# Dirac states of relativistic electrons channeled in a crystal and high-energy channeling electron-positron pair production by photons

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Dirac wave functions for high-energy electrons channeled in crystals are obtained for crystal string potentials. Specifically, we study partial cylindrical wave expansions for ''cylindrical constant and $1/\rho$ potentials. The periodicity along the crystal axis is taken into account as a perturbation to the cylindrical wave functions. We also find a Sommerfeld-Maue-like solution for the $1/\rho$ potential. The cross section for channeling electron-positron pair production in continuum states is obtained for the crystal string potential. For two-dimensional Sommerfeld-Maue-like electron and positron wave functions, matrix elements and cross section are obtained for unpolarized photons. The fact that channeling continuum pair production can only occur when the photon is hitting the crystal string at a small, finite angle is taken into account. [S1050-2947(97)01007-X]

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## I. INTRODUCTION

The phenomenon of channeling states of charged particles in a crystal was introduced by Lindhard in his classical paper [1] in 1965. The existence of bound states of channeled electrons was pointed out [2] by Vorobiev and co-workers as early as in 1973. In fact, energies and angular distributions were calculated and compared for bound states of channeling electrons of energies of a few MeV. In fact, they were the first to use a string potential $V(\rho)\sim 1/\rho$, showing that this potential is very similar to the Lindhard potential [1]. More recent work involves crystal-enhanced bremsstrahlung and pair production, also for much higher energies, which is reviewed in recent books and articles [3].

In the present paper we derive Dirac channeling wave functions including relativistic effects and spin effects, for use in calculations of channeling problems in crystals for very high electron or positron energies. The solutions are given as cylindrical waves for a transverse potential $V(\rho)$, with $(\rho,\phi,z)$ cylindrical coordinates. To obtain specific solutions we consider the $1/\rho$-dependent potential where Dirac solution including the order $\sqrt{\varepsilon/E}$ is obtained, with $\varepsilon$ the transverse energy and $E$ the total energy of the relativistic electron or positron. Both continuous and bound states are considered. We also include the effect of the lattice periodicity, the dependence of the potential on $z$. Since the main effect of the crystal in this high-energy region comes from the transverse degree of freedom, in particular for bound electron states, the effects of the longitudinal potential variation is included as a perturbation. We also give the Dirac wave functions for a (stepwise) constant transverse potential that can be of use for channeling conditions where strong screening effect are important. Perhaps the most convenient wave functions for high-energy applications are Sommerfeld-Maue-like wave functions. These are obtained in Sec. VI for a $1/\rho$ potential.

The production of electron-positron pairs by photons in a crystal is a process that has been studied experimentally and theoretically for many years [4]. The theoretical studies have been confined to semiclassical calculations, which seem to give useful results for experimental applications. The usefulness of approximate semiclassical methods must be considered in the light of the fact that exact calculations are difficult and in general complicated.

The present paper presents a quantum-mechanical calculation of pair production in a crystal for a crystal potential proportional to $1/\rho$, with $\rho=(x^2+y^2)^{1/2}$, the distance to the crystal string. With this potential the Dirac equation has high-energy two-dimensional Sommerfeld-Maue-like solutions, which can be used for exact calculations of the matrix elements. The calculation is similar to the calculation by Bethe and Maximon [5] of pair production on single atoms, although the present two-dimensional calculation proves to be more complicated. It is of course gratifying for the authors to note that matrix element calculations can be performed in two dimensions with methods similar to those used by Sommerfeld [6] and that the integrations give hypergeometric functions as in the three-dimensional case, which is not obvious prior to the calculation. These questions are discussed explicitly in Appendix B.

## II. DIRAC EQUATION IN CYLINDRICAL COORDINATES

The Dirac equation

$$\{i\vec{\gamma}\cdot\vec{\nabla}+\gamma_0[E-V(\rho)]-m\}\psi(\vec{r})=0 \tag{2.1}$$

for a static, cylindrically symmetric potential $V(\rho)$ is, in the standard representation,

$$\vec{\gamma}=\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma_0=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

$$\psi(\vec{r})=N\begin{pmatrix} \varphi(\rho,\phi) \\ \chi(\rho,\phi) \end{pmatrix} e^{ip_z z}, \tag{2.2}$$

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given by

$$(i\vec{\sigma}_\perp\cdot\vec{\nabla}_\perp-\sigma_z p_z)\chi+[E-V(\rho)-m]\varphi=0,$$
$$(i\vec{\sigma}_\perp\cdot\vec{\nabla}_\perp-\sigma_z p_z)\varphi+[E-V(\rho)+m]\chi=0, \qquad (2.3)$$

where

$$\vec{\sigma}_\perp\cdot\vec{\nabla}_\perp=\sigma_\rho\frac{\partial}{\partial\rho}+\sigma_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi}, \qquad (2.4)$$

with $\sigma_\rho=\begin{pmatrix}0 & e^{-i\phi}\\ 0 & 0\end{pmatrix}$ and $\sigma_\phi=\begin{pmatrix}0 & -ie^{-i\phi}\\ ie^{i\phi} & 0\end{pmatrix}$ satisfying the usual SU(2) characteristic equation

$$[\sigma_i,\sigma_j]=2i\varepsilon_{ijk}\sigma_k,$$

with indices $(\rho,\phi,z)=(1,2,3)$.

From Eq. (2.3) follows

$$\chi=(E-V+m)^{-1}(\sigma_z p_z-i\vec{\sigma}_\perp\cdot\vec{\nabla}_\perp)\varphi,$$

and the Dirac wave function is given by

$$\psi(\vec{r})=N\begin{pmatrix}1\\ (E-V+m)^{-1}(\sigma_z p_z-i\vec{\sigma}_\perp\cdot\vec{\nabla}_\perp)\end{pmatrix}\varphi(\rho,\phi)e^{ip_z z}, \qquad (2.5)$$

where the two-component spinor $\varphi(\rho,\phi)$ satisfies the second-order equation

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+(E-V)^2-E_z^2+i\sigma_\rho\frac{1}{(E-V+m)}\frac{dV}{d\rho}(\sigma_z p_z-i\vec{\sigma}_\perp\cdot\vec{\nabla}_\perp)\right]\varphi(\rho,\phi)=0, \qquad (2.6)$$

where we have introduced the ''longitudinal energy'' $E_z$ by

$$E_z^2=p_z^2+m^2. \qquad (2.7)$$

The presence of $\sigma_\rho\vec{\sigma}_\perp\cdot\vec{\nabla}_\perp$ in the last term of Eq. (2.6) shows that $\varphi(\rho,\phi)$ is not an eigenstate of the $z$ component of the angular momentum

$$L_z=-i\frac{\partial}{\partial\phi}.$$

The spinor $\varphi(\rho,\phi)$ is a superposition of states of $L_z=\mu$ and $\mu+1$, as is easily seen since $\sigma_\rho\sigma_z=i\sigma_\phi$ essentially interchanges the $\rho$-dependent wave functions $u(\rho)$ and $v(\rho)$ in $\varphi(\rho,\phi)$:

$$\varphi(\rho,\phi)=\begin{pmatrix}u(\rho)e^{i\mu\phi}\\ v(\rho)e^{i(\mu+1)\phi}\end{pmatrix},$$
$$i\sigma_\phi\varphi(\rho,\phi)=\begin{pmatrix}v(\rho)e^{i\mu\phi}\\ -u(\rho)e^{i(\mu+1)\phi}\end{pmatrix}. \qquad (2.8)$$

In fact, $\varphi(\rho,\phi)$ is an eigenstate of the $z$ component of the total angular momentum

$$J_z=L_z+s_z=\mu+\tfrac{1}{2}.$$

The two differential equations of second order are, from Eq. (2.6),

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{\mu^2}{\rho^2}+(E-V)^2-E_z^2\right)u(\rho)+\frac{i}{(E-V+m)}\frac{dV}{d\rho}\left[i\left(\frac{\partial}{\partial\rho}-\frac{\mu}{\rho}\right)u(\rho)-p_z v(\rho)\right]=0, \qquad (2.9)$$

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{(\mu+1)^2}{\rho^2}+(E-V)^2-E_z^2\right)v(\rho)+\frac{i}{(E-V+m)}\frac{dV}{d\rho}\left[i\left(\frac{\partial}{\partial\rho}+\frac{\mu+1}{\rho}\right)v(\rho)+p_z u(\rho)\right]=0.$$

Equations (2.5), (2.8), and (2.9) are the basis of our further discussion. These equations are exact. Related to channeling we shall discuss the case of very high longitudinal energies as compared to transverse and potential energies. We shall, however, first discuss the case of a free electron $V=0$, described in cylindrical coordinates in order to relate our solution to the plane-wave solution of a free electron. At the same time we include the case of a constant potential

$$V(\rho)=\begin{cases}V_0 & \text{for } \rho<\rho_0\\ 0 & \text{for } \rho>\rho_0,\end{cases} \qquad (2.10)$$

which we shall use as a strong screening potential for channeling process. It is to be noted that in this case the solution of the Dirac equation is exact, valid for all energies. This is a useful check on solutions to the channeling processes for which exact solution may not be obtained.

