, d_T determined above refer to the whole period of the deal. In practice, another type of interest (discount) rate referring to a chosen *base time* interval is used more— usually a year. The choice of the base time interval allows normalizing the interest (discount) rate of the deal according to the formula:

$$i = \frac{i_T}{T}, \; d = \frac{d_T}{T}, \tag{1.1}$$

where T – period (term) of the deal expressed in units of the base period (year, month, etc.). We will operate exactly with the normalized ınterest rate hereafter.

Formula (1.1) and Formula (1.2) may be re-written in a form of:

$$i = \frac{S_1 - P_0}{P_0}, \tag{1.2}$$

$$d = \frac{S_1 - P_0}{S_1}. \tag{1.3}$$

Formula (1.2) and Formula (1.3) imply the existence of two principles of interest calculation. Let us consider the investment of the sum P_0 at the time moment $t = 0$ for one period. As it follows from (1.2), at the moment $t = 1$, i.e. at the period end, the investor will receive the sum $S_1 = P_0 + iP_0$ back. In addition, the sum iP_0 paid at the moment $t = 1$ is the interest $I = S_1 - P_0 = iP_0$ for the time [0, 1] for the loan with an amount of P_0 at the moment $t = 0$. Thus, the interest on the rate i is calculated for the sum of the original debt P_0 at the moment $t = 1$.

According to (1.3), in exchange for the return of the sum S_1 at the moment $t=1$, the investor lends the sum $P_0 = S_1 - dS_1$. In this case, the interest at the rate d is calculated at the initial time point $t=0$ on the sum of the dischargeable debt S_1. The amount P_0 may be considered as a loan of the sum S_1 that will be repaid after a time unit at which the interest of dS_1 is repaid in advance, at the moment $t=0$ and make the profit of the creditor $D = S_1 - P_0 = dS_1$ for the time [0, 1].

Therefore, the interest at the rate i is calculated at the end of the interest calculation period, and the interest at the discount rate d – at the beginning of the interest calculation period.

Simple and *compound* interest types are distinguished. When calculating the simple interest, the base for the calculation is the original sum at the loan term duration. When calculating the compound interest, the base is the sum with the interest calculated at the preceding period.

In financial analysis, the interest rate is used not only as an instrument of accumulating the amount of debt, but also in a wider sense, in particular, as a universal index of the extent of yield of any financial transaction.

1.3. Accumulation with Simple Interest

According to the definition, the accumulated sum is the original sum with interest calculated for this sum. Let us introduce the following symbols. Let

I – the amount of interest for the whole term;
n – the total number of calculation periods (usually in years);
P – the original sum (here we omit the index "0");
S – the accumulated sum (we omit the index "T");
i – the interest rate in the form of a decimal fraction;
d – the discount rate in the form of a decimal fraction.

Then we have

$$S = P + I.$$

At simple interest calculation, the original sum is taken for the base. The interest is calculated n times, therefore $I = P \cdot n \cdot i$ and the formula of simple interest is written as

$$S = P \cdot (1 + n \cdot i). \tag{1.4}$$

The amount $(1 + n \cdot i)$ is called *the accumulation factor* at the simple interest rate, i.e. the accumulation factor indicates the future cost of 1 currency unit accumulated to the moment n invested at the moment $t=0$ for the term n.

The process of increase in the amount of money in connection with the addition of the interest to the original sum is called *accumulation* or the *growth of the original sum*.

The original sum with accumulated interest is called the *accumulated sum.*

Simple interest is used more often when the loan term is less than a year. Then $n = \frac{t}{K}$, where t – is the number of days of the loan, and K – the number of days in a year.

In practice, the simple or commercial interest when K=360 days or exact interest – K=*365* (366) days.

Example 1.3 Let P=1000 rubles, the annual rate i=10%. We will obtain sums accumulated at simple interest.

Year1: 1000 + 0.1·1000 = 1100 rubles; Year 2: 1100 + 100 = 1200 rubles; Year 3: 1200 + 100 = 1300 rubles.

Variable rates

Let us assume that the whole term of the loan n is divided into s intervals with the duration

n_t, each being $n = \sum_{t=1}^{s} n_t$. At every interval the rate i_t rules. Then the formula for accumulating simple interest at the variable rate will have a form of

$$S = P \cdot (1 + n_1 \cdot i_1 + n_2 \cdot i_2 + ... + n_s \cdot i_s)$$

or

$$S = P \cdot (1 + \sum_{t=1}^{s} n_t \cdot i_t). \tag{1.5}$$

Example 1.4 Let P=1000 rubles, the rate in the first year equals i_1=10%, in the second year – i_2=12%, in the third year – i_3=15%. We will obtain a sum accumulated within three years with the simple interest.
1000 + 100 + 120 + 150 = 1370 rubles.

Reinvestment

When reinvestment of funds accumulated at each interval occurs, then it is not the original, but the accumulated sum obtained in the preceding interval that is taken for the base at interest calculation in the subsequent interval. Considering this, the formula of accumulation is written as follows:

$$S = P \cdot (1 + n_1 \cdot i_1) \cdot (1 + n_2 \cdot i_2) \cdot ... \cdot (1 + n_s \cdot i_s).$$

Example 1.5. For the data of Example 1.3. for three years we will obtain the sum:
Year 1: 1000 + 100 = 1100 rubles; Year 2: 1100 + 110 = 1210 rubles; Year 3: 1210 + 121 = 1331 rubles.

1.4. Compound Interest

Calculating Compound Annual Interest (Formula for Accumulation)

In long-term financial transactions, compound interest is used to accumulate the original sum. When calculating compound interest, what is considered for the base is not the original sum, but the sum obtained after interest calculation and after adding the interest to the amount of the debt in the preceding periods.

Adding the calculated interest to the sum that was the base for their calculation is called the *interest capitalization*. The capitalization process happens according to the following model:

1) $P + P \cdot i = P(1+i)$

2) $P(1+i) + P(1+i)i = P(1+i)^2$

3) $P(1+i)^2 + P(1+i)^2 \cdot i = P(1+i)^3$

....

In the general case, the formula for accumulation with compound interest is recorded as:

$$S = P(1+i)^n. \tag{1.6}$$

The factor $(1+i)^n$ is called the *accumulation* factor with the formula of compound interest.

Example 1.5. Let P=1000, i = 10%, i.e. as the fraction i = 0.1. Consequently, the sums accumulated with compound interest are:

1000, 1000+0.1·1000=1000+100=1100, 1100+0.1·1100=1210, 1210+0.1·1210 = 1331.1 и т.д.

Example 1.6. The annual rate of compound interest equals 8%. After how many years will the original sum double?

Solution. An inequality must be solved: $(1+0.08)^n$ > 2. Let us take the logarithm based on the natural logarithms and obtain $n > \ln(2)/\ln(1.08)$.

Answer: after 9 years.

The curves presented on Figure 1.1 illustrate the processes of accumulation with simple and compound interest depending on the term. It is obvious from the figure that if the term $n < 1$, then the factor of accumulation with simple interest $(1+n \cdot i) > (1+i)^n$, if $n > 1$, then $(1+n \cdot i) < (1+i)^n$.

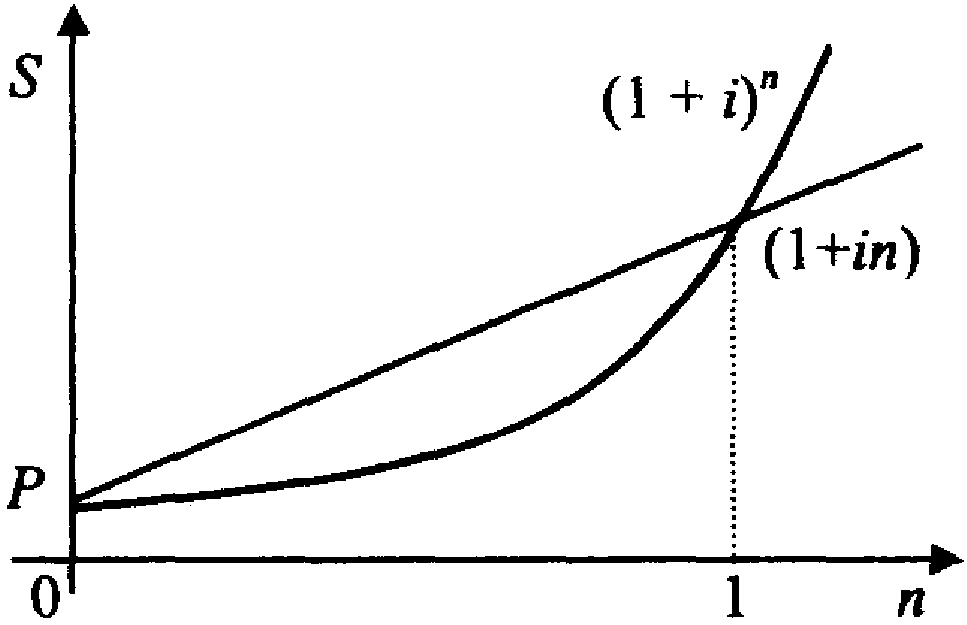


Figure 1.1. The dependence of the accumulated sum on the life of the loan

Rates Varying in Time

Let the loan term n be divided in intervals with the duration n_t,

$t=1,2,...,s,\ n=\sum_{t=1}^{s} n_t$, and in every interval interest calculation is made at the rates

$i_1, i_2, ..., i_s$ correspondingly. Then the formula of accumulation with compound interest will be:

$$S = P(1+i_1)^{n_1}(1+i_2)^{n_2}\cdots(1+i_s)^{n_s}.$$

1.5. Nominal and Effective Interest Rates

Nominal Rate

In practice, during the announcement of the terms of financial transaction, the annual interest rate is often specified and the number of interest payments per year is indicated (for instance, there may be a quarterly calculation – four times a year). In this case, we will indicate the notion of the *nominal rate* by the symbol j . Let the number of interest payments per year equal m. Then the interest calculation is made at the rate j/m, and the total number of payment intervals is n years will equal $m \cdot n$. The accumulated sum is determined by the formula:

$$S = P(1+\frac{j}{m})^{m \cdot n}. \qquad (1.7)$$

Thus, the nominal rate is the annual interest rate at interest calculation m times a year. The higher the number m, the quicker the process of the original sum accumulation.

Effective Rate

To compare various conditions of interest calculation (at various nominal rates and different number of calculations) the notion of *effective rate* is used. The effective rate is the annual rate of interest calculated once a year that gives the same financial result as m–single calculation per year with the use of the nominal rate j. Thus, by definition, the following equality of accumulation factors must be fulfilled:

$$(1+i) = (1+\frac{j}{m})^{m}, \qquad (1.8)$$

where i – effective rate. Hence we obtain

$$i = (1+\frac{j}{m})^{m} - 1. \qquad (1.9)$$

The exchange of the nominal rate j at interest calculation m times a year for the effective rate with Formula (1.7) in the agreement does not change the financial obligations of the parties.

When comparing various offered variants of interest calculation, it is sufficient to calculate and compare the effective rates for each of them.

Example 1.7. The bank calculates the interest out of the nominal value of j =12% per annum. Calculate the effective rate i, if the interest is calculated: a) daily (m=365); b) monthly (m=12); c) quarterly *(m=4).* The calculations by Formula (1.7) give the following results: at daily calculations i =0.1274; at monthly calculations i =0.1268; at quarterly calculations i =0.1255.

If the effective rate i is given, then it is possible to determine the nominal rate j, at the number of payments m:

$$j = m \cdot ((1+i)^{1/m} - 1). \qquad (1.10)$$

Example 1.8. Let the effective annual rate be i =12%. Determine the nominal rate j if the interest is calculated: a) daily (m=365); b) monthly (m=12); c) quarterly *(m=4).*
Solution. By Formula (1.8), we will obtain the following results: at daily calculations j=0,1133; at monthly calculations j=0,1139; at quarterly calculations j=0,1149.

1.6. Determining the Loan Duration and Interest Rates

Simple Rate

Let us consider the following two problems that arise in connection with the notion of interest and that have a practical meaning.

Problem 1. P, S, i values are known*.* Determine the term n*.* The problem must be solved simply: by formula (1.1) we will obtain:

$$n = \frac{S - P}{P \cdot i}. \tag{1.11}$$

Problem 2. P, S, n values are known*.* Determine i*.* The rate equals

$$i = \frac{S - P}{P \cdot n}. \tag{1.12}$$

Example 1.9. The annual rate of simple interest equals 12,5% . In how many years will the original sum double?
Solution. The following inequality must be solved: (1+0.125 n*)>2,* i.e. 0.125n> 1. We obtain n>1/0.125.
Answer: after 8 years.

Compound Rate

Problem 1. P, S, i values are known*.* Determine the term n*.* The term is obtained from the equation (1.3):

$$n = \frac{\ln(S / P)}{\ln(1 + i)}. \tag{1.13}$$

If the nominal rate J with interest calculations m times a year is used, then analogously we will obtain from the equation (1.4):

$$n = \frac{\ln(S/P)}{\ln(1+\frac{j}{m})^m}. \tag{1.14}$$

Problem 2. P, S, n values are known. Determine the rate i. Solving the equation (1.3) in accordance with i, we will obtain:

$$i = (S/P)^{1/n} - 1. \tag{1.15}$$

During interest calculation at the nominal rate m times a year, we will obtain from the equation (1.3):

$$j = m \cdot ((S/P)^{1/m \cdot n} - 1). \tag{1.16}$$

1.7. The Notion of Discounting

Discounting at Simple Rates

Let us consider the following problem. For the given sum S that must be paid after the time n the, original sum P must be determined. In this case it is said that the sum S *is discounted***.**

The term *discounting* in the broad sense means determining the value of the money amount at the given time point under the condition that it will equal the value S in the future. Such a calculation is called *the reduction of cost parameter to the given time point***.**

The value P that was found by the discounting of the sum S is called the *present* or *reduced value* S *.*

It is one of the most important notions at modeling and analysis of financial transactions, since it is exactly with discounting that the time factor is considered. When solving the problem, we will obtain:

$$P = S\frac{1}{1+n \cdot i}. \tag{1.17}$$

This formula is called the formula of *mathematical discounting* (apart from the bank discounting or accounting which is not considered here).

The value $\frac{1}{1+n \cdot i}$ is called the *present value factor***,** the difference $(S-P)$ — the *discount of the sum* S *.*

The notion of discounting will be considered in detail in further sections.

Discounting at a Compound Interest Rate

Let us find he original sum *P* according to the known total amount *S* with a compound rate.

It is obvious that

$$P = \frac{S}{(1+i)^n} = S \cdot v^n, \tag{1.18}$$

where $v^n = \frac{1}{(1+i)^n}$ is the present value factor (the factor of discounting).

If the interest is calculated m times a year with the use of the nominal rate j, then

$$P = \frac{S}{(1+\frac{j}{m})^{m \cdot n}} = S \cdot w^{m \cdot n}, \tag{1.19}$$

where the present value factor is $w^{m \cdot n} = \frac{1}{(1+\frac{j}{m}i)^{m \cdot n}}$.

In the financial and economic literature, the present value *P* is often contracted as PV (Present Value).

It is obvious that the mathematical solution to this problem is elementary, but the problem itself has a deep financial and economic meaning. As it was mentioned above, the present value of the future money amounts is determined by means of discounting. The payment of the S amount after n years is equivalent to the sum P paid at the present time point. Herewith, the question of choosing the interest rate at which discounting is made is of a high importance. The choice of this rate is based on the analysis of the financial market state and the accuracy of evaluating the present value of the future economic amounts depends on the right choice of the rate.

The difference $S-P$ is called the *discount* of the sum S, let us indicate this value by the symbol D. If follows from Formula (1.13) and (1.14) that $D = S(1-v^n)$ if the interest is calculated once a year, and $D = S(1-w^{m \cdot n})$ if the interest is calculated m times a year.

Let us illustrate that the accumulated sum for a certain interim time point $0<t<n$ equals the present value of the payment at the same time point. Let S_t be the accumulated sum at the time point t, and P_t be the present value of the payment at the same moment. Then it is obvious that

$$P_t = \frac{S}{(1+i)^{n-t}} = \frac{P(1+i)^n}{(1+i)^{n-t}} = P(1+i)^t = S_t .$$

1.8. Inflation Accounting at Interest Accumulation

There are many various ways of inflation accounting at compound interest accumulation. Let us consider one of them that is based on using the Fisher formula. Let h be the expected annual rate of inflation in the form of the compound interest (here we do not consider methods of determining this index), i – the interest rate without inflation accounting, r – the real rate with inflation accounting. Then the real rate is determined from the equation which is called Fisher equation:

$$1+r = \frac{1+i}{1+h} . \tag{1.20}$$

Through the solution of this equation in accordance with r, we will obtain

$$r = \frac{i-h}{1+h} . \tag{1.21}$$

The rate without inflation accounting (that is also called the nominal rate) is $i = r + h + r \cdot h$. At small values of h, an approximation formula $i = r + h$ is used, and for the real rate it is $r = i - h$.

1.9. Continuous Accumulation and Discounting (Continuous Interest)

Continuous interest is used in the quantitative analysis of financial transactions, modeling complex industrial and commercial facilities, choosing investment decisions and their justification. It is conditioned by the fact that many economic processes are continuous which is why using continuous interest is more adequate. Models with continuous interest are widely used in modern financial mathematics when describing the yield of securities. By means of continuous interest, complex processes of accumulation and constantly changing interest rates characteristic of financial markets may be accounted. Continuous interest is used for modeling financial streams. Despite the seeming abstractness of this notion, the continuous interest is also used in the practice of financial companies.

A special type of interest rate called the growth rate is used for continuous interest accumulation. It characterizes the relative gain of the accumulated sum in the infinitely small time interval.

Constant Growth Rate

Let us consider the formula $S = P(1+\frac{j}{m})^{m \cdot n}$. It is obvious that at continuous interest calculation $m \to \infty$. We have

$$S = \lim_{m\to\infty} P(1+\frac{j}{m})^{m\cdot n} = P \lim_{m\to\infty} (1+\frac{j}{m})^{m\cdot n}.$$

From the mathematical analysis it is known that $\lim_{m\to\infty} (1+\frac{j}{m})^{m} = e^{j}$, where e is the base of the natural logarithm. Considering this, we will obtain

$$S = P \cdot e^{\delta \cdot n}. \tag{1.22}$$

In the given formula, δ, the growth rate, is the nominal rate at $m \to \infty$.

If the growth rate is known, then its equivalent discrete annual rate may be determined. In essence, it is the effective rate corresponding this continuous rate. Equating the accumulation factors at discrete and continuous rates

$$(1+i) = e^{\delta}$$

we will obtain

$$i = e^{\delta} - 1. \tag{1.23}$$

If the discrete annual rate is known, then its equivalent continuous rate may be determined:

$$\delta = \ln(1+i). \tag{1.24}$$

Let us illustrate a real situation in which continuous interest was used to attract depositors. In 1975 in the USA, the laws imposed restrictions on the fixed annual rate of payments, but did not impose any restrictions on the number of calculation periods per year. The rate of interest payments on the loans and deposits with a term of six to ten years was restricted by the level of 7.75% per annum. In order to attract depositors, some financial institutions declared their readiness to pay 7.75% per annum, but calculate interest every six months, which corresponded the effective rate of 7.9%. This was not a violation of the law.

After this, other institutions offered more profitable terms, in particular, 7.75% per annum, but with quarterly interest calculations. This corresponded the effective rate of 7.978%. Still others offered monthly interest calculations at the same annual rate, that produced the effective rate of 8.031%. The limit was reached when a company offered continuous (daily) interest calculations which corresponded the effective rate of 8.06%.

Discounting at Continuous Interest Rates

The discounting formula at a continuous rate is

$$P = S \cdot e^{-\delta \cdot n}. \qquad (1.25)$$

Variable Growth Rate

Let the growth rate change in time and have the values $\delta_1, \delta_2, ..., \delta_k$ at time intervals $n_1, n_2, ..., n_k$. Then, for n_1 years the original sum P will equal $S_1 = P \cdot e^{\delta_1 \cdot n_1}$. Since the rate is compound, for the next n_2 years, the original sum will grow to $S_1 = S_1 e^{\delta_2 \cdot n_2} = P \cdot e^{\delta_1 \cdot n_1 + \delta_2 \cdot n_2}$. Continuing the discussion, we will obtain that the accumulated sum for all the term will equal

$$S = P \cdot e^{\sum_{t=1}^{k} \delta_t \cdot n_t}. \qquad (1.26)$$

We have considered the case when the growth rate was measured discretely in time.

Let us assume that the growth rate changes continuously and is described with some continuous time function $\delta = \delta(t)$. Then the accumulated sum is

$$S = P \cdot e^{\int_0^k \delta(t)dt}. \qquad (1.27)$$

The discounting formula is

$$P = S \cdot e^{-\int_0^k \delta(t)dt}. \qquad (1.28)$$

1.10. Simple and Compound Interest Rate Equivalency

Rates are equivalent if they lead to one and the same financial result in a particular financial transaction.

The task for determining equivalent rates arises, for instance, at comparing the rates applicable for various deals and agreements, determining the effectiveness of credit and financial transactions, lossless exchange of one rate type with the other or one method of their calculation with another.

In order to determine the equivalency of simple and compound rates, it is necessary to equate the following accumulation factors

$$(1+n\cdot i_n)=(1+i)^n$$

where i_n – simple rate, i – compound rate. From this relation we easily obtain

$$i_n=\frac{(1+i)^n-1}{n}, \qquad (1.29)$$

$$i=(1+n\cdot i_n)^{1/n}-1. \qquad (1.30)$$

If the compound rate is calculated m times a year, then, when also equating accumulation factors, we will obtain

$$i_n=\frac{(1+\frac{j}{m})^{m\cdot n}-1}{n}, \qquad (1.31)$$

$$j=m((1+n\cdot i_n)^{1/m\cdot n}-1). \qquad (1.32)$$

Therefore, for any financial transaction, the exchange of compound interest rates with the equivalent simple rates by Formula (1.29) and (1.31) or simple rate with its equivalent compound rates by Formula (1.30) and (1.32) do not change the final result of the financial transaction.

Example 1.10. What sum is more preferable at the rate of 6%: \$1000 today or \$2000 after 8 years?

Solution. Let us find the present value \$2000 after 8 years at the rate of 6%:

$A=2000\cdot(1+0.06)^{-8}=2000\cdot 0.627=1254$.

Hence, $A=1254>1000$. Consequently, the sum \$2000 after 8 years should be preferred.

1.11. Change of Contract Terms

Under the change of contract terms we will understand the postponement of payment term or payment consolidation, i.e. uniting several obligations into one.

The change of contract terms is based on the principle of financial equivalency of obligations. This principle guarantees break-even condition of change in financial relations for each of the parties. Such payments are considered equivalent that are equal being reduced to one time point at a given interest rate.

The common method for solving such problems is the generation of equivalency equation. In this equation, the sum of payments to be replaced that are reduced to a certain moment is equated to the amount of payments on the new obligation that is also reduced to the same date.

Funding Debt

Let there be payments $S_1, S_2, ..., S_m$. Payment terms are $n_1, n_2, ..., n_m$. These payments are united into a single payment S_0 with the repayment term n_0. Two variants of this problem are possible:

1) the term of the consolidated payments n_0 is given, it is necessary to determine its amount S_0;

2) the amount of payment S_0 is known, it is necessary to determine its term n_0.

Let us first solve the problem with the simple rate.

For Variant 1, it is easier to reduce all the payments to the moment n_0. The payments whose terms are less than n_0 should be accumulated by the formula of simple interest. The payments whose terms are more than n_o, should be discounted. Let

S_j be the amounts of payments whose terms are $n_j < n_0$,

S_k be the amounts of payments whose terms are $n_k > n_0$,

$t_j = n_0 - n_j$ be the time from the payment moment of the j payment with the term $n_j < n_0$ till the moment of repaying the sum S_0,

$t_k = n_k - n_0$ be the time from the moment of repaying the k payment with the term $n_k > n_0$ till the moment of repaying the sum S_0,

i be the simple interest rate. Then

$$S_0 = \sum_j S_j (1 + t_j \cdot i) + \sum_k S_k (1 + t_k \cdot i)^{-1}. \qquad (1.33)$$

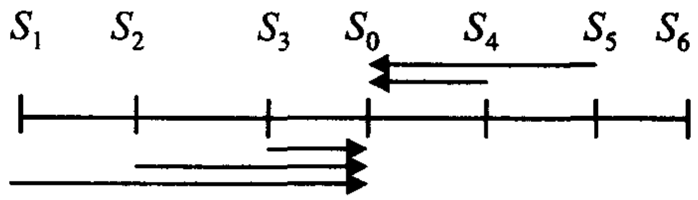


Figure 1.2. Process of funding debt

The first member in this equation unites all the payments whose terms are $n_j < n_0$, the second – those whose term $n_k > n_0$.

Thus, summing all these payments up, we will obtain the sum S_0 which is financially equivalent to all the payments distributed in time. It is important to note that the amount of the sum S_0 and the term of its repayment are interconnected.

Consolidation at compound interest rate leads to the relation

$$S_0 = \sum_j S_j (1+i)^{t_j} + \sum_k S_k (1+i)^{-t_k} . \tag{1.34}$$

Let us consider Variant 2 of the problem, we will first use the simple interest rate. Here it is more convenient to reduce all the payments to the initial date (the time point which is accepted as the start of calculations). The equivalency equation is written as

$$S_0 (1 + n_0 \cdot i)^{-1} = \sum_t S_t (1 + n_t \cdot i)^{-1} .$$

Let us indicate $\sum_t S_t (1 + n_t \cdot i)^{-1} = P_0$ – the present value of substitutable payments.

When solving this equation regarding n_0, we will obtain

$$n_0 = \frac{1}{i} \left(\frac{S_0}{P_0} - 1 \right). \tag{1.35}$$

In this case, the problem does not always have a solution. It is obvious that there is a solution if $S_0 > P_0$, since it is only in this case that $n_0 > 0$.

Let us determine the term of the payment at the compound interest. The equivalence equation at the compound rate is

$$S_0(1+i)^{-n_0} = \sum_t S_t(1+i)^{-n_t} .$$

Let us indicate $Q = \sum_t S_t(1+i)^{-n_t}$ – the sum of payments discounted for the initial date.

Solving the equation, we will obtain

$$n_0 = \frac{\ln(S_0 / Q)}{\ln(1+i)} . \qquad (1.36)$$

The solution of this problem does not always exist. The condition for existence of the solution: $S_0 > Q$, since it is only then that $n_0 > 0$ (formally this follows from the properties of the logarithm function).

A General Case of Change in Contract Terms

In the general case, a few payments may be replaced by other several payments. Let

S_q be the payments with terms n_q that are foreseen by the new conditions,

S_k – substitutable payments with terms n_k.

Let us assume that all the payments are made at the start time.

The equivalency equation for the compound rate is

$$\sum_q S_q v^{n_q} = \sum_k S_k v^{n_k} . \qquad (1.37)$$

This problem does not have a single-valued solution and in some cases may be solved only.

1.12. Discounting and Accumulation at a Discount Rate

1.12.1. Commercial (Bank) Accounting

Let us formulate the problem of bank discounting. On the given sum S that will be paid after n periods, it is necessary to determine the amount of the loan P at the present moment at which the interest for using the loan is paid beforehand, at the $t = 0$ moment of lending the money. The discount rate d is used for interest accumulation and deduction.

Simple Discount Rate d

We have: $t=n$ – the moment of discharging the sum S_n. According to the definition of the discount rate (1.3), the amount necessary to be issued as a loan at the moment $t=n-1$ at a time unit before discharging the sum S_n is

$$P_{n-1}=S_n-dS_n.$$

Then the amount of the discount for the preceding, n discounting period equals $D_n=dS_n$. Since d is the simple discount rate, then the amounts of discount for each discounting period are identical and equal $D_n=D_{n-1}=...=D_1=dS_n$.

The discount value for all the life of the loan n is

$$D(n)=D_n+D_{n-1}+...+D_1=ndS_n.$$

By definition,

$$D(n)=S_n-P_0.$$

Then

$$P_0=S_n(1-nd). \tag{1.38}$$

(1.38) – the formula of the present value of the sum S_n at its bank account with simple discounts at the rate d during n periods. The sums $D_n,\ D_{n-1},...,D_1$ are the discounts for each period (time unit). The expression (1.38) implies that in exchange for the payment of the sum S_n after the time n, the creditor will lend the sum $S_n(1-nd)$ at the beginning of this term. We should note that Formula (1.38) is correct if the life of the loan n and the discount rate d satisfy the condition $nd<1$. Discounting at a simple discount rate is used, as a rule, in the case of short-term deals when $0<n\le 1$.

Example 1.10. A promissory note to be paid off by January 1, 2002, was discharged 10 months before its discharge for a sum of 180 currency units. What is the value of the annual discount rate if the monthly discount equals 2 currency units?

Since interest is deducted for every month, 1 month may be taken as the unit for measuring time. Then, at the beginning of each month, the interest is calculated at the monthly discount rate $d/12$, where d is the annual discount rate. The period to maturity of the promissory note is $n=10$ time units. The sum P_0 = 180 is the reduced (to the moment of promissory note discount t = 0) value of the sum S_n, discharged on the note. Discounts for every period (time unit) are $D_{10}=D_9=...=D_1=2=D$. Consequently, the promissory note is discounted at the simple discount rate. The discount amount for all the term is $D(n)=nD$. Since $P_0=S_n-nD$, then the sum discharged on the note is S_n = 200 currency units. Since $D=S_n\dfrac{d}{12}$, then the annual discount rate is d = 0.12.

Compound Discount Rate d

According to the definition of the compound interest rate, the base for interest calculations in every period is the sum acquired in the preceding discounting period. Since the discount rate d is used for interest calculation, the interest is charged at the beginning of every period. Let us consider the process of discounting the sum S_n in periods, starting from the period n. Such an order of period consideration means that the period n of discounting is preceding to the $(n-1)$ period, and the $(n-1)$ period is preceding to the $(n-2)$ period etc.

