ФЕДЕРАЛЬНОЕ АГЕНТСТВО ПО ОБРАЗОВАНИЮ
Государственное образовательное учреждение высшего профессионального образования
«**ТОМСКИЙ ПОЛИТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ**»

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MATHEMATICS

PREPARATORY COURSE
ALGEBRA

WorkBook

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The workbook is a supplement to the textbook of the same name.

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1. The Real Number System

1.1. True or False Quiz for Properties of Numbers

Recall basic definitions of numbers and their properties.

Which of the following statements are true? Correct erroneous statements and give explanation.

	Statements	Write "true" or correct statements.
a)	Number $\sqrt{5}$ is a positive rational number.	
b)	Number $12/3$ is a real integer.	
c)	A real number is the number that is either natural or negative.	
d)	An even number is any evenly even number.	
e)	Odd numbers are integers that are not divisible by the number three.	
f)	$4(x-y)=4x-y$	
g)	$(3x)^2=3x^2$	
h)	$\frac{a}{0}=0$	
i)	$(x-y)^2=x^2-y^2$	
j)	$(x+y)^2=x^2+xy+y^2$	
k)	$\|-x\|=\|x\|$	
l)	$\sqrt{(-5)^2}=(-5)$	

1.2. True or False Quiz for Fractions

Correct the following wrong statements. Give reasons for your conclusions by the references to the appropriate properties.

	Wrong statements	Your versions of the answers.
a)	If $\frac{x}{y}=\frac{3}{2}$, then $x=3$ and $y=2$.	
b)	$\frac{2x+y}{3x}=\frac{2+y}{3}$	
c)	$\frac{4xy+x}{7x}=\frac{4y}{7}$	
d)	$\frac{2}{x}+\frac{y}{5}=\frac{2+y}{x+5}$	
e)	$\frac{(\frac{x}{3})}{4}=\frac{4x}{3}$	
f)	$\frac{5}{\left(\frac{x}{7}\right)}=\frac{5x}{7}$	
g)	If $\lvert x\rvert>1$ then $x>1$	
h)	If $\lvert x\rvert<1$ then $x<1$	

1.3. True or False Quiz for Exponents

Which of the following statements is true? Correct statements those are false and give reasons for your conclusions.

	Statements	Your conclusions and explanations
a)	$x^3 x^4 = x^{12}$	
b)	$x^7 = x^3 + x^4$	
c)	$(x^3)^2 = x^5$	
d)	$x^2 + 8^2 = (x+8)^2$	
e)	$x^2 x^5 = x^{10}$	
f)	$x^{\frac{1}{2}} = \frac{1}{x^2}$	
g)	$x^{-\frac{1}{2}} = \frac{1}{x^2}$	
h)	$\sqrt{x+y} = \sqrt{x} + \sqrt{y}$	
i)	$\sqrt{x^2+1} = x+1$	

1.4. True or False Quiz for Sets

	Statements	True/False
a)	Two sets in which the same elements are listed in a different order are the same.	
b)	The same element can never appear twice in a set.	
c)	The empty set is a subset of the set $\{4, a, x\}$.	
d)	If two sets are not equal, then one is a subset of the other.	
e)	If $A \cup B = A$, then $A \cap B = B$	
f)	If $A \cup B = A \cap B$, then $A = B$	
g)	If $A = \{1, 3, 4, a\}$ and $B = \{1, 4, 4, 7\}$, then • $A \cup B = \{1, 3, 4, 7, a\}$ • $A \cap B = \{1, 4\}$	

1.5. Which of the following fractions is the least?

a) $\frac{99}{100}$; b) $\frac{247}{250}$; c) $\frac{497}{500}$; d) $\frac{743}{750}$.

Your Explanation:

1.6. In the following problem select from the suggested answers the correct completion.

If $x/5 = y$, then $x/10 =$

a) $2y$; b) $y/2$; c) $50y$; d) $y/50$.

Your Explanation:

1.7. Compute each of the following expressions if it is defined. Do not write the answer as a decimal approximation.

a) $(-8)^{\frac{4}{3}} =$

b) $125^{\frac{2}{3}} =$

c) $((-8)(-2))^{\frac{2}{4}} =$

d) $(-625)^{\frac{3}{6}} =$

e) $(-4)^{\frac{6}{2}} =$

f) $625^{\frac{1}{4}} - 25^{-\frac{1}{2}} =$

g) $7^{-1} + 8^{-1} =$

h) $\dfrac{81^{\frac{1}{2}}}{27^{-\frac{2}{3}}} - \dfrac{32^{\frac{1}{5}}}{125^{\frac{1}{3}}} =$

Hint:

- Simplify each of the expressions using the operations with fractional exponents.
- Then in expressions f)-h) reduce the fractions to a common denominator and add (or subtract) the fractions with the same denominators.