## III. CASE OF A CONSTANT POTENTIAL, INCLUDING $v=0$, A FREE PARTICLE

Equation (2.9) for a constant potential $V=V_0$ shows that $u(\rho)$ and $v(\rho)$ are Bessel functions

$$u(\rho)=u_0 J_\mu(\pi\rho),$$
$$v(\rho)=iv_0 J_{\mu+1}(\pi\rho), \qquad (3.1)$$

with $\pi$ a quantity of dimension momentum

$$\pi=[(E-V)^2-E_z^2]^{1/2}, \qquad (3.2)$$

with $u_0$ and $v_0$ constants. For a free particle, $\pi=p_\perp$ is the transverse momentum. For a bound state, for $V\neq 0$, $\pi$ is imaginary and the Bessel functions in Eq. (3.1) are replaced by McDonald Bessel fucntions [7] $K_\mu(\pi\rho)$ and $K_{\mu+1}(\pi\rho)$.

With $\varphi(\rho,\phi)$ given by

$$\varphi(\rho,\phi)=\begin{pmatrix}u_0 J_\mu(\pi\rho)e^{i\mu\phi}\\ iv_0 J_{\mu+1}(\pi\rho)e^{i(\mu+1)\phi}\end{pmatrix} \qquad (3.3)$$

and with the summation of Bessel functions [7]

$$\sum i^\mu J_\mu(\pi\rho)e^{i\mu\phi}=e^{i\pi\rho\cos\phi},$$

one obtains the plane-wave solution for $\rho > \rho_0$, i.e., for $\pi = p_\perp$,

$$\sum \psi_\mu(\rho,\phi,z) = N \sum i^\mu \begin{pmatrix} 1 \\ \dfrac{-i\vec{\sigma}\nabla}{(E-V+m)} \end{pmatrix} \times \begin{pmatrix} u_0 J_\mu(\pi\rho) e^{i\mu\phi} \\ i v_0 J_{\mu+1}(\pi\rho) e^{i(\mu+1)\phi} \end{pmatrix} e^{ip_z z} = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ \dfrac{\sigma_x p_\perp + \sigma_z p_z}{(E-V+m)} \end{pmatrix} \times \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} e^{ip_\perp \rho \cos\phi + ip_z z}, \tag{3.4}$$

giving also the normalization constant. This is the partial cylindrical wave expansion for a free particle with the momentum $\vec{p}$ in the $x$-$z$ plane. By a rotation $\phi_p$ or by a different choice of constants $u_0$ and $v_0$, $u_0 \exp(-i\mu\phi_p)$ and $v_0 \exp[-i(\mu+1)\phi_p]$ in Eq. (3.1),

$$\varphi(\rho,\phi) = \begin{pmatrix} u_0 J_\mu(\pi\rho) e^{i\mu(\phi-\phi_p)} \\ i v_0 J_{\mu+1}(\pi\rho) e^{i(\mu+1)(\phi-\phi_p)} \end{pmatrix}, \tag{3.5}$$

the asymptotic plane wave in an arbitrary direction with respect to the rotational symmetry $z$ axis is obtained, proportional to $\exp i[p_\perp \rho \cos(\phi-\phi_p) + p_z z]$.

## IV. CHANNELING STATES

From exact considerations so far, we now discuss channeling approximations. For relativistic particles moving in directions close to a crystal axis, the energy $E_z$, defined in Eq. (2.7), is much larger than transverse and potential energies for the most important parts of space. In order to subtract out the longitudinal energy $E_z$ we define $\varepsilon$ by

$$\varepsilon = E - E_z, \tag{4.1}$$

which will be used for continuum as well as for transversely bound states. With the approximations

$$\varepsilon \ll E_z, \quad |V(\rho)| \ll E_z,$$

Eq. (2.9) may be rewritten in the compact form

$$\{\nabla^+\nabla^- + 2E_z[\varepsilon - V(\rho)]\}u(\rho) - i\frac{p_z}{E}\frac{dV}{d\rho}v(\rho) = 0, \tag{4.2}$$

$$\{\nabla^-\nabla^+ + 2E_z[\varepsilon - V(\rho)]\}v(\rho) + i\frac{p_z}{E}\frac{dV}{d\rho}u(\rho) = 0,$$

where we have introduced

$$\nabla^+ = \frac{\partial}{\partial\rho} + \frac{\mu+1}{\rho}, \quad \nabla^- = \frac{\partial}{\partial\rho} - \frac{\mu}{\rho},$$

$$\nabla^+\nabla^- = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{\mu^2}{\rho^2}, \tag{4.3}$$

$$\nabla^-\nabla^+ = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{(\mu+1)^2}{\rho^2}.$$

Equations (4.2) are dominated by the term $2E_z(\varepsilon - V)$. Assuming that $V(\rho)$ is of the order $\varepsilon$ for the most important values of $\rho$, we read from the equations that the terms $\nabla^+\nabla^-$ and $\nabla^-\nabla^+$ are of order $\rho^{-2}$, which must be of order $E_z\varepsilon = (E_z/\varepsilon)\varepsilon^2$. Thus the terms that we have neglected $\varepsilon^2$, $\varepsilon V$, and $V^2$ are all of order $\varepsilon^2$ and therefore negligible. Furthermore, the term $(p_z/E)(dV/d\rho)$ is of order $\varepsilon/\rho = \sqrt{E_z/\varepsilon}\,\varepsilon^2$. This term may therefore be taken into account perturbatively. We shall show that the correction from this term to the wave function indeed is of relative order $\sqrt{\varepsilon/E_z}$.

To highest order in $E_z/\varepsilon$, the wave functions, which we denote $U_\mu$ and $U_{\mu+1}$, satisfy

$$\{\nabla^+\nabla^- + 2E_z[\varepsilon - V(\rho)]\}U_\mu(\rho) = 0, \tag{4.4a}$$

$$\{\nabla^-\nabla^+ + 2E_z[\varepsilon - V(\rho)]\}U_{\mu+1}(\rho) = 0, \tag{4.4b}$$

where we have used the fact that in Eq. (4.4), $v_\mu(\rho) = U_{\mu+1}(\rho)$.

In Eq. (4.2) we introduce

$$u(\rho) = U_\mu(\rho) + \Delta_u(\rho), \quad v(\rho) = U_{\mu+1}(\rho) + \Delta_v(\rho),$$

where $\Delta_u(\rho)$ and $\Delta_v(\rho)$ are small corrections. Keeping highest-order terms in $E_z/\varepsilon$, we obtain, from Eq. (4.2),

$$\{\nabla^+\nabla^- + 2E_z[\varepsilon - V(\rho)]\}\Delta_u(\rho) = i\frac{dV}{d\rho}U_{\mu+1}(\rho),$$
$$\{\nabla^-\nabla^+ + 2E_z[\varepsilon - V(\rho)]\}\Delta_v(\rho) = -i\frac{dV}{d\rho}U_\mu(\rho), \tag{4.5}$$

where we have set $p_z/E = 1$, which is in accordance with our approximation.

