ФЕДЕРАЛЬНОЕ АГЕНТСТВО ПО ОБРАЗОВАНИЮ
Государственное образовательное учреждение высшего профессионального образования
«ТОМСКИЙ ПОЛИТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ»

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MATHEMATICS

PREPARATORY COURSE: TRIGONOMETRY AND GEOMETRY

WorkBook

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The workbook is a supplement to the textbook of the same name.

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TRIGONOMETRY

1. Measurement of Angles

Summary:

- In order to convert a fraction of a degree to minutes multiply this fraction by 60 to get the number of minutes.
- In order to convert a fraction of a minute to seconds multiply this fraction by 60 to get the number of seconds.
- In order to convert degrees to radians, transform the number of degrees-minutes-seconds into a decimal form, then multiply the result by π and divide by 180° to get the angle in radians.
- In order to convert radians to degrees, divide the number of radians by π and multiply by 180°.
- One radian equals approximately 57.296°.

1.1. Write down the following angles in the degree-minute-second notation.

a) $12.27^\circ =$

b) $31.56^\circ =$

c) $44.7^\circ =$

1.2. Convert between degree and radian measurements.

a) $10^\circ =$

b) $175^\circ =$

c) $\frac{3}{4}\pi =$

d) $\frac{11}{12}\pi =$

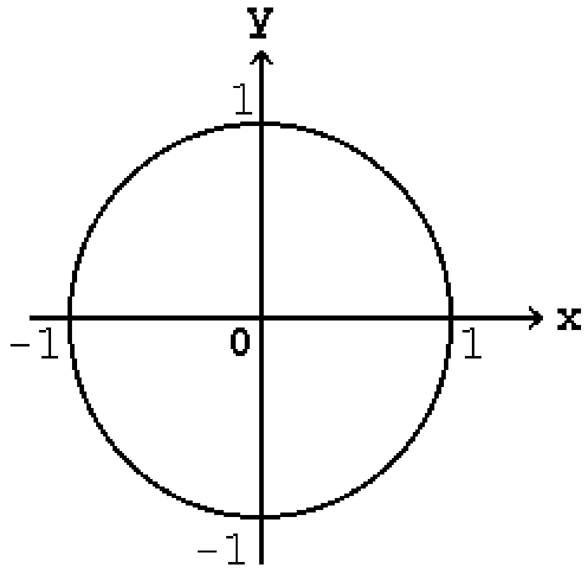
2. Unit Circle and Trigonometric Functions

Summary:

In order to find trigonometric functions of a given angle θ, draw a line from the origin at the angle θ to get the point where this line crosses the unit circle.

- The x-coordinate of this point gives $\cos\theta$.
- The y-coordinate is $\sin\theta$.
- All the other trigonometric functions can be found by using $\sin\theta$ and $\cos\theta$.

2.1. Using the unit circle find $\sin\theta$, $\cos\theta$, $\tan\theta$, and $\cot\theta$ of the angles $\theta = \frac{\pi}{4}n$, where $n = 1, 2, 3, 4$.



$\sin\frac{\pi}{4} =$

$\sin\frac{\pi}{2} =$

$\sin\frac{3\pi}{4} =$

$\sin\pi =$

$\cos\frac{\pi}{4} =$

$\cos\frac{\pi}{2} =$

$\cos\frac{3\pi}{4} =$

$\cos\pi =$

$\tan\frac{\pi}{4} =$

$\tan\frac{\pi}{2} =$

$\tan\frac{3\pi}{4} =$

$\tan\pi =$

$\cot\frac{\pi}{4} =$

$\cot\frac{\pi}{2} =$

$\cot\frac{3\pi}{4} =$

$\cot\pi =$

2.2. Fill in the table below without using a calculator.

θ (radian)	θ (degree)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$
0					
$\frac{\pi}{6}$					
$\frac{\pi}{4}$					
$\frac{\pi}{3}$					
$\frac{\pi}{2}$					

2.3. Calculate without using a calculator:

$\sin 570° =$

$\cos 660° =$

$\tan 405° =$

$\cot(-390°) =$

$\sec(-225°) =$

$\csc 120° =$

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3. Addition and Subtraction Formulas for Trigonometric Functions

Summary:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\sin\beta\cos\alpha$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

3.1. Prove the odd-even properties for sine and cosine using only the definitions of trigonometric functions and the addition or subtraction formulas.