The amount that must be issued as a loan at the moment $t=n-1$, i.e. for a time unit until the repayment of the sum S_n, is

$$P_{n-1} = S_n - D(1) = S_n - D_n .$$

P_{n-1} – is the value of the sum S_n reduced to the moment $t=n-1$; $D(1)$ – the value of the discount for one n period, $D(1) = D_n = dS_n$. Since P_{n-1} is the sum obtained in the n discounting period, then the value of the discount in the (n – 1) discounting period equals $D_{n-1} = dP_{n-1}$.

The sum that must be issued as a loan at the moment $t=n-2$ for two periods before discharging the sum S_n is:

$$P_{n-2} = S_n - D(2) = S_n - D_n - D_{n-1} = P_{n-1} - D_{n-1} .$$

P_{n-2} - the value of the sum S_n reduced to the moment $t=n-2$; D (2) – the value of the discount for 2 periods, n period and (n – 1) period, i.e. $D(2) = D_n + D_{n-1}$. Since P_{n-2} is the sum acquired in the (n – 1) discounting period, the value of the discount in the (n – 2) period makes $D_{n-2} = dP_{n-2}$. And so on.

The value of the sum S_n reduced to the moment t = 0 is the sum P_0 that must be issued as a loan at the moment t = 0 for n periods before discharging the sum S_n:

$$P_0 = S_n - D(n) ,$$

where $D(n)$ - the value of the discount for all the life of the loan. Let us find $D(n)$. We have:

$$D_n = dS_n ,$$

$$D_{n-1} = dP_{n-1} = d(S_n - D_n) = dS_n - dD_n = D_n - dD_n = D_n(1-d) ,$$

$$D_{n-2} = dP_{n-2} = d(P_{n-1} - D_{n-1}) = dP_{n-1} - dD_{n-1} = D_{n-1} - dD_{n-1} = D_{n-1}(1-d) ,$$

……..

$$D_1 = dP_1 = d(P_2 - D_2) = dP_2 - dD_2 = D_2 - dD_2 = D_2(1-d) .$$

Thus,

$$D_{n-1} = D_n(1-d) ,$$

$$D_{n-2} = D_{n-1}(1-d) ,$$

....

$$D_1 = D_2(1-d).$$

Consequently, $D_n, D_{n-1}, ..., D_1$ are the members of a geometric sequence with the first member D_n and the denominator (1 – d). The value of the discount for all the life of the loan n is

$$D(n) = D_n + D_{n-1} + ... + D_1.$$

By formula of the sum n of the members of the geometric sequence, we obtain

$$D(n) = D_n \frac{1-(1-d)^n}{1-(1-d)} = dS_n \frac{1-(1-d)^n}{1-(1-d)} = S_n \left(1-(1-d)^n\right).$$

Since $P_0 = S_n - D(n)$, then

$$P_0 = S_n (1-d)^n. \tag{1.39}$$

(1.39) is the formula of the present value of the sum S_n with its bank accounting of compound interest at the discount rate d within n periods.

Example 1.5. A government bond was discounted five years before its discharge. What is the sum paid for the bond if the discounts for the last and last but one year before discharge were 2000 and 1600 currency units correspondingly?

Let us use the obtained relations for compound discounts. If the unit for measuring time is 1 year, then the life of the loan is n = 5 years, D_4 = 1600 currency units, D_5 = 2000 currency units, $D_4 = D_5(1-d)$, where d is the annual discount rate. Hence d = 0.2. Since $D_5 = dS_5$, then the dischargeable sum is S_5 = 10000 currency units.

Discounting at the Nominal Discount Rate

If discounting at the compound discount rate is made not one, but m times a year, then the annual discount rate is called nominal and is defined through g.

Definition. The annual discount rate g is called nominal if a compound discount rate g/m is used for discounting during the $1/m$ part of the year.

Therefore, if discounting at the compound discount rate is made after equal time intervals m times a year, then at the beginning of every period lasting $1/m$ interest is calculated and reduced at the rate of g/m. If the life of the loan is n years, then nm is the number of periods of applying the rate g/m in the life of the loan. From Formula (1.39) we obtain

$$P_0 = S_n (1-\frac{g}{m})^{nm}, \tag{1.40}$$

where m ≥ 1. If m = 1, then $g = d$, i.e. the nominal discount rate coincides with the annual discount rate of compound interest that is applied once a year. Formula (1.40) is the formula of accounting the sum S_n at the m-time discounting per year at the nominal discount rate g within n years.

Continuous Discounting at the Compound Discount Rate

Continuous discounting is discounting at infinitely small periods of time, i.e. at $\frac{1}{m} \to 0$ (or $m \to \infty$). Since with continuous interest calculations the beginning and the end of interest calculation period coincide, then the nominal interest rates j and g at $m \to \infty$ cease to be distinguished. That is why, at $m \to \infty$, one interest rate is used: the growth rate δ. Then with continuous discounting Formula (1.18) is correct:

$$P_0 = S_n \cdot e^{-\delta \cdot n}. \tag{1.41}$$

Example 1.6. 10 thousand currency units must be repaid after 5 years. Compare the present values of this debt at its discounting at the annual nominal discount rate 0.12 a) in the periods of six months; b) quarterly; c) continuously.

According to the condition, n = 5, S_5 = 10 000, a) m = 2, g = 0.12; b) m = 4, g = 0.12; c) $m \to \infty$, δ = 0.12. From Formula (1.40) and (1.41) we obtain:

$$P_0 = S_5 (1 - \frac{g}{m})^{5m} \text{ for Case a) and b); and } P_0 = S_5 \cdot e^{-5\delta} \text{ for Case c).}$$

Hence, the present value of the amount of 10 thousand currency units whose repayment period is after 5 years, with its discounting at the annual nominal discount rate depending on m makes a) 5386.15 currency units; b) 5437.94 currency units; c) 5488.12 currency units. It is obvious that with an increase in m, the present cost of the amount of 10 000 currency units increases.

Thus, depending on the method of applying the discount rate d, we have four methods of discounting the amount of debt S_n at its rate: at the simple (1.38), compound (1.39), nominal (1.40), continuous (1.41) growth rate.

Variable Discount Rate

Let us consider discrete variable interest rates. Let n be the life of the loan, $n = \sum_{j=1}^{k} n_j$, where n_j is the period in the life of the loan when the discount rate d_j is used; $j = 1, 2, ..., k$; k is the number of periods.

The formulas for the present value of the sum S_n at its bank accounting at the simple, compound variable discount rate, have the form:

$$P_0 = S_n\left(1-\sum_{j=1}^{k} n_j d_j\right), \tag{1.42}$$

$$P_0 = S_n(1-d_k)^{n_k}(1-d_{k-1})^{n_{k-1}}....(1-d_1)^{n_1}. \tag{1.43}$$

1.12.2. Accumulation at Discount Rate

If a problem opposite of bank discounting is solved, then in order to find the sum of the dischargeable debt, the discount rate is used. For example, it is necessary for determining the sum that must be put in the promissory note, if the current amount of the debt is given. From Formula (1.26), (1.27), (1.28), we find

$$S_n = \frac{P_0}{1-nd}, \tag{1.44}$$

$$S_n = \frac{P_0}{(1-d)^n}, \tag{1.45}$$

$$S_n = \frac{P_0}{(1-\frac{g}{m})^{nm}}. \tag{1.46}$$

During the continuous accumulation at a compound discount rate, Formula (1.29) is correct (at $m \to \infty$, the nominal interest rates j and g cease to be distinguished).

If the discount rate is variable, then instead of Formula (1.32) and (1.33) we will obtain

$$S_n = \frac{P_0}{1-\sum_{j=1}^{k} n_j d_j}, \tag{1.47}$$

$$S_n = \frac{P_0}{(1-d_k)^{n_k}(1-d_{k-1})^{n_{k-1}}....(1-d_1)^{n_1}}. \tag{1.48}$$

1.13. Comparison of Accumulation Methods

All the considered accumulation methods are given in the table.

Accumulation method	Formula	Accumulation factor
At the simple interest rate i	$S_n = P_0(1+in)$	$1+in$
At the compound interest rate i	$S_n = P_0(1+i)^n$	$(1+i)^n$
At the nominal interest rate j	$S_n = P_0(1+\frac{j}{m})^{nm}$	$(1+\frac{j}{m})^{nm}$

At the constant growth rate δ	$S_n = P_0 e^{\delta n}$	$e^{\delta n}$
At the nominal discount rate g	$S_n = \dfrac{P_0}{(1-\frac{g}{m})^{nm}}$	$\dfrac{1}{(1-\frac{g}{m})^{nm}}$
At the compound discount rate d	$S_n = \dfrac{P_0}{(1-d)^n}$	$\dfrac{1}{(1-d)^n}$
At the simple discount rate d	$S_n = \dfrac{P_0}{1-nd}$	$\dfrac{1}{1-nd}$

Definition. The number indicating the number of times that the accumulated sum of the debt exceeds the initial one is called the accumulation factor (or cumulation factor).

The economic substance of the accumulation factor is the following. If the life of the loan is n units of time, then the accumulation factor indicates the future cost of 1 currency unit to be accumulated by the n moment, that will be invested at the t = 0 moment for the n term. It is obvious that the accumulation factor is larger than 1. The intensity of accumulation process is determined by the accumulation factor. By comparing these factors for each value of the term n and considering the interest rates equal it is possible to compare the accumulation rates at different interest rates.

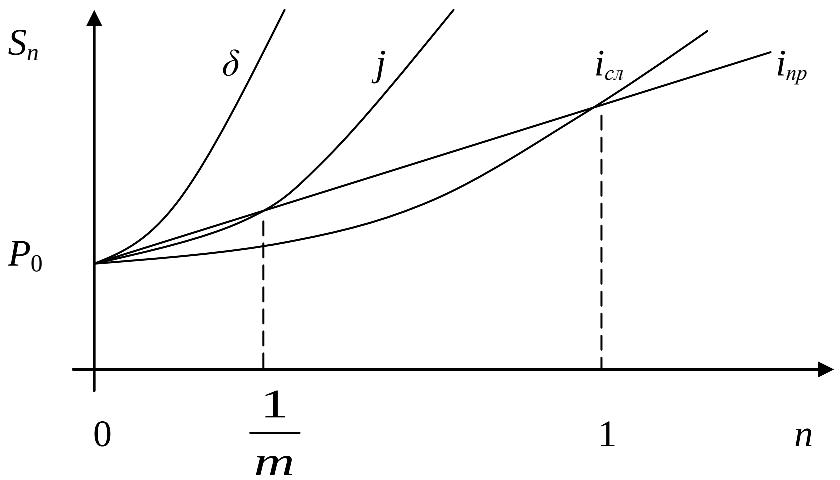


Figure1.3. Accumulation rates at the simple interest rate

Figure 1.3. shows the accumulation curves that correspond four methods of accumulating the amount of the debt at the rate i .

1.14. Comparing Discounting Methods

All the obtained discounting methods are shown in the table.

Discounting method	Formula	Present value factor
At the simple discount rate d	$P_0 = S_n(1-nd)$	$1-nd$
At the compound discount rate d	$P_0 = S_n(1-d)^n$	$(1-d)^n$
At the nominal discount rate g	$P_0 = S_n(1-\frac{g}{m})^{nm}$	$(1-\frac{g}{m})^{nm}$
At the constant growth rate δ	$P_0 = S_n e^{-\delta n}$	$e^{-\delta n}$
At the nominal interest rate j	$P_0 = \frac{S_n}{(1+\frac{j}{m})^{nm}}$	$\frac{1}{(1+\frac{j}{m})^{nm}}$
At the compound interest rate i	$P_0 = \frac{S_n}{(1+i)^n}$	$\frac{1}{(1+i)^n}$
At the simple interest rate i	$P_0 = \frac{S_n}{1+in}$	$\frac{1}{1+in}$

Definition. The number indicating what portion of the dischargeable debt sum is comprised by its present value is called the discount factor.

The economic substance of the discount factor is the following. If the life of the loan is *n* units of time, then the discount factor is the present cost of 1 currency unit that must be paid after the *n* time period. It is obvious that the discount factor is smaller than 1. The intensity of the discounting process is determined by the discount factor. Comparing these factors for each value of the term *n* and considering the interest rates equal at 1 of time, it is possible to compare the discounting rates at different interest rates.

Figure 1.5. shows the discount curves corresponding four methods of mathematical discounting:

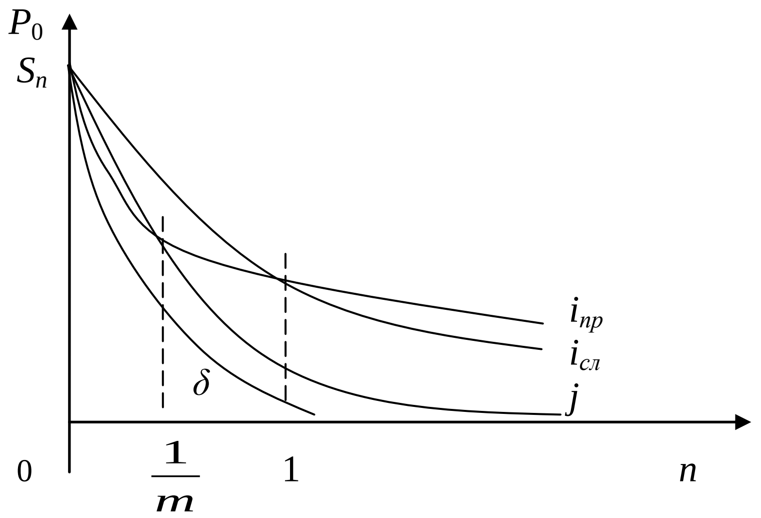


Figure 1.4. Discounting rates at an interest rate

Questions for Self-test

1. What does interest in financial transactions mean?
2. What is interest rate? What is called normalized ınterest rate?
3. What is accumulation?
4. Write the formula of accumulation at simple interest with a constant rate.
5. Write the formula of accumulation at simple interest with a variable rate.
6. Write the formula of accumulation for compound interest with a constant rate.
7. Write the formula of accumulation for compound interest with a variable rate.
8. What is the nominal and effective interest rate? What is the connection between the effective and nominal rates?
9. How to determine the duration of the loan and interest rate for simple interest?
10. How to determine the period and interest rate for compound interest if the original and accumulated sum is known?
11. What is discounting? Write the formula of discounting at simple interest.
12. What is the present value of the sum S? What is the economic meaning of this notion?
13. Write the formula for the present value with interest calculations once a year and with interest calculations m times a year.
14. How is inflation accounted at interest accumulation?
15. What is the growth rate? Write the formula of continuous accumulation and discounting.
16. What is the simple and compound interest rate equivalency?
17. What is funding debt and how is it accounted for simple and compound interest?
18. How is discounting made at a simple discount rate?
19. Determine the formula of discounting at a compound discount rate.
20. Determine the formula of discounting at a nominal discount rate.
21. How is discounting made at a continuous discount rate?
22. How is discounting made at a variable simple discount rate?
23. How is discounting made at a variable compound discount rate?
24. Write the accumulation formula at a simple discount rate
25. Write the accumulation formula at a compound discount rate
26. Write the accumulation formula at a nominal discount rate
27. Write the accumulation formula at a continuous discount rate

28. Write the accumulation formula at a variable simple discount rate
29. Write the accumulation formula at a variable compound discount rate

Chapter 2

Payment, Annuity Streams

2.1. Basic Definitions

In practice, financial operations, as a rule, stipulate payments and cash inflows interspaced in time.

Let us call the sequence (range) of payments and inflows arranged for different time moments a *payment stream.*

The payment stream whose elements are positive values and time intervals between two consequent payments are constant are called a *financial contract* or *annuity* irrespectively of the purpose, destination and origin of these payments.

The annuity is described through the following parameters:

1. Annuity component – the value of each separate payment.
2. Annuity interval – time interval between payments.
3. Annuity term – time from the beginning of the annuity till the end of its last period (interval).
4. Interest rate – the rate used for accumulation or discounting of payments of which the annuity is composed.

Besides, there may be additional parameters: the number of annual payments, the number of annual interest calculations, payment moments (at the beginning or the end of annuity period) etc.

Annuity Types

An annuity is called *annual* if its period equals one year.

An annuity is called *p-due*, if its period is less than a year and the number of annual payments is *p.*

These annuities are *discrete* since their payments are coordinated with discrete time points. There are *continuous* annuities when the payment stream is characterized with the continuous function.

Annuities may be *constant* and *variable*. An annuity is *constant* if all its payments are equal and do not change in time. If the amounts of payment depend on time, the annuity is *variable.*

The annuity is *certain* (*terminating*) if the number of payments is finite; otherwise the annuity is called *indefinite* or *perpetual*. For example, a long-term commitment when the financial transaction term is continuous and is not agreed on in advance makes a perpetual annuity.

Annuities may be *immediate* or *postponed (deferred).* The term of immediate annuities starts from the moment of contract conclusion. If the annuity is postponed, then the starting date of payments is moved aside for a certain time.

If payments are made at the end of a period, then such an annuity is called *ordinary* or *annuity-immediate*. If payments are made at the beginning of a period, then such an annuity is called *annuity-due.*

Generalized Characteristics of Payment Streams

To analyze payments streams, it is necessary to know how to calculate their generalized characteristics. There are two such characteristics: the accumulated sum and the present value of the annuity.

The *accumulated sum* of a payment stream is the sum of all the successive payments with their calculated interest by the end of the annuity.

The *present value* of a payment stream is the sum of all the payments discounted for a certain time point that coincides with the beginning of the payment stream or that predicts it.

The accumulated sum is determined, for instance, in order to be aware of the total amount of indebtedness at a certain time point, the total volume of investments, money reserve accumulated by the evaluation moment etc.

The present value is a most important index when evaluating financial effectiveness of commercial transactions, real and financial investments, etc.

Let us consider the problems for determining the accumulated sums for various types of annuity-immediate.

2.2. The Accumulated Sum of the Annual Annuity

When calculating the formula for accumulated annuity sums it is important to correctly indicate the moment of the consequent payment’s coming and determine the term (in years) during which this payment will be charged with interest. After this, in order to determine the amount of payment with the interest calculated for this term, it is sufficient to use the formula for compound interest accumulation. This problem is solved most simply for the annual annuity with interest calculation once a year. Such an annuity is called ordinary.

Let

S – accumulated sum of annuity;

R – amount of individual payment;

i – interest rate in form of a decimal fraction;

n – annuity term in years.

Let us consider the process of payment accumulation for the annuity-immediate. The first payment of the R amount comes at the end of the first period, it will be charged with interest at the rate i for years (that much money is left from the end of the first period to the term end); as a result, the amount $R(1+i)^{n-1}$ will be obtained at the end of the term.

The second payment will come in at the end of the second year and the amount $R(1+i)^{n-2}$ will be obtained at the end of the term. With this process continued, at the term end we will obtain the sum $R(1+i)$ for the payment coming at the end of the $(n-1)$ year. The last payment will come in at the end of the n year, it will not be charged with interest.

To determine the accumulated value of the annuity S, it is necessary to sum the payment series obtained, i.e. $S = R(1+i)^{n-1} + R(1+i)^{n-2} + ... + R(1+i) + R$ or

$$S = R\sum_{t=0}^{n-1}(1+i)^t . \qquad (2.1)$$

We should note that this series makes a geometric sequence with the first member R and denominator that equals $(1+i)$, and the number of members n.

Let us remind that the geometric sequence is a series whose each member comes out of the preceding one by multiplication by one and the same number.

The formula for the sum n of the geometric sequence members is

$$S_n = a_1(q^n - 1)/(q-1) , \qquad (2.2)$$

where a_1 – the first sequence member, q – the sequence denominator.

Using this formula from (1.1) we will obtain a relation for calculating the accumulated sum of the annual annuity:

$$S = R\frac{(1+i)^n - 1}{(1+i)-1} = R\frac{(1+i)^n - 1}{i} . \qquad (2.1a)$$

The value

$$s(n,i)=\sum_{t=0}^{n-1}(1+i)^t=\frac{(1+i)^n-1}{i} \quad (2.3)$$

is called the *annual annuity accumulation factor*.

Using the value introduced, we may write the expression for the accumulated sum of the annual annuity the following way:

$$S=Rs(n,i). \quad (2.4)$$

If payments come in at the beginning of the term, the annuity accumulation factor equals

$$s(n,i)=\sum_{t=1}^{n}(1+i)^t=(1+i)\cdot\frac{(1+i)^n-1}{i}. \quad (2.6)$$

In the case with simple interest, the accumulated sum of payments equals:

$$R\cdot[1+(n-1)\cdot i]+R\cdot[1+(n-2)\cdot i]+...+R\cdot[1+i]+R$$

$$=R[n+i\cdot(1+2+...+(n-1))]=R\left[n+i\frac{n\cdot(n-1)}{2}\right]=R\cdot s(n,i),$$

where

$$s(n,i)=\left[n+i\frac{n\cdot(n-1)}{2}\right] \quad (2.7)$$

— the annual annuity accumulation factor at the rate of simple interest.

For the annuity-due we will obtain

$$s(n,i)=\left[n+i\frac{n\cdot(n+1)}{2}\right]. \quad (2.8)$$

2.3. Accumulated Sum of Annual Annuity with Interest Calculation m Times a Year

Let payments come in once at the end of the year (meaning, the annuity interval equals one year), and interest calculation occurs m times a year. In this case, interest is calculated at the rate j/m (see 1.5), the accumulation factor of the payment for one year equals $(1+j/m)^m$. The process of forming payments with calculated interest is as follows.

The first payment of the amount R comes in at the end of the first period, it will be charged with interest at the rate i for $(n-1)$ years (the amount of time left from the end of the first period to the term end); as a result, the amount $R(1+j/m)^{m(n-1)}$ will be obtained at the term end.

The second payment will come at the end of the second year and the amount $R(1+j/m)^{m(n-2)}$ will be obtained at the term end. With this process continued, by the end of the term we will obtain an amount of $R(1+j/m)^{m}$ for the payment received at the end of the $(n-1)$ year. The last payment will come in at the end of the year n, it will not be charged with interest.

To determine the accumulated value S it is necessary to sum the payment series obtained., i.e. $S=R(1+j/m)^{m(n-1)}+R(1+j/m)^{m(n-2)}+...+R(1+j/m)+R$ or

$$S=R\sum_{t=0}^{n-1}(1+j/m)^{mt}\ . \tag{2.9}$$

We have a geometric sequence with the first member R and sequence denominator that equals $(1+\frac{j}{m})^{m}$; there are n series members. Using Formula (2.2) for the sum of the geometric sequence members, we will obtain

$$S=R\frac{(1+\frac{j}{m})^{m\cdot n}-1}{(1+\frac{j}{m})^{m}-1}. \tag{2.9a}$$

Let us indicate the following value as the accumulation factor

$$s(n,j/m)=\sum_{t=0}^{n-1}(1+\frac{j}{m})^{mt}=\frac{(1+\frac{j}{m})^{m\cdot n}-1}{(1+\frac{j}{m})^{m}-1}. \tag{2.10}$$

Then the accumulated sum is

$$S=Rs(n,j/m). \tag{2.11}$$

For the annuity-due the annuity accumulation factor equals

$$s(n,j/m)=\sum_{t=1}^{n}(1+\frac{j}{m})^{mt}=(1+\frac{j}{m})^{m}\frac{(1+\frac{j}{m})^{m\cdot n}-1}{(1+\frac{j}{m})^{m}-1}. \tag{2.12}$$

2.4. Accumulated Sum of p – due Annuity

Let the payments come in p times a year with equal amounts, and the interest be calculated once at the end of the year $(m=1)$.

If R is the annual sum, then a single payment equals R/p. Since there are p payments per year, the interval between them will equal $1/p$ years. The first payment will come in at the time moment $1/p$.

Arguing analogously, we will obtain for the accumulated annuity value

$$S=\frac{R}{p}\sum_{t=0}^{np-1}(1+i)^{t(1/p)}. \tag{2.13}$$

We have a geometric sequence whose first member equals R/p, and the denominator is $(1+i)^{1/p}$, with the total number of members $n\cdot p$.

Using the formula for the sum of the geometric sequence members (2.2), we will obtain

$$S=\frac{R}{p}\frac{((1+i)^{1/p})^{np}-1}{(1+i)^{1/p}-1}=R\cdot s(p,n,i), \tag{2.13a}$$

where $s_{n,i}^{(p)}$ – the accumulation factor of the p *-due* annuity with interest calculation once a year, which equals

$$s(p,n,i)=\frac{1}{p}\sum_{t=0}^{np-1}(1+i)^{t(1/p)}=\frac{(1+i)^n-1}{((1+i)^{1/p}-1)}. \tag{2.14}$$

If payments come in at the beginning of the period, then for the accumulation factor of the p-due annuity-due we will obtain:

$$s(p,n,i)=\frac{1}{p}\sum_{t=1}^{np}(1+i)^{t(1/p)}=(1+i)^{1/p}\frac{(1+i)^n-1}{p\cdot((1+i)^{1/p}-1)}. \tag{2.15}$$

2.5. Accumulated Sum of p – due Annuity with $p\neq m,\ m\neq 1$

The parameters of such an annuity are: p payments per year and interest calculation m times a year at the nominal rate j. The annuity with such conditions is called *total*.

The principle of obtaining the formulas for the accumulated sum is analogous to the cases mentioned above.

The first payment equals R/p, the moment of this payment's receipt (the time point from the annuity term beginning) equals $1/p$ and at the end of the annuity term this payment will give a sum with interest

$$\frac{R}{p}(1+\frac{j}{m})^{(n-\frac{1}{p})m}.$$

The second payment with calculated interest will equal

$$\frac{R}{p}(1+\frac{j}{m})^{(n-\frac{2}{p})m}$$

The last but one payment will give a sum of

$$\frac{R}{p}(1+\frac{j}{m})^{(n-\frac{np-1}{p})m}=\frac{R}{p}(1+\frac{j}{m})^{\frac{m}{p}}.$$

The last payment will not be charged with interest and will equal R/p.

Let us sum the members of the series obtained with the use of (2.21) for the sum of the geometric sequence members with the following parameters:

the first sequence member equals $\frac{R}{p}$, the denominator is $(1+\frac{j}{m})^{\frac{m}{p}}$, the number of sequence members being np. We will obtain

$$S=\frac{R}{p}\sum_{t=0}^{np-1}(1+\frac{j}{m})^{t(m/p)}=R\frac{(1+\frac{j}{m})^{mn}-1}{p((1+\frac{j}{m})^{\frac{m}{p}}-1)}. \qquad (2.16)$$

Let us indicate

$$s(p,n,j/m)=\frac{1}{p}\sum_{t=0}^{np-1}(1+\frac{j}{m})^{t(m/p)}=\frac{(1+\frac{j}{m})^{mn}-1}{p((1+\frac{j}{m})^{\frac{m}{p}}-1)} \qquad (2.17)$$

– the accumulation factor of the p – due annuity with interest calculation m times a year. Then the formula for the accumulated sum will be

$$S=Rs(p,n,j/m). \qquad (2.16a)$$

Making analogous arguments, for the annuity-due accumulation factor we will have an expression

$$s(p,n,j/m)=\frac{1}{p}\sum_{t=1}^{np}(1+\frac{j}{m})^{t(m/p)}=(1+\frac{j}{m})^{\frac{m}{p}}\frac{(1+\frac{j}{m})^{mn}-1}{p((1+\frac{j}{m})^{\frac{m}{p}}-1)}. \qquad (2.18)$$

2.6. The Present Value of the Ordinary Annuity

The present value of the annuity is a most important property of the payment stream that determines the value of the future money stream at the present time point. This property serves the base for many methods of financial analysis. By definition, the present value is the sum of all the discounted members of the payment stream at the initial or its preceding time point. Sometimes, instead of the term the *present value*, the terms *reduced* or *capitalized* sum of payments. When determining the present value of the payment stream, it is important to correctly determine the time period from the beginning of the stream (time point for which evaluation is made) to the moment of the payment's receipt (in years). After this, discount formulas may be applied. Let us indicate v – the discount multiplier, $v=\frac{1}{1+i}$, where i is the annual rate.

The discounting process is as follows. The first payment of an amount of R comes in at the end of the first year, its present value at the term beginning equals Rv. The second payment comes in at the end of the second year, its present value equals Rv^2. Continuing this process, we will obtain the present value Rv^n of the last payment that must come in at the term end.

To obtain the present value of the stream, let us sum all the members of the resulting series. It is obvious that we have a geometric sequence with the features: the first member of the sequence equals Rv; the sequence denominator – v; the number of sequence members – n.

As a result we obtain

$$A=R\sum_{t=1}^{n}v^t=R\sum_{t=1}^{n}\frac{1}{(1+i)^t}. \qquad (2.19)$$

The value

$$a(n,i)=\sum_{t=1}^{n}\frac{1}{(1+i)^t}=\frac{1-(1+i)^{-n}}{i} \qquad (2.20)$$

is called the *annual annuity reduction factor*. Expression (2.20) is obtained with the consideration of Formula (2.2).

The diagram on Figure 2.4 illustrates the process of discounted payment stream formation at the beginning of the annuity term. The indices of the degrees – time periods in years from the beginning of the annuity to the moment of the payment's receipt.

If payments are received at the beginning of the period, the annuity reduction factor equals

$$a(n,i)=\sum_{t=0}^{n-1}\frac{1}{(1+i)^t}=(1+i)\frac{1-(1+i)^{-n}}{i}. \qquad (2.21)$$

With this regard, the expression for the present value will be

$$A=Ra(n,i). \qquad (2.19a)$$

In the case of simple interest, we should summarize the stream

$$\frac{R}{1+i}+\frac{R}{1+2\cdot i}+...+\frac{R}{1+n\cdot i}=R\cdot\sum_{k=1}^{n}\frac{1}{1+i\cdot k}$$

or

$$A=R\cdot a(n,i), \qquad (2.19б)$$

where

$$a(n,i)=\sum_{k=1}^{n}\frac{1}{1+i\cdot k} \qquad (2.22)$$

— by the annual annuity reduction factor at the simple interest rate.