When we operate on Eq. (4.4a) by $\nabla^-$ and on Eq. (4.4b) by $\nabla^+$ we find the nice result

$$\{\nabla^-\nabla^+ + 2E_z[\varepsilon - V(\rho)]\}\nabla^- U_\mu(\rho) = 2E_z\frac{dV}{d\rho}U_\mu(\rho),$$
$$\{\nabla^+\nabla^- + 2E_z[\varepsilon - V(\rho)]\}\nabla^+ U_{\mu+1}(\rho) = 2E_z\frac{dV}{d\rho}U_{\mu+1}(\rho). \tag{4.6}$$

Comparing with Eqs. (4.5), we see that $\Delta_u(\rho)$ and $\Delta_v(\rho)$ satisfy the same equations as $(i/2E_z)\nabla^+ U_{\mu+1}(\rho)$ and $(-i/2E_z)\nabla^- U_\mu(\rho)$, respectively. Therefore,

$$\Delta_u(\rho) = \frac{i}{2E_z}\nabla^+ U_{\mu+1}(\rho) + \text{const}\,U_\mu(\rho), \tag{4.7}$$

$$\Delta_v(\rho) = -\frac{i}{2E_z}\nabla^- U_\mu(\rho) + \text{const} U_{\mu+1}(\rho),$$

where the last terms are contributions from the homogeneous equations (4.4). The constants are determined by the requirements that $\Delta_u(\rho)$ and $\Delta_v(\rho)$ must vanish for $V(\rho)=0$. This gives

$$\Delta_u(\rho) = \frac{i}{2E_z}[\nabla^+ U_{\mu+1}(\rho) - \sqrt{2E_z\varepsilon}\,U_\mu(\rho)],$$
$$\Delta_v(\rho) = -\frac{i}{2E_z}[\nabla^- U_\mu(\rho) + \sqrt{2E_z\varepsilon}\,U_{\mu+1}(\rho)]. \tag{4.8}$$

The spinor $\varphi(\rho,\phi)$ then becomes

$$\varphi(\rho,\phi) = \left(1 - i\sqrt{\frac{\varepsilon}{2E_z}} + i\,\frac{\sigma_z\vec{\sigma}_\perp\cdot\vec{\nabla}_\perp}{2E_z}\right)\varphi_\mu(\rho,\phi), \tag{4.9}$$

with

$$\varphi_\mu(\rho,\phi) = \begin{pmatrix} U_\mu(\rho)e^{i\mu\phi} \\ U_{\mu+1}(\rho)e^{i(\mu+1)\phi}\end{pmatrix}, \tag{4.10}$$

and where we have used

$$\begin{pmatrix} \nabla^+ U_{\mu+1}(\rho)e^{i\mu\phi} \\ \nabla^- U_\mu(\rho)e^{i(\mu+1)\phi}\end{pmatrix} = \vec{\sigma}_\perp\cdot\vec{\nabla}_\perp\varphi_\mu(\rho,\phi)$$

in Eq. (4.9). The Dirac channeling wave function is then

$$\psi_{\mu,p_z}(\vec{r}) = N\left(1 - i\sqrt{\frac{\varepsilon}{2E_z}} + i\gamma_0\,\frac{\sigma_z\vec{\sigma}_\perp\cdot\vec{\nabla}_\perp}{2E_z}\right) \times\begin{pmatrix} 1 \\ (E-V+m)^{-1}(\sigma_z p_z - i\vec{\sigma}_\perp\cdot\vec{\nabla}_\perp)\end{pmatrix} \times\varphi_\mu(\rho,\phi)e^{ip_z z}. \tag{4.11}$$

We have used here that $\sigma_z\vec{\sigma}_\perp\cdot\vec{\nabla}_\perp$ anticommutes with $(\sigma_z p_z - i\vec{\sigma}_\perp\cdot\vec{\nabla}_\perp)$ and that $dV(\rho)/d\rho$ gives a negligible term in our approximation. As seen from Eq. (4.11), the correction terms are indeed of relative order $\sqrt{\varepsilon/E_z}$, as stated above.

## V. $1/\rho$ POTENTIAL

The wave equations (4.4) can be written in the compact form

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{\kappa^2}{\rho^2} + 2E_z[\varepsilon - V(\rho)]\right]U_\kappa(\rho) = 0, \tag{5.1}$$

with $\kappa=\mu$ and $\mu+1$ for the upper and lower spinor components, respectively, in Eq. (4.10). We shall discuss in this section the solutions for the approximate crystal potential

$$V(\rho) = -\frac{Z\alpha}{\rho}\frac{a}{b}\,c = -V_0 a/\rho, \tag{5.2}$$

with $b$ the interatomic distance in the crystal row, $a$ the screening (e.g., Thomas-Fermi) length, and $c$ an empirical constant. The potential has been used in several calculations [2(b),8]. The potential may be considered reliable for $\rho$, distances of the order of the interatomic distance along the crystal row.

With the substitution

$$U_\kappa(\rho) = \sqrt{\rho}\,f_\kappa(\rho), \tag{5.3}$$

the wave equation (5.1) is of the same form as the equation in a spherical symmetric potential, namely,

$$\left[\frac{d^2}{d\rho^2} + \frac{2}{\rho}\frac{d}{d\rho} - \frac{\kappa^2-\frac{1}{4}}{\rho^2} + 2E_z[\varepsilon - V(\rho)]\right]f_\kappa(\rho) = 0, \tag{5.4}$$

compared to the Schrödinger equation for a spheric symmetrical potential $V(r)$,

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + 2m[\varepsilon - V(r)]\right]R_l(r) = 0. \tag{5.5}$$

At the same time the normalization for bound states are identical

$$\int U^2_{\kappa,n}(\rho)\rho\,d\rho = \int f^2_{\kappa,n}(\rho)\rho^2 d\rho = \int R^2_{l,n}(r)r^2 dr = 1. \tag{5.6}$$

This shows that the cylindrical wave functions and energy levels may be obtained directly from the corresponding sperical wave functions, if these are known. The substitutions are

$$l\to\kappa-\tfrac{1}{2},\quad m\to E_z,\quad E\to\varepsilon,\quad V(r)\to V(\rho), \tag{5.7}$$

for potentials of identical functional dependences.

For the potential (5.2) the substitution in the potential is

$$Z\alpha/r\to V_0 a/\rho,\quad V_0 = cZ\alpha/b. \tag{5.8}$$

The continuum states are then obtained from the hydrogenlike states [9] as

$$U^\pm_\kappa(\rho) = \exp\left(\mp i\sigma_{|\kappa|} + \frac{\pi}{2}\frac{V_0 a E_z}{p_\perp}\right)\sqrt{\frac{2}{\pi}\rho}\,\frac{|\Gamma(-\eta+\kappa+\frac{1}{2})|}{\Gamma(2\kappa+1)} \times\exp(-ip_\perp\rho)(2p_\perp)(2p_\perp\rho)^{\kappa-(1/2)} \times F(-\eta+\kappa+\tfrac{1}{2};2\kappa+1;2ip_\perp\rho), \tag{5.9}$$

where $\sigma_{|\kappa|} = \arg\Gamma(-\eta+|\kappa|+\frac{1}{2})$, $\eta = -iV_0 aE_z/p_\perp$, $p_\perp = \sqrt{E^2-E_z^2}\approx\sqrt{2E_z\varepsilon}$, and $F(a;c,z)$ is the confluent hypergeometric function, the Kummer function. $U^+_\kappa(\rho)$ and $U^-_\kappa(\rho)$ are solutions with an asymptotic form plane wave plus outgoing and ingoing cylindrical waves, respectively, as seen from the asymptotic form

$$U^\pm_\kappa(\rho) = \exp(\mp i\sigma_{|\kappa|})\sqrt{\frac{2}{\pi\rho}}\cos\left(p_\perp\rho + \frac{V_0 aE_z}{p_\perp}\ln(2p_\perp\rho) - \frac{\pi}{2}(\kappa+\tfrac{1}{2}) - \sigma_{|\kappa|}\right). \tag{5.10}$$

As usual with a Coulomb potential, in order to obtain pure outgoing or ingoing solutions, one has to assume a formal screening at $\rho\to\infty$ that removes the logarithmic function at large distances.