$\sin(-\theta) =$

$\cos(-\theta) =$

$\tan(-\theta) =$

$\cot(-\theta) =$

3.2. Prove the fundamental trigonometric identity making use of the addition formulas for cosines:

$$\sin^2\theta + \cos^2\theta = 1$$

Proof:

3.3. Calculate without using a calculator:

a) $\sin 75°$ b) $\cos 165°$ c) $\sin 345°$

d) $\cos 195°$ e) $\tan 255°$ f) $\cot 285°$

Solution:

a) $\sin 75° =$

b) $\cos 165° =$

c) $\sin 345° =$

d) $\cos 195° =$

e) $\tan 255° =$

f) $\cot 285° =$

3.4. Calculate without using a calculator:

a) $\sin\alpha$, if $\cos\alpha = 4/5$ and $0 < \alpha < \pi/2$

b) $\sin\alpha$, if $\cos\alpha = 4/5$ and $\pi/2 < \alpha < \pi$

c) $\cos\alpha$, if $\sin\alpha = 2/3$ and $-\pi/2 < \alpha < 0$

d) $\cos\alpha$, if $\sin\alpha = 2/3$ and $\pi/2 < \alpha < \pi$

Solution:

a) $\sin\alpha =$

b) $\sin\alpha =$

c) $\cos\alpha =$

d) $\cos\alpha =$

3.5. Calculate $\sin(\alpha+\beta)$, if $\sin\alpha = \frac{\sqrt{3}}{4}$, $\cos\beta = -\frac{\sqrt{13}}{4}$, $0 < \alpha < \frac{\pi}{2}$, and $\frac{\pi}{2} < \beta < \pi$.

Solution:

$\sin(\alpha+\beta) =$

3.6. Calculate $\cos(\alpha-\beta)$, if $\sin\alpha = -\frac{\sqrt{5}}{5}$, $\cos\beta = \frac{1}{3}$, $-\frac{\pi}{2} < \alpha < 0$, and $0 < \beta < \frac{\pi}{2}$.

Solution:

$\cos(\alpha-\beta) =$

3.7. Calculate $\tan(\alpha+\beta)$, if $\tan\alpha=\sqrt{7}$, $\tan\beta=-\frac{1}{\sqrt{7}}$, $\pi<\alpha<\frac{3}{2}\pi$, and $\frac{\pi}{2}<\beta<\pi$.

Solution:

$\tan(\alpha+\beta)=$

3.8. Find all solutions of the following equations

a) $\sin 2x+\sqrt{3}\cos 2x=0$

b) $\sqrt{3}\sin 3x-\cos 3x=0$

Solution:

a) $\sin 2x+\sqrt{3}\cos 2x=0$

Hint:

- First divide both sides of the equation by the number two.
- Next replace the factors $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ by equal values $\cos\frac{\pi}{3}$ and $\sin\frac{\pi}{3}$ respectively.
- Then use the addition formula for sine to simplify the equation.
- Finally recall that the equation $\sin\alpha=a$ has the following solution set:

$$\alpha=(-1)^n\arcsin a+\pi n,\, n\in N$$

b) $\sqrt{3}\sin 3x - \cos 3x = 0$

Hint:

Transform the equation in a similar way as above. Then recall that the solution set of the equation $\cos\alpha = a$ is

$$\alpha = \pm\arccos a + 2\pi n, \qquad n \in N$$

3.9. Provide arguments for the validity of the following formulas:

a) $1 + \tan^2\theta = \dfrac{1}{\cos^2\theta}$

b) $1 + \cot^2\theta = \dfrac{1}{\sin^2\theta}$

c) $\sin(\theta + \pi/2) = \cos\theta$

d) $\cos(\theta + \pi/2) = -\sin\theta$

e) $\cos(\theta + \pi) = -\cos\theta$

f) $\sin(\theta + \pi) = -\sin\theta$

Proof:

a) $1 + \tan^2\theta =$

b) $1 + \cot^2\theta =$

c) $\sin(\theta + \pi/2) =$

d) $\cos(\theta + \pi/2) =$

e) $\cos(\theta + \pi) =$

f) $\sin(\theta + \pi) =$

3.10. Proof the following double-angle formulas:

a) $\sin 2\alpha = 2\sin\alpha \cdot \cos\alpha$

b) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

c) $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

Hint: Use the addition formulas.