2.7. The Present Value of the Annual Annuity with Interest Calculation m Times a Year

Let us insert the multiplier $(1+\frac{j}{m})^{-m}$ into the obtained formula for the present value of the annual annuity instead of the discount multiplier $(1+i)^{-1}$. We will obtain

$$A=R\frac{1-(1+j/m)^{-mn}}{(1+j/m)^{m}-1}. \qquad (2.23)$$

Let us indicate

$$a(n, j/m)=\sum_{t=1}^{n}\frac{1}{(1+j/m)^{mt}}=\frac{1-(1+j/m)^{-mn}}{(1+j/m)^{m}-1}, \tag{2.24}$$

then

$$A=Ra(n, j/m). \tag{2.23a}$$

For the annuity-due the reduction factor equals

$$a(n, j/m)=\sum_{t=0}^{n-1}\frac{1}{(1+j/m)^{mt}}=(1+j/m)^{m}\frac{1-(1+j/m)^{-mn}}{(1+j/m)^{m}-1}. \tag{2.25}$$

2.8. The Present Value of the p – Due Annuity $(m=1)$

The interval between payments of such an annuity equals $1/p$, the amount of payment is R/p. Let us consider the order of forming discounting payments.

The present value of the first payment equals $\frac{R}{p}v^{1/p}$, of the second payment – $\frac{R}{p}v^{2/p}$ etc., until the last payment – $\frac{R}{p}v^{n/p}$.

We have a geometric sequence with the number of members that equals np, the first sequence member is $\frac{R}{p}v^{1/p}$, the sequence denominator – $v^{1/p}$. Using Formula (2.2) for the sum of the geometric sequence members, we will obtain

$$A=\frac{R}{p}\frac{v^{\frac{1}{p}}(v^{\frac{1}{p}np}-1)}{v^{\frac{1}{p}}-1}=\frac{R}{p}\frac{(v^{n}-1)}{(1-\frac{1}{v^{1/p}})}=R\frac{1-(1+i)^{-n}}{p((1+i)^{1/p}-1)}. \tag{2.26}$$

Let us indicate

$$a(p,n,i)=\frac{1}{p}\sum_{t=1}^{np-1}\frac{1}{(1+i)^{t/p}}=\frac{1-(1+i)^{-n}}{p((1+i)^{1/p}-1)} \tag{2.27}$$

– the reduction factor of the p – due annuity.

Then the formula for calculating the present value of the p – due annuity will be

$$A=Ra(p,n,i). \tag{2.26a}$$

For the annuity-due we have

$$a(p,n,i)=\frac{1}{p}\sum_{t=0}^{np}\frac{1}{(1+i)^{t/p}}=(1+i)^{1/p}\frac{1-(1+i)^{-n}}{p((1+i)^{1/p}-1)}. \qquad (2.28)$$

2.9. The Present Value of the p – Due Annuity with $m \neq 1,\ p \neq m$

In this case, the discounting factor is $v=\frac{1}{1+j/m}$. Then the first discounted payment equals $\frac{R}{p}v^{m/p}$, the second discounted payment equals $\frac{R}{p}v^{2m/p}$, the last but one discounted payment is $\frac{R}{p}v^{\frac{(np-1}{p}m}$ and, finally, the last payment being $\frac{R}{p}v^{mn}$. We should remind here that the exponent of the discount multiplier is the time interval (measured in years) from the beginning of the annuity to the moment of the payment's receipt with adjustments for m – interest calculation once a year. We have a geometric sequence with the number of members np, whose first member is $\frac{R}{p}v^{m/p}$ and the sequence denominator is $v^{m/p}$. The sum of the members of this sequence is

$$A=\frac{R}{p}\frac{v^{m/p}(v^{mn}-1)}{v^{m/p}-1}=\frac{R}{p}\cdot\frac{1-(1+j/m)^{-mn}}{(1+j/m)^{m/p}-1}. \qquad (2.29)$$

Let us indicate

$$a(p,n,j/m)=\frac{1}{p}\sum_{t=1}^{np}\frac{1}{(1+j/m)^{t(m/p)}}=\frac{1-(1+j/m)^{-mn}}{p((1+j/m)^{m/p}-1)} \qquad (2.30)$$

– the ordinary annuity reduction factor.

Then it is definitely that the formula for the present value of this annuity is

$$A=Ra(p,n,j/m). \qquad (2.29a)$$

In practice, the number of interest calculations and payments on them (or on the receipts) often coincides, i.e. $p=m$. Then

$$a(p,n,j/m)=\frac{1}{p}\frac{1-(1+j/m)^{-mn}}{j/m}. \qquad (2.31)$$

For the annuity-due, instead of (2.30) we will obtain:

$$a(p,n,j/m)=\frac{1}{p}\sum_{t=0}^{np-1}\frac{1}{(1+j/m)^{t(m/p)}}=(1+j/m)^{m/p}\frac{1-(1+j/m)^{-mn}}{p((1+j/m)^{m/p}-1)}. \qquad (2.32)$$

2.10. The Relation between the Accumulated and Present Values of Annuity

Let A be the present value of the annual annuity for the beginning of the term with interest calculation once a year, S – the accumulated sum of this annuity. Then it is possible to show that $A(1+i)^n = S$, i.e. interest calculation on the sum A for n periods gives an accumulated sum of the annuity. Indeed,

$$A(1+i)^n = R\frac{1-(1+i)^{-n}}{i}(1+i)^n = R\frac{(1+i)^{+n}-1}{i} = Rs(n,i) = S.$$

Besides, it is obvious that we have another range of relations:

$$Sv^n = A, \text{ where } v = \frac{1}{1+i},$$

$$a_{n,i}(1+i)^n = s(n,i),\ s(n,i)v^n = a(n,i). \qquad (2.31)$$

2.11. Determining Annuity Parameters

In this section, let us consider several problems that arise in connection with the notion of annuity.

1. Determining the amount of payment of the annuity (member).
2. Determining the term of the annuity.
3. Determining the interest rate.

In order not to overload this topic's presentation by lengthy computations and formulas, let us consider the most simple annuity type, in particular, the annual annuity with interest calculation once a year. For other annuity types the solutions will be analogous.

Determining the Amount of Payment

It is obvious that if the accumulated sum is given, then

$$R = \frac{S}{s(n,i)}. \qquad (2.32)$$

If the present value of the annuity is given, then

$$R = \frac{A}{a(n,i)}. \qquad (2.33)$$

Determining the Term of the Annuity

If the accumulated sum S is known, then the term n is determined from the solution of this equation:

$$R\frac{(1+i)^n-1}{i}=S$$

After simple transformations we will have

$$(1+i)^n=\frac{S}{R}i+1$$

Taking the logarithm of the right and left parts and expressing out of the obtained relation n, we will have

$$n=\frac{\ln(1+\frac{S}{R}i)}{\ln(1+i)}. \tag{2.34}$$

If the reduced value of annuity is known, we obtain analogously

$$n=-\frac{\ln(1-\frac{A}{R}i)}{\ln(1+i)}. \tag{2.35}$$

Determining the interest rate

The estimated value of the rate is of importance in the financial and economic analysis when determining the financial and commercial transaction yield.

If the accumulated sum is known, to determine the rate, it is necessary to solve the equation in relation to the unknown value i

$$S=R\frac{(1+i)^n-1}{i}. \tag{2.36}$$

If the present value of the annuity is known, then we solve the equation

$$A=R\frac{1-(1+i)^{-n}}{i}. \tag{2.37}$$

We have nonlinear equations, whose accurate solution (as a calculation formula for i) is not possible. The data of the equation allow only an approximate solution with the use of special numerical techniques, for instance, Newton's method that is easily realized with the help of such tabular processing units as Microsoft Excel.

2.12. Annuity Conversion

Converting the annuity means changing the conditions of the financial contract that stipulates payment of this annuity.

Simple types of conversion

To simple types of conversion belong:

1. Annuity buyout – a replacement of the annuity with a lump sum payment. It follows from the principle of financial equivalency that in this case, instead of the annuity, its present value is paid.

2. Payment by instalments – a replacement of a lump sum payment with an annuity.

In this case, the present value of the annuity must be equated with the value of the substitutable payment and either the amount of payment of such an annuity at the given term or the annuity term (if the amount of payment is known) must be determined. The data of the problem reduce themselves to the problems of determining the annuity parameters that were considered earlier (see Subsection 2.11).

Compound types of conversion

Compound types of conversion include the replacement of one annuity with another that means changing annuity parameters. It follows from the conditions of financial equivalency that at such a replacement, the present values of these annuities must be equal. In other words, if $A1$ is the present value of the annuity to be replaced,, $A2$ *is the present value of the replacing annuity at the same time point, then the following condition must be observed:* $A1 = A2$.

Examples of conversion

1. Replacing the immediate annuity with the deferred one.

Let there be an annual immediate annuity with the parameters of R_1, n_1, i. This annuity is replaced with the one deferred for t years with the term n_2 and rate i. The present problem is reduced to determining the amount of payment of R_2 of the second annuity. From the condition of financial equivalency we have an equation

$$R_1 a(n_1, i) = R_2 a(n_2, i) v^t, \tag{2.38}$$

where v – the discount multiplier at the rate i, from which we obtain

$$R_2 = R_1 \left(\frac{a(n_1, i)}{a(n_2, i)} \right) (1+i)^t.$$

If interest is calculated m times a year, then the problem is solved by analogy.

Let us suppose further that $R_1 = R_2$, and since the annuity is deferred, its term also changes that must also be determined. Solving the equation of equivalency (2.44) in relation to n_2, we will obtain

$$n_2 = \frac{-\ln\left\{1-\left[1-(1+i)^{-n_1}\right](1+i)^t\right\}}{\ln(1+i)}.$$

2. Changing the duration and the term structure of annuity.

Let us consider an ordinary annual annuity with the parameters R_1, n_1, i. This annuity is replaced with another one whose parameters are R_2, n_2, i

Let us suppose that the payment R_2 is known, it is necessary to determine the term n_2. Making the present values of annuities equal, we will obtain the equation

$$R_1 a(n_1, i) = R_2 a(n_1, i),$$

from which we find the term of the second annuity n_2. This problem is solved analogously with the problem of determining the annual annuity term (see Subsection 2.11). If the term is known, then it is possible to estimate R_2 from the same equation.

Let us consider this problem in the case when the term structure changes. Let the p–due annuity with the parameters p_1, R_1, n_1, i_1 be replaced with the p-due annuity with the parameters p_2, R_2, n_2, i_2. It is necessary to determine the parameter of the second annuity – the amount of payment R_2. From the equation condition of the payment stream present values we have

$$R_1 a(p_1, n_1, i) = R_2 a(p_2, n_2, i),$$

from which R_2 is easily determined. If R_2 is known, then by solving this equation it is also possible to determine n_2.

3. Annuity integration.

Let us suppose that k annual annuities with interest calculation once a year, whose parameters are known, are replaced with a single annual annuity. Let A be the present value of the replacing annuity; A_q, $q = 1, 2, ..k$ – the present values of the substitutable annuities. Then the condition of financial equivalency is written as:

$$A = \sum_{q=1}^{k} A_q .$$

If the moments of annuity beginning do not coincide, the present values of these annuities are determined for the starting moment of the earliest annuity. When integrating annuities two tasks arise:

1) determining the payment amount of the replacing annuity if its term is given;

2) determining the term of the replacing annuity with other parameters given.

Since the present value of the replacing annuity is known, both these problems are reduced to the corresponding problems of determining the annual annuity parameters.

Questions for Self-test

1. What is the financial contract? What parameters is it described with?
2. List financial contract types.
3. Give the definition of accumulated sum and present value of payment stream.
4. Determine the formula for accumulated sum of payment stream.
5. Determine the formula for accumulated sum of payment stream with interest calculations m times a year.
6. Determine the formula for accumulated sum of p - due annuity.
7. Determine the formula for accumulated sum of p - due annuity with interest calculations m times a year.
8. Determine the formula for the present value of payment stream.
9. Determine the formula for the present value of payment stream with interest calculations m times a year.
10. Determine the formula for the present value of p - due annuity.
11. Determine the formula for the present value of p - due annuity with interest calculations m times a year.
12. Write the connection between the factors of annuity accumulation and annuity discounting factor.
13. How to determine the amount of payment, annuity period and amount of interest rate?
14. Write the formula for accumulated value of discrete annuity with continuous interest calculation.

15. Write the formula for the present value of discrete annuity with continuous interest calculation.
16. Write the formula for accumulated value of continuous annuity with discrete interest calculation.
17. How should the immediate annuity be replaced with the deferred one?
18. How should one annual annuity be replaced with another?

Chapter 3

Financial Transaction Yield

A transaction is called financial if its beginning and ending have a money valuation – $P(0)$ and $S(T)$ correspondingly and the purpose of its performance is in maximizing the difference $S(T)-P(0)$ or another similar index. A most important property of the transaction is its yield.

In the definition, $P(0)$ is the really invested funds at the moment t = 0, $S(T)$ – monetary funds really returned as the result of a transaction whose term is T time units. It is natural to measure the effect from investing in the form of the interest accumulation rate, which in this case is called the yield.

3.1. The Absolute and Average Annual Transaction Yield

Two types of financial transaction yield are distinguished – absolute and average annual.

Absolute yield d *(*the yield for all the term) of the transaction is determined from the equation $P(0)(1+d)=S(T)$ or $d=(S(T)-P(0)/P(0)=S(T)/P(0)-1.$ The value $S(T)/P(0)$ is called *accumulation factor* or *multiplier.* It is obvious that $S(T)/P(0)=1+d$.

Average annual yield r of the financial transaction is the simple or compound interest rate that is used to measure the effectiveness of a financial transaction.

By definition, the average annual financial transaction yield is the positive value r that satisfies the equality:

$$P(0)(1+r\cdot T)=S(T) \tag{3.1}$$

or

$$P(0)(1+r)^T=S(T). \tag{3.2}$$

From (3.1) and (3.2) we find the connection between d and r:

$1+r\cdot T=1+d$ (for the simple rate) и

$(1+r)^T=1+d$ (for the compound rate).

Thus, the financial transaction is associated with an equivalent transaction of accumulating the sum $P(0)$ at the rate r within the time T . Such an approach enables comparing the obtained yield value with the yields on alternative invested funds.

3.2. Tax and Inflation Accounting

Taxes and inflation remarkably influence the effectiveness of the financial transaction. Let us analyze tax accounting. The tax is charged, as a rule, on the interest acquired when placing an amount of money into growth. Let us assume that the sum P_0 within the time n was charged with interest at the rate i , g - is the tax rate on interest. Then the value of interest is

$$I(n)=S_n-P_0\ ,$$

and the amount of tax is $G_n=g\cdot I(n)$. The accumulated sum after paying the tax makes

$$S(n)=S_n-G_n.$$

Since $S(n) < S_n$, then tax accounting actually reduces the accumulation rate. Hence,

$$S(n) = S_n - G_n = S_n - g \cdot I(n) = S_n - g \cdot (S_n - P_0) = S_n(1-g) + gP_0$$

If i is the simple interest rate, then $S_n = P_0(1 + i \cdot n)$. Thus,

$$S(n) = P_0(1 + i \cdot (1-g)n).$$

We see that in the reality, accumulation is made according to the rate $i(1-g) < i$.

If i is the compound interest rate, then $S_n = P_0(1+i)^n$. Thus,

$$S(n) = P_0\left((1+i)^n(1-g) + g\right).$$

Example 3.1. When granting a credit for 2 years with an annual compound interest rate of 0.08, the creditor retains the commission fee at the rate of 0.5% of the credit amount. The tax amount on interest is 10%. What is the transaction yield for the creditor?

If P_0 is the credit amount, and S_n is the sum of the dischargeable debt, then $S_n = P_0(1+i)^n$, where i = 0.08 , n = 2. The commission fee cP_0, where c = 0.005. Then the sum actually issued as a loan will make $P(0) = P_0(1-c)$. After paying the tax, the creditor will have $S(n) = P_0\left((1+i)^n(1-g) + g\right)$, where g = 0.1 is the tax rate. The equation for the yield is $S(n) = P(0)(1+r)^n$. When solving this equation in accordance with r, we will obtain

$$r = \left(\frac{S(n)}{P(0)}\right)^{\frac{1}{n}} - 1 = \left(\frac{(1+i)^n(1-g) + g}{1-c}\right)^{\frac{1}{n}} - 1 = 0.07496.$$

It must be noted that without tax accounting (g = 0) the transaction yield would be 0.08271.

Inflation – depreciation of money that is revealed in the growth of the prices for goods and services that is followed by the reductions of purchasing capacity of money.

Inflation is characterized by two quantitative indices – price index and inflation rate. Let us suppose that a time unit is chosen. Let us consider a time segment $[0, n]$, whose length is t time units from the starting moment t = 0.

The price index for the time $[0, n]$ is the number

$$J(n) = \frac{K(n)}{K(0)},$$

that indicates the number of times by which the cost of the consumer goods basket has grown for the time period $[0, n]$.

The inflation rate for the time $[0, n]$ is the number

$$H(n) = \frac{K(n) - K(0)}{K(0)},$$

indicating by how much percent the cost of the consumer goods basket has grown in the time period $[0, n]$. Since $H(n) = \frac{K(n)}{K(0)} - 1$, then the relations between the inflation rate and the price index are:

$$H(n) = J(n) - 1 \tag{3.3}$$

and

$$J(n) = 1 + H(n) \qquad (3.4)$$

for any time period $[0, n]$.

Let $[0, nt] = \bigcup_{k=1}^{m} [t_{k-1}, t_k]$, where $[0, t_1]$, … , $[t_{m-1}, t_m]$ - time segments in the term $[0, n]$ ($t_0 = 0,\ t_m = n$), whose lengths are $t_1, (t_2 - t_1), ..., (t_m - t_{m-1})$ time units.

$j(0, t_1), ..., j(t_{n-1}, t_n)$ и $h(0, t_1), ..., h(t_{n-1}, t_n)$ – are the price indices and inflation rates for the periods $j(0, t_1), ..., j(t_{n-1}, t_n)$ correspondingly. According to (3.4),

$$j(t_{k-1}, t_k) = 1 + h(t_{k-1}, t_k),\ \ k = 1, 2, ..., n.$$

where $h(t_{k-1}, t_k)$ - inflation rate for the period $[t_{k-1}, t_k]$. The price index $j(t_{k-1}, t_k)$ for the period $[t_{k-1}, t_k]$ demonstrates, by the number of times that the prices for this period have increased in relation to the price level of the preceding period.

Let j_k and h_k - be the price index and inflation rate for 1 time unit at the time segment $[t_{k-1}, t_k]$. Then

$$j_k = 1 + h_k, \quad k = 1, 2, ..., m,$$

and the price index for $[t_{k-1}, t_k]$ equals

$$j(t_{k-1,} t_k) = j_k^{(t_k - t_{k-1})} = (1 + h_k)^{(t_k - t_{k-1})}, \quad k = 1, 2, ..., m.$$

By the price index definition we have

$$J(n) = j_1^{t_1} \cdot j_2^{(t_2 - t_1)} \cdots j_m^{(t_m - t_{m-1})}.$$

Then

$$J(n) = (1 + h_1)^{t_1} \cdot (1 + h_2)^{(t_2 - t_1)} \cdots (1 + h_m)^{(t_m - t_{m-1})}, \qquad (3.5)$$

$$1 + H(n) = (1 + h_1)^{t_1} \cdot (1 + h_2)^{(t_2 - t_1)} \cdots (1 + h_m)^{(t_m - t_{m-1})}. \qquad (3.6)$$

If $h_1 = h_2 = ... = h_n = h$, то

$$J(n) = (1 + h)^n \qquad (3.7)$$

$$1 + H(n) = (1 + h)^n\ . \qquad (3.8)$$

Here h is the inflation rate for 1 time unit at the time segment $[0, n]$, $J(n)$ и $H(n)$ - the price index and inflation rate for the time period $[0, n]$.

Let us suppose that an accumulated sum of the deposit S_n is acquired for n time units. The price index for the period $[0, n]$ has grown to the value $J(n)$. Then the real amount of the deposit as the result of the decrease in the purchasing capacity of money will be

$$S(n) = \frac{S_n}{J(n)}.$$

The price index $J(n)$ is calculated by one of the formulas given above: (3.5) or (3.7) depending on the initial data. Since $J(n) > 1$, then $S(n) < S_n$, which means the actual decrease in the accumulation rate.

Example 2.2. The expected annual inflation rate of the first two years of the deposit is 3%, and of the next three – 4%. What minimum annual compound interest rate should the bank offer the client in order that the real annual yield of the deposit be no less than 8% ?

Here $t=0$ - is the moment of placing the deposit, 1 year – is the unit of measuring time, the deposit term n = 5 years. h_1 = 0.03 и h_2 = 0.04 – the average annual inflation rates on the time segments [0,2], [2,5]. For the yield on the deposit r, the condition $r \geq 0.08$ must be fulfilled. Let i be the annual compound interest rate, at which the sum P_0 is deposited. Then the accumulated sum of the deposit after n years $S_n = P_0(1+i)^n$. Adjusted for inflation, the real sum of the deposit will equal $S(n) = \dfrac{S_n}{J(n)}$, where the price index, according to (3.8), equals $J(t) = (1+h_1)^2 \cdot (1+h_2)^3$. The yield equation is: $S(n) = P(0)(1+r)^n$. Solving this equation in accordance with r and taking the required condition for the yield into consideration, we will obtain:

$$r = \frac{1+i}{(1+h_1)^{\frac{2}{5}}(1+h_2)^{\frac{3}{5}}} - 1 \geq 0.08.$$

Hence $i \geq$0.11887. It means that the minimum interest rate of placing the deposit makes 0.11887 versus 0.08 without inflation accounting.

3.3. Payment Stream and its Yield

Let $\{R_k, t_k\}$ be the payment stream with t_k – time points, and R_k – payments. Let us say that the stream under consideration has the present value A at the yield level j, if $\sum_k R_k/(1+j)^{t_k} = A$. If the stream is the annual annuity with an annual payment R and the duration n, then the annuity has the present value A at the yield level j, if $R \cdot a_{n,j} = A$. Let us fix A, then with the increase in R the yield of the annuity increases. It is also possible to say it in another way: in order to increase the annuity yield, the annual payment must be increased.

All these considerations are especially well seen with the example of the perpetual annuity since for this $A = R/j$, or, otherwise: the perpetual annuity yield is $j = R/A$. It is important to note that the yield of the payment stream defined this way does not depend on the interest rate, but on the value and moments of payments themselves, due to which it is often called the domestic return of the payment stream.

More precisely, the domestic return of the payment stream is such its yield in the recently defined sense at which the present value of this stream equals zero (such is a characteristic is not of every payment stream). The equality $A=0$ is possible only then when there are negative values in the payment stream.

Example 3.1. A promissory note was discounted at rate of i =10% 160 days before its maturity (the base annual number is 360 days). At the execution of the transaction a commission fee was retained from the note owner that equaled 0.5% of the principal amount of the note. Find the yield of the transaction

Solution. When calculating the yield of the promissory note, its face value does not often play a role. The absolute yield of the transaction without accounting the commission:

$$d=\frac{S}{P}-1=\frac{N}{N(1-i\cdot m)}-1,$$

where S,P — the final and initial cost of the promissory note; N – the face value of the promissory note; $m=160/360$. Let us insert the given data, we will obtain: d = 0.046, i.e. d = 4.6%.

With consideration of the commission, the absolute yield equals:

$$d=\frac{S}{P}-1=\frac{N}{N(1-i\cdot m-0.005)}-1 = 5.2\%$$

The transaction effectiveness (the relative yield), i.e. the yield in the interest per annum,

$(1.046)^{360/160}$ – 1 = 0.106, i.e. 10,6% —, with commission,

$(1.052)^{360/160}$ – 1 = 0.1208, i.e. 12,08% — without commission.

3.4. Instant Profit

Let at the moment t the capital is $K(t)$, and after some time Δt the capital equals $K(t+\Delta t)$, then the average yield d on the segment $[t,t+\Delta t]$ in interest per annum (in amounts) equals

$$K(t+\Delta t)/K(t)=(1+d)^{\Delta t},$$

at the small Δt the value $(1+d)^{\Delta t}$ with an accuracy to infinitely small of the second order equals $1+d\cdot\Delta t$. Directing Δt to zero, we obtain

$$d=\lim_{\Delta t\to 0}[K(t+\Delta t)-K(t)]/[K(t)\cdot\Delta t]=K'(t)/K(t)=[\ln K(t)]'.$$

Thus, instant profit is a derivative with respect to time of the capital's base logarithm or, as they say, logarithmic derivative.

In particular, at the constant instant profit *d,* the capital grows in time on the exponent: $K(t)=K(0)\cdot e^{d\cdot t}$.

Example 3.2. The capital grows in time with a constant speed v, i.e. $K(t)=K_0(1+vt)$. Find the instant profit at an arbitrary time point.

Solution. Let us indicate the unknown instant profit $d(t)$, then $d(t)=K'(t)/K(t)=K_0v/K_0(1+vt)=v/(1+vt)$.

Thus, the yield changes with time. It is also clear – the capital gain per time unit is constant and equals K_0v, and the capital itself grows.

Questions for Self-test

1. What is a financial transaction? What types of financial transaction yields are there and how are they determined?
2. How are the inflation rate and price index determined? How do taxes and inflation influence the yield?
3. How is the yield of the payment stream determined?

Chapter 4
Credit Calculations

A borrowing, credit, loan are the earliest financial transactions. *Creditum* means 'loan' in Latin; the Russian word *credit* has a stress on the second syllable (*credit* with the first syllable stressed is the right member of accounting entries).

All three words — *borrowing*, *credit*, *loan* — mean one and the same – giving money or goods as a loan on conditions of recurrence and, as a rule, with interest payment. The one who sells money or goods on credit is called creditor, the one who takes – loan debtor (or debtor). The conditions for credit (borrowing, loan) issue and repayment are quite versatile. Here, only the simplest and most widespread ways of repaying borrowings are considered.

4.1. Total Yield Index of a Financial and Credit Transaction

The income from financial and credit transactions and different commercial transactions may have a different form, in particular, it may be:

- interest on the loans issued,
- commission fees,
- return on bonds and other securities etc.

As a rule, one and the same transaction implies a few source of income: the loan brings the creditor interest and commission payments, besides the interest on the bond, the bond holder receives the difference between the buy-back price of the bond and its purchase price.

In connection with this, there arises a problem of measuring the effectiveness (yield) of the transaction with regard to all the income sources. The generalized characteristic of yield must be universal and applicable to all the financial transaction types.

The degree of financial effectiveness of such transactions is usually measured in the form of the annual compound interest rate. The given rate as an index of effectiveness (total yield, i.e. yield with regard to all the income sources stipulated in the transaction) is received on account of the general principle, in particular: all the income types discounted at the unknown rate (capitalized value of income), stipulated by the given transaction, are equated to the expenses reduced at the same rate and at the same time point. The unknown rate is determined from the equation obtained. For the loan transaction it means the equivalence of the debit amount of the credit, i.e. the credit with the deduction of commission fees, to the sum of discounted inflows. The higher the rate, the higher the effectiveness of the transaction.

This interest rate does not appear in contracts directly and depending on the transaction type bears different titles: in loan operations – the term *effective interest rate*, in analysis of transaction yield – *yield at the repayment moment*, in the analysis of industrial investments an analogous index is called the *internal rate of return* (intrinsic interest norm).

The base for calculating the total yield of the financial transaction is the relation called *the balance of a financial and credit transaction*.

4.2. The Balance of a Financial and Credit Transaction

The necessary condition for any financial and credit transaction is an equilibrium of investments and return. Let us consider the notion of financial and balance transaction balance.

Let a credit of a K_0 amount be issued for a term T, the interest on the credit equals i. Let two payments whose amounts are R_1 and R_2 be made for all this term on account of discharging the debt, and at the term end the final amount R_3 is paid. All the term is divided into three periods with the duration t_1, t_2, t_3. For the time period t_1, the debt increases to the amount D_1, (since interest is charged on the credit amount). With termination of this time, the sum R_1 is deposited on account of repaying the credit and the amount of debt becomes equal with K_1. Then, for the time t_2, the sum K_1 will increase to the value D_2. With termination of the time interval t_2 the subsequent sum R_2 is paid, the amount of debt decreases and becomes equal with the sum K_2. Finally, for the time t_3 the amount of debt increases to the value D_3 and with the termination of this time interval, at the time point *T,* the sum R_3 is paid. For a balanced transaction, the amount of payment R_3 must be so that the debt be discharged. Figure 4.1 illustrates the described process as a graph that is called *financial and credit transaction contour*.

A balanced transaction must have a closed contour.

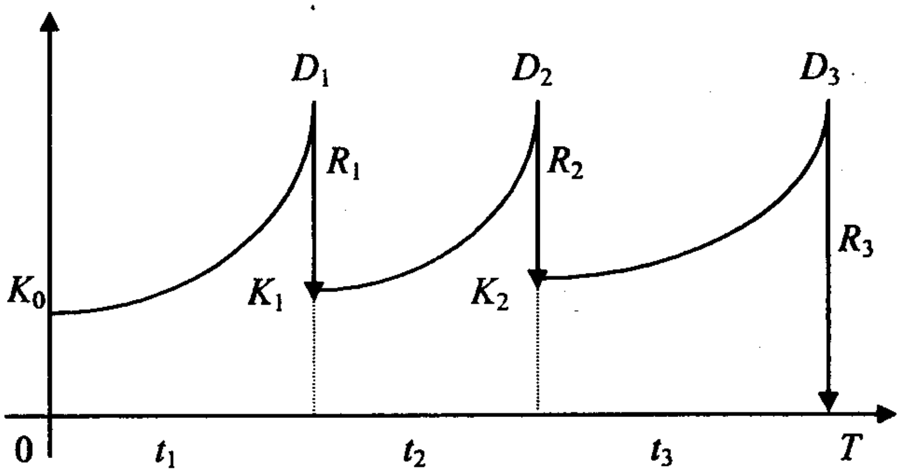


Figure. 4.1. Financial and credit transaction contour

Mathematically, the process of repaying the debt may be described with an equation:

$$K_1 = K_0 q^{t_1} - R_1,\ K_2 = K_1 q^{t_2} - R_2,$$

$$K_2 q^{t_3} - R_3 = 0, \qquad (4.1)$$

where $q = (1+i)$ – the accumulation factor at the compound rate.

The latter equation is called accounting equation and describes the condition of transaction equilibrium.