For bound states one obtains from hydrogenlike atoms [9]

$$U_{n,\kappa}(\rho)=\sqrt{\rho}\,\frac{1}{\Gamma(2\kappa+1)}\left(\frac{\Gamma(\eta+\kappa+\frac{1}{2})}{\Gamma(\eta-\kappa+\frac{1}{2})2\eta}\right)^{1/2}\times(2\sqrt{-2\varepsilon_n E_z})^{3/2}\exp(-\sqrt{-2E_z\varepsilon_n}\rho)\times(2\sqrt{-2\varepsilon_n E_z}\phi)^{\kappa-(1/2)}F(-\eta+\kappa+\tfrac{1}{2};2\kappa+1;2\sqrt{-2E_z\varepsilon_n}\rho),\tag{5.11}$$

with the energy eigenvalues, for $-\eta+\kappa+\frac{1}{2}=-n$,

$$\varepsilon_n=-\frac{(Z\alpha)^2E_z}{2(n+\kappa+\frac{1}{2})^2}\left(c\,\frac{a}{b}\right)^2,\tag{5.12}$$

which gives a $2n+1$ degeneracy $-n<\kappa<n$.

As shown in Appendix A, continuum and bound states (5.9) and (5.11) are valid for positive and negative values of $\kappa$, i.e., $\mu$ and $\mu+1$. There are no singularities for negative values of $\kappa$.

## VI. TWO-DIMENSIONAL SOMMERFELD-MAUE-LIKE WAVE FUNCTIONS

Operating with $\gamma_0\{-i\vec{\gamma}\cdot\vec{V}+\gamma_0[E-V(\rho)]+m\}\gamma_0$ on Eq. (2.1), one obtains the second-order Dirac equation

$$[\nabla^2+p^2-2EV(\rho)]\psi(\vec{r})=[-i\gamma_0\vec{\gamma}\cdot\vec{\nabla}V(\rho)-V^2(\rho)]\psi(\vec{r}),\tag{6.1}$$

with the usual Sommerfeld-Maue-type [10] approximate solution

$$\psi(\vec{r})=e^{i\vec{p}\cdot\vec{r}}\left(1-\frac{i}{2E}\,\gamma_0\vec{\gamma}\cdot\vec{\nabla}\right)e^{-i\vec{p}\cdot\vec{r}}\psi_0(\vec{r}),\tag{6.2}$$

where $\psi_0(\vec{r})$ is the solution of the equation

$$[\nabla^2+p^2-2EV(\rho)]\psi_0(\vec{r})=0.\tag{6.3}$$

The usual Sommerfeld-Maue solution is for the Coulomb potential. For a string $1/\rho$ potential the situation is different; still a solution similar to the Coulomb case is obtained.

With the substitution $\psi_0(\vec{r})=\exp(ip_z z)\varphi(\rho,\phi)$ and the coordinates

$$\xi=\rho(1+\cos\phi),\quad \eta=\rho(1-\cos\phi),$$

we find

$$\left(2\xi\,\frac{\partial^2}{\partial\xi^2}+\frac{\partial}{\partial\xi}+2\eta\,\frac{\partial^2}{\partial\eta^2}+\frac{\partial}{\partial\eta}+\frac{p_\perp^2}{2}(\xi+\eta)+2EV_0a\right)\varphi(\xi,\eta)=0,\tag{6.4}$$

with the Sommerfeld-Maue-like solution

$$\varphi(\xi,\eta)=Ne^{i(p_\perp/2)(\xi-\eta)}F\left(iV_0a\,\frac{E}{p_\perp};\,\frac{1}{2};ip_\perp\eta\right),\tag{6.5}$$

where $F(\eta)$ is the Kummer function. The wave function $\psi(\vec{r})$ is then

$$\psi(\vec{r})=Ne^{i\vec{p}\cdot\vec{r}}\left(1-\frac{i}{2E}\,\gamma_0\vec{\gamma}\cdot\vec{\nabla}_\perp\right)\times F\left(iV_0a\,\frac{E}{p_\perp};\,\frac{1}{2};i(p_\perp\rho-\vec{p}_\perp\vec{\rho})\right)u.\tag{6.6}$$

The Dirac spinor effect has been taken into account by multiplication with the free-particle. Dirac spinor $u$. The asymptotic wave function is of the form

$$\psi_0(\vec{r})=N\sqrt{\pi}\,\frac{e^{-\pi d/2}}{\Gamma(\frac{1}{2}-id)}\,e^{ip_z z}\left[e^{i\vec{p}_\perp\vec{\rho}-id\,\ln(p_\perp\rho-\vec{p}_\perp\vec{\rho})}+\frac{\Gamma(\frac{1}{2}-id)}{\Gamma(id)}\,e^{-i\pi/4}\,\frac{e^{ip_\perp\rho+id\,\ln(p_\perp\rho-\vec{p}_\perp\vec{\rho})}}{\sqrt{p_\perp\rho-\vec{p}_\perp\vec{\rho}}}\right)u,\tag{6.7}$$

with $d=V_0aE/p_\perp$. This shows that $\psi(\vec{r})$ [Eq. (6.6)] describes a plane wave plus a cylindrical outgoing wave. The ingoing cylindrical wave solution is obtained by replacing $ip_\perp\rho$ in Eq. (6.6) by $-ip_\perp\rho$. Equation (6.6) also shows that the normalizations constant $N$ is given by

$$N=\frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}-id\right)e^{\pi d/2}=\frac{1}{\sqrt{\pi}}\left|\Gamma\left(\frac{1}{2}-id\right)\right|e^{\pi d/2+i\lambda}=(\cosh\pi d)^{-1/2}e^{\pi d/2+i\lambda},\tag{6.8}$$

with

$$\lambda=\arg\,\Gamma(\tfrac{1}{2}-id),$$

which gives the plane-wave part of $\psi(\vec{r})$ [Eq. (6.7)], $\exp(i\vec{p}\cdot\vec{r})u$ with the normalized free-particle spinor $u$. It should also be noted that the asymptotic cylindrical wave [Eq. (6.7)] has the $\rho$ dependence $1/\sqrt{\rho}$, as it should.

## VII. PERIODICITY ALONG THE CRYSTAL AXIS

We take into account the periodic variation of the potential along the crystal axis $V(\rho,z)$ by an expansion in Fourier series

$$V(\rho,z)=V(\rho)+\sum_{k=1}^{\infty}V_k(\rho)\cos(g_kz).\tag{7.1}$$

Here $V(\rho)$ is the potential used in the previous sections, which can be written as

$$V(\rho)=\int_{z_0}^{z_0+a_z}V(\rho,z)\,\frac{dz}{a_z},\tag{7.2}$$

while the $z$ dependence is taken into account by the coefficients $V_k(\rho)$,

$$V_k(\rho)=2\int_{z_0}^{z_0+a_z}V(\rho,z)\cos(g_k z)\,\frac{dz}{a_z}, \tag{7.3}$$

where $g_k=2\pi k/a_z$ are reciprocal lattice vectors, with $k$ an integer, and $a_z$ is the atomic distance along the crystal axis and $z_0$ is arbitrary.

The Dirac equation (2.1) now becomes

$$\{i\vec{\gamma}\cdot\vec{\nabla}+\gamma_0[E-V(\rho)-Z(\rho,z)]-m\}\psi_Z(\vec{r})=0, \tag{7.4}$$

where the index $Z$ indicates the $z$-dependent potential

$$Z(\rho,z)=\sum_{k=-\infty}^{\infty}V_k(\rho)e^{ig_k z},\quad k\neq 0,$$

and $\psi'_{Z=0}(\vec{r})$ is given by Eq. (2.5) for $Z(\rho,z)=0$ with the $z$ dependence $\exp(ip_z z)$.