Proof:

a) $\sin 2\alpha =$

b) $\cos 2\alpha =$

c) $\tan 2\alpha =$

3.11. Proof the following half-angle formulas:

a) $$\sin^2\frac{\alpha}{2}=\frac{1}{2}(1-\cos\alpha)$$

b) $$\cos^2\frac{\alpha}{2}=\frac{1}{2}(1+\cos\alpha)$$

c) $$\tan\frac{\alpha}{2}=\frac{1-\cos\alpha}{\sin\alpha}$$

d) $$\tan\frac{\alpha}{2}=\frac{\sin\alpha}{1+\cos\alpha}$$

Hint: To prove formulas a)-b) transform the expressions on the right sides using the double-angle formula for cosine.

In cases c)-d) use also the double-angle formula for sine.

Proof:

a) $1-\cos\alpha=$

b) $1+\cos\alpha=$

c) $$\frac{1-\cos\alpha}{\sin\alpha}=$$

d) $$\frac{\sin\alpha}{1+\cos\alpha}=$$

3.12. Calculate without using a calculator:

a) $\sin\frac{\pi}{8} =$

b) $\tan(\frac{21}{8}\pi) =$

c) $\cot(\frac{31}{8}\pi) =$

3.13. Calculate $\sin\alpha$, if $\tan\frac{\alpha}{2} = \sqrt{5}$ and $0 < \alpha < \frac{\pi}{2}$.

Solution:

3.14. Calculate $\cos 2\alpha$, if $\tan\alpha = -\sqrt{2}$ and $-\frac{\pi}{2} < \alpha < 0$.

Solution:

3.15. Find all solutions of the following equations

a) $\sin x + \cos 2x = 1$

b) $1 + \sin 2x + \cos 2x = 0$

c) $\sin 2x = \sqrt{2} \cos x$

d) $\sin^4 x - \cos^4 x = \sin x$

Solution:

a) $\sin x + \cos 2x = 1$

Hint:

- First subtract number 1 from both sides; next use the formula

$$1 - \cos\alpha = 2\sin^2 \frac{\alpha}{2}$$

to transform the equation.

- Next factor the expression on the left-hand side and equate each factor with zero.
- Then solve each equation separately.

b) $1 + \sin 2x + \cos 2x = 0$

Hint:

- Use the double-angle formula for sine and the above formula

$$1 + \cos\alpha = 2\cos^2 \frac{\alpha}{2}.$$

- Then transform and solve the equation in a similar way as above.

c) $\sin 2x = \sqrt{2}\cos x$

Hint:

Use the double-angle formula for sine.

d) $\sin^4 x - \cos^4 x = \sin 2x$

Hint:

- First consider the expression on the left-hand side of the equation as the difference between squares.
- Then simplify the equation transforming it into the form $\tan\alpha = a$.

  Recall that the last equation has the following solution set:

$$\alpha = \arctan a + \pi n, \qquad n \in N$$

4. Other Trigonometric Identities

Summary:

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{a-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{a-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{a+\beta}{2}$$

$$\tan\alpha + \tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$

$$\tan\alpha - \tan\beta = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$$

$$\cot\alpha + \cot\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$$

$$\cot\alpha - \cot\beta = -\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta}$$

$$\tan\alpha + \cot\beta = \frac{\cos(\alpha-\beta)}{\cos\alpha\sin\beta}$$

$$\tan\alpha - \cot\beta = -\frac{\cos(\alpha+\beta)}{\cos\alpha\sin\beta}$$

$$\tan\alpha + \cot\alpha = \frac{2}{\sin 2\alpha}$$

$$\tan\alpha - \cot\alpha = -2\cot 2\alpha$$

4.1. Find all solutions of the equations in the interval $0 \le x \le \pi$.

a) $\sin x - \sin 3x = 2\cos x$

b) $1 - \sin x + \sin^2 x - \sin^3 x + \ldots = \frac{2}{3}$

c) $\sin^4 x + \cos^4 x = \frac{5}{8}$

d) $6\cos^2 x + 5\sin x = 2$

e) $\sin 4x \cdot \cos 2x = \sin 5x \cdot \cos x$

f) $\sin x \sin 2x \sin 3x = \frac{1}{4}\sin 4x$

Note: You have first to find all solutions of the equation and then select all the solution angles that lie in the interval $0 \le x \le \pi$.

Solution:

a) $\sin x - \sin 3x = 2\cos x$

Hint:

Transform the sum of the sines into the product using the above formula. Then simplify the equation and solve it in the usual way.

b) $1 - \sin x + \sin^2 x - \sin^3 x + \ldots = \frac{2}{3}$

Hint: First use the formula of the sum of an infinite number of terms of decreasing geometric progression; then simplify the equation.

c) $\sin^4 x + \cos^4 x = \frac{5}{8}$

Hint: Get a perfect square on the left-hand side

d) $6\cos^2 x + 5\sin x = 2$

Hint: Express $\cos x$ through $\sin x$ and then solve the quadratic equation for $\sin x$.

e) $\sin 4x \cdot \cos 2x = \sin 5x \cdot \cos x$

Hint: You can use the following formulas whenever it is necessary to transform the product of sines and cosines into their sum.