Example:

- $4^{-\frac{1}{2}} - 9^{-\frac{1}{2}} = \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{3-2}{6} = \frac{1}{6}$

1.8. Write each of the following as a product or quotient of powers in which each variable occurs but once, and all exponents are positive. Assume that all variables represent positive real numbers only.

a) $(x^2 y^{-1} x^{\frac{-3}{2}})^{\frac{-1}{2}} =$

b) $(\frac{x^{-1} y^2}{2x^0 y^5})^{-2} =$

c) $\sqrt[4]{\frac{x^{-2} y^{-2} z}{xy^{-3} z}} =$

1.9. Find the factor A.

a) $x^{-\frac{3}{2}} + x^{-\frac{1}{2}} = Ax^{-\frac{1}{2}}$ $\Rightarrow$ $\Rightarrow$ $A =$

b) $x - x^{\frac{2}{3}} = Ax^{\frac{1}{3}}$ $\Rightarrow$ $\Rightarrow$ $A =$

c) $b^{\frac{3}{4}} = Ab^{\frac{1}{4}}$ $\Rightarrow$ $\Rightarrow$ $A =$

Example:

- $x^{\frac{3}{5}} + x^{-\frac{1}{5}} = Ax^{\frac{2}{5}}$ $\Rightarrow$ $x^{\frac{2}{5}}(x^{\frac{1}{5}} + x^{-\frac{3}{5}}) = Ax^{\frac{2}{5}}$ $\Rightarrow$ $A = x^{\frac{1}{5}} + x^{-\frac{3}{5}}$

1.10. In the following problems write the given expressions as a single fraction involving positive exponents only. Assume that all variables represent positive real numbers only.

a) $x^{-2} - y^{-2} =$

b) $x^{-1}y + yx^{-1} =$

c) $\dfrac{x^{-1} + y^{-1}}{x^{-1} - y^{-1}} =$

1.11. Write each of the given expressions as a single fraction.

a) $\dfrac{(\frac{a}{b})}{c} =$

b) $\dfrac{a}{(\frac{b}{c})} =$

c) $\dfrac{(\frac{a}{b})}{(\frac{c}{d})} =$

d) $\dfrac{(\frac{a}{b})}{(\frac{c}{d})}(\frac{e}{f}) =$

1.12. Let numbers $n, n+1,$ and $n+2$ be consecutive integers. Is the average of the three integers an even number?

Note: The average of the three numbers $a, b,$ and c is $(a+b+c)/3$.

Solution:

Is the sum of the three integers divisible by the middle integer?

Solution:

Is $(n+1)(n+2)-n(n+1)$ an even number?

Solution:

1.13. Solve the following proportions.

a) $\frac{2}{7}=\frac{3}{x} \quad \Rightarrow$ $\qquad x=$

b) $\frac{4}{9}=\frac{3x}{5} \Rightarrow$ $\qquad x=$

c) $\frac{3}{2x}=\frac{8}{7} \Rightarrow$ $\qquad x=$

d) $\frac{5x}{4}=\frac{2}{5} \Rightarrow$ $\qquad x=$

Hint. Use the cross product property of proportions:

$$\frac{a}{b}=\frac{c}{d} \qquad \Rightarrow \qquad ad=bc.$$

1.14. If $a<b$ and $|a-c|=|b-c|$, which of the following inequalities could be true?

a) $c<a$; b) $a<c<b$; c) $b<c$.

Hint: Recall that $|a-c|$ represents the distance between a and c on the number line. Use the number line and find the point c such that its distance from the point a is the same as from the points b.

You can also solve this problem algebraically using the definition of absolute values.

Geometric Solution:

Algebraic Solution:

1.15. Find (without using a calculator) the integer n that is represented by the following difference:

$$n = \sqrt{24\sqrt{3}+43} - \sqrt{|24\sqrt{3}-43|}.$$

Solution:

1) State the sign of the expression $24\sqrt{3}-43$ to drop the absolute symbol under the sign of the radical:

$24\sqrt{3}-43 =$

Therefore, $|24\sqrt{3}-43| =$

2) Transform the difference $\sqrt{24\sqrt{3}+43} - \sqrt{|24\sqrt{3}-43|}$.

3) Conclusion:

Example: Let us consider the difference:

$$n = \sqrt{|12\sqrt{5} - 29|} - \sqrt{12\sqrt{5} + 29} \qquad (1)$$

1) In order to drop the absolute symbol under the sign of the radical we have to take into account the sign of the expression $12\sqrt{5} - 29$.

Let us transform the expression $12\sqrt{5} - 29$ in the following way:

$$12\sqrt{5} - 29 = \frac{(12\sqrt{5} - 29)(12\sqrt{5} + 29)}{12\sqrt{5} + 29}$$

$$= \frac{(12\sqrt{5})^2 - 29^2}{12\sqrt{5} + 29} = \frac{144 \cdot 5 - 841}{12\sqrt{5} + 29} = \frac{-121}{12\sqrt{5} + 29} < 0$$

If $a < 0$, then $|a| = -a$ by definition of absolute values.

Therefore, $|12\sqrt{5} - 29| = -(12\sqrt{5} - 29) = 29 - 12\sqrt{5}$.

2) Let us square the both sides of equality (1) and then simplify it in view of the above:

$$n^2 = (\sqrt{29 - 12\sqrt{5}} - \sqrt{12\sqrt{5} + 29})^2$$

$$= (29 - 12\sqrt{5}) + (12\sqrt{5} + 29) - 2\sqrt{(29 - 12\sqrt{5})(29 + 12\sqrt{5})}$$

$$= 58 - 2\sqrt{29^2 - (12\sqrt{5})^2} = 58 - 2\sqrt{841 - 720}$$

$$= 58 - 2\sqrt{121} = 58 - 22 = 36$$

Hence, $|n| = 6$.

It is evident that $n < 0$, so $n = -6$.

2. Algebraic Transformations

Recall the following methods of manipulating and simplifying algebraic expressions: factoring, expanding and rationalizing the denominator.