We shall solve the Dirac equation (7.4) assuming a solution

$$\psi_Z(\vec{r})=[1+\hat{Z}(\vec{r})]\psi(\vec{r}), \tag{7.5}$$

with $\hat{Z}(\vec{r})$ a small perturbation. Introducing $\psi_Z(\vec{r})$ into Eq. (7.4), we find

$$\{i\vec{\gamma}\cdot\vec{\nabla}+\gamma_0[E-V(\rho)-Z(\rho,z)]-m\}\hat{Z}(\vec{r})\psi(\vec{r})=\gamma_0 Z(\vec{r})\psi(\vec{r}). \tag{7.6}$$

When we neglect the small terms $V\hat{Z}(\vec{r})$, $Z\hat{Z}(\vec{r})$, and $i\vec{\sigma}_\perp\nabla_\perp\hat{Z}(\vec{r})\psi(\vec{r})$ we find

$$\left(i\gamma_z\frac{\partial}{\partial_Z}+\gamma_0 E-m\right)\hat{Z}(\vec{r})\psi(\vec{r})=\gamma_0 Z(\vec{r})\psi(\vec{r}).$$

Expanding $\hat{Z}(\vec{r})$,

$$\hat{Z}(\vec{r})=\sum_{k=-\infty}^{\infty}Z_k(\rho)e^{iq_k z}, \tag{7.7}$$

we find

$$Z_k(\rho)=[\gamma_0 E-\gamma_z(p_z+g_k)-m]^{-1}\gamma_0 V_k(\rho), \tag{7.8}$$

and the wave function including the periodicity along the lattice string is given by Eq. (7.5), with

$$\hat{Z}(\vec{r})=\sum_{k=-\infty}^{\infty}\frac{1}{2E_z(\varepsilon-g_k)}\times[\gamma_0 E-\gamma_z(p_z+g_k)+m]\gamma_0 V_k(\rho)e^{ig_k z},\quad k\neq 0, \tag{7.9}$$

in our approximations $E_z\gg\varepsilon$ and $g_k$. The wave function for the potential $V(\rho,z)$ [Eq. (7.1)] is therefore given by Eq. (7.5),

$$\psi_Z(\vec{r})=[1+\hat{Z}(\vec{r})]\psi(\vec{r}), \tag{7.10}$$

with $\hat{Z}(\vec{r})$ given by Eq. (7.9) and $\psi(\vec{r})$ by Eqs. (4.10) and (4.11),

$$\psi(\vec{r})=\psi_{\mu,p_z}(\vec{r}), \tag{7.11}$$

where $U_\mu(\rho)$ and $U_{\mu+1}(\rho)$ are obtained from Eqs. (5.9) and (5.11) for continuum and bound states, with $\kappa=\mu$ or $\mu+1$.

## VIII. MATRIX ELEMENTS

With a potential $V(\rho)$ that does not take into account the crystal structure along the crystal string, no momentum can be transferred in the string direction, which we take as the $z$ axis,

$$q_z=k_z-p_z^+-p_z^-=0, \tag{8.1}$$

where $\vec{q}$ is the momentum transfer and $\vec{k},\vec{p}^{\,+},\vec{p}^{\,-}$, are the momenta of the photon, positron, and electron, respectively. Now if the photon momentum is parallel to the crystal string, $k_z=\omega$, the momentum and energy balance cannot be maintained. In order to obtain pair production then, $\vec{k}$ must have a transverse component $\vec{k}_\perp$, giving $\vec{q}_\perp=\vec{k}_\perp-\vec{p}_\perp^{\,+}-\vec{p}_\perp^{\,-}$. Only photons hitting the crystal string at a small angle $\delta=k_\perp/\omega$ larger than

$$\delta_{\min}=\frac{k_{\perp\,\min}}{\omega}>\frac{m}{\sqrt{E_+E_-}} \tag{8.2}$$

can produce pairs. The minimum value $k_{\perp\,\min}$ is obtained from the useful high-energy, small-angle relation

$$\frac{\omega}{E_+}(p_\perp^+)^2+\frac{\omega}{E_-}(p_\perp^-)^2=k_\perp^2-\frac{\omega^2}{E_+E_-}m^2. \tag{8.3}$$

These considerations do not seem to have been considered to be of importance and taken into account in the published papers on semiclassical calculations.

The cross section for pair production is given by

$$d^4\sigma=\frac{1}{(2\pi)^4}\frac{\alpha}{\omega}|M|^2\delta^4(k-p_+-p_--q)d^3p_+d^3p_-d^3q, \tag{8.4}$$

with $\alpha$ the fine-structure constant and $\omega$ the photon energy. $M$ is the matrix element

$$M=\int d^3x\ \overline{\psi_-}(\vec{r})\vec{\gamma}\cdot\vec{e}e^{i\vec{k}\cdot\vec{r}}\psi_+(\vec{r}), \tag{8.5}$$

with $\psi_\pm(\vec{r})$ the positron and electron wave functions, $\vec{e}$ the photon polarization, and $\vec{\gamma}$ the Dirac vector matrix.

It is convenient to factor out the $z$-dependent part of the matrix element, which is

$$|M_z|^2=\left|\int_0^L e^{i(k_z-p_z^+-p_z^-)z}dz\right|^2=2\pi L\,\delta(q_z) \tag{8.6}$$

for large coherence lengths $L$. Integrating out the redundant coordinates in Eq. (2.4), we find the physical cross section per unit length of the crystal

$$d^4\sigma/L=\frac{1}{(2\pi)^3}\frac{\alpha}{\omega}|M_\perp|^2\frac{E_+E_-}{|p_z^+E_--p_z^-E_+|}\times p_\perp^+dp_\perp^+d\varphi^+p_\perp^-dp_\perp^-d\varphi^-, \tag{8.7}$$

with the transverse part of the matrix element

$$M_\perp=\overline{u_-}\int d^2\rho\ \overline{F_-}(\vec\rho)\vec\gamma\cdot\vec e E^{i\vec q\cdot\vec\rho}F_+(\vec\rho)u_+, \tag{8.8}$$

where $F_\pm(\vec\rho)$ is obtained from Eq. (6.6),

$$F_\pm(\vec\rho)=N_\pm\left(1\pm\frac{i}{2E_\pm}\gamma_0\vec\gamma_0\cdot\vec\nabla_\perp\right)\times F(-id_\pm;\tfrac12;\pm i(p_\perp^\pm\rho+\vec p_\perp^\pm\vec\rho)), \tag{8.9}$$

$u_\pm$ are the free-particle positron and electron spinors, $d_\pm=d(E_\pm/p^\pm)$, and $d=Z\alpha(a/b)c$, with the parameters defined in Sec. VI. The electron and positron wave functions describing produced particles are accordingly asymptotically given by plane waves plus cylindrical ingoing waves.