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta))$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$$

f) $\sin x \sin 2x \sin 3x = \frac{1}{4}\sin 4x$

5. Graphs of Trigonometric Functions

5.1. Plot the graphs of the following functions:

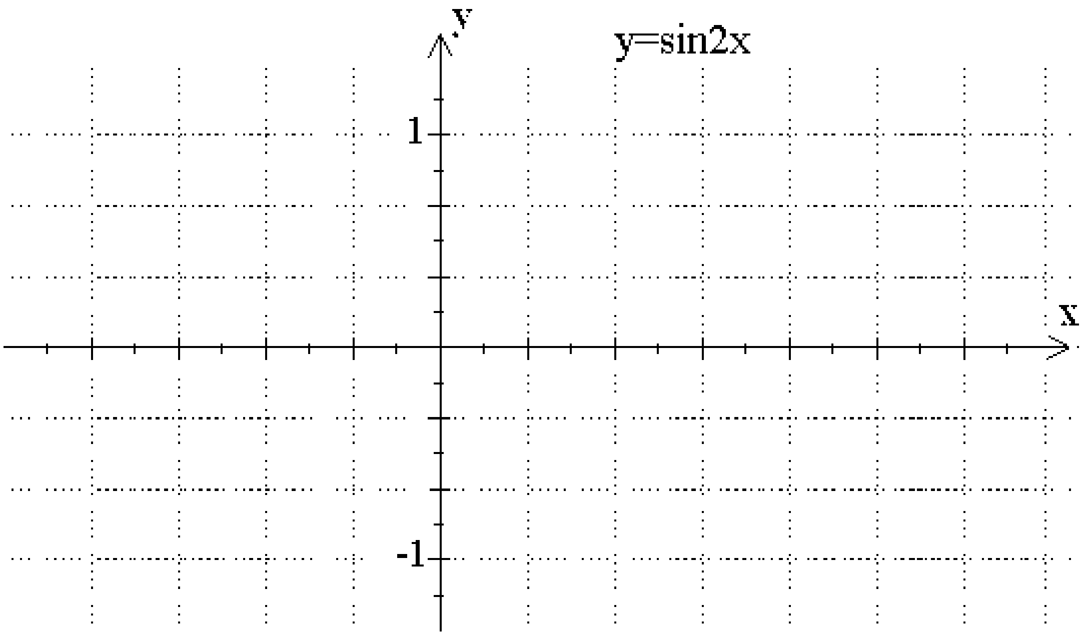
a) $y = \sin 2x$

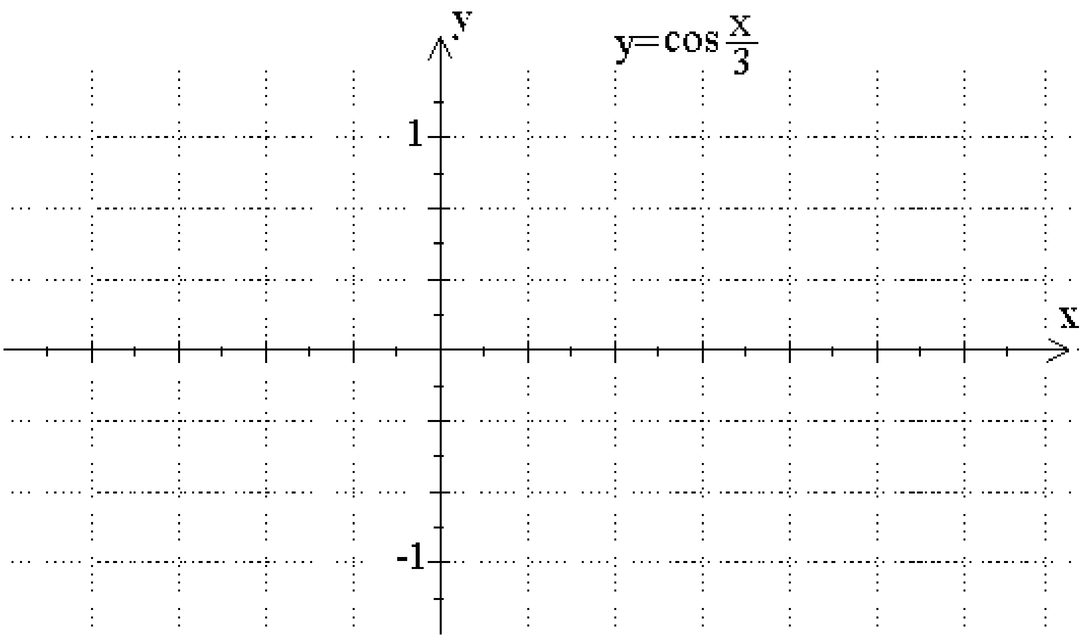
b) $y = \cos(x/3)$

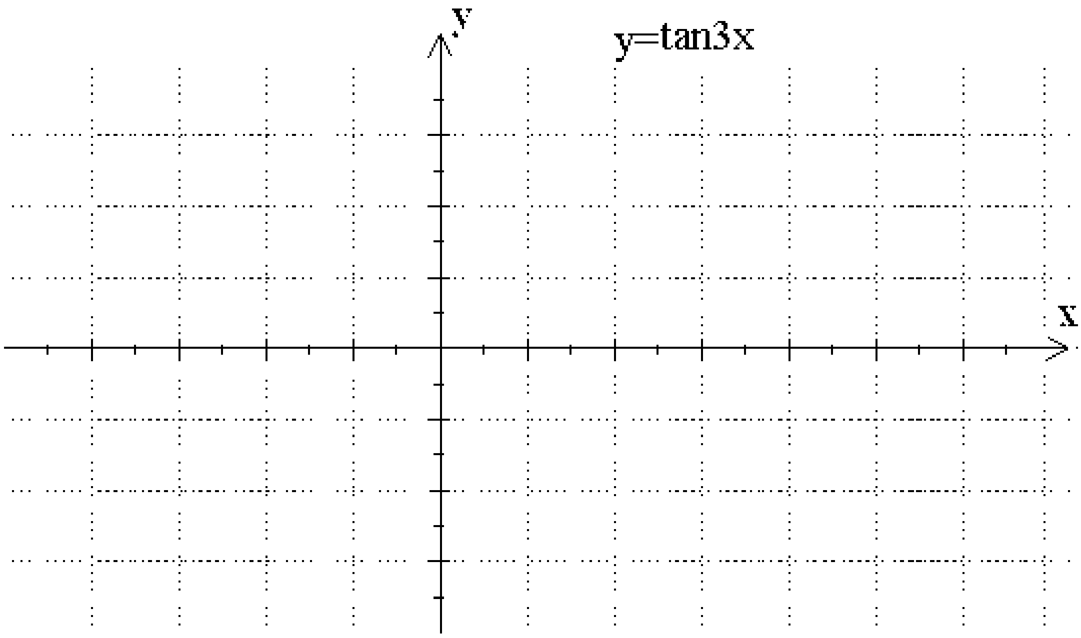
c) $y = \tan 3x$

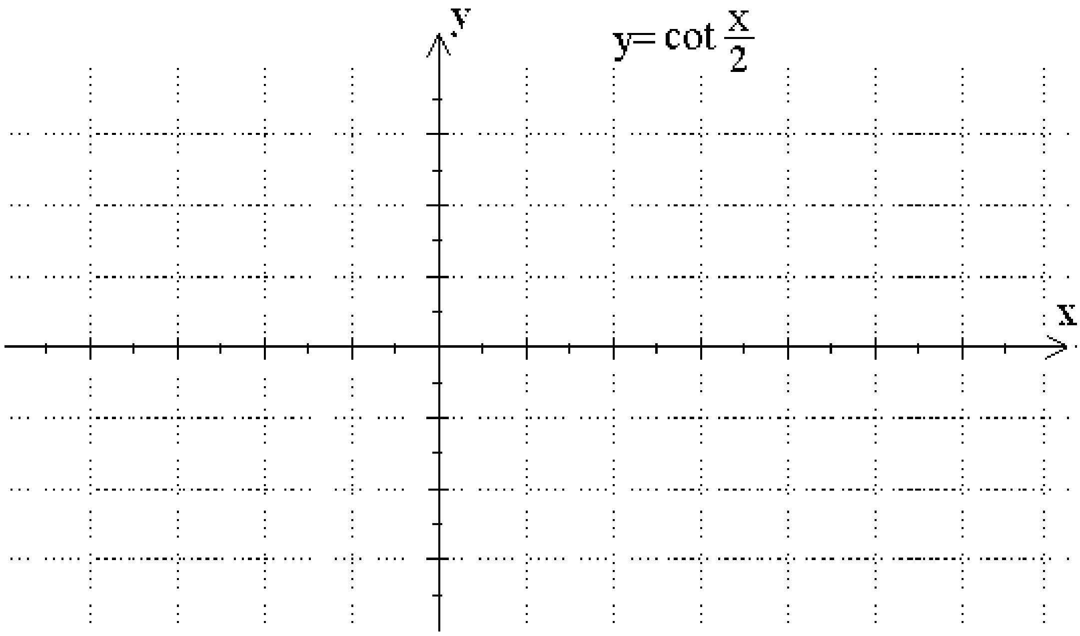
d) $y = \cot(x/2)$

Solution:






y
y=tan3x
x


y
y= cot x/2
x

GEOMETRY

1. Basic Terms of Geometry

1.1. What is the difference between postulates and theorems?

What is a proof?

What is an indirect proof?

1.2. Give some examples of figures in geometry. Illustrate your answer by the drawings.

1.3. Match the following angles with their drawings:

a) an acute angle;

b) an obtuse angle;