Let us define K_2 through K_0 and insert the result into the accounting equation (4.1). We will obtain

$$K_2 = (K_0 q^{t_1} - R_1) q^{t_2} - R_2,$$

$$[(K_0 q^{t_1} - R_1) q^{t_2} - R_2] q^{t_3} - R_3 = 0,$$

$$K_0 q^T - (R_1 q^{t_2+t_3} + R_2 q^{t_3} + R_3) = 0, \qquad (4.2)$$

Where $T = \sum_i t_i$.

From the latter equation it is obvious that the financial and credit transaction may be conventionally divided into two counter processes:

1) accumulation of the initial debt for all the time period;

2) accumulation of discharging payments for the period from the payment moment to the end of the transaction.

This equation may be transformed through multiplication by a discount factor

$$v^T = \left(\frac{1}{1+i}\right)^T .$$

As a result, we will obtain

$$K_0 - (R_1 v^{t_1} + R_2 v^{t_1+t_2} + R_3 v^T) = 0 . \tag{4.3}$$

Thus, the sum of the present values of discharging payments at the moment of credit issue with the total equilibrium equals the credit amount.

For a general case n of discharging payments, the accounting equation will be analogous and as follows

$$K_0 q^T - \sum_{j=1}^{n} R_j q^{T_j} = 0 , \tag{4.4}$$

where $T_j = \sum_{r=j+1}^{n} t_r$ – time from the payment moment R_j to the term end.

On the basis of accounting equations, it is possible to measure the financial and credit transaction yield. An accounting equation must be formed for this where accumulation or discounting is made at an unknown rate that characterizes the total yield; then this equation must be solved with regard to the rate sought for.

4.3. Determining the Total Yield of Loan Operations With Commission

The yield index of a loan operation without considering commission fees is the annual compound interest rate that is equivalent to the interest rate used in this operation. The creditor often takes a commission fee for opening a credit and other services and this obviously increases the yield of the transaction for them because the sum actually issued decreases.

Let the loan of the sum D be issued for the term n. At its issue, a commission of the G amount is retained. This means, the actually issued loan equals $(D-G)$. The deal stipulates charging simple interest at the rate i. The total yield rate will be defined by i_e. When determining the yield of this transaction as the annual compound interest rate, we proceed from the premise that the accumulation of the value $(D-G)$ at this rate must give the same result as the accumulation of the value D at the simple interest rate i.

The equilibrium equation for this transaction is (see Figure 4.2)

$$(D-G)(1+i_e)^n = D(1+n \cdot i)\,. \qquad (4.5)$$

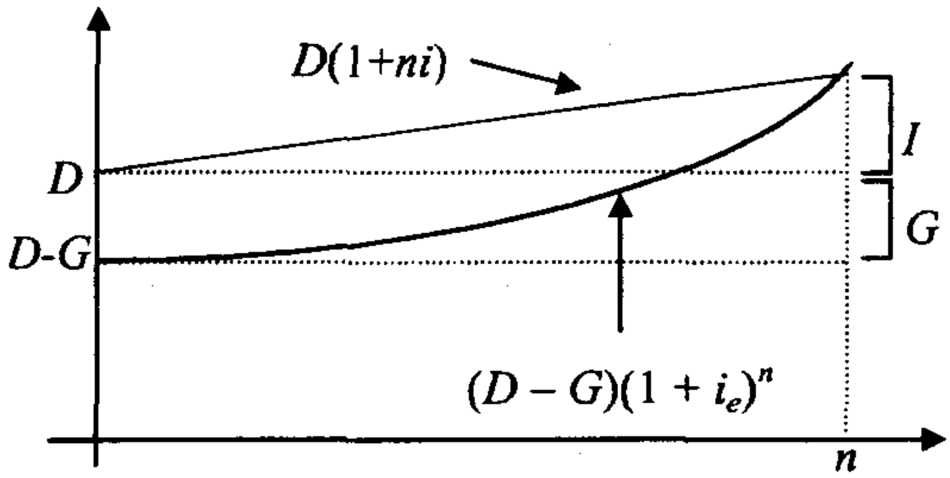


Рис 4.2. A graphic illustration of accounting equation

The higher the amount of the commission, the more effective the rate (total yield rate). Figure 3.2 graphically illustrates this equation.

Let $G = D \cdot g$, where g – commission interest. Then, when solving Equation (4.1) with regard i_e we will obtain

$$i_e = \left(\frac{1+n \cdot i}{1-g} \right)^{1/n} - 1. \qquad (4.6)$$

If the deal stipulates compound interest calculation, then the accounting equation will be

$$(D-G)(1+i_e)^n = D(1+i)^n\,, \qquad (4.7)$$

from which an expression for the effective rate follows:

$$i_e = \frac{1+i}{(1-g)^{1/n}} - 1\,. \qquad (4.8)$$

The effective rate i_e does not appear in the conditions for the transaction directly, it is fully determined by the interest rate on the credit and the value of commission.

Loans with Regular Interest Payments

If the commission is not paid, then the yield of such a transaction equals the compound interest rate that is equivalent to any rate applicable in this deal.

Let us suppose that commission is stipulated. Let the loan D be discharged after n years, and the interest on the simple rate i be paid regularly once at the end of the year. Then the amount of the interest paid equals $D \cdot i$. On account of the commission, the loan amount equals $D(1-g)$. Let us write the accounting equation:

$$D(1-g)-(D \cdot i \cdot a(n,i_e)+Dv^n)=0, \tag{4.10}$$

where $v=\frac{1}{1+i_e}$; $a(n,i_e)=\frac{1-(1+i_e)^{-n}}{i_e}$ – the annual annuity reduction factor.

This equation equals the following:

$$f(i_e)=v^n+i \cdot a(n,i_e)-(1-g)=0. \tag{4.11}$$

We have a nonlinear equation that must be solved with regard to the variable i_e – the total yield rate. The accurate solution for this equation in the form of a calculation formula for i_e is impossible to obtain. The rate sought for may be determined relatively with the use of some numerical method.

If the interest on the credit is paid p times a year, then the equation to determine the transaction yield will be

$$f(i_e)=v^n+i \cdot a(p,n,i_e)-(1-g)=0, \tag{4.11a}$$

where $a(p,n,i_e)=\frac{1-(1+i_e)^{-n}}{p((1+i_e)^{1/p}-1)}$ –is the p – due annuity reduction factor.

Loans with Recurrent Expenses

Let interest be paid on the loan regularly and the principal balance of the debt be discharged, moreover, the amount of expenses be constant. We suppose that the payments will be made once at the end of the year. Let R be the annual amount on the debt service.
If the debt equals D, then we will obtain from the condition for transaction equilibrium

$$D=Ra(n,i),$$

where i – compound interest rate on the credit. Hence

$$R=\frac{D}{a(n,i)}. \tag{4.12}$$

If the commission is not paid, then the effectiveness of such a transaction as a compound annual interest rate coincides with the rate on the credit i. If the commission is paid, then the loan debtor does in fact receive the sum equivalent to $D(1-g)$. Then the accounting equation will be

$$D(1-g)-Ra(n,i_e)=0, \tag{4.13}$$

or

$$f(i_e)=a(n,i_e)-a(n,i)(1-g)=0.$$

The rate i_e that is sought for may be determined with the use of numerical methods.

If the maturing payments are made p times a year, then in this equation, the annual annuity reduction factor $a(n,i)$ must be replaced with the reduction factor of the p - due annuity $a(p,n,i)$.

Repaying a loan with an irregular payment stream

So far we have supposed that the debt is discharged with equal payments. Let us consider the case when the debt is discharged by repaying an irregular payment stream: $R_1, R_2, ..., R_n$.

In this case, the accounting equation is written as:

$$f(i_e)=D(1-g)-\sum_{j=1}^{n} R_j v^j=0, \tag{4.14}$$

where $v=\dfrac{1}{1+i_e}$ - the discount multiplier at the total yield rate of the transaction. The amount of the final payment R_n here depends on the amount of the preceding payments and must be determined from the condition for equilibrium of the financial transaction (the final payment must discharge the debt), i.e.

$$R_n=Dq^n-\sum_{j=1}^{n-1} R_j q^{T_j}=0, \tag{4.15}$$

where T_j is the term of paying the j payment till the end of the deal, $q=1+i$ – accumulation factor with the use of the rate on the credit i.

4.4. Method of Comparing and Analyzing Commercial Contracts

There are often cases in the commercial practice when one and the same type of goods may be bought from different suppliers each of whom offers their conditions for sales (price, credit conditions, payment of the deal etc.). Since both the moments of paying sums of money and their amounts do not, as a rule, coincide for different variants, these conditions can often mismatch directly. For a reasonable choice of the most profitable variant, the consumer must have a certain analytical procedure that allows to determine the comparable indices of contract costs that account all the conditions stipulated in the contracts.

We will consider the classical approach that is based on comparing the present values of all the payments stipulated by the contracts. Payments are reduced to one time point, as a rule, to the beginning of the agreement validity. The variant with the lowest present value from the financial viewpoint is considered preferable to the consumer along with the acceptance of all other conditions.

When calculating the present values for contract comparison, the central moment is choosing the level of the interest rate on which discounting is made. This rate is called the *effective percentage rate*. The higher the rate, the less impact the outer payments have on the present value of expenditures. Therefore, the increase in the effective percentage rate makes contracts with longer terms for debt repayment preferable for investors. Indices calculated at the accepted effective percentage rate are conventional, however, the contract rating adjusted in this way turns out to be stable.

Let us first consider a problem with competitive conditions of debt repayment. Let th price of the goods be constant in all the variants. In the general case, the rates on the credit may be different. The longer the life of the credit, the higher the rate, since it is necessary to compensate the decrease in the cost of more outer payments. Each variant of the contract states the following basic conditions for debt repayment:

1) advance payments, their amount and moments of repayment;

2) duration and conditions for interest payments in the grace period, if it is stipulated (in the grace period, the principal balance of the debt is not repaid, it is only the interest on the principal balance of the debt that is paid);

3) term for debt repayment;

4) method of debt repayment.

The problem involves writing out the equation for calculating the present values of stream payments stipulated by the competing contracts. Since in the real practice, a set of different variants may appear, it is not possible to write a single formula applicable for all the cases. That is why, let us consider several most characteristic variants.

Example 4.1. A shipbuilding company offers to pay 8 million dollars for its order. Two payment options are offered:

1) 5% at the conclusion of the contract, 5% at the ship launch after 6 months and further on within 5 years equal expenses for the debt service, i.e. the remainder of the debt is to be discharged within 5 with equal sums.

This option does not stipulate any grace period.

2) 5% at the conclusion of the contract, 10% at the ship launch after 6 month; a grace period is stipulated that starts right after the repayment of the second advance payment and equals six months. Payment of interest calculated with the compound rate is stipulated at the end of the grace period. After the end of the grace period the debt repayment occurs within 8 years with equal payments.

The interest on the credit in both variants is i = 10%.

Let us accept the effective percentage rate q = 15%. It is the rate with which discounting of all payments stipulated by the contract occurs.

Let us write an equation for the present value of payments on the first contract. Two advance payments are stipulated in it: Q_1 and Q_2, the latter to be paid at the time point t = 0.5 years from the beginning of the contract validity. Consequently, it must be discounted through multiplying it by the factor v^t, where $v = 1/(1+q)$ - is the discount multiplier at the rate q. The process of discharging the balance of the debt $D_1 = P - (Q_1 + Q_2)$ starts from the moment of paying the second advance payment. Taking into account that payments are received once at the end of the year, the stream of payments discharging the debt may be considered as a constant annual annuity postponed for t years with parameters R_1, n_1, whose present value is defined with the use of the rate equal to the effective percentage rate q. Here n_1 is the time period during which the principal balance of the debt for the first variant is discharged (n_1=5). It is necessary to define the reduced value of this annuity for the initial time point. The present value of the annuity at the moment of paying the second advance payment equals $R_1 a(n_1, q)$, and at the initial time point it is $R_1 a(n_1, q) v^t$. The amount of a single discharging payment is calculated by the formula $R_1 = D_1 / a(n_1, i)$. Note that for calculating R_1, the rate on the credit i is used.

Thus, mathematically the process of discharging the debt (the present value of expenses) for the first variant is described by the equation:

$$A_1 = Q_1 + Q_2 v^t + R_1 a(n_1, q) v^t. \tag{4.16}$$

Let us consider the second variant. A grace period is stipulated by this variant. Consequently, in the equation for the present value of payments it is necessary to account the interest paid at the end of this period. Let L be the duration of the grace period; $D_2 = P-(Q_3+Q_4)$, here Q_3, Q_4 are advance payments. Then the sum of the interest in the grace period equals $D_2[(1+i)^L-1]$. In order to reduce this payment to the zero moment, it must be multiplied by the discount multiplier v^{t+L}. Discharge of the principal balance of the debt starts after the end of the grace period. The stream of payments discharging the debt is an annuity postponed for $t+L$ years. The present value of the discounted at the initial moment discharging payments equals $R_2 a(n_2, q)v^{t+L}$, where $R_2 = D_2 / a(n_2, i)$. The equation to determine the present value of payments in the second variant is formed as in the first and has the following form

$$A_2 = Q_3 + Q_4 v^t + D[(1+i)^L - 1]v^{t+L} + R_2 a(n_2, q)v^{t+L}, \tag{4.17}$$

here n_2 is the time period during which the principal balance of the debt is discharged for the second variant (n_2=8).

Making calculations, we will obtain: A_1 = $ 6 710 000, A_2 = $ 6 382 000.

Therefore, in accordance with the method described above, the second variant has turned more profitable because $A_2 < A_1$.

The sums obtained are conventional and are used only in order to determine the contract rating. It is possible to show that if the present value of payments on one of the contracts compared is higher than that on the other, then a similar relation is preserved for other values of the effective percentage rate, in the case when they exceed the highest of the rates of the contracts under comparison or when the effective percentage rates are smaller than the smallest of these rates. In other words, if for any two contracts $A_2 < A_1$, besides, in the first contract, the rate on the credit is i_1, and in the second – i_2, $i_1 > i_2$ at a certain effective percentage rate q, then the same relation between A_2 and A_1 will be preserved for all the other values of q, such that $q > i_1$ or for all the values $q < i_2$.

Let us consider one more typical variant of the contract. Let the contract conditions are as follows: an advance payment in the sum Q is paid once at the beginning of the deal, while a single supply of the goods after the time period t from the moment of contract conclusion is stipulated, a grace period of the duration of L is stipulated that starts from the moment of supplying the goods; the discharge of the debt occurs by equal payments, what is more, the process of discharging the debt starts from the moment of the grace period's termination, the interest in the grace period is paid regularly once at the end of the year, the total contract term is $n+t+L$ years.

Considering all these conditions, let us write the equation that determines the present value of payments on the contract:

$$A = Q + (P-Q)\left(\frac{a(n,q)}{a(n,i)}v^{t+L} + i \cdot a(L,q)v^{t}\right). \tag{4.18}$$

In this equation, the sum of annual interest payments in the grace period equals $(P-Q)i$. Since the interest is paid regularly from the beginning of supplying the goods, the stream of these payments may be considered as an annual annuity postponed for t years with the parameters R, i, L, q. The size of the discharging payments equals $R = (P-Q)/a(n,i)$. The stream of these payments is the annual annuity postponed for $t+L$ years.

4.5. Planning Long-Term Debt Repayment

Expenses on the Debt Service

The notion of regular expenses connected with the borrowing is called *debt service*. A single sum for the debt service is called *a due payment***.** Due payments include:

- the current interest payments;
- the means intended for repayment (amortization) of the principal balance of the debt (the principal).

The methods of determining the amount of due payments depends on the conditions of the borrowing. These conditions stipulate the term, the duration of the grace period (if it exists), the level of the interest rate, the method of repayment and payment of the interest and the principal balance of the debt.

Usually the interest is paid at the duration of the whole borrowing regularly. Sometimes it is charged and added to the principal balance of the debt. The main amount is usually repaid in parts, besides, the amounts of discharging payments (due payment) may be equal or change.

The principal balance of the debt is not discharged in the grace period, the interest may be paid by regular payments or at the end of the period by one payment, or not paid at all but added to the principal balance of the debt.

Let D be the amount of the debt, I - the interest on the borrowing, L - the duration of the grace period, R - annual expenses on discharging the principal balance of the debt, g - the interest rate on the conditions of the borrowing, Y - due payment.

In the period when the principal balance of the debt is discharged, the due payment contains two elements: $Y = R + I$. In the grace period it is $Y = I$.

The calculation methods substantially depend on the type of the borrowing. The main feature of borrowing classification is the method of loan repayment. The following types of borrowings are distinguished:

1. Borrowings without mandatory redemption (*perpetual bonds*) – the loan debtor is committed to pay the creditor a profit in a form of fixed interest in certain terms, the borrowed sum is not to be returned. The loan debtor keeps the right to buy out or discharge all the certificates of indebtedness at any time.

2. Borrowings with mandatory redemption at one term. The loan debtor gives the borrowed amount back at the specified time limits and pays interest regularly or at the term end.

3. Borrowings with mandatory redemption in several terms. The loan debtor gives the borrowed sum back in parts and regularly pays a profit on the borrowing in form for interest.

Forming a Fund

To secure the repayment of debt, a repayment fund is usually created. Sometimes it is stated in the agreement on the borrowing issue. The fund is formed from subsequent payments that are charged with interest (for example, a bank account). It clearly has a meaning if the loan debtor has an opportunity of receiving larger interest on the repayment fund money than that on which they have taken the loan. Different variants of forming the fund and repaying the debt are possible. We will consider three models of forming the repayment fund:

1. The principal balance of the debt is discharged from the fund at the term end by a lump sum payment. The sum of payments with interest in the fund must be identical with the debt at the moment of its payment. The interest on the debt is paid not from the fund.

2. The conditions for financial commitment instead of regular interest payment stipulate their addition to the principal balance of the debt.

3. The fund is formed in such a way that regular payment of interest on the debt (from the fund) and the repayment of the principal balance of the debt at the term end are secured.

Let us consider the first model. Let the accumulation of finances in the fund be made by regular annual payments whose amount equals R, and let interest at the rate i be charged on these payments. Payment of interest charged on the debt at the rate g occurs simultaneously (the interest is paid not from the fund). Then the due payment is $Y = Dg + R$, where Dg - is the interest on the debt, R - payments in the fun.

The fund must be accumulated for n years. Payments in the fund are an annuity with the parameters R, n, i. *Knowing* the amount of the debt, it is easy to determine a single payment in the fund:

$$R = \frac{D}{s(n,i)},$$

where $s(n,i)$ – annual annuity accumulation factor that equals $s(n,i) = \frac{(1+i)^n - 1}{i}$.

On account of this, the due payment is

$$Y = D(g + \frac{1}{s(n,i)}). \tag{4.19}$$

In the second model, the conditions for the financial commitment stipulate adding the interest to the principal balance of the debt, which is why the payments in the fund at the term end must secure the accumulation of the sum $D(1+g)^n$. In this case, the due payment equals

$$Y = \frac{D(1+g)^n}{s(n,i)}, \tag{4.20}$$

The rate i characterizes the speed of the repayment fund growth, and the rate g – the amount of interest paid for the borrowing.

Let us consider the third model of forming the fund. Let the credit be discharged the following way:

- at the beginning of each year, equal amounts R are deposited to the repayment fund;
- the interest on the credit starts to be discharged after the first calculation of interest on payments in the fund;
- the creditor receives payments on interest from the fund;

at the term end the principal balance of the debt will be paid plus the interest on the last year of the credit.

The peculiarity of this problem is that: a) the payment in the fund is made not at the end of the year, but as we have supposed before, at its beginning; b) the interest on the credit is paid regularly from the fund.

Let us consider the process of fund formation (see Figure 4.3). The first payment into the fund of the sum R is made at the moment of taking the credit. For the sum R interest for n years (n - the total credit term) at the compound rate i is calculated. At the term end this sum will equal $R(1+i)^n$. At the beginning of the second year, the sum R is deposited into the fund and

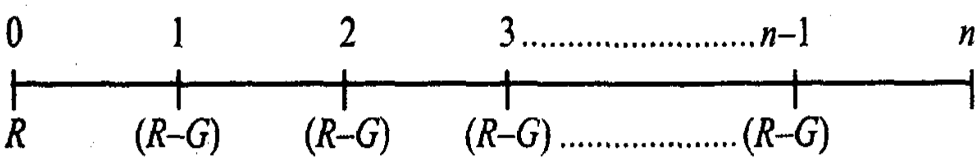


Figure 4.3. The process of forming the repayment fund on Model 3.

and interest on the debt G is paid from the fund. Therefore, in fact it is the sum $(R-G)$ that is deposited into the fund. It will be charged with the interest during $(n-1)$ years, by the end of the term we will obtain an amount $(R-G)(1+i)^{n-1}$. At the beginning of the third year we deposit the sum R into the fund and simultaneously we take the sum G from the fund. This deposit (after the deduction of interest on the debt) will be charged with the interest within $(n-2)$ years. As the result, by the end of the term we will obtain the sum $(R-G)(1+i)^{n-2}$. The deposit into the fund is made at the time point $(n-1)$ and with the calculated interest (with consideration of withdrawing the amount of interest on the debt) it will give the sum $(R-G)(1+i)$. At the term end it is necessary to secure payment of the sum whose amount equals $(D+G)$ (the principal balance of the debt plus the interest on the debt for the previous year).

Let us derive an accounting equation in which all the payments into the fund after the deduction of the interest on the debt, with consideration of their interest calculation in the fund will be equated to the debt amount plus the interest for the preceding year. We have

$$R(1+i)^n + (R-G)(1+i)^{n-1} + ... + (R-G)(1+i)^2 + (R-G)(1+i) = D+G.$$

It is necessary to determine the amount of deposits into the fund R from this equation

After quite simple transformations the accounting equation may be written as

$$R\sum_{k=1}^{n}(1+i)^k - G\sum_{k=0}^{n-1}(1+i)^k = D.$$

Note that $\sum_{k=0}^{n-1}(1+i)^k$ is the accumulation factor of the annual annuity that may be written as: $\frac{(1+i)^n - 1}{i} = s(n,i)$. Besides, we have

$$\sum_{k=1}^{n}(1+i)^k=(1+i)\sum_{k=0}^{n-1}(1+i)^k .$$

Considering these relations, the accounting equation will be written as follows:

$$R\frac{(1+i)((1+i)^n-1)}{i}-G\frac{(1+i)^n-1}{i}=D.$$

When solving it, let us determine the value of the deposit into the fund:

$$R=\frac{D}{(1+i)s(n,i)}+\frac{G}{1+i}. \tag{4.21}$$

As it is seen from the obtained relation, the deposit into the fund is formed from two constituents: the first item is the sum that secures repayment of the principal balance of the debt (note, there is the factor $(1+i)$ reflecting the occurrence of deposits into the fund at the beginning of the year); the second item is the sum that secures regular discharge of the interest on the debt by payments from the fund.

Repaying the Debt by Equal Instalments

One of the ways to repay the debt by instalments is to pay it by equal amounts.

Let the borrowing D be discharged within n years by equal payments. Then the sum spent on discharge of the debt annually equals d, where $d=D/n$. Apart from paying the principal balance of the debt, the debtor pays the interest on the *remainder* of the debt. Let us assume that the interest is paid once at the end of the year at the rate g. Then the first interest payment at the end of the first year equals Dg. At the end of the second year the interest amount equals $(D-d)g$. At the end of the third year it is $(D-2d)g$ etc.

Due payments are formed the following way:

The first payment equals $Y_1=d+Dg$, the second payment – $Y_2=d+(D-d)g$, the third payment – $Y_3=d+(D-2d)g$ etc.

The due payment at an arbitrary time point t equals:

$$Y_t=d+D_t g, \tag{4.22}$$

where D_t is the remainder of the debt at the beginning of the year t $(t=1,2,...,n)$ that is defined through the following recursion relation:

$$D_{t+1}=D_t-d=D-td,\ D_1=D. \tag{4.23}$$

Repaying the Debt by Instalments with Equal Due Payments

Along all the discharging term a constant due payment is paid regularly, its one part counts for the repayment of the principal balance of the debt, and the other part – for the repayment of the interest on the borrowing. The value of the principal balance of the debt decreases after each payment. However, in connection with the decrease in payments on the interest, the amounts going for discharging the principal balance of the debt increase in time. Such a method is called *progressive repayment*. The due payment

$$Y = D_t g + d_t = const\,. \tag{4.24}$$

The regularly paid amounts Y may be viewed as a constant annual annuity whose member is determined by the formula:

$$Y = \frac{D_1}{a(n,g)}\,. \tag{4.25}$$

Let us determine the structure of the due payment, i.e. its part that goes for discharging the principal balance of the debt and the part that goes for discharging the interest.

The amount of the first maturing payment is $d_1 = Y - D_1 g$.

From Formula (3.25) it follows that $D_1 = Ya(n,g)$, consequently,

$$d_1 = Y - D_1 g = Y(1 - g \cdot a(n,g)) = Yv^n\,, \tag{4.26}$$

where $v = \dfrac{1}{1+g}$.

Then

$d_2 = Y - D_2 g = Y - (D_1 - d_1)g$ (here it is taken into account that $D_2 = D_1 - d_1$) or

$$d_2 = Yv^n + Yv^n g = Yv^{n-1} = d_1(1+g)\,. \tag{4.27}$$

For the payment at the time point t we have $d_t = Yv^{n-(t-1)}$, or

$$d_t = d_1(1+g)^{t-1}. \tag{4.28}$$

Therefore, it is possible to write the following system of relations.

The amount of the maturing payment in the year t:

$$d_t = Y - D_t g = d_{t-1}(1+g),\ t = 1,2,...,n\,, \tag{4.29}$$

$$d_1 = Y - D_1 g = Yv^n = D_1 v^n / a(n,g) = D_1 / s(n,g)\,. \tag{4.30}$$

The remainder of the debt for the beginning of the year $t+1$:

$$D_{t+1} = D_t - d_t = D_t(1+g) - Y . \tag{4.31}$$

The amount of the repaid debt for the beginning of the year $t+1$:

$$W_t = \sum_{k=0}^{t-1} d_1(1+g)^k = d_1 s(t,g) , \tag{4.32}$$

where $s(t,g)$ - is the annual annuity accumulation factor whose term equals t years.

If the maturing payments and interest is paid p times a year, then the given problem is solved analogously, only with the use of the corresponding formulas for the p–due annuities.

Questions for Self-test

1. What characteristic is used to evaluate the yield of the financial and credit transaction?
2. What is the balance of the credit transaction? Write the balance equation.
3. How to determine the total yield of the financial and credit transaction without commission?
4. How to determine the total yield of a loan operation with commission?
5. How to determine the total yield of a loan operation with regular interest payments?
6. How to determine the total yield of a loan operation with recurrent expenses?
7. How to determine the total yield of a loan operation with irregular payment stream?
8. What is the essence of the method of comparing different contracts based on payment capitalization?
9. How are the loans classified in accordance with the method of their repayment?
10. Name the main methods of forming a repayment fund.
11. How is the repayment fund formed to discharge the principal balance of the debt from the fund at the term end by a lump sum payment , and the interest on debt is paid not from the fund?
12. How is the repayment fund formed to discharge the principal balance of the debt and the interest at the term end by a lump sum payment?
13. How is the repayment fund formed to discharge the principal balance of the debt at the term end by a lump sum payment and secure regular interest payments on the debt (from the fund)?
14. How is the debt discharged through repayment by equal instalments?
15. How is the debt discharged through repayment by instalments with equal due payments?

Chapter 5
Analysis of Real Investments
5.1. Introduction

The task for analyzing and evaluating the financial effectiveness of investment projects with the goal of choosing the most effective project is one of the basic ones in financial management.

In its broad sense the term *to invest* means any deposit of financial resources with the goal of gaining income in the future. Two types of investments are distinguished – real and financial.

Real investments imply investments into any type of material assets such as the land, equipment, factories.

Financial investments are contracts recorded on paper such as ordinary shares and bonds.

In developed economies, most investments belong to the financial ones.

The main object of mathematical modeling and analysis in this case is the payment stream, in particular, the sums of financial expenses and inflows distributed in time and supposed as the result of the investment project realization. Herewith, an important role is played by two factors connected with the investment process — time and risk. For example, with investments into fixed income securities (bonds), time will be a most important factor, with investments into wildcat securities (ordinary shares) both time and risk are essential.

From the financial viewpoint, an investment project unites two contrary processes:

depositing financial resources (with the purpose of creating an industrial facility, capital accumulation etc.);

subsequent obtainment of income.

These processes may be parallel, with or without an interval between them, parallel at some time segments (in this case, the return from the investment starts before the termination of deposits). Both these processes may have a different distribution in time.

The analysis of investments means basically evaluating and comparing the effectiveness of alternative investment projects. The evaluation of effectiveness is made through calculating an index system. The methods of evaluating the project effectiveness viewed in this chapter are based on a principle of payment stream discounting, which is fundamental in financial analysis, in particular, on drawing investment expenses and income to one time point (usually to the beginning of the project realization). Besides, a most important moment is the choice of the interest rate level on which discounting is made. In the analysis of real investments, this rate is called the effective percentage rate.

Let us consider this notion in detail through Example [2].

Let a company that produces household appliances consider acquiring a robotic center for painting the appliance surface with a total cost of $1 000 000. Taking the salary of the operator that maintains the center into account, as well as the salary of workers that must be replaced with it, and the expenses on the center maintenance, the company evaluates savings caused by the introduction of the center as $120 000 per annum.

Question: is it worthy to spend S1 000 000 today in order to annually save $120 000 but in the future?

The answer to this question depends on the opportunity cost of gaining $1 000 000 by this company. If the company should take a loan for this, then the opportunity cost will reflect the interest rate that is taken by the creditor.

If somebody's own financial resources are planned to be spent, then the opportunity cost is determined by the income that this enterprise could obtain, with the use of $1000000 for alternative (possibly, more profitable) purposes. For instance, if the company obtained bonds, then the opportunity cost of investing the capital into the new center would be identical with the market interest rate of the profit from these securities.