When the sum over electron and positron polarizations and the average over photon polarizations are performed we find

$$\begin{aligned}\frac12\sum_{\rm pol}|M_\perp|^2=&\frac{(N_+N_-)^2}{E_+E_-}([E_+E_-+m^2-(\vec p_+\cdot\hat{\vec k})(\vec p_-\cdot\hat{\vec k})]|I_1|^2\\&+[E_+E_--m^2+(\vec p_+\cdot\hat{\vec k})(\vec p_-\cdot\hat{\vec k})](|I_-|^2+|\vec I_+|^2)\\&+2\,\mathrm{Re}\{E_-I_1^*[(\vec I_+\cdot\vec p_+)-(\vec p_+\cdot k)(\vec I_-\cdot\hat{\vec k})]\\&+E_+I^*[(\vec I_-\cdot\vec p_-)-(\vec p_-\cdot\hat{\vec k})(\vec I_+\cdot\hat{\vec k})]\}),\end{aligned} \tag{8.10}$$

where the integrals $I_1$, $I_+$, and $I_-$ are given by

$$\begin{aligned}I_1&=\int d^2\rho\ F_-(\vec\rho_-)e^{i\vec q_\perp\cdot\vec\rho}F_+(\vec\rho),\\I_+&=\frac{i}{2E_+}\int d^2\rho\ F_-(\vec\rho)e^{i\vec q_\perp\cdot\vec\rho}\nabla_\perp F_+(\vec\rho),\\\vec I_-&=\frac{i}{2E_-}\int d^2\rho[\nabla_\perp F_-(\vec\rho)]e^{i\vec q_\perp\cdot\vec\rho}F_+(\vec\rho),\end{aligned} \tag{8.11}$$

where

$$F_\pm(\vec\rho)=F(\mp id_\pm;\tfrac12;i(p_\perp^\pm\rho+\vec\rho^{\,\pm}\rho)). \tag{8.12}$$

As first used by Sommerfeld [6] and later by Nordsieck [11], an integral

$$I_0=\int d^2\rho\ \rho^{-1}F_-(\vec\rho)e^{i\vec q_\perp\cdot\vec\rho-\varepsilon\rho}F_+(\vec\rho) \tag{8.13}$$

is defined that makes it possible to derive all integrals from $I_0$ by the use of the relation

$$\nabla_\perp(p_\perp\rho+\vec p_\perp\cdot\vec\rho)=(p_\perp/\rho)\nabla_{p_\perp}(p_\perp\rho+\vec p_\perp\cdot\vec\rho),$$

which gives

$$\begin{aligned}I_1&=-\frac{\partial}{\partial\varepsilon}I_0(\vec p_\perp^{\,\pm},\varepsilon)_{\varepsilon=0},\\\vec I_\pm&=(ip_\perp^\pm/2E_\pm)\nabla_{p_\perp^\pm}I_0(\vec p_\perp^{\,\pm},\varepsilon)_{\varepsilon=0}.\end{aligned} \tag{8.14}$$

The calculation of the integral $I_0$, following Sommerfeld's [6] method of integration, also used by Nordsieck [11], is performed in Appendix B.

Performing the derivations in Eq. (2.14), we find the final results for the integrals

$$\begin{aligned}I_1=C\Bigg\{&\left(\frac{p_\perp^-d_-}{D_-}-\frac{p_\perp^+d_+}{D_+}\right)F(x)\\&+i\left[\mu^2\left(\frac{p_\perp^-}{D_-}+\frac{p_\perp^+}{D_+}\right)-p_\perp^+-p_\perp^-\right]G(x)\Bigg\}\end{aligned}$$

$$\begin{aligned}\vec I_\pm=C\,\frac{p_\perp^\pm}{2E_\pm}\Bigg\{&\mp d_\pm\frac{\vec q_\perp}{D_\pm}F(x)\\&+i\left[\left(\frac{\mu^2}{D_\pm}-1\right)\vec q_\perp-\left(\vec p_\perp^{\,\mp}-\frac{p_\perp^\mp}{p_\perp^\pm}\vec p_\perp^{\,\pm}\right)\right]G(x)\Bigg\}.\end{aligned} \tag{8.15}$$

Here

$$C=\frac{4\pi}{q_\perp}\left(\frac{q_\perp^2}{D_\pm}\right)^{-id_+}\left(\frac{q_\perp^2}{D_-}\right)^{id_-},\quad D_\pm=q_\perp^2+2(\vec q_\perp p_\perp^\pm),$$

$$\mu^2=\vec k_\perp^2-(p_\perp^++p_\perp^-)^2$$

and

$$\begin{aligned}F(x)&={}_2F_1(-id_+,id_-;\tfrac12;x),\\G(x)&=2q_\perp^2\frac{d_+d_-}{D_+D_-}{}_2F_1(1-id_+,1+id_-;\tfrac32;x),\end{aligned} \tag{8.16}$$

with ${}_2F_1$ the hypergeometric function and

$$x=(2/D_+D_-)\{q_\perp^2(p_\perp^+p_\perp^--\vec p_\perp^{\,+}\cdot\vec p_\perp^{\,-})+2(\vec q_\perp\cdot\vec p_\perp^{\,+})(\vec q_\perp\cdot\vec p_\perp^{\,-})\}. \tag{8.17}$$

## IX. CROSS SECTION

In the further calculation it is convenient to introduce the vectors $V_+^\perp$ and $V_-^\perp$,

$$\vec V_\pm=\vec p_\perp^{\,\pm}-E_\pm(\vec k_\perp/\omega) \tag{9.1}$$

and note that

$$\vec{q}_\perp = \vec{k}_\perp - \vec{p}_\perp^{\,+} - \vec{p}_\perp^{\,-} = -(\vec{V}_+ + \vec{V}_-). \tag{9.2}$$

In addition,

$$D_\pm = q_\perp^2 + 2(\vec{q}_\perp \vec{p}_\pm^{\,+}) = \frac{\omega}{E_\mp}(m^2 + V_\mp^2), \tag{9.3}$$

which shows the convenience of introducing, in analogy to Ref. [5],

$$\xi = (m^2 + V_+^2)^{-1}, \quad \eta = (m^2 + V_-^2)^{-1}. \tag{9.4}$$

With these notations, equations in Sec. VIII simplify considerably.

Equation (2.10) becomes, for high energies and small angles,

$$\frac{1}{2}\sum_{\text{pol}} |M_\perp|^2 = \frac{|N_+ N_-|^2}{E_+ E_-}\left\{\frac{1}{2E_+E_-}(E_+^2 V_-^2 + E_-^2 V_+^2 + \omega^2 m^2)|I_1|^2 + 2E_+E_-(|\vec{I}_+|^2 + |\vec{I}_-|^2) + 2\,\mathrm{Re}\{I_1^*[E_+(\vec{I}_-\cdot\vec{V}_+) + E_-(\vec{I}_+\cdot\vec{V}_-)]\}\right\}. \tag{9.5}$$

Likewise, the integrals (8.15) simplify

$$I_1 = C\left\{d\,\frac{E_+E_-}{\omega}(\xi-\eta)F(x) + i\left[\frac{\mu^2}{\omega}(p_\perp^- E_+\xi + p_\perp^+\xi_- - \zeta) - p_\perp^+ - p_\perp^-\right]G(x)\right\},$$

$$\vec{I}_+ = C\,\frac{p_\perp^+}{2E_+}\left\{-d\,\frac{\vec{q}_\perp}{p_\perp^+}\,\frac{E_+E_-}{\omega}\,\eta F(x) + i\left[\left(\frac{\mu^2 E_-}{\omega}\,\eta - 1\right)\vec{q}_\perp - \left(\vec{p}_\perp^{\,-} - \frac{p_\perp^-}{p_\perp^+}\vec{p}_\perp^{\,+}\right)\right]G(x)\right\},$$

$$\vec{I}_- = C\,\frac{p_\perp^-}{2E_-}\left\{-d\,\frac{q_\perp}{p_\perp^-}\,\frac{E_+E_-}{\omega}\,\xi F(x) + i\left[\left(\frac{\mu^2 E_+}{\omega}\,\xi - 1\right)\vec{q}_\perp - \left(\vec{p}_\perp^{\,+} - \frac{p_\perp^+}{p_\perp^-}\vec{p}_\perp^{\,-}\right)\right]G(x)\right\}, \tag{9.6}$$

where $d = d_\pm \rho_\perp^\pm / E_\pm$, while $x$ can be written as

$$x = 4p_\perp^+ p_\perp^-\,\frac{E_+E_-}{\omega^2}\,\xi\eta(\vec{V}_+ + \vec{V}_-)^2\cos^2\left(\frac{\phi_+ + \phi_-}{2}\right), \tag{9.7}$$

where $\phi_+$ and $\phi_-$ are the angles in the $\vec{\rho}^{\,\perp}$ plane given by

$$\hat{q}_\perp \hat{p}_\perp^\pm = \cos\phi_\pm .$$

Alternatively, $x$ may be expressed as

$$x = 4p_\perp^+ p_\perp^-\,\frac{E_+E_-}{\omega^2}\,\xi\eta\left[k_\perp\cos\left(\frac{\varphi_+ + \varphi_-}{2}\right) - (p_\perp^+ + p_\perp^-)\cos\left(\frac{\varphi_+ - \varphi_-}{2}\right)\right]^2, \tag{9.8}$$

where the angles $\varphi_+$ and $\varphi_-$ refer to the fixed vector $\vec{k}_\perp$,

$$\hat{k}_\perp \hat{p}_\perp^\pm = \cos\varphi_\pm .$$

In order to obtain the cross section, we want the polarization-independent matrix element squared written in terms of $F(x)$ and $G(x)$. We define the coefficients $f$, $g$, and $h$, rewriting Eq. (9.5) in the form

$$\frac{1}{2}\sum_{\text{pol}} |M_\perp|^2 = \frac{|N_+N_-|^2}{E_+E_-}|C|^2\{f|F(x)|^2 + g|G(x)|^2 + h\ \mathrm{Im}[F^*(x)G(x)]\}. \tag{9.9}$$