c) a right angle.

What is the difference between oblique angles and right angles?

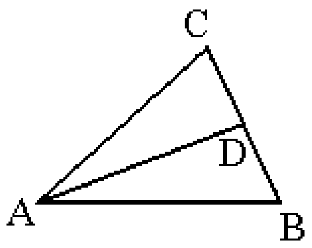
.

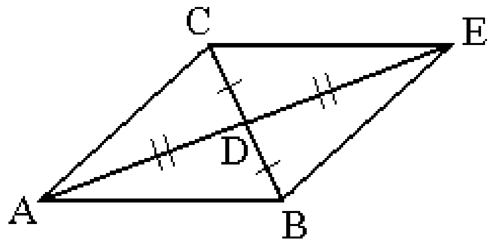
2. Triangles

2.1. Given are two sides, AB and BC, and the median AD in the triangle ABC. Find the area of the triangle, if $AB = 27$, $BC = 29$, and $AD = 26$.

Solution:

Hint: Look at the drawings below:





Recall that the area of a triangle is equal to half the area of the parallelogram with the same base and height. Therefore, the triangle ABC has the same area as ΔAEC with given sides.

2.2. An angle of an isosceles triangle is equal to $70°$. Find the rest angles. How many solutions has the problem?

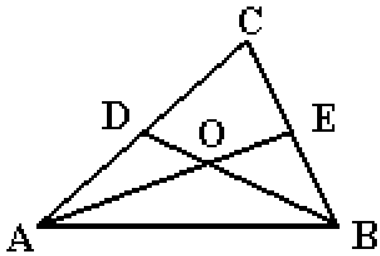
Solution:

2.3. The midpoints of AC and BC are D and E, respectively. Find the area of the triangle DEC if the area of the triangle ABC is equal to 80 units.