2.1. Factor the following quadratic expressions:

a) $3x^2 - 5x - 2 =$

b) $x^2 - 4x - 21 =$

c) $2x^2 - 5ax - 3a^2 =$

2.2. In the following problems factor the given polynomials as completely as possible:

a) $2x^3 - 5x^2 - x + 6$

b) $x^3 + 5x^2 - 4x - 20$

c) $2x^3 + 3x^2 - 8x + 3$

Solution:

a) $2x^3 - 5x^2 - x + 6$

b) $x^3 + 5x^2 - 4x - 20$

c) $2x^3 + 3x^2 - 8x + 3$

2.3. Write down each of the following expressions as a single fraction in which the denominator is rationalized and simplify when it is possible.

a) $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} =$

b) $\sqrt{x+1}-\dfrac{x}{\sqrt{x+1}} =$

c) $\dfrac{x}{\sqrt{x^2+1}}-\dfrac{\sqrt{x^2+1}}{x} =.$

2.4. Complete the square for each of the following quadratic polynomials:

a) $2x^2+3x+1=$

b) $x^2+4x-3=$

c) $x^2+4ax+b=$

Example:

$$x^2-5x+4=(x^2-2\frac{5}{2}x+\frac{25}{4})-\frac{25}{4}+4=(x-\frac{5}{2})^2-\frac{9}{2}$$

2.5. Write the given expression in the form $\sqrt{a}\sqrt{b^2 \pm (cx+d)^2}$ and simplify when b is a fraction:

a) $\sqrt{x^2 + 3x} =$

b) $\sqrt{9x - 4x^2} =$

c) $\sqrt{-x^2 - 3x + 8} =$.

3. Quadratic Equations

3.1. Solve the following quadratic equations by completing the square:

a) $4x^2 - 4x - 1 = 0$

b) $(x-2)^2 + 3x - 5 = 0$

c) $3x(3x-2) = 6x + 5$.

3.2. Use the quadratic formula to solve the following equations:

a) $2x^2 - 6x - 1 = 0$

b) $x(2x - 3) = 2x - 6$

c) $x^2 + 2ax - b^2 = 0$

3.3. For the following equations $f(x) = 0$ determine whether the real roots are unequal, equal, or do not exist.

a) $f(x) = 4x^2 - x - 5$

b) $f(x) = 9x^2 - 12x + 4$

c) $f(x) = x^2 + x + 1$

Solution:

a) $4x^2 - x - 5 = 0$

b) $9x^2 - 12x + 4 = 0$

c) $x^2 + x + 1 = 0$

3.4. Determine for what values of x the function $f(x)$ is greater than zero and for what values of x the function is less than zero. Express your answers in interval notation.

a) $f(x) = 2x^2 - x$

b) $f(x) = 4x^2 + 4x + 1$

c) $f(x) = x^2 + x + 1$

Hints:

- A product ab of real numbers is positive if and only if both factors are nonzero and have the same sign.
- If $a > 0$ and $b^2 - 4ac < 0$, then $ax^2 + bx + c > 0$ for all $x \in R$.

  If $a < 0$ and $b^2 - 4ac < 0$, then $ax^2 + bx + c < 0$ for all $x \in R$.

Solution:

a) $f(x) = 2x^2 - x =$

b) $f(x) = 4x^2 + 4x + 1 =$

c) $f(x) = x^2 + x + 1 =$

4. Quadratic Functions

A quadratic function is a function of the form $f(x) = ax^2 + bx + c$ with a non-zero coefficient a. The graph of this function is a parabola.

4.1. Prove the following properties of the parabola:

a) Its vertex occurs at the point on the graph with x-coordinate that is equal to $b/2a$.

b) It is symmetric around the vertical line through the vertex.

c) If the coefficient a of x^2 is positive, then the parabola opens up; if a is negative, then it opens down.

d) Its y-intercept is equal to c.

e) Its x-intercepts are equal to the solutions of the quadratic equation $ax^2 + bx + c = 0$ if they are exist.

Proof:

Hint: Complete the square to get standard form $f(x) = a(x - x_0)^2 + y_0$, where $x_0 = b/2a$ and $y_0 = f(x_0)$.

Using this form one can easily prove the statements a)-.c).

The proof of the statements d)-e) can be based on the definition of intercepts.

4.2. Graph the function $y = f(x)$, state the coordinates of the vertex and the equation of the axis of symmetry for each of the following functions:

a) $f(x) = 2x^2 - 5x - 3$

b) $f(x) = -x^2 - 4x + 5$

c) $f(x) = x^2 + 4x - 12$

Hints: In order to graph a parabola $y = ax^2 + bx + c$ and state its main characteristics transform the quadratic functions to the standard form $f(x) = a(x - x_0)^2 + y_0$.

The axis of symmetry is the vertical line $x = x_0$, and the coordinates of the vertex are (x_0, y_0). The value $f(x_0) = y_0$ is either the minimum or maximum of the function; it depends on the sign of the coefficient a. Mark the point of the vertex.

Next select a few x values to the right or the left of the axis of symmetry, substitute these in for x in the original equation and solve for $f(x)$.

Then plot these points and another points at equal distances from the axis of symmetry on the opposite side.

Draw the parabola through the points you have plotted.