It is possible to say that irrespective of the character of resources, i.e. if they are borrowed or somebody's own, their opportunity cost corresponds to the rate of interest domineering on the market. In the market economy, there is no common rate of interest; there is rather a whole collection of interest rates (for instance, the rates on short and long term loans, the rates on bonds of various types etc.). This market rate should be viewed as a certain imputed value reflecting the characteristic rate of interest out of all its manifolds that exist on financial markets.

Let us suppose that the rate of interest equals 10% per annum. This implies that through any way the company may gain $ 100000 per annum having invested its resources of $1000000 that were saved for buying the center. Since it is less than $120000, the given investment project deserves attention. Besides, if the rate the interest equals 13% per annum, then the company may gain $ 130000 per annum and investing in the robotic center becomes unprofitable.

It is possible to draw the following conclusion from the given example: the effectiveness of investments, and consequently, the activity of investors depend on the rate of interest domineering on the market. The lower the rate of the interest, the higher the level of investment expenses that are profitable to the investor.

The choice of a particular rate must be made on the basis of an economic analysis by the investor. The higher the rate, the more the reflection of the time factor – the more distant payments have a smaller impact on the present value of the payment stream. The amounts of the present values of the earnings from investment are conventional characteristics since they substantially depend on the effective percentage rate accepted for the future – the market rate of interest.

As it was mentioned above, an important factor at investment analysis is the allowance for risk. In the investment process, the risk is revealed as a possible decrease of the real return from investments compared with the expected one. The evaluation of project effectiveness depends on the right choice of the interest rate at which payment capitalization occurs. As a recommendation on the allowance for risk, it is suggested that a correction to the level of the interest rate be introduced, i.e. a risk premium be added.

In this topic, methods for modeling and analyzing the investment processes connected with real investments are viewed. In the subsequent topics, models and methods for analyzing financial investments are described.

In the financial analysis of real investments four main indices are used:

1) net present value;
2) inner rate of return;
3) payback period;
4) profitability index.

5.2. Net Present Value

A contraction NPV is frequently used for this index in the financial literature.

NPV is used in the practice of large and medium enterprises. The conditions accepted when defining the NPV are:

- streams of financial resources for the whole period of project realization are known;
- the evaluation of interest market rate for the future is defined (for all the period of project realization).

The net present value (NPV) – is a difference of streams of income and investment discounted at the effective percentage rate for one time point (usually for the beginning of project realization).

This value characterizes the absolute result of investment activity. Let us sequentially consider the models of various most characteristic variants of investment processes on whose basis the NPV is calculated, starting from the simplest.

1. Let investments be made by one payment at the starting time point. The return from the investments comes once at the end of the year within n years. Let us mark K - the amount of investments, R - the amount of the annual earnings, W - the net present value. Thus, it is obvious that the stream of the earnings flow may be viewed as an annual annuity. Then, in accordance with the definition, the net present value is

$$W = R\frac{1-(1+q)^{-n}}{q} - K\,, \tag{5.1}$$

where q - the effective percentage rate, the first summand – the present value of the earnings flow.

The rule of making a decision based on the NPV is the following: if $W > 0$, then the project is accepted for consideration. If $W = 0$, then the earnings will only cover the investments and do not bring any profit. If $W < 0$ less than zero, then the project is unprofitable. When analyzing several alternative projects, with other equal conditions, a preference is given to the project with the highest NPV.

2. Let the investments and earnings be uniform discrete payment streams that are received once at the end of the year. The process of return starts right after the termination of investments.

Let us mark n_1 as the duration of investment period (in years), n_2 -the duration of the period of return from investments, K - the amount of annual investments, R - the amount of annual return.

The streams of earnings and expenses are discounted at the starting time point. The expenses make the annual annuity with the term n_1,. Consequently, the reduced value of expenses equals $Ka(n_1,q)$. The earnings is the annual annuity postponed for n_1 year whose all payments must be reduced by the zero time point. The present value of such an annuity is determined by the formula $Ra(n_2,q)v^{n_1}$, where v is the discount multiplier at the rate q. Therefore, the equation to determine the NPV will be in this case:

$$W = Ra(n_2,q)v^{n_1} - Ka(n_1,q)\,. \tag{5.2}$$

Note. If processes of investment and return are described through the p-due annuities (their parameters p may be different), then it is obvious that in the given equation it is necessary to use the reduction factor of the corresponding p-due annuities.

3. Let us suppose that the investment expenditures and earnings are divided into two nonuniform payment streams, where the process of return from investments starts right after the end of investing and all the payments are received at the end of the year. Let R_j – be the amounts of the earnings in the year j $\left(j=1,2,...,n_2\right)$, K_t - investment expenditures in the year t $(t=1,2,...,n_1)$. The amount of the discounted investments is determined by the formula for the present value of the variable payments stream (see Subsection .2.12) and equals $\sum_{t=1}^{n_1} K_t v^t$. Let us consider a separate payment R_j. Let us discount it for the moment of the investment project start. It is obvious that the discounted value of this payment equals $v^{n_1} R_j v^j$. The amount of the discounted payments R_j will give the present value of the payment stream $v^{n_1} \sum_{j=1}^{n_2} R_j v^j$. The equation for the NPV is

$$W = v^{n_1} \sum_{j=1}^{n_2} R_j v^j - \sum_{t=1}^{n_1} K_t v^t . \tag{5.3}$$

We have supposed that the process of return would start right after the termination of investments. If the return starts n years after the beginning of project implementation and $n > n_1$, then it is necessary to use the multiplier v^n instead of the multiplier v^{n_1} in the given formula.

4. Let us suppose that the processes of investing and return are given as a single nonuniform payment stream that are received once at the end of the year. This means that the processes of investing and receiving earnings may be both subsequent and parallel.

Let us mark R_t as the amount of a single payment. Then the net present value will be determined by the formula

$$W = \sum_t R_t v^t ,\ t = 1,2,...,n_1 + n_2 , \tag{5.4}$$

where $n_1 + n_2$ – is the total term of project implementation. In this formula, the payments R_t corresponding the investments are taken with the ‘minus’ sign. In practice, for more trustworthy conclusions, it is recommended to determine the value of the net present value for different values of the effective percentage rate q. Figure 5.1. illustrates the dependence of the net present value W on the effective percentage rate q for the case when investments are made by a single payment at the beginning of the term, and the inflows are a uniform payment stream. With $q = q_{in}$, the net present value equals zero $W = 0$.

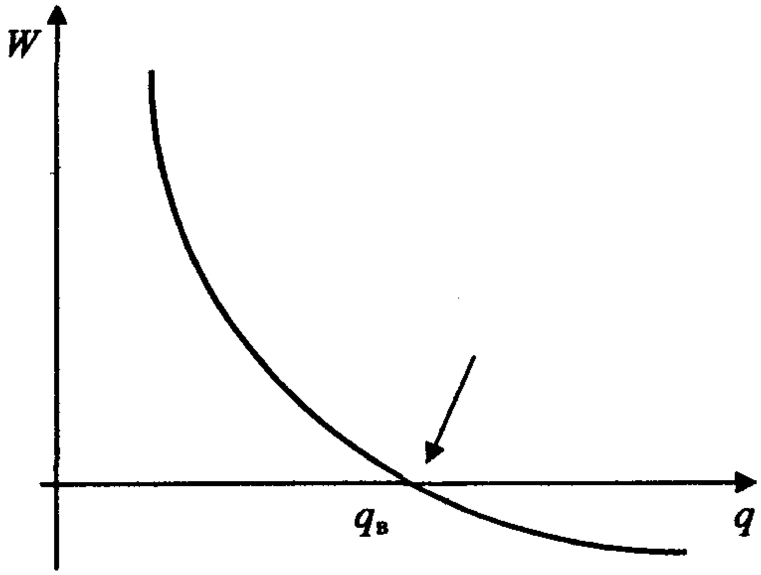


Figure 5.1. The dependence of the net present value on the effective percentage rate q.

5.3. Internal Rate of Return

This index is often marked by an abbreviation IRR in the financial literature.

IRR is understood as a computed interest rate with which capitalization of the income received gives a sum equal to the investments made and, consequently, the investments will pay off.

The economic sense of the given index consists in the fact that if the investments precede the earnings flow, it gives the marginal value of the rate of discounting at which the project still remains profitable.

In other words, if the project is funded only by the attracted funds, then the value of IRR indicates the upper boundary of the acceptable level of the bank interest rate, whose exceeding makes the project unprofitable. Let us mark the internal rate of return through q_{in}. If the credit was taken at the rate i, then the difference $(q_{in} - i)$ characterizes the effectiveness of the investment activity. If $(q_{in} - i)$=0, then the income will only pay off the investments, if $q_{in} < i$, then the investments are unprofitable. When comparing different projects, let us choose the one with this index being the highest. Let us consider the method of IRR.

For the first model of the investment process (see Expression (5.1)) the internal rate of return q_{in} is determined as the root of equation

$$R\frac{1-(1+q_{in})^{-n}}{q_{in}}-K=0, \tag{5.5}$$

where n – the term of project implementation.

For the second and the third models of an investment process, the index q_{in} is determined from the equations

$$R\frac{1-(1+q_{in})^{-n_2}}{q_{in}}\cdot\frac{1}{(1+q_{in})^{n_1}}-K\frac{1-(1+q_{in})^{-n_1}}{q_{in}}=0, \tag{5.6}$$

$$\frac{1}{(1+q_{in})^{n_1}}\sum_{j=1}^{n_2}R_j\frac{1}{(1+q_{in})^{j_1}}-\sum_{t=1}^{n_1}K_t\frac{1}{(1+q_{in})^t}=0. \tag{5.7}$$

If investments and their return are given as a single payment stream, then q_{in} is determined as the smallest positive root of equation:

$$\sum_{t=1}^{n}R_t\frac{1}{(1+q_{in})^t}=0. \tag{5.8}$$

If investments are preceded by the process of return, Equation (5.8) has the only positive solution and the calculation of the IRR index is reasonable. If investments alternate with the return, then there is no single-valued solution. In this case, it is possible to use another method for calculations based on continuous interest.

It is possible to solve Equations (5.6) – (5.8) only numerically.

5.4. Payback Period

The payback period is viewed as a period duration within which the amount of income discounted at the moment of investment termination becomes identical with the amount of investments normalized to the same time point. Discounting is made at the effective percentage rate. We should underline that when determining the payback period, in contrast to other indices, all the payments are made for the moment of investment termination (or, what is the same, for the starting point of return period).

Let us consider the ways of determining this index, starting from the simplest investment model corresponding Variant One of Subsection 5.2. In this case, the payback period may be determined from the equation

$$R\frac{1-(1+q)^{-n_{pb}}}{q}=K\,. \tag{5.9}$$

Solving this equation with respect to the variable n_{pb}, we will obtain

$$n_{pb}=-\frac{\ln(1-\frac{K}{R}q)}{\ln(1+q)}\,. \tag{5.10}$$

It is visible from this formula that not all the income level leads to return on investments. The payback period will be the final value, if $R>qK$ (formally it follows from the properties of the logarithmic function).

Let us consider how the payback period is determined n_{pb} for the second and the third models of the investment process.

Let K_0 - be the value of investment normalized to the start of the period of return, i.e. $K_0=\sum_{t=1}^{n_1}K_t(1+q)^t$ - the accumulated sum of all payments that make up the investments. Let the earnings be an arbitrary stream of inflows. Then the payback period n_{pb} is determined through summation of earnings discounted at the rate q until we obtain a sum identical with the amount of investment. The algorithm of definition n_{pb} is the following. We subsequently determine the values $S_m=\sum_{t=1}^{m}R_t\frac{1}{(1+q)^t}$ for $m=1,2,...,n_2$ (here n_2 - is the duration of the period of return on investments). As soon as at a certain value m the inequality $S_m<K_0<S_{m+1}$ is made, then we suppose that $n_{pb}=m+\Delta$, where m - is the a whole number of years, Δ - a fraction of the year that is approximately evaluated by the formula

$$\Delta\approx\frac{K_0-S_m}{R_{m+1}v^{m+1}}, \tag{5.11}$$

where $v^{m+1}=\frac{1}{(1+q)^{m+1}}$.

The main disadvantage of the index n_{pb} as a measure of investment effectiveness is that it does not count all the period of project implementation and is not influenced by the return that is beyond this period. For this reason, this index is recommended for use only as a limitation at decision making. The investor determines for themselves a certain critical amount of the payback period n_{cr} that still satisfies them, and if the payback period of the project under analysis is $n_{pb} > n_{cr}$, then the project is not accepted trivially. Only after this the comparison of projects is made with other indices.

5.5. Profitability Index

The profitability index shows how many currency units of the present cost of the future income money flows is accounted for one currency unit of normalized investments.

Let us suppose that the investment expenses and earnings are variable payment streams that are received once at the end of the year. Then the profitability index is determined by the formula

$$U = \frac{v^{n_1} \sum_{j=1}^{n_2} R_j v^j}{\sum_{t=1}^{n_1} K_t v^t}.$$

Here the numerator is the present value of the earnings stream for the starting point of the investment project, the denominator illustrates the investment expenses discounted for the same time point.

If $U > 1$, then the project is accepted for consideration; If $U = 1$, then the project does not bring profit, but only pays off; with $U < 1$ the project is unprofitable. Out of several projects, the one with the highest profitability index is chosen. It is obvious that the use of this index is correct if investments and earnings are subsequent.

The disadvantage of all the indices. All the considered indices presuppose that the parameters of the future expenses and earnings, their amount and time of payments or inflows are known at the moment of their calculation. In the real situation, these values may be determined and predicted only approximately. Besides, the result of the effectiveness index calculation, except for the internal rate of return, depends on the effective percentage rate substantially. The rate is evaluated subjectively by a financial analyst and, consequently, all the calculations with this rate are also conventional. In connection with this, in order to decrease the uncertainty at taking decisions based on the calculation of the indices viewed, it is recommended to use the scenario approach. It is as follows. First, the evaluations of effectiveness indices are obtained for a certain base scenario. In this scenario, the most probable conditions for the given investment project's functioning are used. Then, analogous evaluations are made for the worst (pessimistic) and the best (optimistic) variants.

5.6. Model of Human Capital Investment

On the basis of the investment analysis approach considered above, we will try to answer the following question in this subsection: whether education is worthy to be paid for from somebody's own pocket, and how much should be paid.

Let us consider the model of human capital investment. We will build the model with respect to the following suppositions.

1. The labor of the educated and professionally trained human is more productive than that of the untrained one. Consequently, investments in education make a so-called human capital that must pay off in the future and bring profit.

2. People as consumers are interested in maximizing the profit of life in total, but not of a separate period.

3. There is a direct interdependence between the education level of a worker and their potential earnings.

4. People make a decision on investing in their education on the basis of comparing the expenses and profit connected with that.

The profit from education consists in the expected higher income in the future.

The expenses have two forms:

а) obvious expenses for the education course;

б) hidden expenses, in particular, the earnings missed during education.

The earnings and expenses belong to different time periods, which is why every human taking decision on education should compare the present value of the expected profit with the present value of the expected expenses.

Let us consider an individual deciding if they should take an extra education within one more year. Let us mark C - education expenditures within the extra year (tuition fee plus the lost profit). This value must be compared with the expected profit of higher earnings provided by the labor market. Let B_t - be the expected extra annual return in the year t, g - the market rate of interest, N - the duration of the upcoming labor life of the given individual. Then the normalized value of profit is determined as

$$P = \sum_{t=1}^{N} \frac{B_t}{(1+g)^t}.$$

The difference $(P-C)$ is the net present value from the education.

If $(P-C) > 0$, then from the financial viewpoint, there is sense in studying one more year. It is obvious that the smaller C and g, and the higher the income and the larger N, the more profitable it is to invest money in education.

Let the salary of the individual be 48000 per annum. If they study one extra year, with a tuition fee of 20000, then their salary will increase. Let us determine the value by which their salary must increase in order for it to be profitable to invest in education. Let the market rate value be g = 15%. The education expenditures are

$$C = 48000 + 20000 = 68000.$$

After the education, the salary will increase by B currency units per annum (we suppose it to be a constant value for the remaining life period). Let us determine the value of B. With the formula for the present value of the annual annuity, we obtain the equation

$$C = B\left[\frac{1-(1+0.15)^{-N}}{0.15}\right],$$

from which it is easy to determine the value of B. Let $N = 40$, then $B = 10200$. This means that the education will be financially profitable if the future profit of the individual grows no less than by 10200. If $N = 5$. then the calculations will give the following result: $B = 20300$.

Questions for Self-test

1. What are real and financial investments?
2. What most important factors are connected with investment process?
3. What is investment analysis?
4. Name four main factors applied in financial analysis of real investments.
5. What is the net present value? Write the model of net present value for the case when investments are made once, and the return is received annually at the end of the year.
6. Write the model of net present value for the case when investments and inflows are uniform discrete payment streams that come in once at the end of the year. The process of return starts right after the completion of investments.

7. Write the model of the net present value for the case when investment expenses and returns are divided into two nonuniform payment streams, where the process of investment return starts right after the completion of investments and all the payments come in at the end of the year.
8. Write the model of the net present value for the case when investment and return processes are given as a single nonuniform stream of payments that come in once at the end of the year.
9. What is the internal rate of return and how is it determined?
10. What is the payback period and how is it determined?
11. What is the profitability index and how is it determined?
12. Name the disadvantages of the basic indices applied in financial analysis of real investments.

Chapter 6
Quantitative Financial Analysis of Fixed Income Securities

6.1. Introduction

This section considers methods for quantitative financial analysis of fixed income securities. The securities bringing a fixed income in form of the interest belong to this type of financial commitments - bonds, various certificates, preferred stock and other securities on which a preconditioned earning is paid. The investments in securities belong to the financial investments. The main type of fixed income securities is bonds.

The bond is a security that testifies that its holder has provided a loan to the emitter of the security. The bond, as a rule, secures its possessor with a regular receipt of a fixed income in form of interest on the face value and at the term end – of some buy-back price, usually identical with the face value.

The main parameters of the bond:

1. The nominal price or buy-back price, if it is different from the face value.
2. The date of repayment.
3. The rate of return or coupon rate. This is the interest rate on which an earning is regularly paid to the possessor of the bond.
4. The terms of interest payments.

The presence or absence of a ban on a prior redemption of the bonds is of a certain importance. The presence of the emitter's prior redemption right lowers the quality of the bond because the degree of uncertainty increases for the investor.

Depending on the methods of paying the earnings and discharging the borrowings, the bonds may be classified as.

1. The bonds on which only interest payments are made, and the capital is not paid back. The emitter only indicates the possibility of their redemption without binding themselves with a certain term.
2. The bonds on which interest is not paid. These are the so-called bonds with a zero coupon.
3. The bonds on which interest is calculated for the holders and paid together with the face value at the moment of repayment.
4. The bonds allowing their possessors to obtain a regularly paid income in form of interest and the future repayment price at the discharge. This type of bonds is the most widespread.

Bonds is a very important object of financial investments; from the moment of their emission and until their repayment they are sold and bought with the prices formed on the market.

The market price at the moment of emission may be lower, higher or identical with the face value. Since the face values of different bonds may be substantially different, a comparable index of the bond market price is required. Such an index is the bond rate. This rate is a purchase price of a single bond with reliance on 100 currency units of the face value.

Let P - be the market price, N - the face value, K - the rate. Then, by definition

$$K = \frac{P}{N}100. \tag{6.1}$$

The main problems of bond analysis:

1) determining the total yield of the bond;

2) determining the intrinsic value of the bond and disclosing securities incorrectly evaluated by the market;

3) evaluating the risk connected with investing in bonds.

This topic considers methods of solving the problems listed above.

6.2. Determining the Total Yield of Bonds

The total yield of bonds is formed by three elements:

1) the regularly paid coupon yield;

2) the change in the market price of the bond for a certain period of time. If the bond was bought with a price lower than the face value or, as the phrase goes, with a discount, then this element of the yield is a positive value. If the bond was bought with a price higher than the face value, or as the phrase goes, with at a premium, then this element of the yield will be a negative value. If the purchase price equals the face value, then this element of the yield is missing;

3) the yield on reinvesting the inflows from coupons.

There are a few methods and indices for measuring the bond yield. For instance, the bond yield with a regular interest payment may be measured in form of the coupon yield, but in this case, the second element of the yield will not be accounted for. This section considers the methods for determining the total yield that take the first two elements of the yield into account (it is obvious that the use of the coupon yield depends on the individual preferences of the investor and in the general case is not possible to be taken into account). The total yield index measures the real financial effectiveness of the bond for the investors and is usually determined in form of the annual compound interest rate. Different titles are used for this rate in the financial literature - *yield to maturity*, *promised yield to maturity*, as well as the term *the rate of investment*. The term *the promised yield to maturity* reflects the conceptual meaning of this index. On the securities market this rate does not play part directly. This is a calculation value that may be determined only out of the market price of the bond.

The method for calculating the total bond yield is based on determining the present worth of the payment stream acquired by the bond holder. Along with that, the discounting of payments is made at the rate of the yield to maturity.

The Yield of the Bond without Interest Payment

Such a bond has one source of income for the investor – the difference between the buy-back price of the bond (face value) and the purchase price (market price).

Let P be the market price of the bond (purchase price), N - the face value, n - the term before its maturity. At the end of the term, the holder of such a bond will obtain a price identical with the face value. This value must be discounted and its present worth must be equated with the market price of the bond. As the result, we will obtain the equation

$$Nv^n = P, \tag{6.2}$$

where $v = \dfrac{1}{1+i}$, i – the rate of return sought for. Hence, we have

$$i = (N/P)^{1/n} - 1. \tag{6.3}$$

Using the concept of the rate, the expression (6.3) may be re-written as

$$i = \frac{1}{(K/100)^{1/n}} - 1. \tag{6.4}$$

If the rate $K < 100$ or, which is the same, the market price $P < N$, then the rate i – is a positive value and the given bond will bring some profit.

Determining the Yield of the Bond without Mandatory Redemption with Regular Interest Payments

The yield from this bond type is acquired only in form of regularly paid interest on the face value. Let the interest be paid once at the end of the year, g – be the coupon rate купонная ставка, then gN is the yield received annually. The payment of the interest payment steam may be viewed as a perpetual annuity. The present value of this annuity must be equated with the purchase price of the bond.

Let us obtain the formula for the present value of the perpetual annuit. Let us mark A_∞ as the present value of such an annuity. Then we have

$$A_\infty = \lim_{n\to\infty} A = \lim_{n\to\infty} R\frac{1-(1+i)^{-n}}{i} = \frac{R}{i}.$$

Let us use this formula for determining the total bond yield. It is obvious that the present value of payments on the bond equals $\frac{gN}{i}$. Let us equate this value with the purchase price. We will obtain the equation

$$\frac{gN}{i} = P = \frac{K}{100}N, \tag{6.5}$$

with whose solution we will determine

$$i = \frac{g}{K}100. \tag{6.6}$$

If the rate is $K < 100$, then the yield to maturity is $i > g$; if $K = 100$, then $i = g$; if $K > 100$, then $i < g$.

If the interest payments are made p times a year, then the stream of these payments may be viewed as a perpetual p-due annuity. By equating the present value of this annuity to the purchase price, we will obtain the equation with whose solution we will determine the total yield rate i:

$$i = \left(\frac{g}{p}\frac{100}{K} + 1\right)^p - 1. \tag{6.7}$$

The Yield of the Bond with Interest Payments at the End of the Term

For this bond type, the interest is charged and paid at the term end in form of a single sum together with the face value. This bond has two sources of income:

1) the interest on all the period of the borrowing;

2) the capital gain, i.e. the difference between the face value and the purchase price.

At the term end, the holder of such a bond will receive the sum $N(1+g)^n$.

Let us discount this value and equate the result with the purchase price. We will obtain the equation

$$N(1+g)^n \nu^n = P = \frac{KN}{100},$$

from which we will obtain the rate

$$i = \frac{1+g}{(K/100)^{1/n}} - 1. \qquad (6.8)$$

If the rate is $K < 100$, then $i > g$; if $K > 100$, then $i < g$.

Determining the Yield of the Bond with Regular Interest Payments to be Discharged at the End of the Term

The total income on this bond type is acquired by two elements:

1) the current income realized by the coupons;

2) the income received at the term end (that equals the face value or the purchase price, if it is different from the face value).

Let us suppose that the coupon payments are made once a year. Then the stream of these payments may be viewed as an annual annuity. By discounting all the payments and equating the result with the purchase price, we will obtain the equation

$$N(1+i)^{-n} + Nga(n,i) = P \qquad (6.9)$$

or, using the concept of the rate,

$$((1+i)^{-n} + ga(n,i))100 = K. \qquad (6.10)$$

Here $a(n,i) = \frac{1-(1+i)^{-n}}{i}$ – annual annuity reduction factor.

This equation must be solved in accordance with the rate i. It has the only positive solution that may be determined only numerically.

If the coupon payments are made p times a year, then the equation will be

$$((1+i)^{-n} + ga(p,n,i))100 = K. \qquad (6.11)$$

And in this case, the rate sought for will be determined numerically.

Bond Yield Adjusted for Taxation

Let us consider a bond with equal interest payments and the discharge at the term end. Let us introduce two tax rates: m - the tax rate on capital gain, h - the tax rate on the current income. The amount of the tax on capital gain equals $m(N-P)$. With account of this tax, the holder will receive not the face value, but the sum of $(N - m(N-P))$ at the term end. The amount of tax for the current income equals hNg. With account of this tax, the holder will regularly receive a sum that equals $(1-h)Ng$.

Let us discount the payment stream with account of taxes and equate the result to the purchase price of the bond. We will obtain the equation

$$(N - m(N - P))v^n + (1-h)Nga(n,i) = P\,. \tag{6.12}$$

If the notion of the rate is used, then the given equation may be transformed as:

$$\frac{100}{1 - mv^n}((1-m)v^n + (1-h)ga(n,i)) = K\,. \tag{6.13}$$

In this case, the rate of the yield to maturity may be defined also only numerically. If payments are received p times a year, then in this equation the reduction factor $a_{n,i}$ will be replaced with the reduction factor of the p-due annuity $a_{n,i}^{(p)}$.

6.3. Bond Portfolio Return

A sensible investor invests resources not into a single security type, but forms a bond portfolio that includes bonds different in types and terms. This section considers the simplest analysis of the portfolio. It consists in evaluating the total portfolio return. The portfolio return is changed in the form of an annual compound interest rate. This rate is determined by the solution of the equation in which the total cost of the bonds included in the portfolio is equated with the sum of the present values of all type payments on the bonds. Let C_t be the element of the payment stream at the time point t, x_j - the amount of j-type bonds included in the portfolio, P_j - purchase price for one j-type bond. The equation to determine the return is

$$\sum_t C_t v^t - \sum_j x_j P_j = 0\,. \tag{6.14}$$

Here $\sum_j x_j P_j$ – the market price of the portfolio, $\sum_t C_t v^t$ – the sum of the present values of all the payments on all the bonds that are included in the portfolio.

The rate is determined by the numerical method.

6.4. Bond Evaluation

As it was noted before, one of the main goals for the financial analysis of securities is the disclosure of securities incorrectly evaluated by the market.

Let us suppose that it is possible to disclose securities incorrectly evaluated by the market based on the open access information. For this, it is possible to have a certain analytical procedure on the basis of which the investor could disclose such bonds and with consideration of this take reasoned decisions regarding the sale or purchase of these bonds.

Two approaches to solve the given problem are possible:

1. The rate of the yield to maturity of the bonds that are analyzed by the investor is compared with the value of the rate that is "fair", according to the investor. The investor forms their opinion on the basis of the analysis of both bond characteristics and the current market situation. If the bond yield is higher than the fair one, then it is said that the bond is undervalued and is, in this case, a candidate for purchase. If the yield to maturity is lower than the fair one, then the bond is called overvalued, and it becomes a candidate for sale.
2. The investor evaluates the true or the intrinsic value of the bond and compares it with the market price. If the current market price is lower than the intrinsic value, then the bond is undervalued by the market, and contrariwise, if current market price is higher than the intrinsic value, then the bond is overvalued.

Both procedures of the analysis and evaluation of the bond are based on the method of income capitalization, i.e. on the reduction of all the payments on the bond to the present time point. Let us consider the first approach.

Yield to Maturity

The methods for determining the yield to maturity (more precisely the promised yield to maturity) for various bond types were considered in the previous subsection. Let us consider the general model for determining this index. Let P be the current market price of the bond with the residual circulation period of n years and the assumed monetary payments to the investor $C_1, C_2, ..., C_n$. Then the promised yield to maturity is the interest rate determined from the equation

$$P = \frac{C_1}{1+i} + \frac{C_2}{(1+i)^2} + ... + \frac{C_n}{(1+i)^n}$$

or

$$P = \sum_{t=1}^{n} \frac{C_t}{(1+i)^t}. \tag{6.15}$$

Let us compare the obtained rate i with a certain fair, according to the investor, rate i^*.

If $i > i^*$, then the given bond is undervalued.

If $i < i^*$, then the bond is overvalued on the market.

If $i = i^*$, then the bond is valued fairly by the market.

Example. The current bond price is P = \$ 900, the residual maturity equals 3 years, the nominal value is N = \$ 1000. Coupon payments are \$ 60 per annum, i.e. C_1 = \$ 60, C_2 = \$ 60, C_3 = \$ 1060. Let us insert these into the equation and obtain

$$\$900 = \frac{\$60}{(1+i)} + \frac{\$60}{(1+i)^2} + \frac{\$1060}{(1+i)^3},$$

Hence $i = 10.02\%$. If the subsequent analysis shows that the interest rate must equal i^* = 9%, then the bond is undervalued, since $i > i^*$.

Intrinsic Value

The method based on determining the intrinsic value implies that the intrinsic value of any asset, including the bond, is determined by the discounted values of the payments that the investor expects to receive in the future due to possessing this asset.

Let us determine the intrinsic value of the bond by the following way:

$$V = \sum_{t=1}^{n} \frac{C_t}{(1+i^*)^t}. \qquad (6.16)$$

The given model is called the basic bond valuation model.

The net present value (NPV) of the bond:

$$NPV = V - P = \sum_{t=1}^{n} \frac{C_t}{(1+i^*)^t} - P. \qquad (6.17)$$

If NPV > 0, then the bond is undervalued by the market; if NPV < 0, then it is overvalued. Any bond with $i > i^*$ will have an NPV > 0 and vice versa.