Solution:

2.4. Given are two medians, AE and BD, and the side AC in the triangle ABC. Find the area of the triangle AOB, if $AE = 5$, $BD = 6$, and $AB = 7$.

Solution:



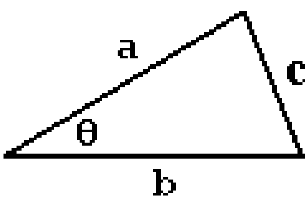
2.5. Given are sides a, b and c in a triangle. Prove that the area S of the triangle is the following:

$$S = \sqrt{p(p-a)(p-b)(p-c)}$$

where p is half the perimeter of the triangle.

Proof:

Hint::



Use the following formulas

$$p = (a+b+c)/2$$

$$S = \frac{1}{2}ab\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{2S}{ab}$$

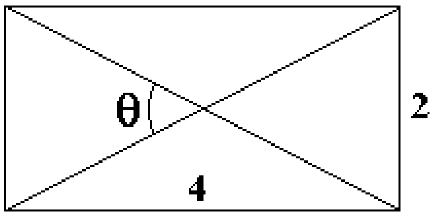
$$c^2 = a^2 + b^2 - 2ab\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\sin^2\theta + \cos^2\theta = 1$$

3. Quadrilaterals

3.1. The sides of a rectangle are equal to 2 and 4 units. Find $\cos\theta$, where θ is the acute angle between diagonals of the rectangle.