Example: Let the function $f(x)$ be equal to the following quadratic polynomial:

$$f(x) = 2x^2 - 6x + 5.$$

1) First we complete the square for this polynomial:

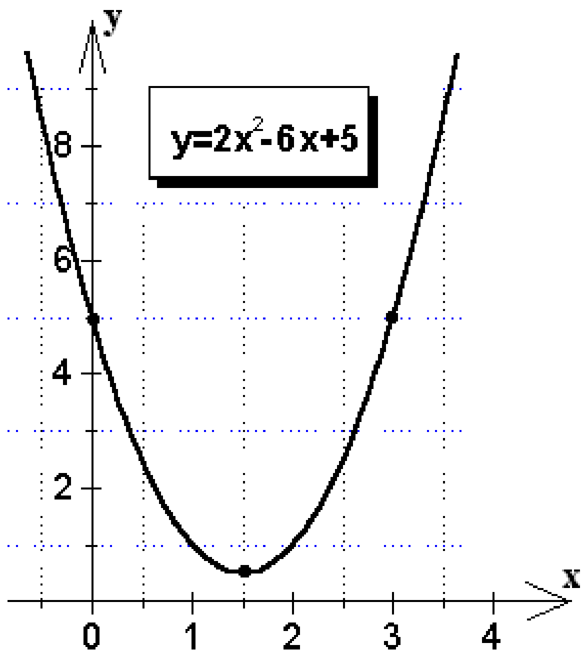
$$f(x) = 2x^2 - 6x + 5 = 2(x^2 - 3x) + 5$$

$$= 2(x^2 - 3\frac{3}{2}x + \frac{9}{4}) - \frac{9}{2} + 5 = 2(x - \frac{3}{2})^2 + \frac{1}{2}$$

There are no real roots for the equations $f(x)=0$, because $f(x)=2(x-\frac{3}{2})^2+\frac{1}{2}>0$ for any $x\in R$.

3) The graph of the function $y=2x^2-6x+5$ is the parabola.

The vertical line $x=\frac{3}{2}$ is its axis of symmetry. The coordinates of the vertex are $(\frac{3}{2},\frac{1}{2})$. It is also clear that $f(0)=5$.



Solution:

a) The standard form of the function $2x^2-5x-3$ is the following:

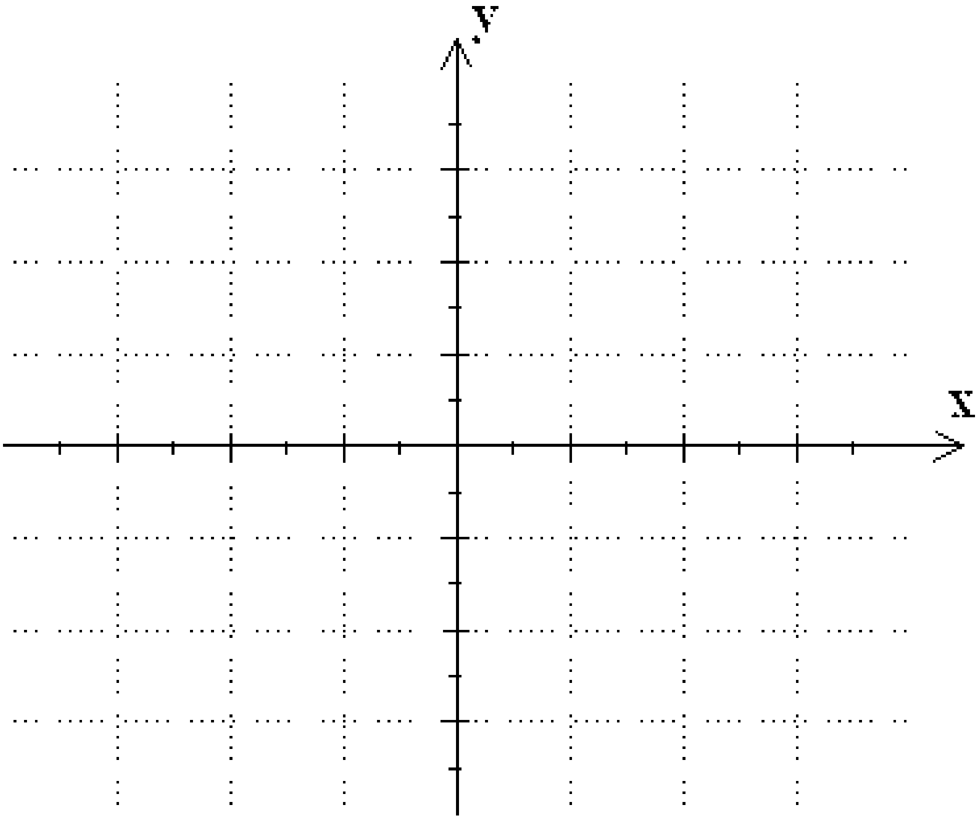
$2x^2-5x-3=$

The axis of symmetry is the vertical line $x=$

The coordinates of the vertex are

A few additional points:

x				
$f(x)$				



b) The standard form of the function $(-x^2 - 4x + 5)$ is the following:

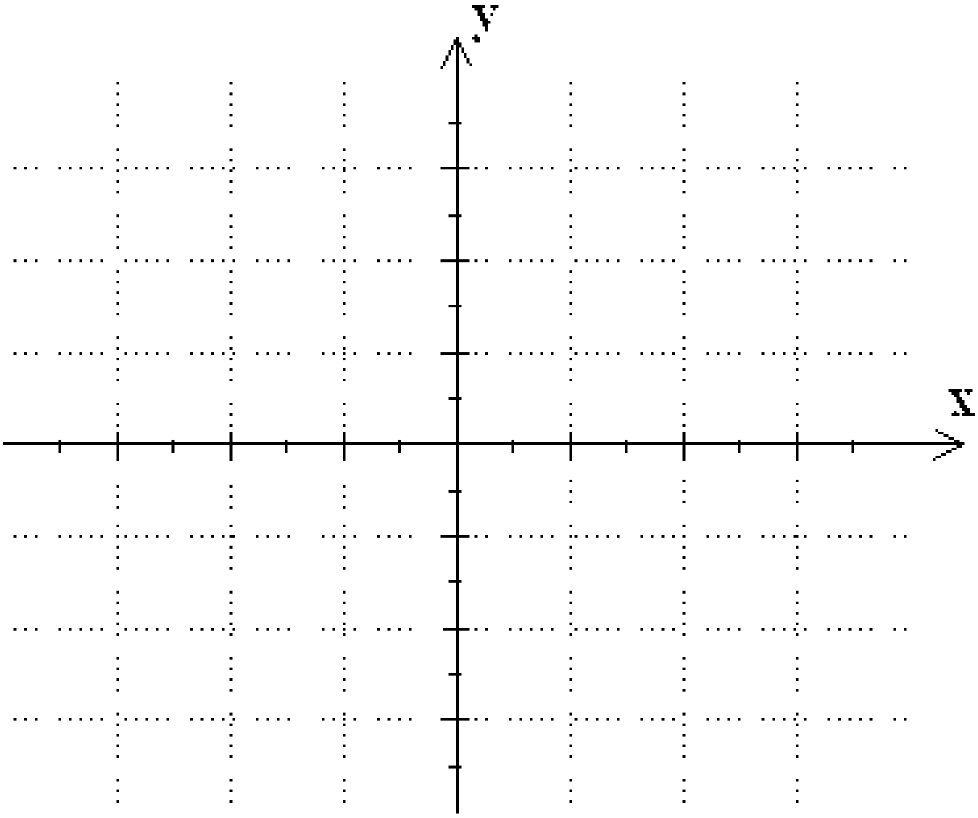
$-x^2 - 4x + 5 =$

The axis of symmetry is the vertical line $x =$

The coordinates of the vertex are

A few additional points:

x				
$f(x)$				



c) The standard form of the function $x^2 + 4x - 12$ is the following:

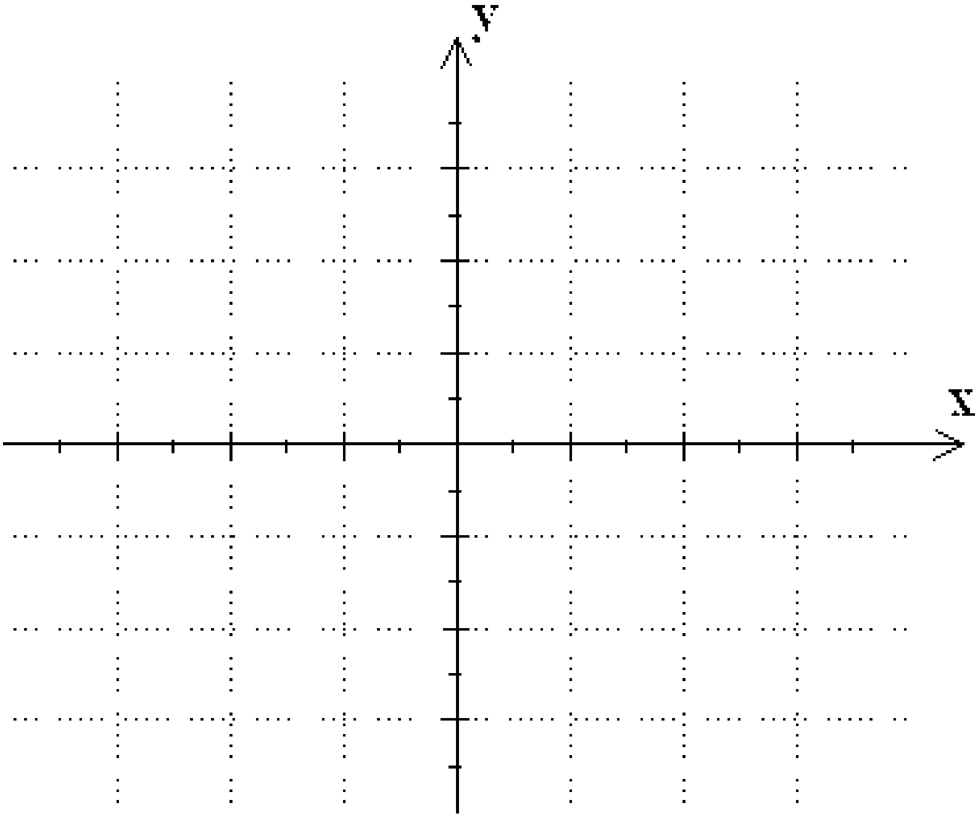
$x^2 + 4x - 12 =$

The axis of symmetry is the vertical line $x =$

The coordinates of the vertex are

A few additional points:

x				
$f(x)$				



4.3. Determine the values of A for which the equation has real and equal roots.

a) $5x^2 - 4x - (5 + A) = 0$

b) $x^2 + 3x - A(2x - 2) = 0$

c) $(A + 2)x^2 + 5Ax - 2 = 0$

Solution:

a) $5x^2 - 4x - (5 + A) = 0$

b) $x^2 + 3x - A(2x - 2) = 0$

c) $(A + 2)x^2 + 5Ax - 2 = 0$

5. Polynomials

5.1. For each of the following equations determine all values of the constant A such that the equation has no real roots. Write your answer in the form of an interval (or union of intervals).

a) $x^2 + 3x = A$

b) $x^4 - 4x^2 = A$

c) $x^4 - 2x^2 - 8 + A = 0$

Solution:

a) $x^2 + 3x = A$

b) $x^4 - 4x^2 = A$

Hint: Use a change of variable, $y = x^2$, to get a quadratic equation.

c) $x^4 - 2x^2 - 8 + A = 0$

Hint: Make the same change of the variable as above.