Example. From the previous example we have

$$NPV = \left[\frac{60}{(1+0.09)} + \frac{60}{(1+0.09)^2} + \frac{1060}{(1+0.09)^3} \right] - 900 = \$24.06.$$

Since this bond has a positive NPV, it is undervalued. This will be always when the yield to maturity of the bond is higher than the one that the investor views as correct.

To evaluate the bond by income capitalization, the values C_t, P, i^* must be calculated. The values of payments C_t are known, the value of the market price P is also known. The rate i^* depends on the investor's subjective evaluation of both the bond characteristics, and the current conditions on the market. Thus, the main constituent of bond analysis is the determination of fair, according to the investor, value of the rate i^*.

6.5. The Evaluation of the Intrinsic Value of Bonds

To evaluate the intrinsic value of various bond types it is convenient to use formulas obtained in accordance with the peculiarities of the reception of payments on this bond type. With a consideration of these peculiarities it is possible to obtain the special cases o the formula (6.16).

Bonds without mandatory redemption with regular interest payments

The payments on such a bond may be viewed as a perpetual annuity. Determining the intrinsic value is reduced to defining the present value of such an annuity. The corresponding formula has a form

$$V = \frac{gN}{i^*}.$$

If the return on the bond is paid p times a year, then

$$V = \frac{gN}{p[1+i^*)^{1/p} - 1]}.$$

Bonds without regular interest payments

The interest on such a bond is paid at the moment of borrowing discharge. The holder receives the sum $V = N(1+g)^n$. The formula to calculate the intrinsic value:

$$V = N\left(\frac{1+g}{1+i^*}\right).$$

Bonds with a zero coupon

The holder receives only the face value or buy-back price, the interest is not paid. The calculation formula is:

$$V = N\left(\frac{1}{1+i^*}\right)^n.$$

Bonds with a maturity at one term and regular income payment

Let the interest be paid regularly once at the end of the year, and the discharge be made on the face value. By discounting the payments we will obtain

$$V = Nv^n + gNa(n, i^*),$$

where $v = \dfrac{1}{1+i^*}$.

If the return is paid p times a year, and g - is the annual coupon rate, then

$$V = Nv^n + gNa(p, n, i^*).$$

6.6. Valuation of Risk Connected with Investments in Bonds

Methods for analyzing the bonds connected with determining the total yield and the intrinsic value of the given securities were considered above. However, only these characteristics are not sufficient for a reasonable decision making on the investments into one or another bond type. This is connected with the fact that investments into any securities are connected with a certain risk. There are two main risk types:

1) credit risk;

2) market risk.

Credit risk is connected with the possibility of emitter's refusal from their commitments, which may result in the total termination of paying the current payments and buy-back price, or violation of specified payment time limits.

Market risk is connected with variations in the market interest rate that substantially influences the change of the intrinsic value of the bond and, correspondingly, of the bond's market price.

Market and credit risk types are connected with the bond term – the longer the term, the higher the risk. However, only the bond term, i.e. the period of time from its acquisition to discharge does not take the peculiarity of different bond type return distribution in time – the so-called *return profile*. It is obvious that at other equal conditions, the risk of investing in bonds on which interest is paid regularly is lower than the risk of investing in bonds without interest payment.

Let us consider two indices that allow measuring the risk with account of the return profile – i.e. the average term and Duration.

Average Term

This index considers the payment term of all bond types in form of the weighted arithmetic mean value. The payment amounts are taken as a scale. Thus, the higher the amount of payment, the higher impact on the average term is made by the term of paying this payment.

Let us consider the bond whose coupon payments are received once at the end of the year. Let C_t – be the expected payment on the bond, t - the payment term, $t=1,...,n$. Let us include the payment on the face value into the payment stream. At these conditions, the average term is determined by the formula

$$T=\frac{\sum_{t=1}^{n} t\cdot C_t}{\sum_{t=1}^{n} C_t}. \tag{6.18}$$

The average term is $T<n$, if coupon payments are paid, i.e. if the coupon rate is $g>0$.

For the bond without coupon payments $g=0$ (we should remind that the holder of such a bond will receive only the face value at the term end) $T=n$. The higher the current return on the bond as regard to the face value, the smaller the average term T and, consequently, the lower the risk connected with the investment in this bond type.

This formula will be as follows for the bond with an annual interest payment and discharge at the term end:

$$T=\frac{Ng\sum_{t=1}^{n} t+nN}{nNg+N},$$

since in this case $C_t=gN$ with $t=1,2,...n-1$; $C_n=gN+N$. Let us note that

$\sum_{t=1}^{n} t=\frac{n(n+1)}{2}$ – is the sum of the members of the arithmetic sequence.

In consideration with this, it is possible to bring the formula for the average term to the form

$$T=n\frac{g(n+1)/2+1}{gn+1}.$$

It is possible to acquire an analogous formula for the case when coupon payments are received several times a year.

Questions for Self-test

1. Name the main parameters used for quantitative analysis of bonds.
2. How are bonds classified on the method of return payments and methods of loan repayment.
3. How is the bond rate determined?
4. How is the yield of bond without interest payment determined?
5. How is the yield of bond without maturity but with regular interest payments determined?
6. How is the yield of bond with interest payment at the end of the term determined?
7. How is the yield of bond with regular interest payments that will be discharged at the term end determined?
8. How is the bond portfolio return determined?
9. What is the method of income capitalization at bond evaluation?
10. How to evaluate the yield to bond maturity?
11. Write the model of the bond intrinsic value.
12. Write the model of bond evaluation with regular interest payments without maturity.
13. Write the model of bond evaluation with interest payment at the moment of maturity.
14. Write the model of bond evaluation with zero coupon.
15. Write the model of bond evaluation with regular interest payments with maturity at one term.
16. How to evaluate the average bond term with regular coupon payments?
17. How to evaluate the average bond term without regular coupon payments?
18. How to evaluate the average bond term with regular coupon payments and discharge of the face value at the term end?

Chapter 7.

Bond Duration

7.1. The Notion of Duration

In its general case, the duration is determine by the formula

$$D = \frac{\sum_{i=1}^{n} t_i C_i v^{t_i}}{\sum_{i=1}^{n} C_i v^{t_i}}, \qquad (7.1)$$

where v - is the discount multiplier at the rate of the yield to maturity r, i.e. $v = \frac{1}{1+r}$; $C_1, C_2, ..., C_n$ – payments on the bond after the time moments $t_1, t_2, ..., t_n$. The maturity is $T = t_n$. Apart from the average bond term, when calculating the index of *duration* the scale becomes not the payments, but their discounted values. Consequently, when determining the duration, the time factor is considered. Thus, duration is an average continuation of payments.

Since the rate of the yield to maturity is taken as a discount rate, then

$$P = \sum_{i=1}^{n} C_i v^{t_i} = \sum_{i=1}^{n} \frac{C_i}{(1+r)^{t_i}}, \qquad (7.2)$$

where P – is the current market price of the bond.
With account of this relation, we will have

$$D = \sum_{i=1}^{n} \left(\frac{C_i v^{t_i}}{P(r)} t_i \right) = \sum_{i=1}^{n} \left(\frac{C_i(0)}{P(r)} t_i \right). \qquad (7.3)$$

Here $C_{t_i}(0)$ – is the current present worth of the payment that was received at the time point t_i. Thus, the weight coefficients in Formula (7.3) are the relations of the present worth of each payment to the market price $P(r)$, i.e. the weight coefficients $\frac{C_{t_i}(0)}{P(r)}$ reflect the part of the bond's market price that will be received after t_i years, $i = 1, 2, ..., n$. The sum of coefficients in Formula (7.3) equals the unit:

$$\sum_{i=1}^{n} \frac{C_i(0)}{P(r)} = \frac{1}{P(r)} \sum_{i=1}^{n} C_i(0) = \frac{1}{P(r)} \sum_{i=1}^{n} \frac{C_i}{(1+r)^{t_i}} = 1. \qquad (7.4)$$

The considered index *duration* has the following high spot. As it was noted in the previous section, the cost of the bonds with identical maturities, but different coupon payments changes in a different way at the same change of the interest rate. However, the bonds with the identical duration react on the change of the rate in a similar way. In connection with this, when forming the bond portfolio the investors aim at including bonds with an identical duration into the portfolio. This method is called ***immunization*** of the portfolio and enables limitation of the impact of future market interest price variations on the expected earnings.

7.2. Connection of Duration with Bond Price Change

Let us consider the connection of duration with the relative bond price change $\Delta P(r)/P(r)$ with a change in the rate of return r.

The market price of the bond is determined by Formula (7.2).

Let us suppose that the rate of return has changed by Δr Then the cost of the bond will be

$$P(r+\Delta r)=\sum_{i=1}^{n}\frac{C_i}{(1+r+\Delta r)^{t_i}}\,. \qquad (7.5)$$

$\Delta r>0$ means an increase in interest rates, $\Delta r<0$ – an decrease. The bond price increment $\Delta P(r)=P(r+\Delta r)-P(r)$ is a positive value with $\Delta r<0$ and means the growth of the bond price at a decrease in interest rates on the market. The negative value $\Delta P(r)=P(r+\Delta r)-P(r)$ means a drop of the bond price with an increase in interest rates by the value $\Delta r>0$. The sign of the relative bond price increment $\dfrac{\Delta P(r)}{P(r)}$ is of the same meaning. The relative bond price increment with a change in interest rates for the value Δr equals

$$\frac{\Delta P(r)}{P(r)}=\frac{P(r+\Delta r)-P(r)}{P(r)}\,, \qquad (7.6)$$

where $P(r)$ and $P(r+\Delta r)$ are calculated by Formula (7.2) and (7.5). Let us consider how the value $\dfrac{\Delta P(r)}{P(r)}$ may be evaluated without the use of accurate calculations by Formula (7.6).

Considering Δr fairly small on the absolute value, we will obtain by the Taylor's formula

$$\Delta P(r)=P(r+\Delta r)-P(r)\approx P'(r)\cdot\Delta r$$

or with consideration of the expansion terms of the second order

$$\Delta P(r)=P(r+\Delta r)-P(r)\approx P'(r)\cdot\Delta r+\frac{1}{2}P''(r)\cdot(\Delta r)^2.$$

The members of a higher order are considered insignificant when determining the sensitivity of the bond price to the change in interest rates on the market. For the relative bond price increments we have

$$\frac{\Delta P(r)}{P(r)}\approx\frac{P'(r)}{P(r)}\Delta r \qquad (7.7)$$

or

$$\frac{\Delta P(r)}{P(r)}\approx\frac{P'(r)}{P(r)}\Delta r+\frac{1}{2}\frac{P''(r)}{P(r)}(\Delta r)^2\,. \qquad (7.8)$$

From (7.2) we will obtain $P'(r)=-\dfrac{1}{1+r}\sum_{i=1}^{n}t_i\dfrac{C_i}{(1+r)^{t_i}}=-\dfrac{1}{1+r}\sum_{i=1}^{n}t_iC_i(0)$ and

$$P''(r) = \frac{1}{(1+r)^2}\sum_{i=1}^{n} t_i(t_i+1)\frac{C_i}{(1+r)^t} = \frac{1}{(1+r)^2}\sum_{i=1}^{n} t_i(t_i+1)C_i(0),$$

Let us insert the number

$$C = \sum_{i=1}^{n} t_i(t_i+1)\frac{C_i(0)}{P(r)}, \tag{7.9}$$

that is called the *factor of bond convexity*.
Then with account of Formula (7.3) and (7.9) we will obtain

$$\frac{P'(r)}{P(r)} = -\frac{1}{1+r}\sum_{i=1}^{n} t_i \frac{C_i(0)}{P(r)} = -\frac{D}{1+r},$$

$$\frac{P''(r)}{P(r)} = \frac{1}{(1+r)^2}\sum_{i=1}^{n} t_i(t_i+1)\frac{C_i(0)}{P(r)} = \frac{C}{(1+r)^2}.$$

Let us substitute these expressions in Formula (7.7) and (7.8), and obtain the following expressions for the relative bond price change:

$$\frac{\Delta P(r)}{P(r)} \approx -D\frac{\Delta r}{1+r} \tag{7.10}$$

or

$$\frac{\Delta P(r)}{P(r)} \approx -D\frac{\Delta r}{1+r} + \frac{1}{2}C\left(\frac{\Delta r}{1+r}\right)^2. \tag{7.11}$$

Let us analyze these expressions. Since the sensitivity of the bond price to the change in the interest rates is characterized by the value $\frac{\Delta P(r)}{P(r)}$, then from (7.10) it follows that the bond duration evaluates the sensitivity of the bond price to the change in the time structure of the interest rates. Consequently, the bond duration may be considered as a measure of the bond's interest risk – the longer the duration, the higher the bond's interest risk.

Let $\frac{\Delta P(r)}{P(r)} \approx -D\frac{\Delta r}{1+r} + \frac{1}{2}C\left(\frac{\Delta r}{1+r}\right)^2$ and the factor of convexity C is such that the second summand must not be neglected in comparison with the first one. Consequently, the higher the factor of convexity, the worse does the bond duration evaluate the amount $\frac{\Delta P(r)}{P(r)}$. And contrariwise – the lower C, the more correct is the approximate equality (7.10). Consequently, the lower C, the better does the bond duration evaluate the sensitivity of the bond price to the change in the rate of return. Thus, the factor of bond convexity may be interpreted as an indicator of how accurately the bond duration evaluates the amount $\frac{\Delta P(r)}{P(r)}$.

Thus, at the moment $t = 0$ the bond duration is a measure of its interest risk.

Example 7.1. A 6% coupon bond with a face value of 1000 currency units is given, on which coupon payments are promised to be made every 6 months within 3 years. The risk-free interest rate (rate of return) is 8% per annum. Determine:

1. The duration and factor of bond convexity;

2. The relative bond price change $\frac{\Delta P(r)}{P(r)}$ with a change of interest rates by the value Δr = 0.01; 0.02; – 0.01 by the formulas: (7.6) – the exact value, (7.10) – an approximate value with account of the bond duration only, (7.11) - – an approximate value with account of the bond duration and factor of bond convexity.

Here the values of bond parameters are the following: N = 1000 currency units, g = 0.06, m = 2, T = 3 years, r = 0.08.

The results of duration calculation and factor of bond convexity are given in Table 7.1.

Table 7.1.

Payment number	Payment term t_i	Payment amount C_i	$C_i(0)$	$\frac{C_i(0)}{P(r)}$	$t_i \frac{C_i(0)}{P(r)}$	$t_i(t_i+1)\frac{C_i(0)}{P(r)}$
1	0.5	30	28.867513	0.030339	0.015170	0.022754
2	1	30	27.777778	0.029194	0.029194	0.058388
3	1.5	30	26.729179	0.028092	0.042138	0.105345
4	2	30	25.720165	0.027031	0.054063	0.162189
5	2.5	30	24.749240	0.026011	0.065028	0.227596
6	3	1030	817.647208	0.859332	2.577997	10.311990
		Sum	951.491083	1.000000	2.783589	10.888262

Thus, the bond price is $P(0.08)$ = 951.491 currency units, its duration is D = 2.784 years, the factor of convexity C = 10.888 years^2.

2. Calculations in accordance with the price change by Formula (7.6), (7.10), (7.11) for the three values of Δr are given in the table:

Δr		0.01	0.02	-0.01
$\frac{\Delta P(r)}{P(r)}$	Formula (7.7)	-0.025314	-0.049736	0.026248
	Formula (7.11)	-0.025774	-0.051548	0.025774
	Formula (7.12)	-0.025307	-0.049681	0.026241

The negative values $\frac{\Delta P(r)}{P(r)}$ correspond to the decrease in the price with an increase of the interest rates, the positive – to its growth at the decrease of the interest rates. From the calculations it is obvious that the smaller the value Δr on the absolute value, the closer the values obtained by Formula (7.10) and (7.11). It means that the error in evaluating the bond price change is smaller only with an assistance of the bond duration.

7.3. Properties of the Duration and Factor of Bond Convexity

1. Bond duration does not surpass the term before its maturity T.

Indeed,

$$D = \sum_{i=1}^{n} t_i \frac{C_i(0)}{P(r)} < \sum_{i=1}^{n} t_n \frac{C_i(0)}{P(r)} = t_n \sum_{i=1}^{n} \frac{C_i(0)}{P(r)} = t_n = T,$$

where $P(r)$ – is the market value of the bond at the moment t = 0, r – its domestic return.

2. Bond duration without interest payments (pure discount bond) equals the term before its maturity, i.e. $D = T$. Indeed, since this bond has only one payment $C_T = N$, then its present value equals $C_T(0) = \frac{N}{(1+r)^T}$, where N – is the face value of the bond. The market value of the bond equals $P(r) = \frac{N}{(1+r)^T}$.

   Then the bond duration equals

$$D = T \frac{\frac{N}{(1+r)^T}}{P(r)} = T.$$

3. If the bond is a coupon bond, then the higher the domestic return of the bond, the smaller are its duration and factor of convexity. Thus, the duration $D(r)$ and factor of convexity $C(r)$ of the coupon bond are a decreasing function of the domestic return r.

4. If payments on the bond are postponed for t_0 years without any change of the domestic return r, then the bond duration will increase by t_0 years, and the factor of convexity – by $(t_0^2 + 2t_0 D + t_0)$ years.

Proof. The duration of the given bond is $D=\sum_{i=1}^{n} t_i f_i$, where $f_i=\frac{1}{P(r)}\frac{C_i}{(1+r)^{t_i}}=\frac{C_i(0)}{P(r)}$. The bond duration with deferred payments:

$$D_{t_0}=\sum_{i=1}^{n}(t_i+t_0)\,f_i=\sum_{i=1}^{n}t_i\,f_i+t_0\sum_{i=1}^{n}f_i=D+t_0$$

Thus,

$$D_{t_0}=D+t_0 \tag{7.12}$$

The factor of convexity of the given bond is $C=\sum_{i=1}^{n}t_i(t_i+1)\,f_i$.

The factor of convexity of the bonds with deferred payments equals:

$$C=\sum_{i=1}^{n}(t_i+t_0)(t_i+t_0+1)\,f_i=\sum_{i=1}^{n}(t_i^2+2t_0t_i+t_i+t_0+t_0^0)\,f_i=$$

$$=\sum_{i=1}^{n}t_i(t_i+1)\,f_i+(t_0^2+t_0)\sum_{i=1}^{n}f_i+2t_0\sum_{i=1}^{n}t_i\,f_i=C+t_0^2+t_0+2t_0D$$

Thus,

$$C_{t_0}=C+(t_0^2+2t_0D+t_0) \tag{7.13}$$

The property is proven.

5. If there are more than one coupon periods left before the bond maturity, then with the given value of the domestic return *r* the bond duration and factor of convexity are the larger as the coupon rate is smaller, i.e. the duration $D(g)$ and factor of convexity $C(g)$ are decreasing functions of the coupon rate g.
6. Let us formulate the dependency of the bond duration on the term before its maturity with constant g and r, where g and r – are the coupon rate and the domestic return of the bond respectively, in the form of the following expositions. Let D_n be the bond duration whose payments are made p times a year and until the maturity of which there are n coupon periods. Then

   6a. If $g \ge r$, then the sequence $\{D_n\}$ is ascending.

   6b. If $g<r$, then such a number n_0 can be specified that for the bonds with the number of periods before maturity $n<n_0$ the sequence $\{D_n\}$ is ascending.

6c. $\lim_{n\to\infty} D_n \approx \frac{r+p}{rp}$.

7.4. Time Dependence of the Value of Investment in the Bond. Immunization Property Of Bond Duration

The problem of evaluating the bond exists not only when the bond is bought or sold on the market, but also when it is kept by the holder. In order to evaluate the cost of the bond after t years from the purchase, where $t \in [0,T]$, T years – is the term to bond maturity, the notion of investment value at the moment t is used.

Let us consider the bond on which monetary sums $C_1, C_2, ..., C_n$ are promised to be made after $t_1, t_2, ..., t_n = T$ years correspondingly from the current time point t = 0.

Definition. The value of investment in a bond at the time point $t \in [0,T]$ is the value of the payment stream $C_1, C_2, ..., C_n$ on the bond $P(t)$ at the time point t.

Let us indicate the value of investment in the bond t years after the purchase via $P(t)$. As it follows from the definition, $P(t)$ – is the sum of all the members of the payment stream on the bond normalized to the time point t. Let $t_1, t_2, \ldots t_m\ t_{m+1}\ , \ldots t_n$ be the moments of $C_1, C_2, ..., C_m, C_{m+1}, ..., C_n$ payment inflows correspondingly and $t_m \le t \le t_{m+1}$. Then

$$P(t) = \sum_{k=1}^{m} C_k (1+r)^{t-t_k} + \sum_{k=m+1}^{n} C_k \frac{1}{(1+r)^{t_k - t}}, \tag{7.14}$$

where $(1+r)^{t-t_k}$ - the accumulation factor of k payment at the time interval $[t_k, t]$, $k = 1, 2, ..., m$; $\frac{1}{(1+r)^{t_k - t}}$ - the discount factor of k payment at the interval $[t, t_k]$, $k = m+1, ..., n$.

Thus, the value of investing in the bond at the moment t has two constituents – the result of reinvesting the payments on the bond received before the moment t:

$$R_t = \sum_{k=1}^{m} C_k (1+r)^{t-t_k}$$

and the market price of the bond at the moment t:

$$P_t = \sum_{k=m+1}^{n} C_k \frac{1}{(1+r)^{t_k - t}}.$$

As it goes from these expressions, the value of investment at the moment t = 0 is the market price of bond purchase, i.e. $P(0) = P$.

Thus, the value of investing in the bond after t years from the purchase is received out of the following suppositions:

1) all the payments received from the bond to the moment t are reinvested;

2) at the moment t the bonds of this issue are on the market. The bond bought t years ago may be sold on the market with the market price existing at that moment P_t.

Then

$$P(t) = R_t + P_t \ . \qquad (7.15)$$

If every period has its own active risk-free rate r_i, then

$$R_t = \sum_{k=1}^{m} C_k (1 + r_k)^{t - t_k}, \ P_t = \sum_{k=m+1}^{n} \frac{C_k}{(1 + r_k)^{t_k - t}} . \qquad (7.16)$$

Example 7.2. A bond with the following payment stream at the moment of purchase (t = 0) is given:

Term, years	1	2	3	4	5	6
Payment, currency units	20	20	20	15	15	135

Determine the value of investment in this bond 3.5 years after purchase for risk-free interest rates given in the table:

Rate, %	17	16	15	15	15.5	16
Term of investment, years	2.5	1.5	0.5	0.5	1.5	2.5
Moment of investment	1	2	3	4	5	6

The result of reinvestment of payments on the bond that came before the t = 3,5 moment is

R_t = 20(1 + 0.17)$^{2.5}$ + 20(1 + 0.16)$^{1.5}$ + 20(1 + 0.15)$^{0.5}$ = 76.0486 (currency units)

The market value of the bond 3.5 years after its purchase will be

$P_t = \frac{15}{(1+0.15)^{0.5}} + \frac{15}{(1+0.155)^{1.5}} + \frac{135}{(1+0.16)^{2.5}}$ = 119.2231(currency units)

Thus, the value of investment in the bond 3.5 years after its purchase will be 76.0486 + 119.2231 = 195.2717 (currency units).

Now let us suppose that at the moment of bond purchase t = 0 the time structure of interest rates is such that the risk-free interest rates for all terms are identical and equal r. Let us consider the value of investing in the bond after t from the purchase for two cases:

1) the time structure of interest rates remains the same until the discharge of the bond;
2) right after the purchase of the bond, the risk-free interest rates for all the terms have immediately changed for one and the same value and become $\tilde{r}$, and have not changed henceforth.

The value of investment in the bond at the moment t in the first case is called *planned* and is defined with $P(r,t)$, in the second case – *actual* and is defined with $P(\bar{r},t)$.

7.5. Properties of the Planned and Actual Value of Investments

1. $P(r,t)$ and $P(\bar{r},t)$ – continuous increasing functions of time:

$$P(r,t) = P(r)(1 + r)^t , \qquad (7.17)$$

$$P(\bar{r},t) = P(\bar{r})(1 + \bar{r})^t . \qquad (7.18)$$

where

$$P(r) = \frac{C_1}{(1+r)^{t_1}} + ... + \frac{C_n}{(1+r)^{t_n}} \tag{7.19}$$

– the market price of bond purchase at the moment t = 0, corresponding to the time structure of the interest rates existing at that time point, $P(\bar{r},t)$ – the market price of the bond calculated at the new rate.

Indeed, according to (7.15),

$$P(r,t) = R_t + P_t = \sum_{k=1}^{m} C_k (1+r)^{t-t_k} + \sum_{k=m+1}^{n} \frac{C_k}{(1+r)^{t_k - t}} =$$

$$= (1+r)^t \sum_{k=1}^{m} C_k (1+r)^{-t_k} + (1+r)^t \sum_{k=m+1}^{n} \frac{C_k}{(1+r)^{t_k}} =$$

$$= (1+r)^t \sum_{k=1}^{n} \frac{C_k}{(1+r)^{t_k}} = P(r)(1+r)^t \text{ что and т.д.}$$

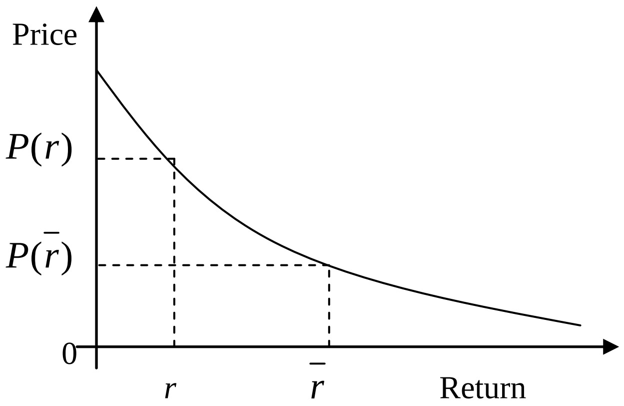
Equality (7.18) is proven analogously.

Formula (7.17) and (7.18) are the exponential functions of time, whose bases are more than one. From the elementary mathematics it is obvious that such a function is constant and increasing.

2. Besides, there is a unique time point t^*, when the actual value of investment equals the planned one.

Proof. Let $\bar{r} > r$. Consider the time point t = 0. Then $P(\bar{r}) < P(r)$ (see Formula (7.19) and Figure 7.1.), or

$$P(\bar{r},0) < P(r,0). \tag{7.20}$$



Figue 7.1. The dependence of the bond price on the return r.

Let us now consider the moment of bond discharge $t = t_n$. Then

$$P(r,t_n) = \sum_{i=1}^{n} C_i (1+r)^{t_n - t_i} ,$$

$$P(\bar{r},t_n) = \sum_{i=1}^{n} C_i (1+\bar{r})^{t_n - t_i} .$$

Since $\bar{r} > r$, then

$$P(\bar{r}, t_n) > P(r, t_n). \tag{7.21}$$

From Equality (7.20) and (7.21) it follows that there is such a time point t^* when $P(\bar{r}, t_n^*) = P(r, t_n^*)$. It is possible to show that the moment t^* is the only one (see Figure 7.2.). We will find the value t^* from the equality $P(\bar{r})(1+\bar{r})^{t^*} = P(r)(1+r)^{t^*}$.

Hence

$$t^* = \frac{\ln\left(\dfrac{P(r)}{P(\bar{r})}\right)}{\ln\left(\dfrac{1+\bar{r}}{1+r}\right)}. \tag{7.22}$$

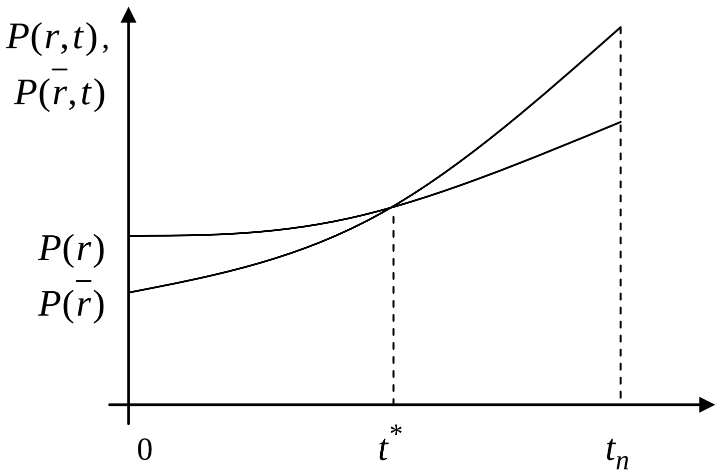


Figure 7.2. The dependence of the value of investment in the bond on the time.

The case with $\bar{r} < r$ is proven analogously.

3. Theorem (on immunization property of bond duration).

Let $D = D(r)$ be the bond duration at the time point $t = 0$, when risk-free interest rates for all the terms are common and equal r. Then at the time point $t = D$, that equals the bond duration, the actual value of investment in the bond is smaller than the planned one, i.e.

$$P(\bar{r}, D) \geq P(r, D) \tag{7.23}$$

for any value $\bar{r}$.

Proof. If after the bond purchase the time structure of interest rates has not changed, then $\bar{r} = r$ and $P(\bar{r}, D) = P(r, D)$.

If right after the purchase of the bond the risk-free interest rates have changed and become identical with $\bar{r}$, then at the time point $t = D$ the actual value of investment in the bond $P(\bar{r}, D)$ according to (7.18) equals:

$$P(\bar{r}, D) = P(\bar{r})(1+\bar{r})^D .$$

Let us differentiate this expression by $\bar{r}$:

$$\left(P(\bar{r},D)\right)'_{\bar{r}} = P'(\bar{r})(1+\bar{r})^{D} + D\cdot P(\bar{r})(1+\bar{r})^{D-1}.$$

Since $\dfrac{P'(\bar{r})}{P(\bar{r})} = -D(\bar{r})\dfrac{1}{1+\bar{r}}$ (see Subsection 7.2), then

$$\left(P(\bar{r},D)\right)'_{\bar{r}} = P(\bar{r})(1+\bar{r})^{D-1}\left(D - D(\bar{r})\right).$$

Let $\bar{r} > r$. Then by Property 3 of the bond duration $D(\bar{r}) < D(r) = D$. Hence $\left(P(\bar{r},D)\right)'_{\bar{r}} > 0$. It means, $P(\bar{r},D)$ is the increasing function $\bar{r}$ with $\bar{r} > r$.