Solution:



3.2. Prove the following statements:

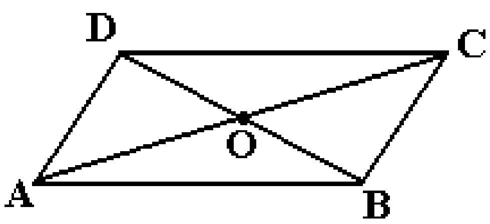
a) The diagonals of a parallelogram bisect each other.

b) If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

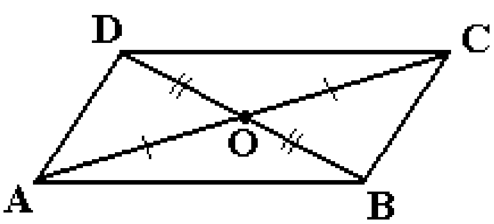
Prove:

a)

Hint: Compare with each other ΔAOB and ΔCOD.



b)



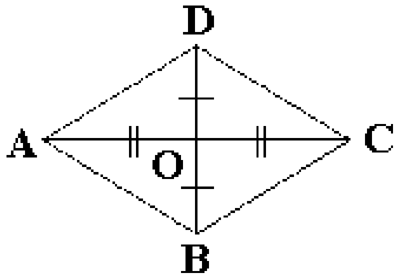
3.3. Prove the following statements:

a) The diagonals of a rhombus are perpendicular.

b) Each diagonal of a rhombus bisects two angles of the rhombus.

Prove:

Hint: Compare with each other ΔAOB and ΔAOD in view of the theorems proven above.

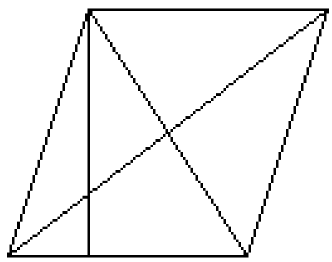


a)

b)

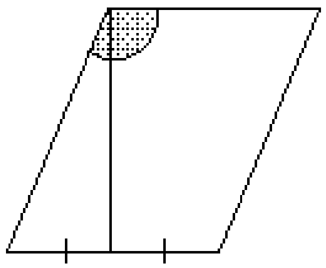
3.4. The diagonals of a rhombus are equal to 5 and 10 units. Find the height of the rhombus.

Solution:



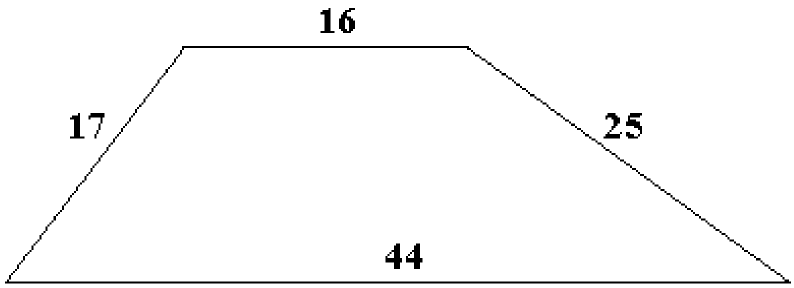
3.5. An altitude from the vertex of an obtuse angle of a rhombus is a bisector of the base. Find the obtuse angle.

Solution:



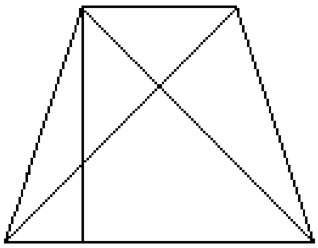
3.6. The bases of a trapezoid are equal to 16 and 44 units, and its legs equal 17 and 25 units. Find the area of the trapezoid.

Solution:



3.7. The diagonals of an isosceles trapezoid are perpendicular and the height equals 5 units. Find the area of the trapezoid.

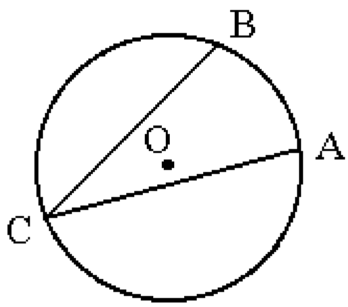
Solution:



4. Circles

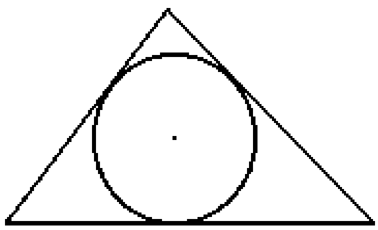
4.1. Given is a circle with the radius of the unit length. An inscribed angle ACB equals $15°$. Find the length of the arc AB.