5.2. How many roots has each of the following equations for positive values of the constant A?

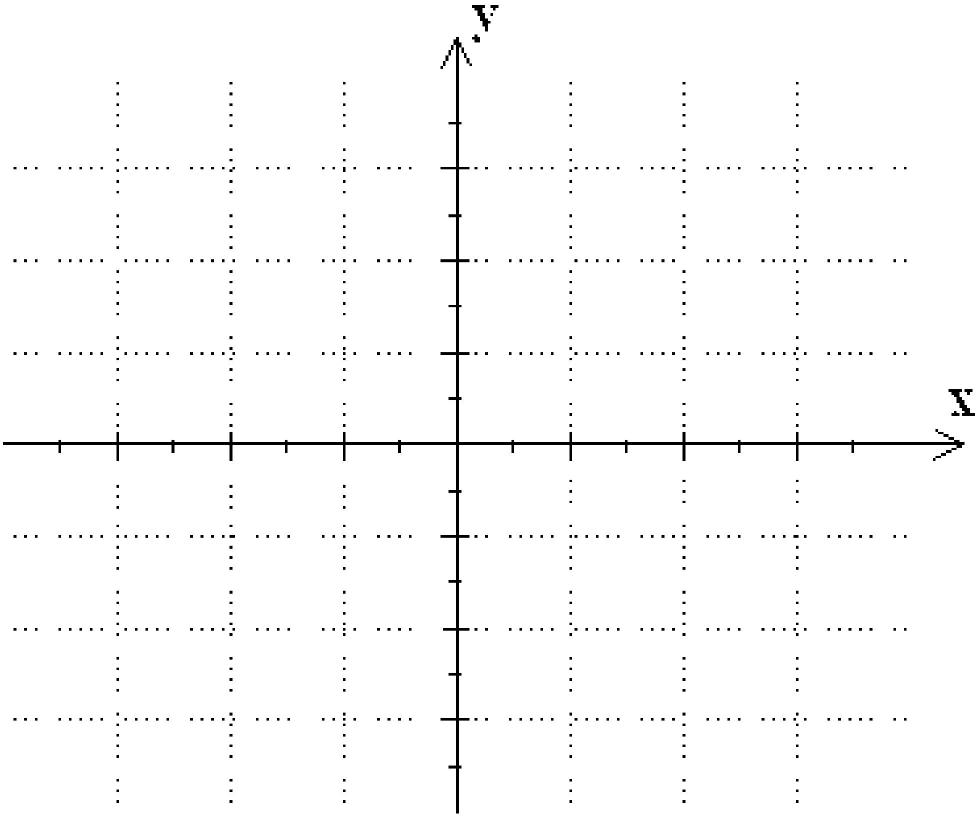
a) $x^3 - 48x + 128 + A = 0$

b) $2x^4 - 5x^2 = A$

Illustrate your answer in diagram form.

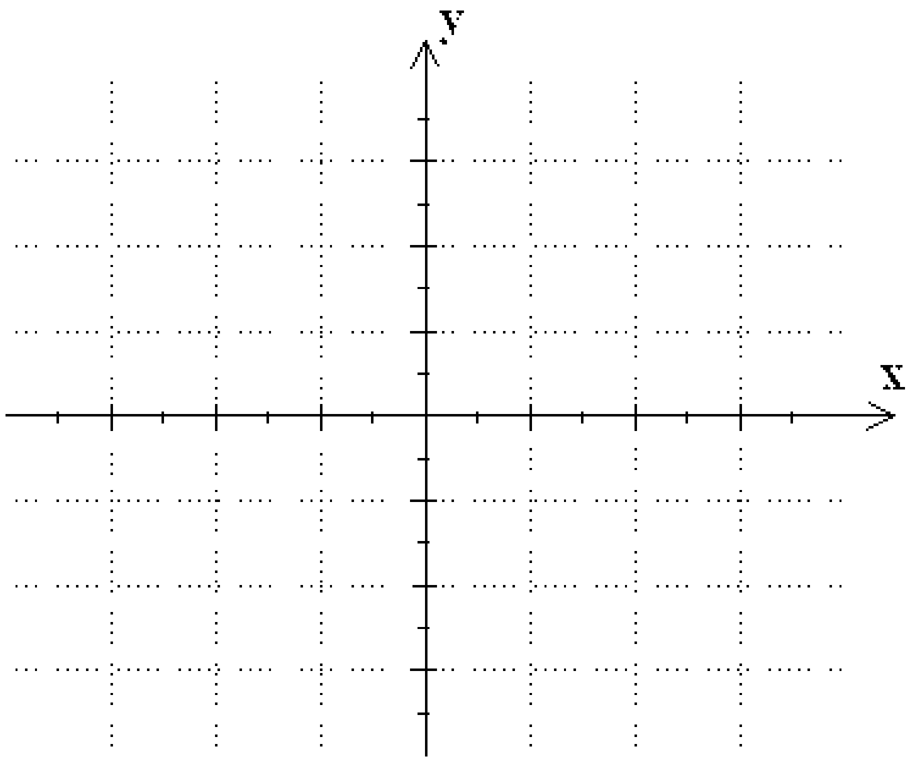
Solution:

a) $x^3 - 48x + 128 + A = 0$



b) $2x^4 - 5x^2 = A$

Hint: Use a change of the variable, $y = x^2$ to get a quadratic equation.