After some algebra we find

$$f = d^2\,\frac{E_+E_-}{\omega^2}\left\{\frac{1}{2}\,\omega^2(V_+^2 + V_-^2)\xi\eta + (E_+^2 + E_-^2)\vec{V}_+\cdot\vec{V}_-\xi\eta - E_+E_-(V_+^2\xi^2 + V_-^2\eta^2)\right\}, \tag{9.10a}$$

$$g = \frac{\mu^4}{\omega^2}\left\{\frac{m^2\omega^2}{2E_+E_-}(p_\perp^+E_- \eta + p_\perp^- E_+\xi)^2 - p_\perp^+ p_\perp^-(E_+^2 + E_-^2)\vec{V}_+\cdot\vec{V}_-\xi\eta + \frac{E_+^2 + E_-^2}{2E_+E_-}(p_{\perp+}^2 E_-^2 V_-^2\eta^2 + p_{\perp-}^2 E_+^2 V_+^2\xi^2)\right\}$$
$$- \frac{\mu^2}{\omega}\left\{(p_\perp^+E_-\eta + p_\perp^-E_+\xi)(p_\perp^+ + p_\perp^-)\,\frac{m^2\omega^2}{E_+E_-} - \frac{1}{\omega E_+E_-}(p_\perp^+E_- - p_\perp^-E_+)(E_+^2\vec{V}_-\cdot\vec{k}_\perp - E_-^2\vec{V}_+\cdot\vec{k}_\perp)\right\}$$
$$+ \frac{1}{2}(p_\perp^+ + p_\perp^-)^2\,\frac{m^2\omega^2}{E_+E_-} + \frac{E_+^2 + E_-^2}{2E_+E_-}(p_\perp^+E_- - p_\perp^-E_+)^2\,\frac{k_\perp^2}{\omega^2}, \tag{9.10b}$$

$$h=\frac{d}{2}\left[\mu^2\left\{\frac{E_+^2+E_-^2}{\omega^2}\,(p_\perp^- E_+ V_+^2\xi^2-p_\perp^+ E_- V_-^2\,\eta^2)+m^2(p_\perp^- E_+\xi^2-p_\perp^+ E_-\eta^2)+\frac{E_+E_-}{\omega^2}\,\xi\eta\vec{V}_+\cdot\vec{V}_-\left(\frac{E_+^2}{E_-}\,p_\perp^- -\frac{E_-^2}{E_+}\,p_\perp^+\right)\right.\right.$$
$$\left.+\frac{E_+E_-}{\omega^2}\,\xi\eta\left(\frac{m^2\omega^2}{E_+E_-}+\vec{V}_+\cdot\vec{V}_-\right)(p_\perp^+E_- -p_\perp^- E_+)\right\}-(p_\perp^+ +p_\perp^-)m^2\omega(\xi-\eta)$$
$$\left.-\frac{E_+^2+E_-^2}{\omega^2}\,(p_\perp^+E_- -p_\perp^- E_+)(\vec{V}_+\cdot\vec{k}_\perp\xi+\vec{V}_-\cdot\vec{k}_\perp\,\eta)\right]. \tag{9.10c}$$

Note that $f$ and $g$ are symmetric in $+\leftrightarrow-$, while $h$ is antisymmetric.

The cross section (8.7), averaged over photon polarizations and summed over electron and positron polarizations, then becomes

$$d^4\sigma/L=\frac{1}{\pi q_\perp^2}\,\frac{2\alpha}{|p_z^+E_- -p_z^- E_+|}\,\frac{|N_+N_-|^2}{\omega}\,\{f|F(x)|^2+g|G(x)|^2+h\ \mathrm{Im}[F^*(x)G(x)]\}\times p_\perp^+dp_\perp^+d\varphi^+p_\perp^-dp_\perp^-d\varphi^-, \tag{9.11}$$

with $f$, $g$, and $h$ given in Eqs. (9.10a), (9.10b), and (9.10c), respectively, and $F(x)$ and $G(x)$ in Eqs. (8.16). The normalization factors are, according to Eq. (6.8),

$$N_\pm=(\cosh\pi d_\pm)^{1/2}e^{\mp\pi d_\pm/2+i\lambda},$$

where $\lambda$ is a phase. This gives

$$|N_+N_-|^2=\frac{e^{-\pi d_+}}{\cosh\pi d_+}\,\frac{e^{+\pi d_-}}{\cosh\pi d_-}. \tag{9.12}$$

The cross section is then given by

$$d^4\sigma/L=\frac{1}{\pi q_\perp^2}\,\frac{2\alpha}{\omega|p_z^+E_- -p_z^- E_+|}\,\frac{e^{-\pi d_+}}{\cosh\pi d_+}\,\frac{e^{\pi d_-}}{\cosh\pi d_-}$$
$$\times\{f|F(x)|^2+g|G(x)|^2+h\ \mathrm{Im}[F^*(x)G(x)]\}$$
$$\times p_\perp^+dp_\perp^+p_\perp^-dp_\perp^-d\varphi_+d\varphi_-. \tag{9.13}$$

This is the exact high energy cross section for production of electron-positron pairs in continuum states in a chaneling $1/\rho$ potential given by Eq. (5.2).

## APPENDIX A

The easily proved theorem [7]

$$\frac{1}{\Gamma(c)}\,F(a;c;z)=\frac{z^m\Gamma(a+m+1)}{\Gamma(a)}\,F(a+m+1;m+2;z),\quad c=-m, \tag{A1}$$

for the Kummer function

$$F(a;c;z)=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(c)z^n}{\Gamma(a)\Gamma(c+n)n!}$$

shows that the particular combination $[\Gamma(c)]^{(-1)}F(a;c;z)$ is finite for $c=-m$. One then easily finds

$$z^{-\kappa-(1/2)}\,\frac{\Gamma(-\eta-\kappa+\frac{1}{2})}{\Gamma(-2\kappa+1)}\,F(-\eta-\kappa+\tfrac{1}{2};-2\kappa+1;z)$$
$$=z^{\kappa-(1/2)}\,\frac{\Gamma(-\eta+\kappa+\frac{1}{2})}{\Gamma(2\kappa+1)}\,F(-\eta+\kappa+\tfrac{1}{2};2\kappa+1;z), \tag{A2}$$

with no singularities, and positive and negative values of $\kappa$ ($\mu$ or $\mu+1$) are equivalent and may be summed over. In fact, $\kappa$ may be replaced by $|\kappa|$.