If $\bar{r} < r$, then $D(\bar{r}) > D(r) = D$. Then $\left(P(\bar{r},D)\right)'_{\bar{r}} < 0$. It means, $P(\bar{r},D)$ is a decreasing function $\bar{r}$ with $\bar{r} < r$. Thus, at the point $\bar{r} = r$ the function $P(\bar{r},D)$ reaches its minimum. Consequently,

$$P(\bar{r},D) > P(r,D). \tag{7.24}$$

Thus, at any values $\bar{r}$ Inequality (7.23) is fulfilled. We should note that with $\bar{r} \neq r$ the inequality is strict, i.e. it has a form of (7.24). The theorem is proven.

It is possible to formulate the **immunization property** of bond **duration** on the basis of the proven theorem. Let at the investment time point t = 0 the risk-free interest rates for all the terms be identical. Then at the time point that equals the bond duration the investment in the bond is immunized (protected) against changes of risk-free interest rates right after t = 0 per one and the same value (or until the moment t_1 – the first payment on the bond, which is easy to be convinced of).

Example 7.3. A 10% coupon bond with the face value of номиналом 100 currency units is given on which coupon payments are promised to be made within three years. Risk-free interest rates for all the terms are common and equal 10% per annum. Right after purchasing the bond, the interest rates a) decreased to 9% per annum; b) increased to 11% per annum. Find:

1) the planned and actual values of investment in the bond at the time point equal to the bond duration;

2) time points when the planned and actual values of investment coincide.

The table displays the calculations of price $P(r)$ and bond duration $D = D(r)$ for the moment of bond purchase, and also the values $P(r_1)$ and $P(r_2)$, where r = 10% per annum, r_1 = 9%, r_2 = 11% per annum.

Payment number	Payment term t_i	Payment amount C_i	$C_i(0)$			$\dfrac{C_i(0)}{P(r)}$	$t_i\dfrac{C_i(0)}{P(r)}$
			r = 0.1	r_1 = 0.09	r_2 = 0.11		
1	1	10	9.090	9.174	9.009	0.0909	0.0901
2	2	10	8.264	8.416	8.116	0.0826	0.16529
3	3	110	82.644	84.940	80.431	0.826	2.47934
		Сумма	100.000	102.531	97.556	1.000	2.73554

Thus, the bond duration at the moment of its purchase is D = 2.735 years. The purchase price is $P(0.1)$ = 100.00 currency units. The values $P(0.09)$ = 102.531 currency units and $P(0.11)$ = 97.556 currency units – the bond evaluations for the time point t = 0 that correspond to the new time structure of the interest rates after t = 0. Then the planned value of investment in the bond at the time point $t = D$ equals:

$$P(0.1, D) = P(0.1)(1+0.1)^D = 129.787.$$

Actual values

$$P(0.09, D) = P(0.09)(1+0.09)^D = 129.789.$$

$$P(0.11, D) = P(0.11)(1+0.11)^D = 129.789.$$

In both cases the actual value of investment at the time point $t = D$ is higher than the planned one. In the first case at the time point $t = D$ the decrease in the reinvestment rate is compensated by the growth of the market price of the bond at the time point $t = 0$ in comparison with the planned one. In the second case the decrease in the market price at the time point $t = 0$ due to the growth of the interest rates is compensated by the increased reinvestment rate in comparison with the planned one.

2) The time points when the planned and the actual values of investment coincide equal correspondingly

$$t^*(0.09) = \frac{\ln\left(\dfrac{P(0,1)}{P(0,09)}\right)}{\ln\left(\dfrac{1,09}{1,1}\right)} = 2.737$$

$$t^*(0.11) = \frac{\ln\left(\dfrac{P(0,1)}{P(0,11)}\right)}{\ln\left(\dfrac{1,11}{1,1}\right)} = 2.733.$$

Questions for Self-test

1. Give the definition of bond duration and write the formula for its calculation.
2. How is duration connected with bond price change? Obtain the formula.
3. What is the factor of bond convexity and how does it influence the bond price change?
4. Give the first two properties of bond duration.
5. The property of coupon bond duration and factor of convexity.
6. The property of duration and factor of convexity of the bond with deferred payments.
7. The dependence of bond duration and the factor of convexity on the coupon rate.
8. The dependence of bond duration on the period to maturity.

9. Give the definition of the value of investment in a bond. How is it determined?
10. How are the planned and actual values of investment in a bond determined? The properties of the planned and actual values of investment in a bond.
11. The properties of the planned and actual values of investment in a bond.
12. Formulate and prove the theorem on the immunization property of bond duration.
13. What is the immunization property of bond duration?

Chapter 8

Securities Portfolio Optimization

8.1. Problem of Choosing the Investment Portfolio

This section considers the approach to forming a portfolio of risk-laden securities suggested by an American Nobel prize winner in economics H. Markowitz in 1952 and being the basis for the present theory of securities portfolio.

Let us consider financial transaction, consisting in the purchase of risk-laden securities at a known price and in their purchase in the future at a price unknown in advance. It is supposed that the investor at the present time point invests a certain sum of money in securities. This money will be invested for a certain time interval which is called the *hold period*. At the end of this period the investor sells the securities that were bought at the beginning of the period. Thus, at the time point $t=0$ the investor must take the decision on the purchase of securities that will be in their portfolio until the time point $t=1$. This problem is called the *problem of choosing the investment portfolio*.

Quantitative characteristics of the portfolio

The yield (effectiveness) of the risk-laden security depends on three factors: the purchase price that is known with certainty; intermediary payments for the hold period (dividends) that are not known with certainty; the sale price which is unknown. Thus, the financial transaction consisting in the purchase of the security with the purpose of receiving a certain profit in the future is risk-laden. The main hypothesis that enables analyzing such a transaction consists in the following: we suppose that each specific value of the yield of such a financial transaction is the realization of the random value

$$R=\frac{C_1-C_0+D}{C_0}, \tag{8.1}$$

where C_0 – the purchase price, C_1 – the sale price, D – the dividends paid for a hold period.

Forming a securities portfolio, the investor would like to maximize the expected portfolio return at a minimum risk. As a rule, these two goals contradict each other. Making a decision, the investor aims at making it so that these two goals be balanced.

The portfolio return is also a random value:

$$R_p = \frac{W_1 - W_0}{W_0}, \quad (8.2)$$

where W_0 – the aggregate price of the securities purchase that are included in the portfolio at the time point $t=0$, W_1 – the aggregate market price of securities at the time point $t=1$ and the the aggregate monetary return of these securities (dividends) that the holder will receive for the hold period from the moment $t=0$ till $t=1$.

Any random value may be characterized by two parameters: the expected or average value (mathematical expectation) and the standard deviation (mean square deviation).

In accordance with the model under consideration it is supposed that the investor bases their decision on choosing the portfolio only on these two parameters. Consequently, the investor must evaluate the expected return and standard deviation of each possible portfolio. Then they must choose the best of the portfolios on the basis of the relation of these two parameters. Along with that, the expected return is considered as a measure of the potential reward connected with the specific portfolio, and the standard deviation – the measure of the risk connected with this portfolio.

So, let us consider the financial transaction consisting in the securities purchase at a known price at the time point $t=0$ and their sale at a price unknown in advance at the time point $t=1$. Along with that, the investor may count on obtaining intermediary payments. Let us mark $m = M(R)$ – the expected value of securities effectiveness – the mathematical expectation of the random value R (it is an average on all the realizations (values) of the random value calculated with consideration of the frequency of their possible appearance), $V = M\{(R-m)^2)\}$ – the dispersion or variation of the random value – the deviation measure on the average the random value R from its expected value. It is often that instead of dispersion the mean square or standard deviation is used $\sigma = \sqrt{V}$.

The covariance $V_{12} = M\{(R_1 - m_1)(R_2 - m_2)\}$ characterizes the statistical interconnection of two random values R_1 and R_2.

The risk of investing in the specific securities is connected with the uncertainty of the future earnings and, consequently, with the uncertainty of the effectiveness of the given transaction. The higher the standard deviation, the more on the average the random value may deviate from its expected value, the bigger the uncertainty and higher the risk.

On the other hand, if $\sigma = 0$, then the effectiveness does not deviate from its expected value, it takes certain not random values, and the risk is missing. Thus, the standard deviation characterizes the risk level connected with the specific security, and is accepted as the *risk measure*.

Let us consider the activity of the investor that makes the decision on buying the securities. Let us illustrate each security type by a dot on the coordinate plane (m_j, σ_j) (see Figure 8.1)

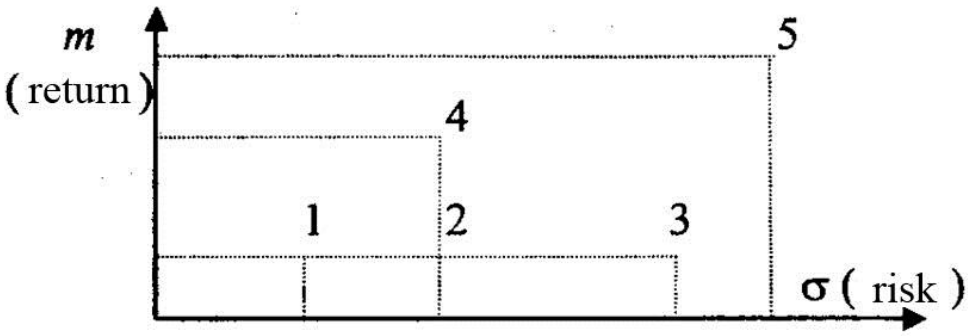


Figure 8.1. The securities with various earnings and risks

It is obvious that a sensible investor will prefer an investment represented by Dot 1, Investment 2 and 3; and prefer Investment 4 to Investment 2. Their choice of investments represented by Dot 1, 4, 5 depends on *this investor's risk proneness*.

Let us suppose that the investor invests money not in one security type, but into several. Let $x_j \, (j = 1, 2, ..., n)$ – be the share of the total investment accounted for j security type; n – the amount of security types that the investor includes in the portfolio. It is obvious that the equality must be fulfilled

$$\sum_{j=1}^{n} x_j = 1. \tag{8.3}$$

Let R_p be the effectiveness of the portfolio, R_j – the effectiveness of the j of the security. Then

$$R_p = \sum_{j=1}^{n} R_j x_j \, . \tag{8.4}$$

The expected effectiveness of the portfolio:

$$m_p = M\{R_p\} = \sum_{j=1}^{n} x_j M\{R_j\} = \sum_{j=1}^{n} x_j m_j \ , \qquad (8.5)$$

where $m_j = M\{R_j\}$ – the expected effectiveness of j-type security. The deviation from the expected effectiveness

$$R_p - m_p = \sum_{j=1}^{n} x_j (R_j - m_j) .$$

The dispersion of the portfolio effectiveness:

$$V_p = M\{(R_p - m_p)^2\} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j M\{(R_i - m_i)(R_j - m_j)\} = \sum_{i=1}^{n}\sum_{j=1}^{n} V_{ij} x_i x_j \ , \qquad (8.6)$$

where $V_{ij} = M\{(R_i - m_i)(R_j - m_j)\}$ – the covariance of the random effectiveness R_i and R_j of the i and j security types. It is easy to note that

$$V_{jj} = M\{(R_j - m_j)^2\} = \sigma_j^2 .$$

The value V_p (or $\sigma_p = \sqrt{V_p}$) characterizes the uncertainty of the portfolio as a whole and is called *portfolio risk*.

Portfolio Diversification

By investing money in various security types, the investor thus diversifies their portfolio. Let us consider the effect of such a diversification.

Let us suppose first, that the random effectiveness of various security types are mutually uncorrelated. This means that $V_{ij} = 0$ with $i \neq j$. Then the variation of the portfolio and standard deviation are identical:

$$V_p = \sum_{j=1}^{n} x_j^2 \sigma_j^2 \ , \ \sigma_p = \sqrt{\sum_{j=1}^{n} x_j^2 \sigma_j^2} \ . \qquad (8.7)$$

Let us suppose that the investor has invested money in equal shared into all the securities. Then

$$x_j = \frac{1}{n},\ m_p = \frac{1}{n}\sum_{j=1}^{n} m_j,\ \sigma_p = \frac{1}{n}\sqrt{\sum_{j=1}^{n}\sigma_j^2}\ .$$

Let us indicate $\sigma_m = \max_j \sigma_j$. Then we have

$$\sigma_p \le \frac{1}{n}\sqrt{\sum_{j=1}^{n}\sigma_m^2} = \frac{1}{n}\sqrt{n\sigma_m^2} = \frac{\sigma_m}{\sqrt{n}}\ . \tag{8.8}$$

Thus, with the growth of the number n of security types that are included in the portfolio, the portfolio risk is limited and tends toward zero ($\sigma_p \to 0$ with $n \to \infty$). In the financial risk theory this result is known as an effect of diversification of the portfolio. Consequently, in order to decrease the risk of the portfolio it is reasonable not to make investments in one security type, but to create a portfolio that contains probably a large variation of securities, whose effectiveness is not arbitrary but random deviations are independent.

Let us consider the case when there is a functional dependence between the effectiveness of the securities. For the further analysis it is convenient to use the notion of the coefficient of correlation that is determined in the form

$$\rho_{ij} = \frac{V_{ij}}{\sigma_i \sigma_j}\ .$$

It is obvious that coefficient of correlation is $\rho_{ij} \in [-1,1]$. Then, considering that $V_p = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} x_i x_j V_{ij}\right)$, we will obtain

$$V_p = \sum_{i=1}^{n}\sum_{j=1}^{n}(\sigma_i x_i)(\sigma_j x_j)\rho_{ij}\ . \tag{8.9}$$

Let us suppose that all $\rho_{ij} = 1$, which means the functional dependence between the i and j security types. Then

$$V_p = \sum_{i=1}^{n}\sum_{j=1}^{n}(\sigma_i x_i)(\sigma_j x_j) = \left(\sum_{i=1}^{n}(\sigma_i x_i)\right)\left(\sum_{j=1}^{n}(\sigma_j x_j)\right) = \left(\sum_{j=1}^{n}\sigma_j x_j\right)^2\ . \tag{8.10}$$

If the capital is invested in all security types equally, then we will obtain

$$V_p = \left(\frac{1}{n} \sum_{j=1}^{n} \sigma_j \right)^2 . \qquad (8.11)$$

The standard deviation in this case is $\sigma_p = \frac{1}{n} \sum_{j=1}^{n} \sigma_j$.

Thus, with a full correlation, the diversification is not effective. In this case the portfolio risk equals the average risk of separate investments and does not increase with the increase of the number of security types, that are included in the portfolio.

The positive correlation between the effectiveness of two securities takes place if the rate of both is determined by one and the same external factor, besides, the changes of this factor act into one and the same direction on the effectiveness of the securities (for instance, the increase in the oil price leads to the decrease of stock prices of oil and energetic companies).

Let us consider the case of the total inverse correlation, when $\rho_{ij} = -1$. This means that if the rate of one security increases, then that of another decreases to the same degree. Let us consider two security types. The variance of such a portfolio

$$V_p = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 - 2\sigma_1 x_1 \sigma_2 x_2 = \left(\sigma_1 x_1 - \sigma_2 x_2 \right)^2 . \qquad (8.12)$$

If the capital is distributed in the proportion $x_2 = \frac{\sigma_1}{\sigma_2} x_1$, then $V_p = 0$ and the given portfolio will be risk-free.

Thus, with the total inverse correlation such a distribution of investments between various security types is possible that the risk is totally missing. Unfortunately, on the real financial markets there may be neither full straight, nor inverse correlations. However, it is possible to expect that in any case the diversification of the portfolio will lead to the decrease of the risk at the desired level of the expected effectiveness.

8.2. Optimization of the Wildcat Security Portfolio

In this section the problem of problem of an optimal choice of the wildcat security portfolio.

The main suppositions taken at the building of the portfolio optimization models.

1. The investors make an evaluation of investment portfolios on the basis of the expected return and their standard deviations for the hold period.

2. When choosing between two portfolios the investors will prefer the one that gives the highest expected return at other equal conditions.
3. When choosing between two portfolios the investor will prefer the one that has the smallest standard deviation at other equal conditions.
4. Private assets are infinitely divisible. This means that the investor may optionally buy part of the share.
5. Taxes and transaction expenses are unessential.

With these suppositions it is possible to formulate the following optimization problem: determine the investment shares x_j ($j = 1, 2, ..., n$) minimizing the variance (risk) of the portfolio

$$V_p = \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} x_i x_j , \tag{8.13}$$

at the condition that the given value m_p is secured by the expected effectiveness of the portfolio

$$m_p = \sum_{j=1}^{n} m_j x_j . \tag{8.14}$$

Besides, extra limitations of the following type must be fulfilled

$$\sum_{j=1}^{n} x_j = 1, \ x_j \geq 0 \tag{8.15}$$

with all $j = 1, 2, ..., n$.

The given problem with account of the last limitation is called the problem of quadratic programming. The solution of the formulated problem does not cause any principle difficulties and for its search the known numerical procedures may be used, and also the packages of applications realizing these procedures such as Microsoft Excel, Mathcad.

We will obtain the solution of the problem without consideration of limitations of the type $x_j \geq 0$. If $x_j > 0$, then this implies the recommendation to insert the share x_j of the capital into the securities of the j type. If $x_j < 0$, then this implies the recommendation to borrow the securities of the j type of the amount x_j per unit of the available capital. When working on the stock exchange this means that the client authorizes the broker to participate in the transaction of the short sale type (an uncovered sale). In principle, such a transaction is acceptable and its meaning is the following. The investor’s order on the short sale for a certain amount of securities means that the broker must borrow some amount of securities, and then after some small time put them on sale, then after some time buy these securities again and return the debt. If the client supposes that the borrowed securities will decrease in price after a while, then when repaying the debt as a fixed amount of these securities they may purchase them for a lower price than the one acquired at the first sale right after borrowing them. The client's income (without considering transaction expenses and payment for broker's services) will make a difference between the price for sale of the borrowed securities and the price of their purchase after some time for debt repayment. The client will receive benefit from participating in the given transaction if they foresee the probability of price decrease of these shares correctly. On the other hand, the given transaction is connected with significant risk. If the share price does not decrease, but on the contrary, increases, then the client will have a loss.

Let us insert the following indications. Let V - the matrix whose elements are covariances V_{ij}:

$$V = \begin{pmatrix} V_{11} & V_{12} & \dots & V_{1n} \\ V_{21} & V_{22} & \dots & V_{2n} \\ \dots & \dots & \dots & \dots \\ V_{n1} & V_{n2} & \dots & V_{nn} \end{pmatrix},$$

besides $V_{ij} = V_{ji}$, $i \neq j$, i.e. V - the symmetrical matrix;

$m = \begin{pmatrix} m_1 \\ m_2 \\ \dots \\ m_n \end{pmatrix}$ – the column matrix of the expected effectiveness;

$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ – the column matrix of investment shares;

I – the column matrix with single dimension elements $n \times 1$.

With account of the values introduced, the problem of optimization of the portfolio structure may be written in a vector-matrix form: minimize

$$V_p = x^T V x \to \min_x \tag{8.16}$$

at two limitations of the type

$$m^T x = m_p, \tag{8.17}$$

$$I^T x = 1. \qquad (8.18)$$

Here the superscript T means matrix transposition.

We have a problem on the constrained extremum (minimum). This problem may be solved by using the method of Lagrange factors. Let us introduce the Lagrange function

$$L = x^T V x + \lambda_1 (I^T x - 1) + \lambda_2 (m^T x - m_p),$$

where λ_1, λ_2 – are the factors of Lagrange.

The solution of the problem on the constrained extremum must satisfy the following relations:

$$\frac{\partial L}{\partial x} = 2Vx + \lambda_1 I + \lambda_2 m = 0,$$

$$\frac{\partial L}{\partial \lambda_1} = I^T x - 1 = 0 ,$$

$$\frac{\partial L}{\partial \lambda_2} = m^T x - m_p = 0.$$

The solution of this system gives an obvious expression of the optimal structure of the portfolio. Let us indicate such a structure x^*. We have

$$x^* = V^{-1} \left[m_p (I J_{12} - m J_1) + m J_{12} - I J_2 \right] \left[J_{12}^2 - J_1 J_2 \right]^{-1}, \qquad (8.19)$$

where the indications are used: $J_1 = I^T V^{-1} I$, $J_2 = m^T V^{-1} m$, $J_{12} = I^T V^{-1} m$.

We should note that the acquisition of solutions is linear in accordance with the expected effectiveness of the portfolio m_p. Consequently, the minimal variation value of the portfolio $V_p^* = x^{*^T} V x^*$ is with the convex function of the argument m_p. It is obvious that this is correct and for the function $\sigma_p^* = \sqrt{V_p^*}$ – the standard deviations of the portfolio.

The character of the change σ_p^* in relation to the expected effectiveness is illustrated on the graph (Figure 8.2).

Figure 8.2. The dependence of the optimal portfolio risk on its return

Thus, when changing the expected effectiveness of the portfolio the investments in every security change linearly if the transaction of borrowing the securities is acceptable. Along with that the portfolio with any expected effectiveness may be formulated, but the risk of such a portfolio will increase.

8.3. Optimization of the Portfolio with Risk-free Investment Possibility

On the security market, apart from the wildcat securities there are risk-free securities like the short-term government commitments with fixed income. Considering this, the investor's problem may be formulated as: it is necessary to distribute the capital between risk-free and risk-laden securities and simultaneously determine the structure of risk-laden investments. Let us consider the solution to this problem.

Let m_0 be the effectiveness (the rate of return) of the risk-free investment, the determinated value. Let us formulate the problem of optimizing the portfolio with account of risk-free investments the following way: minimize the the portfolio risk (variance of the risk-laden part of the portfolio)

$$V_p = x^T V x \to \min_x \tag{8.20}$$

with limitations

$$m^T x + m_0 x_0 = m_p , \tag{8.21}$$

$$I^T x + x_0 = 1 , \tag{8.22}$$

where x_0 – the share of the capital, that is invested in risk-free securities. If additional limitations of the type $x_j \geq 0$ ($j = 1, 2, ..., n$) are imposed, then we have a problem of quadratic programming.

Let us solve the given problem without consideration of limitations for nonnegativeness of the variables x_j. We have a problem for conventional extremum. we will solve it with the method of factors of Lagrange. It is necessary to determine the optimal values of the vector x and variable x_0.

Let us build the function of Lagrange:

$$L = x^T Vx + \lambda_1 (I^T x + x_0 - 1) + \lambda_2 (m^T x + m_0 x_0 - m_p)\,. \qquad (8.23)$$

Minimum conditions of the Lagrange function:

$$\frac{\partial L}{\partial x} = 0\,,\ \frac{\partial L}{\partial x_0} = 0\,,\ \frac{\partial L}{\partial \lambda_1} = 0\,,\ \frac{\partial L}{\partial \lambda_2} = 0\,.$$

Taking the derivatives by dx, dx_0, $d\lambda_1$ and $d\lambda_2$ we will obtain the following system of linear equations:

$$2Vx + \lambda_1 I + \lambda_2 m = 0\,,$$

$$\lambda_1 + \lambda_2 m_0 = 0\,,$$

$$m^T x + m_0 x_0 = m_p\,,$$

$$I^T x + x_0 = 1\,.$$

When solving this system, we will obtain an obvious expression for the optimal structure of the portfolio:

$$x^* = V^{-1}(m - m_0 I)\left[(m - m_0 I)^T V^{-1}(m - m_0 I)\right]^{-1}(m_p - m_0)\,, \qquad (8.24)$$

where x^* – is the structure of the risk share of the optimal portfolio.

We should note that the value m_p – the expected effectiveness of the portfolio – enters this expression linearly as a scalar factor, consequently, the structure of risk-laden investments does not depend on the value m_p. Introducing expressions for x^* in the formula for value variance of the portfolio, we will obtain an expression for the minimum variance of the portfolio:

$$V_p^* = x^{*T} V x^* = \frac{(m_p - m_0)^2}{d^2}\,,$$

$$d^2 = (m - m_0 I)^T V^{-1} (m - m_0 I)\,.$$

The standard deviation of the portfolio equals $\sigma_p^* = g^{-1}(m_p - m_0)$.

Hence we have

$$m_p = m_0 + g\sigma_p^*. \tag{8.22}$$

Thus, the expected effectiveness of the optimal portfolio and its standard deviation are interconnected linearly, at the condition that there are risk-free investments. It is obvious that with x_0 = 0 the optimal portfolio consists of only risk-laden securities, with x_0 = 1 only risk-free securities are included in the portfolio. Since with the lack of risk-free deposits, the optimal solution of Problem (8.22) must coincide with the solution of Problem (8.19), what is more, this solution is unique, then right line (8.22) must touch the curve $\sigma_p^* = \sigma_p^*(m_p)$ at the single dot (Figure 8.3). Any intermediary value 0 < x_0 < 1 corresponds to the dot on the tangent described by the equation (8.22).

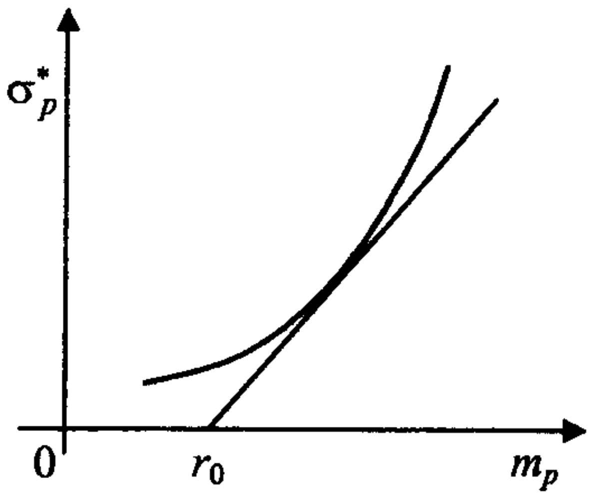


Figure 8.3. The dependence of the optimal portfolio risk on its return.

The main conclusion from the obtained result is formulated as the *separation theorem*: *If there is a possibility to divide the capital not only between the given risk-laden portfolio and the risk-free securities, but also simultaneously choose the structure of the risk-laden portfolio, then only one such structure not depending on the investor's risk proneness will be optimal*.

Or, in other words, the combination of risk-laden securities optimal for any investor does not depend on their risk proneness.

The main qualitative conclusions obtained above remain in power also under condition that the conditions for nonnegativeness are imposed for the variables x_j (we should remind that in this case, the task for optimization turns into the mathematical problem of the quadratic programming).

8.4. Valuating Security Contribution to the Total Expected Portfolio Return

Let us determine the values

$$\beta_j^* = \frac{1}{V_p^*} M\left\{(R_j - m_j)(R_p^* - m_p)\right\}, \tag{8.23}$$

that are called j investments with a Beta regarding the optimal portfolio.

The given coefficients characterize the statistical interconnection between the effectiveness of the portfolio as a whole and of the security of the j type and play an important role in the theory of the optimal portfolio. Let us introduce the column vector $\beta^* = (\beta_1^*, \beta_2^*, ..., \beta_n^*)^T$. Then it is possible to write

$$.\beta^* = \frac{1}{V_p^*} M\left\{(R - m)(R_p^* - m_p)\right\}. \tag{8.23}$$

The effectiveness of the optimal portfolio:

$$R_p^* = m_0 x_0 + \sum_{j=1}^{n} R_j x_j^* = m_0 x_0 + R^T x^*;$$

Let us write the difference

$$R_p^* - m_p = m_0 x_0 - m_p + R^T x^* = m_0 x_0 - (m_0 x_0 + m^T x^*) + R^T x^* = (R - m)^T x^*,$$

where R – the column vector of the security effectiveness whose components are the effectiveness R_j. Returning to the expression for β^* and substituting the obtained expression for $(R_p^* - m_p)$ in it, the obtained expression is possible to be written as:

$$\beta^* = \frac{1}{V_p^*} M\left\{(R - m)(R - m)^T x^*\right\} = \frac{1}{V_p^*} V x^*, \tag{8.24}$$

where $M\left\{(R - m)(R - m_p)^T\right\} = V$.

Considering the formulas for x^* and V_p^*:

$$x^* = \frac{(m_p - m_0)}{d^2} V^{-1}(m - m_0 I),\ d^2 = (m - m_0 I)^T V^{-1}(m - m_0 I),$$

$$V_p^* = x^{*T} V x^* = \frac{(m_p - m_0)^2}{d^2},$$

we will obtain

$$\beta^* = \frac{m - m_0 I}{m_p - m_0}. \tag{8.25}$$

This relation is usually written the following way:

$$m - m_0 I = \beta^* (m_p - m_0) \tag{8.26}$$

or the scalar form

$$m_j - m_0 = \beta_j^* (m_p - m_0). \tag{8.27}$$

The excess of the expected effectiveness of the risk-laden security or of the security portfolio over the effectiveness of the risk-free investment is called *risk premium*. In this case, the difference $(m_j - m_0)$ – is the risk premium of the j security, the difference $(m_p - m_0)$ – the risk premium of the portfolio.

Thus, the risk premium of any security included in the optimal portfolio is proportional to the risk premium for the portfolio as a whole. The coefficient of proportionality is the j investment with a Beta regarding the portfolio. The higher the coefficient of the investment with a Beta of the given security, the higher the share of the total risk connected with the investment exactly into this security, but, on the other hand, the higher the risk premium.

8.5. A Pricing Model on the Competitive Financial Market

The given section considers the pricing model on the market of the financial assets (the capital asset pricing model, CAPM). When building this model, it is supposed that the investors use the method described above for forming their portfolios.

Let us introduce the notion of the ideal competitive market. The ideal competitive market is the market whose all members possess equal share of information and make the best decisions on its basis. This means that all the participants of the market are familiar with the statistical forecast of the effectiveness of all the securities on the market, and are aware of evaluating the variance and covariance of the security effectiveness, and any participant of the market aims at forming the optimal structure of their own security portfolio.