5.3. For each of the following equations determines all values of the constant A such that the equation has exactly two real roots. Write your answer in the form of a graph on the number line.

a) $x^2 - 8x = A$

b) $x^4 - 4x^2 - 5 + A = 0$

c) $2x^4 - 5x^2 + 2 = A$

Solution:

a) $x^2 - 8x = A$

b) $x^4 - 4x^2 - 5 + A = 0$

c) $2x^4 - 5x^2 + 2 = A$

5.4. For each of the following equations determine even one value of the constant A such that the equation has exactly three real roots.

a) $x^4 - 2x^2 + 8 - A = 0$

b) $3x - x^3 = A$

c) $x^3 - 48x = A$

Explain how you arrived at your answer.

Solution:

a) $x^4 - 2x^2 + 8 - A = 0$

b) $3x - x^3 = A$

c) $x^3 - 48x = A$

5.5. For each of the following equations determine even one value of A such that the equations has exactly four real roots.

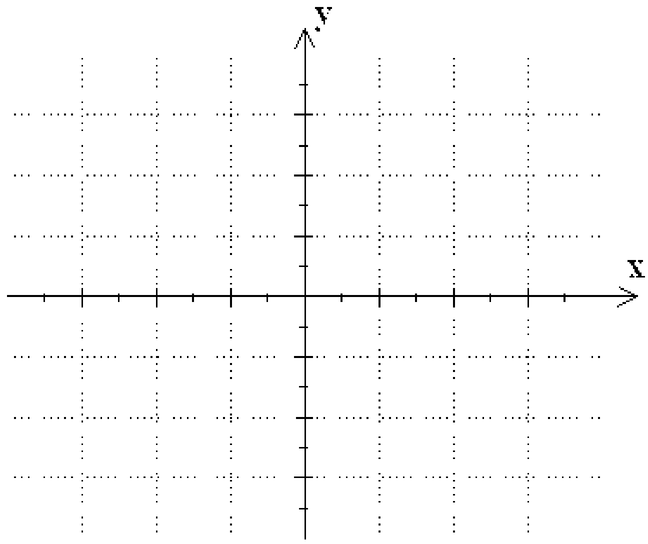
a) $x^4 - 4x^2 + 5 = A$

b) $2x^4 - 5x^2 + A = 0$

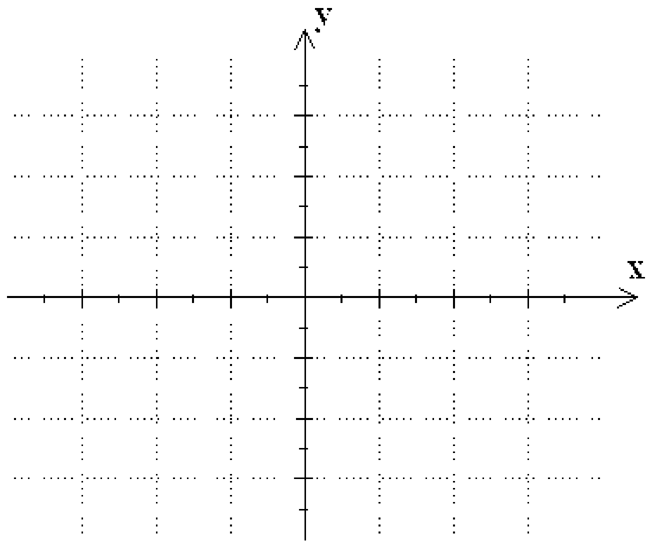
Explain how you arrived at your answer and illustrate it in diagram form.

Solution:

a) $x^4 - 4x^2 + 5 = A$



b) $2x^4 - 5x^2 + 2 = A$



6. Inequalities

6.1. Solve the following inequalities involving absolute values. Give your answer in the form of a graph on the number line.

a) $(x-1)|x+1|>0$

b) $\sqrt{2|x|-1}>0$

c) $\sqrt{x^2-6|x|+9}\geq 1$

d) $|2-x|\leq 5x-6$

e) $|5x-1|<4x$

Hint: The absolute value symbol gives us two possible solutions; so an inequality is split into two inequalities (without absolute value bars) and each of them is solved ordinarily.

Solution:

a) $(x-1)|x+1|>0 \quad \Rightarrow$

Hint for the problems b)-c):

$\sqrt{x}\geq 0$ and $\sqrt{x}$ is defined only on the interval $x\geq 0$. Therefore, any expression containing this factor is restricted to a subset of the interval $x\geq 0$.

b) $\sqrt{2|x|-1}>0 \quad \Rightarrow$

c) $\sqrt{x^2 - 6|x| + 9} > 1 \Rightarrow$

d) $|2 - x| \le 5x - 6$

Hint: Consider two cases, $2 - x \ge 0$ and $2 - x < 0$. Then solve two ordinary linear inequalities and select solutions to satisfy the corresponding conditions.

e) $|5x - 1| < 4x$

Hint: Solve in a similar way as above.

6.2. In the following problems solve the rational inequality using chart method.

a) $\frac{(x^2+9x+18)}{x+3} > 0$

b) $\frac{x-3}{x-1} > x-4$

c) $\frac{(2x^2-3x+5)}{x+5} \geq 0$

d) $\frac{(x^3-2)(x^2-x-6)}{x^2-25} \leq 0$

e) $\frac{(x^2+6x+9)\ln x}{x^5} \geq 0$

f) $\ln(x+1)-\ln(x-1)+\ln(x^2-3x+2) > 0$

g) $\frac{e^x(x-1)(x+2)}{\sqrt{x}} \geq 0$

Hints:

An effective stepwise procedure for solving rational inequalities is the following:

- Move all the terms to one side (usually the left side) of the inequality with zero on the other side.
- Combine the non-zero terms into a single rational expression.
- Factor the numerator and denominator of this expression.
- Find the critical points. That is, the points where the numerator or denominator is zero.