Solution:



4.2. Given are two sides in a triangle that are equal to 8 and 10 units, respectively. The triangle is circumscribed about a circle with radius 3 units. The area of the triangle equals 60 units. Find the third side of the triangle.

Solution:

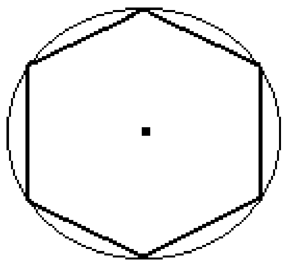


4.3. Let a circle be inscribed in a circle sector whose radius is 9 units and the central angle equals $60°$. Find the radius of the circle.

Solution:

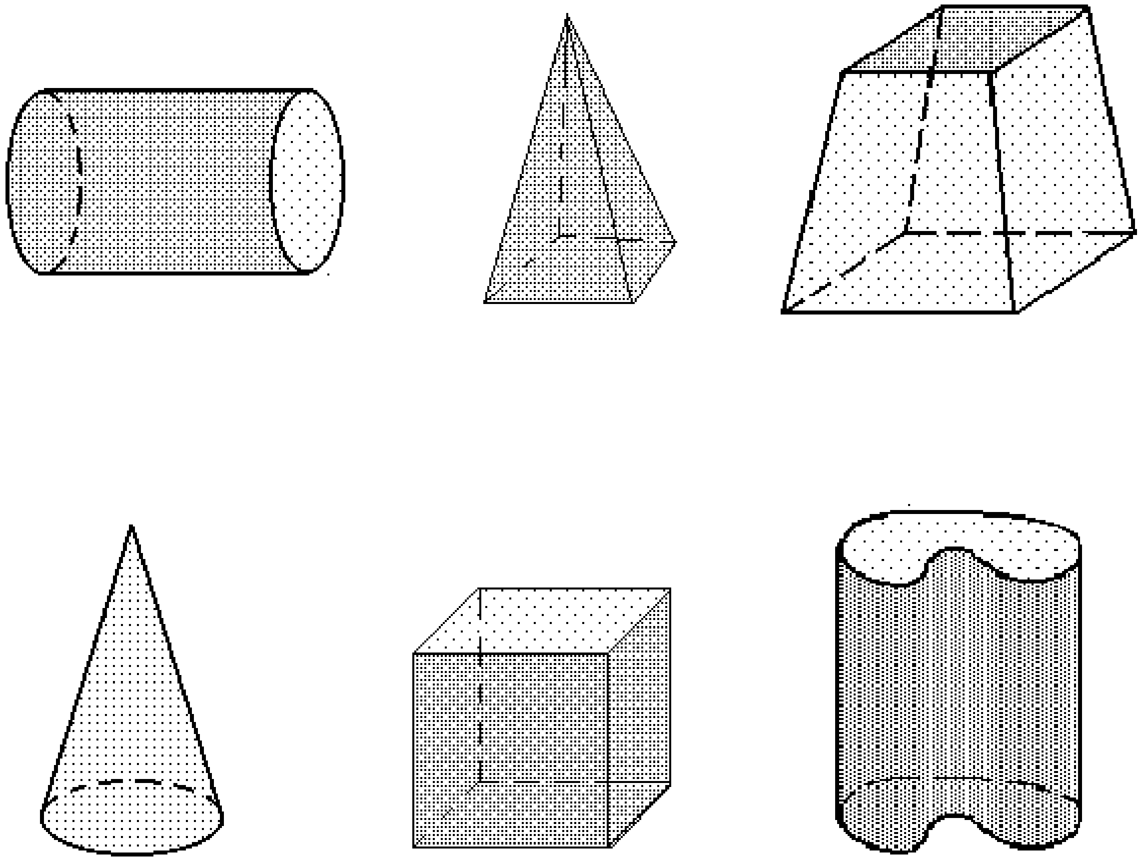
4.4. Find the area of a regular six-sided polygon that is inscribed in the circle with radius 2 units.

Solution:



5. Solids

5.1. The volume of which of the solids shown below can be found by applying the formula $V = S\,h$, where V represents the volume, S represents the area of a base, and h represents the altitude to the base that has area S? Explain your answer. Write down the correct formulas when the above is false.



5.2. The pyramid has a square base with sides of length 4 units. If the altitude of the pyramid is 12, what is the value of the lateral edges?

5.3. Find the volume of a layer between two concentric spheres with radii 4 and 5 units.

5.4. Find the volume of a sphere if its area equals 9π units.

Solution:

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