## APPENDIX B

In order to calculate the integral $I_0$ [Eq. (2.13)] we follow the method of Sommerfeld [10], who calculated the corresponding integral in three dimensions. The function $I_0$

$$I_0=\int d\rho\ d\varphi\ F(id_-;\tfrac{1}{2};i(p_\perp^-\rho+\vec{p}^-\cdot\vec{\rho}))e^{i\vec{q}_\perp\vec{\rho}-\varepsilon\rho}F$$
$$\times(-id_+;\tfrac{1}{2};i(p_\perp^+\rho+\vec{p}^+\cdot\vec{\rho})) \tag{B1}$$

becomes, when the integral representation of the Kummer function

$$F(a;c;x)=B\int_0^1 e^{xt}t^{a-1}(1-t)^{c-a-1}\,dt,$$
$$B=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \tag{B2}$$

is introduced,

$$I_0=B_+B_-\int_0^1 dt\ t^{-id_+-1}(1-t)^{-(1/2)+id_+}\int_0^1 du\ u^{id_+-1}(1-u)^{-(1/2)-id_-}\int d\rho\ d\varphi\ \exp[i\vec{q}_\perp\cdot\vec{\rho}-\varepsilon\rho+i(p_\perp^+\rho+\vec{p}^+\vec{\rho})t+i(p_\perp^-\rho+\vec{p}^-\vec{\rho})u],$$

where

$$B_{\pm}=\frac{\Gamma(\frac{1}{2})}{\Gamma(\mp id_{\pm})\Gamma(\frac{1}{2}\pm id_{\pm})}. \tag{B3}$$

The $\rho$ and $\varphi$ integrations give

$$I_0=2\pi B_+B_-\int_0^1 dt\ t^{-id_+-1}(1-t)^{-(1/2)+id_+}\times\int_0^1 du\ u^{id_--1}(1-u)^{-(1/2)-id_-}(A^2+B^2)^{-1/2}, \tag{B4}$$

where

$$A=\varepsilon-i(tp_\perp^+ +up_\perp^-),\quad \vec{B}=\vec{q}_\perp+t\vec{p}_\perp^{\,+}+u\vec{p}^{\,-}.$$

It is important that the quadratic terms in $A^2+B^2$ cancel, giving a linear function in $t$ and $u$,

$$(A^2+B^2)^{-1/2}=(a-bu)^{-1/2}=\frac{1}{\sqrt{a}}\left(1-\frac{b}{a}u\right)^{-1/2}, \tag{B5}$$

with

$$a=q_\perp^2+2(\vec{q}_\perp\cdot\vec{p}_\perp^{\,+}-i\varepsilon p_\perp^+)t,$$

$$b=2(p_\perp^+p_\perp^- -\vec{p}_\perp^{\,+}\cdot\vec{p}_\perp^{\,-})t+2(i\varepsilon p_\perp^- -\vec{q}_\perp\cdot\vec{p}_\perp^{\,-}),$$

where we have neglected $\varepsilon^2$ terms since they do not contribute to $I_1$. The introduction of Eq. (B5) in Eq. (B4) gives a hypergeometric function

$$\int_0^1 du\ u^{id_--1}(1-u)^{-(1/2)-id_-}\left(1-\frac{b}{a}u\right)^{-1/2}=B_-^{-1}\,{}_2F_1\left(\frac{1}{2},\quad id_-;\frac{1}{2};\frac{b}{a}\right)=B_-^{-1}\left(1-\frac{b}{a}\right)^{-id_-},$$

and this is a crucial point in the integration; the index 1/2 in the Kummer function matches the power 1/2 from the spatial integration, reducing the hypergeometric function to a simple function, which makes it possible to obtain a hypergeometric function as a result of the final $t$ integration. In the three-dimensional case of Sommerfeld the crucial index is 1.

The integral is now

$$I_0=2\pi B_+\int_0^1 dt\ t^{-id_+-1}(1-t)^{-(1/2)+id_+}a^{-(1/2)+id_-}\times(a-b)^{-id_-}.$$

Considered as a loop integrand, the integral has the four branch points

$$t_1=0,\quad t_2=1,\quad t_3=-q_\perp^2/2(\vec{q}_\perp\cdot\vec{p}_\perp^{\,+}-i\varepsilon p_\perp^+)\quad (a=0),$$

and

$$t_4=\frac{q_\perp^2+2(\vec{q}_\perp\cdot\vec{p}^{\,-}-i\varepsilon p_\perp^-)}{2[p_\perp^+p_\perp^- -\vec{p}^{\,+}\cdot\vec{p}^{\,-}-\vec{q}_\perp\cdot\vec{p}_\perp^{\,+}+\varepsilon p_\perp^+]}\quad (a-b=0),$$

while the integrand vanishes as $t^{-2}$ at infinity.

The change of variable, conserving the limits (0,1),

$$t=\frac{t_3v}{v-1+t_3},$$

changes the integral into

$$I_0=B_+C\int_0^1 dv\ v^{-id_+-1}(1-v)^{-(1/2)+id_+}(1-xv)^{-id_-}, \tag{B6}$$

with

$$x=2\,\frac{q^2(p_\perp^+p_\perp^- -\vec{p}_\perp^{\,+}\cdot\vec{p}_\perp^{\,-})+2(\vec{q}_\perp\cdot\vec{p}_\perp^{\,+})(\vec{q}_\perp\cdot\vec{p}_\perp^{\,-})-2i\varepsilon[p_\perp^+\vec{q}_\perp\cdot\vec{p}_\perp^{\,-}+p_\perp^-(\vec{q}_\perp\vec{p})]}{(D^+-2i\varepsilon p_\perp^+)(D^--2i\varepsilon p_\perp^-)}$$

and

$$C=\frac{4\pi}{q_\perp}\left(\frac{q_\perp^2}{D_+-2i\varepsilon p_\perp^+}\right)^{-id_+}\left(\frac{q_\perp^2}{D_--2i\varepsilon p_\perp^-}\right)^{id_-}, \tag{B7}$$

where we always neglect $\varepsilon^2$ terms. The integrand in terms of the new variable has the branch points $\nu_1=0$, $\nu_2=1$, $\nu_3=1/x$, and $\nu_4=\infty$, and the integral $I_0$ is a hypergeometric function

$$I_0=2\pi K\ {}_2F_1(-id_+,id_-;\tfrac{1}{2};x).$$

Following the prescriptions in Eq. (8.14) and remembering that

$$\frac{d}{dx}\,{}_2F_1(-id_+,id_-;\tfrac{1}{2};x)=2d_+d_-\,{}_2F_1(1-id_+,1+id_-;\tfrac{3}{2};x),$$

one finds the integrals $I_1$ and $I_\pm$ given in Eq. (8.15).

[1] Jens Lindhard, Kgl. Dan. Vidensk. Selsk. Mat. Fys. Medd. **34**, No. 14 (1965).

[2] (a)A. J. Bobudaev, V. V. Kaplin, and S. A. Vorobiev, Phys. Lett. **45A**, 71 (1973); (b) V. V. Kaplin, S. V. Plotnikov, I. A. Tsekhanovsky, and S. A. Vorobiev, *ibid.* **54A**, 447 (1975).

[3] J. C. Kimball and N. Cue, Phys. Rep. **125**, 70 (1985); *Coherent Radiation Sources*, edited by A. V. Sáenz and H. Überall (Springer, Berlin, 1985); *Relativistic Channeling*, edited by R. A. Carrigan and J. A. Ellison (Plenum, New York, 1987).

[4] For a review see J. C. Kimball and N. Cue, Phys. Rep. **125**, 69 (1985).

[5] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954); see also H. A. Olsen, L. C. Maximon, and H. Wergeland, *ibid.* **106**, 27 (1957); H. A. Olsen and L. C. Maximon, *ibid.* **114**, 887 (1959).

[6] A. Sommerfeld, *Atombau und Spektrallinien II* (Vieweg, Braunschweig, 1960).

[7] We use the definitions of W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer, Berlin, 1966).

[8] V. V. Kaplin and S. A. Vorobiev, Zh. Eksp. Teor. Fiz. **73**, 583 (1987) [Sov. Phys. JETP **46**, 304 (1977)].

[9] H. A. Bethe and E. E. Salpeter, in *Quantum Mechanics of One- and Two-Electron Systems*, edited by S. Flügge, Handbuch der Physik Vol. XXXV (Springer, Berlin, 1957), Chap. I.

[10] A. Sommerfeld, Ann. Phys. (Leipzig) **11**, 257 (1931); A. Sommerfeld and A. W. Maue, *ibid.* **22**, 629 (1935).

[11] A. T. Nordsieck, Phys. Rev. **93**, 784 (1954).