From the separation theorem it follows that the structure of the risk share of the optimal portfolio does not depend on the investor’s risk proneness and is totally determined by the probability characteristics of the risk-laden securities. Hence, it follows that based on the identical information, all the investors will aim at choosing one and the same structure of the risk share of their own portfolio. Thus, if such conditions have been formed on the real market when their activity within a long period of time is determined by the members that possess almost identical information in an identical amount and making the best possible decisions on forming their security portfolio, then the distribution of risk-laden securities on such a market will have the properties close to those of the optimal portfolio. Consequently, the investor may trust the market and choose the portfolio with the same structure of the risk share as the market portfolio as a whole. Depending on their risk proneness, each investor decides for themselves, in what proportion the capital must be divided into the risk-laden and risk-free shares.

An equation was obtained for the optimal portfolio earlier:

$$m_j - m_0 = \beta_j (m_p - m_0).$$

Since, according to the CAPM, the market portfolio has the same structure as the optimal portfolio, it is possible to claim that an analogous equation will take place also for the market as a whole. Thus, we obtain *the basic equilibrium market equation*:

$$m_j - m_0 = \beta_j (m_m - m_0), \tag{8.28}$$

where m_m is the expected effectiveness of the market as a whole. The coefficient β_j in this equation is called the *Beta of the j type security in accordance with the market*.

The real market is different from the ideal one. The statistical analysis of the market indicates the fairness of the more general relation, in particular:

$$m_j - m_0 = \beta_j (m_m - m_0) + \alpha_j, \tag{8.29}$$

where α_j is a certain parameter that is called the Alpha of the j deposit. On the ideal market, this parameter for all the securities equals zero. However, since not all the members on the real market are equally informed and not all are aimed at acting optimally, then it is also the portfolio of the real market that is different from the optimal one. Besides, the CAPM supposes that the rates at the purchase and sale and the rates on the credits are identical; however, it is not so for the real market.

All these parameters may be evaluated based on the previous statistical data on the effectiveness of the securities (the technical analysis of the market) with the use of the methods of mathematical statistics.

A fundamental analysis implies a forecast of effectiveness coming out of the analysis and evaluation of economic factors influencing the effectiveness of securities.

8.6. The Statistical Analysis of the Financial Market

Single Factor Market Model

The theory of the optimal portfolio is based on the fact that the investor knows the statistical characteristics of the securities: their expected effectiveness m_i and covariance V_{ij}. These prices may be evaluated by the methods of mathematical statistics based on the data on the previous values of effectiveness of securities that have been fairly long on the market. Let us consider the method based on building a single factor market model.

Observations indicate that the effectiveness of the common stock for some time period (for instance, a month) is connected with the effectiveness of the market as a whole. However, since it is not possible to accurately determine the structure of the market portfolio, to measure the effectiveness of the market, market indices are used –the weighted with account of the capital effectiveness amounts of stock of the leading companies (in the United States, it is, for example, the Dow Jones index or $SP500$ index – Standard and Poor's 500 Stock Index, with calculating which, the stock of five hundred leading corporations are taken into account). The given interconnection may be expressed in form of the *market model:*

$$R_j = a_j + \beta_j R_m + e_j, \tag{8.30}$$

where R_j – the return of the j security for the given period; R_m – the effectiveness of the market (the market index); a_j – the coefficient of displacement; β_j – the obliquing factor; e_j – the random error.

The obliquing factor of the market model is called the Beta coefficient and is determined by the formula

$$\beta_j = \frac{V_{jm}}{V_m},$$

where $V_{jm} = M\left\{(m_j - R_j)(R_m - m_m\right\}$, $V_m = M\left\{(R_m - m_m)^2\right\}$. If for any shares $\beta_j > 1$, then such shares are called *aggressive*, if $\beta_j < 1$, then such shares are called *defensive*. We should note that in the market model the coefficient β_j is measured in accordance with the market index, whereas, in the equilibrium market model CAPM, the analogous coefficients are determined with regard to the market as a whole. In practice, the Beta coefficients with account of the market index are used to evaluate the coefficients β_j in CAPM.

The random error in this equation reflects the fact that the market model describes the return of the securities inaccurately. It is usually supposed that the random variable e_j has a distribution of probability with the zero mathematical expectation and standard deviation σ_{ej}^2, besides, the random errors of various securities are not mutually correlated with R_m, i.e. $M\left\{e_i e_j\right\}=0$, $i \neq j$, and $M\left\{e_j R_m\right\}=0$.

From the equation of the market model we obtain that the expected value of effectiveness of the j security is

$$m_j = a_j + \beta_j m_m,$$

where m_m - is the expected value of market effectiveness (market index). Hence

$$R_j - m_j = \beta_j (R_m - m_m) + e_j.$$

Using this relation, we will obtain the expression for variance and covariance

$$V_j = M\left\{(R_j - m_j)^2\right\} = \beta_j^2 M\left\{(R_m - m_m)^2\right\} + M\left\{(e_j^2)\right\} = \beta_j^2 V_m + V_{ej},$$

$$V_{ij} = M\left\{(R_i - m_i)(R_j - m_j)\right\} = \beta_i \beta_j M\left\{(R_m - m_m)^2\right\} = \beta_i \beta_j V_m,$$

where V_m – is the market effectiveness variance, V_e – the random error variance.

Thus, to evaluate the expected effectiveness, variance and covariance of securities it is necessary to evaluate the parameters a_j, β_j, the expected value of market effectiveness m_m, and the market variance V_m and random error V_e. In order to evaluate these values, it is possible to use statistical procedures, for instance, the method of the least squares. If inconsistencies of security effectiveness observations are known: $R_j(t)$, $j = 1, 2, ..., n$, and inconsistencies of market effectiveness $R_m(t)$, where $t = 1, 2, ..., T$, *then* the method of the least squares gives the following evaluations:

$$a_j^e = m_j^e - \beta_j^e m_j^e; \ \beta_j^e = \frac{V_{jm}^e}{V_m^e};$$

$$m_j^e = \frac{1}{T}\sum_{t=1}^{T} R_j(t); \ m_m^e = \frac{1}{T}\sum_{t=1}^{T} R_m(t);$$

$$V_{jm}^{e} = \frac{1}{T-1}\sum_{t=1}^{T}[(R_m(t) - m_m^e)(R_j(t) - m_j^e)];$$

$$V_m^e = \frac{1}{T-1}\sum_{t=1}^{T}[(R_m(t) - m_m^e)^2];$$

$$V_{ej}^e = \frac{1}{T-1}\sum_{t=1}^{T}[R_j(t) - a_j^e - \beta_j^e R_m(t)]^2 .$$

Correspondingly, the evaluations of variance and covariance of securities have a form

$$V_j^e = (\beta_j^e)^2 V_m^e + V_{ej}^e ,$$

$$V_{ij}^e = \beta_i^e \beta_j^e V_m^e .$$

Multifactor models

The state of economics depends on many factors, among which several basic ones that influence all the economic spheres may be distinguished:

1) the growth rate of the gross domestic product;
2) the level of interest rates;
3) the rate of inflation;
4) the rate of oil prices.

The consideration of these (and possibly, other) factors enables building more accurate models of security yield. The common multifactor model is written in the form

$$R_i(t) = a_i + b_{i1}F_1(t) + b_{i2}F_2(t) + ... + b_{ik}F_k(t) + e_i , \qquad (8.31)$$

where $F_i(t)$ – are the factors, influencing the yield of all securities;

a_i , b_{is} $(s = 1, 2, ..., k)$ – the parameters (parameters b_{is} are called sensitivities of the security i to the corresponding factors), $e_i(t)$ – random error.

It is obvious that building the factor models considered must be based on the reliable statistical information.

Questions for Self-test

1. How are the yield of a risk-laden security and its risk determined?
2. How are the portfolio return and its risk determined?
3. What is the portfolio diversification?
4. How does the correlation of securities influence the portfolio diversification?
5. Write the optimization model of the portfolio of risk-laden securities.
6. Write the expression for the optimal structure of the portfolio.
7. How is the risk of the optimal portfolio connected with efficiency?
8. Write the optimization model of the portfolio of risk-laden securities with risk-free investments.
9. How is the risk of the optimal portfolio with risk-free investments connected with the portfolio return?
10. What is the investment with a Beta of the j security?
11. What is the premium for the security and portfolio risk?
12. How is the premium for the security risk connected with the premium for portfolio risk?
13. Write the equilibrium market equation.
14. What is the investment with an alpha of the j security?
15. Write the single factor market model.
16. What is the connection between the effectiveness of the j security with the market efficiency?
17. Write the connection of the variance of the j security with the market variance.
18. Write the connection between the covariance of the j security with the market covariance.
19. Write the multifactor market model.

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APPENDIX 1

Sample Solutions of Routine Problems

Problem 1. A deposit A = 5000 rubles is placed in a bank. The interest calculated on this deposit will be i_1 = 10% per annum in the first year , i_2 = 12% per annum in the second, i_3= 15% per annum in the third, $i_4 = i_5$ = 16% per annum in the fourth and the fifth. How much will there be on the account at the end of the fifth year? How much should have been placed on the account at the constant interest rate of i = 13% to secure an equal sum? Make calculations at the simple and compound interest rates.

***Solution*.** The accumulation formula with a simple and compound interest model at the variable rate is as follows:

a) $S = A(1+i_1)^{n_1} \cdot (1+i_2)^{n_2} \cdot (1+i_3)^{n_3} \cdot (1+i_4)^{n_4}$,

b) $S = A(1+\sum_{k=1}^{4} n_k i_k)$.

where n_i — is the i interest calculation period ($n_1 = n_2 = n_3 = 1$, $n_4 = 2$, $n = \sum_{i=1}^{4} n_i = 5$).

Let us put in the given data:

$$A := 5000 \qquad i := \begin{pmatrix} 10\% \\ 12\% \\ 15\% \\ 16\% \end{pmatrix} \qquad n := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \qquad i1 := 13\% \qquad n1 := 5$$

The MathCAD solution at the simple rate:

$$S := A \cdot \left(1 + \sum_{k=1}^{4} i_k \cdot n_k\right) \qquad S = 8.45 \times 10^3$$

$P := 1$

Given

$S = P \cdot (1 + i1 \cdot n1)$

$P := \mathrm{Find}(P) \qquad P = 5.121212 \times 10^3$

The MathCAD solution at the compound rate:

$$S1 := A \cdot \prod_{k=1}^{4} \left(1 + i_k\right)^{n_k}$$

$$S1 = 9.53223 \times 10^3$$

$$P1 := 1$$

Given

$$S1 = P1 \cdot (1 + i1 \cdot n1)$$

$$P1 := \mathrm{Find}(P1) \qquad P1 = 5.777109 \times 10^3$$

Answer: at the end of the 5th year there will be either 8450 rubles or 9352 rubles on the account, if the interest calculation is performed in accordance with the simple interest model or the compound interest model. To obtain the sum of 8450 rubles by the end of the 5th year at the rate of i = 13%, it is necessary to place a deposit of 5121 rubles (with the simple interest model) or 5173 rubles (with the compound interest model) at the beginning of the period in order to eventually obtain the sum of 9352 rubles.

Problem 2. Calculate the amount of payment of an n - year loan for buying an apartment for A rubles at an annual rate of i percent and a down payment of q percent. Make calculations for monthly and annual payments.

Make calculations with the following data: n = 20 years; A = 1 400 000 rubles; i = 18%; q = 30%.

Make calculations at the compound interest rate.

***Solution*.** The sum to be paid on the loan equals $A - q \cdot A = A \cdot (1-q)$. Let us calculate the annual payment R of loan repayment out of the equation:

$A \cdot (1-q) = R \cdot \dfrac{1-(1+j/m)^{-n \cdot m}}{(1+j/m)^{m/p} - 1} = R \cdot a(p,n,j/m)$, hence $R = \dfrac{A \cdot (1-q)}{a(p,n,j/m)}$. Here p = 12 (number of annual payments), m = 12 (amount of annual interest calculations).

Let us put in the given data.

$$A := 1400000 \qquad j := 18\% \qquad q := 30\% \qquad n1 := 20$$

$$p := 12 \qquad m := 12$$

The MathCAD solution:

$$a := \frac{1 - \left(1 + \frac{j}{m}\right)^{-n1 \cdot m}}{\left(1 + \frac{j}{m}\right)^{\frac{m}{p}} - 1}$$

$$a = 64.795732$$

$$R := \frac{A \cdot (1 - q)}{a} \qquad R = 1.512445 \times 10^{4}$$

Answer. Monthly payments will be 15 124.45 rubles.

Problem 3. A family wants to save $12000 for a car by depositing $1000 in the bank annually. The annual interest rate in the bank is 7%. How long will it have to be saving?

Solution. Let us use the formula of the accumulated value of the annuity to solve this problem.

$$.\, S = R \cdot (1+i)\frac{((1+i)^{n} - 1)}{i}$$

Hence:

$$n = \frac{\ln(\frac{S \cdot i}{R(1+i)} + 1)}{\ln(1+i)}$$

Let us write the given data:

$$S := 12000 \qquad R := 1000 \qquad i1 := 7\%$$

The MathCAD solution

$$n1 := \frac{\ln\left[1 + \frac{S \cdot i1}{R \cdot (1 + i1)}\right]}{\ln(1 + i1)} \qquad n1 = 8.564235$$

$$S1 := R \cdot (1 + i1) \cdot \frac{(1 + i1)^{9} - 1}{i1} \qquad S1 = 1.281645 \times 10^{4}$$

Answer. The family must be saving money for 9 years. By the end of Year 9 there will be 12816.5 rubles on the account.

Problem 4. A loan was taken at i_1=16% per annum; an amount of 500 currency units (R_1=500 currency units) per must be paid quarterly within n=2 years. Due to a change of situation in the country, the interest rate was decreased to i_2=6% per annum. The bank agreed with the necessity of recalculating quarterly payments. What size must the new amount of payment be?
Make calculations at the compound interest rate.

Solution. In order to solve this problem, it is necessary to put in the present value of the outstanding sum at the rate of i_1=16% and equate it to the present value of the payment stream at the rate of i_2=6%.

We have $R_1 \dfrac{1-(1+i_1/m)^{-n\cdot m}}{(1+i_1/m)^{m/p}-1} = R_2 \dfrac{1-(1+i_2/m)^{-n\cdot m}}{(1+i_2/m)^{m/p}-1}$, where m = 4 (the number of annual interest calculations), p = 4 (number of annual payments). We find the amount of payment R_2 from this equation.

The given data for MathCAD:

$$R1 := 500 \qquad n := 2 \qquad m := 4 \qquad p := 4$$

$$i1 := 16\% \qquad i2 := 6\%$$

The MathCAD solution:

$$R2 := 1$$

Given

$$R1 \cdot \frac{1 - \left(1 + \frac{i1}{m}\right)^{-n \cdot m}}{\left(1 + \frac{i1}{m}\right)^{\frac{m}{p}} - 1} = R2 \cdot \frac{1 - \left(1 + \frac{i2}{m}\right)^{-n \cdot m}}{\left(1 + \frac{i2}{m}\right)^{\frac{m}{p}} - 1}$$

$$R2 := \mathrm{Find}(R2) \qquad R2 = 449.693578$$

Answer. The amount of the new payment will be 449.7 rubles.

Problem 5. A loan commitment at an amount of 50 000 currency units should be accounted 4 before its maturity. For the commitment accounting the bank uses a compound interest rate of 5 %. The interest may be calculated 1, 2 or 4 times a year. Indicate the terms of the contract on which this commitment may be accounted.

Solution. It is necessary to find the present value of the sum S that will equal 50 000 currency units 4 years later depending on the number of annual interest calculations. Let us make the calculation by the formula $P = \frac{S}{(1 + j/m)^{n \cdot m}}$, where j - the annual rate, m - the number of annual interest calculations.

The given data:

$$S := 50000 \qquad i1 := 5\%$$

$$n1 := 4$$

The MathCAD solution:

$$P1(m1) := \frac{S}{\left(1 + \frac{i1}{m1}\right)^{n1 \cdot m1}}$$

$$P1(1) = 4.113512 \times 10^{4}$$

$$P1(2) = 4.103733 \times 10^{4}$$

$$P1(4) = 4.098732 \times 10^{4}$$

Answer. The commitment will be discounted for a sum of 41135 currency units with an interest calculation once a year, for a sum of 41037 currency units – with an interest calculation two times a year, for a sum of 41987 currency units – with an interest calculation four times a year.

Problem 6. How will the payback period of a project change with a change of the values of investments, annual returns, interest rate? Build graphs and give explanations.

Solution. Let us consider the following investment project model. Project investments with an amount of K are made through a lump sum payment at the beginning of the term, the R return is annually regular within n years, the interest rate is j . The payback period in this case is calculated by the formula $n = -\dfrac{\ln(1 - \frac{K \cdot j}{R})}{\ln(1 + j)}$.

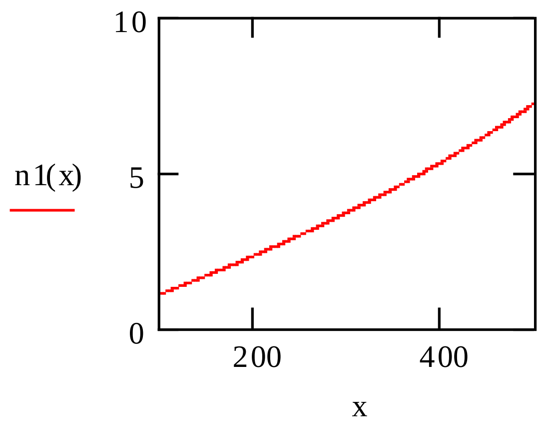
The given data:

$$K := 500 \qquad R := 100 \qquad j := 10\%$$

The MathCAD solution:

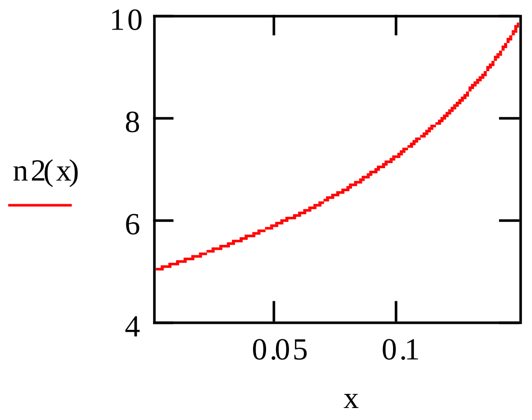
The dependency of the payback period on the investment amount

$$n1(x) := \frac{-\ln\left(1 - \frac{x \cdot j}{R}\right)}{\ln(1 + j)}$$



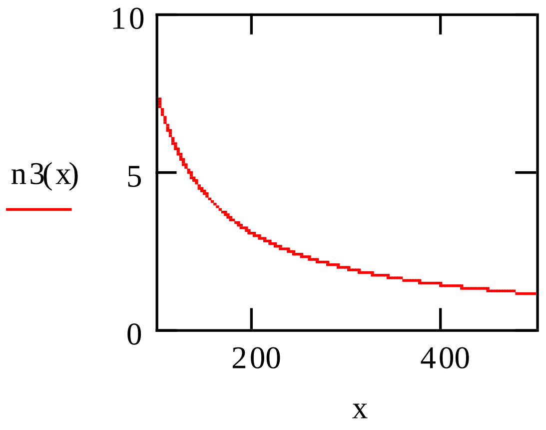
The dependency of the payback period on the rate

$$n2(x) := \frac{-\ln\left(1 - \frac{x \cdot K}{R}\right)}{\ln(1 + x)}$$



The dependency of the payback period on the value of the return

$$n3(x) := \frac{-\ln\left(1 - \frac{j \cdot K}{x}\right)}{\ln(1 + j)}$$



Answer. The payback period with a growth in the volume of investments increases because for return on investments, a larger amount of time is required to receive return from the project.

Along with the growth of the interest rate the payback period also grows. From the economic point of view, it may be explained the following way. If a loan is taken from a bank at the interest rate j for investments, then along with the growth of the rate, interest on the loan also grows, and consequently, the debt amount of the loan debtor grows, too. Hence, a larger amount of time is required to receive return from the project in order to discharge the loan.

With the growth of the return from the project the payback period decreases.

Problem 7. Check the plan of discharging the principal balance of the debt by equal annual payments if the amount of the loan D is 600 currency units, and the interest rate i is 8%.

Payments	168.0	158.4	148.8	139.2	129.6
Years	1	2	3	4	5

Solution. The amount of the loan D = 600 currency units is discharged by equal portions within 5 years. The interest on the loan is paid annually in accordance with the remainder of a debt.

Thus, the amount of the due payment in the t year equals $Y_t = d + (D - (t-1)d) \cdot i$, where $d = D/n$, n – life of the loan.

The given data:

$$D := 600 \qquad n1 := 5 \qquad j := 0.08$$

The MathCAD solution:

$$d := \frac{D}{n1} \qquad d = 120$$

$$t := 1..n1$$

$$Y_t := d + [D - (t-1)\cdot d]\cdot j$$

$$Y = \begin{pmatrix} 168 \\ 158.4 \\ 148.8 \\ 139.2 \\ 129.6 \end{pmatrix}$$

Answer. The loan discharge plan has been worked out correctly.

Problem 8. Check the calculations. For an investment project with a duration of 6 with the planned annual return of 400 currency units and an annual return of 10% the necessary investments are found: 1742 currency units.

Solution. An investment project with a given duration that equals the payback period is considered here. The project must provide the defined annual return.

In the general case, the solution of the problem is as follows: let R, n, i be the amount of the consequent annual return (it is assumed that the income on investments will begin after the completion of investing), project duration and interest rate. What minimal investments are required for securing this?

It is obvious that the investment required is $K = R\dfrac{1-(1+i)^{-n}}{i}$. This means that the net present value of the project equals 0, and the domestic return coincides the interest rate.

Let us make computer calculations based on these conclusions.

Let us put in the given data:

$$R := 400 \qquad n1 := 6 \qquad j := 0.1$$

The MathCAD calculation:

$$K := R \cdot \frac{1-(1+j)^{-n1}}{j} \qquad K = 1.742104 \times 10^{3}$$

Answer: The value of investment required will be 1742 currency units, i.e. the project calculations are correct.

Problem 9. Somebody has received inheritance in a form of a fat bank account K and now is “eating it away” by taking a sum of R from the account annually and spending it within a year. In essence, it is an “inverse” investment process. The bank rate equals i =10% per annum. What are the investments, payback period, inner rate of return, net present value here? What measures should the heir take at the inflation rate increase? Calculate these characteristics with the following given data: K = 30 000 currency units, R =10 000 currency units, rate i =10%.

Solution. In this case we have an “inverse” investment project. The bank account K works as an investment on which the bank calculates the interest at the rate i. As the income on investments we have the sum R that the heir withdraws from the bank annually. Thus, the payback period of this “project” will equal the time period during which the heir will have withdrawn the whole sum from the bank. The internal rate of return is the marginal interest rate, at which the heir will have withdrawn the whole sum from the bank.

Thus, the payback period n, i.e. the term within which the heir will have withdrawn the whole sum, will be determined from the equation $K = R \cdot \frac{1-(1+i)^{-n}}{i}$, where i is the rate of the bank interest. From here we obtain $n = \frac{-\ln(1 - i \cdot \frac{K}{R})}{\ln(1+i)}$. The internal rate of return of the “project” is determined from the same equation where the term n is given, and i must be determined. If the term of the "project" equals the payback period, then the internal rate of return coincides with the bank’s interest rate, and the net present value will equal zero.

The given data:

$$K := 30000 \qquad R := 10000 \qquad j := 0.1$$

The MathCAD calculation:

$$no := \frac{-\ln\left(1 - \frac{K \cdot j}{R}\right)}{\ln(1 + j)} \qquad no = 3.742254$$

i.e. the heir will have eaten the inheritance away within 4 years; besides, in the last, 4th, year the heir will get a sum smaller than 10000 currency. Let us find the sum.

$$R \cdot \frac{1 - (1 + j)^{-4}}{j} = 3.169865 \times 10^{4}$$

We obtain from here 31699 – 30000 = 1699 currency. Therefore, in the last year the heir will get not 10000 currency units, but a sum less by 1699 currency units, i.e. 10000 – 1699 = 8301 currency units.

Let us find the internal rate of return of «project».

$$q := 0.001$$

$$f(q) := \left[R \cdot \frac{1 - (1 + q)^{-no}}{q}\right] - K$$

$$\text{root}(f(q), q) = 0.1$$

Therefore, the internal rate of return coincides with the bank's interest rate.

If the interest rate is more than 10%, for example, 13%, then the net present value will equal:

$$j := 13\%$$

$$W := \left[R \cdot \frac{1 - (1 + j)^{-no}}{j}\right] - K$$

$$W = -1.76511 \times 10^{3}$$

For the investment project this would mean that the project is unprofitable, and for this "inverse project" it means that the heir will have 1765 currency units remaining on the account.

Problem 10. Find the bond rate without discharging with a regular – once a year – interest payment with q = *8%* , i = *5%* . Calculate the yield of such a bond if its rate equals K =120.

Solution. Payments on such a bond may be considered a perpetual annuity. The bond rate by definition equals $K = \frac{P}{N}100$, where P is the market price of the bond (purchase price), N is the face value of the bond. The market price of the bond with no maturity date equals $P = \frac{N \cdot q}{i}$, where q is the coupon rate. Thus, the rate equals $K = \frac{q}{i}100$. The yield of the bond equals $j = \frac{q}{K}100$.

The given data:

$$q := 0.08 \qquad j := 0.05 \qquad K1 := 120$$

The MathCAD solution:

$$K := 100 \cdot \frac{q}{j} \qquad K = 160$$

$$j1 := 100 \cdot \frac{q}{K1} \qquad j1 = 0.066667$$

Answer. The bond rate at q = 8% , i = 5% equals 160, the yield of the bond at the rate 120 equals 6,67%.

Problem 11. An 8% coupon bond with the face value of 1000 currency units is considered; coupon payments are promised to be made on this bond twice a year within 3 years. Risk-free interest rates are the same for all the intervals and equal 10% per annum. Calculate the bond duration and factor of convexity.

Solution. Duration is an average period of payments. Bond duration characterizes the measure of the bond's interest risk – the longer the duration, the higher the bond's interest risk. The factor of bond convexity is an auxiliary response. The factor of bond convexity may be interpreted as a factor of how exactly the bond duration evaluates the value of the bond's interest risk.

Duration D and the factor of convexity C of the bond are calculated by the following formulas:

$$D = \frac{1}{P}\sum_{k=1}^{m} t_k \frac{R_k}{(1+i)^{t_k}}, \quad C = \frac{1}{P}\sum_{k=1}^{m} t_k (t_k + 1)\frac{R_k}{(1+i)^{t_k}},$$

where $P = \sum_{k=1}^{m} \frac{R_k}{(1+i)^{t_k}}$ is the market price of the bond; R_k – the amount of payment at the time point t_k; i – the risk-free interest rate; $m = n \cdot p$ – the number of payments on the bond until its maturity date (here n – the life of the loan, p – the number of annual payments). The amount of the k payment equals $R_k = \frac{N \cdot q}{p}$, where N – the face value of the bond.

The given data:

$n1 := 3 \qquad p := 2 \qquad q := 8\%$

$N := 1000 \qquad j := 10\%$

The MathCAD solution:

$n2 := n1 \cdot p$

$k := 1..n1 \cdot p$

$t_k := \frac{k}{p} \qquad R_k := N \cdot \frac{q}{p} \qquad R_{n2} := R_{n2} + N$

$$P := \sum_{k=1}^{n2} \frac{R_k}{(1+j)^{t_k}} \qquad D := \frac{\sum_{k=1}^{n2} \frac{t_k \cdot R_k}{(1+j)^{t_k}}}{P} \qquad C := \frac{\sum_{k=1}^{n2} \frac{t_k (t_k + 1) \cdot R_k}{(1+j)^{t_k}}}{P}$$

$D = 2.718466 \qquad C = 10.555657$

Answer. The bond duration equals 2.718 years, the factor of convexity equals 10.555.

Problem 12. Calculate the Markowitz optimal portfolio of the minimum risk for three securities with yields and risks: (4,10); (10,40); (40,80); the lower boundary of the portfolio return m_p is given equal 15.

Solution. The Markowitz portfolio of the minimum risk is formed by the following model.

The ratios of investment x_j $(j = 1, 2, ..., n)$ in risk-laden securities that minimize the portfolio variation (risk) must be determined:

$$V_p = \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} x_i x_j$$

on condition that the given value m_p of the expected effectiveness is provided

$$\sum_{j=1}^{n} m_j x_j \geq m_p.$$

Besides, additional constraints of the following type must be satisfied:

$$\sum_{j=1}^{n} x_j = 1, \; x_j \geq 0$$

for all $j = 1, 2, ..., n$.

Here V_{ij} – entries in the matrix of risk-laden security yield covariance $(i, \; j = 1, 2, ..., n)$; n – number of security types; m_j – the yield of the j security.

The given data:

$$mp := 15 \qquad m1 := 3$$

$$m2 := \begin{pmatrix} 4 \\ 10 \\ 40 \end{pmatrix} \qquad r := \begin{pmatrix} 10 \\ 40 \\ 80 \end{pmatrix}$$

The MathCAD solution:

$$Vp(x) := \left(r1_1\right)^2 \cdot x_1 + \left(r1_2\right)^2 \cdot x_2 + \left(r1_3\right)^2 \cdot x_3$$

$$x_3 := 0$$

Given

$$m2_1 \cdot x_1 + m2_2 \cdot x_2 + m2_3 \cdot x_3 \geq mp$$

$$\sum_{k=1}^{m1} x_k = 1$$

$$x \geq 0$$

$$x1 := \mathrm{Minimize}(Vp, x) \qquad x1 = \begin{pmatrix} 0.694444 \\ 0 \\ 0.305556 \end{pmatrix}$$

$$m2^T \cdot x1 = (15)$$

$$Vp(x1) = 2.025 \times 10^3$$

$$\sqrt{Vp(x1)} = 45$$

Answer. The share of securities in the portfolio makes (0,69;0;0,31), the risk of the portfolio equals 45.