- Draw the real line and divide it into intervals separated by the critical points. Circle each point on this line where the expression is undefined.
- Analyze each factor to determine where it is negative, zero, positive or undefined. Mark each place on the line where the factor changes sign. Record their values on the number line. Record this information in the interval with a "+" or a "-".
    - The sign of an expression is unaffected by any factor that is never negative. Therefore, it may be omitted in the chart providing you record the value of x that makes the expression zero or undefined on the number line.
    - The sign of an expression is unaffected by canceling a factor that appears in the numerator and denominator. Therefore omit on the chart but record on the number line the value of x that makes the factor zero. The expression is undefined for this value of x.
- Select the intervals that satisfy the requirement of the inequality.
- Use the chart to answer the question asked.

  If the problem also involves an equation, select the appropriate critical points that satisfy the given equation. Usually, these are the values of x which make the numerator equal to zero.
- Write your answer in the form of a set (or union of sets), or in the interval notation (or union of intervals), or a graph on the real line.
- Remember that you should not multiply both sides of an inequality by an expression that contains the unknown and can change sign.

Solution:

a) $\dfrac{(x^2+9x+18)}{x+3} > 0 \quad \Rightarrow$

b) $\dfrac{x-3}{x-1} > x-4 \quad \Rightarrow$

c) **Hint**:

Recall the following statements:

- If $a > 0$ and $b^2 - 4ac < 0$, then $ax^2 + bx + c > 0$ for all $x \in R$.
- If $a < 0$ and $b^2 - 4ac < 0$, then $ax^2 + bx + c < 0$ for all $x \in R$.

Therefore,

$\dfrac{(2x^2-3x+5)}{x+5} \geq 0 \quad \Rightarrow$

d) $\dfrac{(x^3-2)(x^2-x-6)}{x^2-25} \leq 0 \quad \Rightarrow$

e) **Hint**: $\ln x$ is defined only on the interval $x > 0$. Hence, any expression containing $\ln x$ as a factor is restricted to a subset of the interval $x > 0$.

$\dfrac{(x^2+6x+9)\ln x}{x^5} \geq 0 \quad \Rightarrow$

f) **Hint**: See the above remark.

$\ln(x+1) - \ln(x-1) + \ln(x^2-3x+2) > 0 \quad \Rightarrow$

g) **Hints**:

- Recall that $e^x > 0$ for all $x \in R$.
- $\sqrt{x} \geq 0$ and $\sqrt{x}$ is defined only on the interval $x \geq 0$. Therefore, any expression containing this factor is restricted to a subset of the interval $x \geq 0$.

$\dfrac{e^x(x-1)(x+2)}{\sqrt{x}} \geq 0 \quad \Rightarrow$

7. Functions

7.1. What is the domain and range of each of the following functions?

a) $f(x) = \sqrt{16 - x^2}$

b) $f(x) = \dfrac{1}{\sqrt{16 - x^2}}$

c) $f(x) = \dfrac{x}{(x^2 - 3)(x^2 + 1)}$

d) $f(x) = \ln |x|$

e) $f(x) = |\ln |x + 5||$

f) $f(x) = \dfrac{\sqrt[3]{x}}{\ln x + 1}$

Solution:

a)

b)

c)

d)

e)

f)

7.2. Prove that any function $f(x)$ defined for all real numbers can be written as the sum of an even function and an odd function.

Solution:

Hints:

- Prove that $f(x)+f(-x)$ is an even function.

- Prove that $f(x)-f(-x)$ is an odd function.

- Represent the function $f(x)$ as a linear combination of the above even and odd functions.

7.3. Which of the following formulas expresses the relationship between x and y in the table below?

a) $y=3x+2$

b) $y=2|x|-3$

c) $y=x^2-2$

d) $y=-x+2$

x	–1	1	2	3
y	–1	–1	2	7

7.4. Give a formula for the inverse function of $f(x)$. Check whether function found is inverse of $f(x)$:

a) $f(x) = 5 - 4x$

b) $f(x) = 3x + 1$

c) $f(x) = 2\ln x$

d) $f(x) = e^{-7x}$

Solution:

Hint: Recall that the functions, $f(x)$ and $f^{-1}(x)$, are said to be inverse of each other if $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

a) $f(x) = 5 - 4x \quad \Rightarrow$

b) $f(x) = 3x + 1 \Rightarrow$

c) $f(x) = 2\ln x \quad \Rightarrow$

d) $f(x) = e^{-7x} \quad \Rightarrow$

7.5. True or False Quiz for Functions

	Statements	True/False
a)	The graph of any equation in two variables is a straight line.	
b)	The graph of any linear equation in two variables is a straight line.	
c)	If $f(x)=(2-x)^2$, then $f(3)=-1$.	
d)	If $f(x)=\frac{\lvert x\rvert}{x}$, then $f(0)=1$.	
e)	If $f(x)=-5$, then $f(0)$ is not defined.	
f)	Every straight line has a slope.	
g)	To find the equation of a straight line, all you need is a point and the slope.	
h)	The graph of a quadratic function $f(x)=ax^2+bx+c$ (with a nonzero) is always a parabola.	
i)	Some parabolas cross neither axis.	
j)	The vertex is the lowest point on a parabola.	
k)	Every parabola crosses both axes.	
l)	$\ln\frac{x}{y}=\frac{\ln x}{\ln y}$	
m)	$\log_a 1 = a$	
n)	$\log_a a = 1$	
o)	If $\log_a x = 2\log_a y$, then $x = 2y$.	

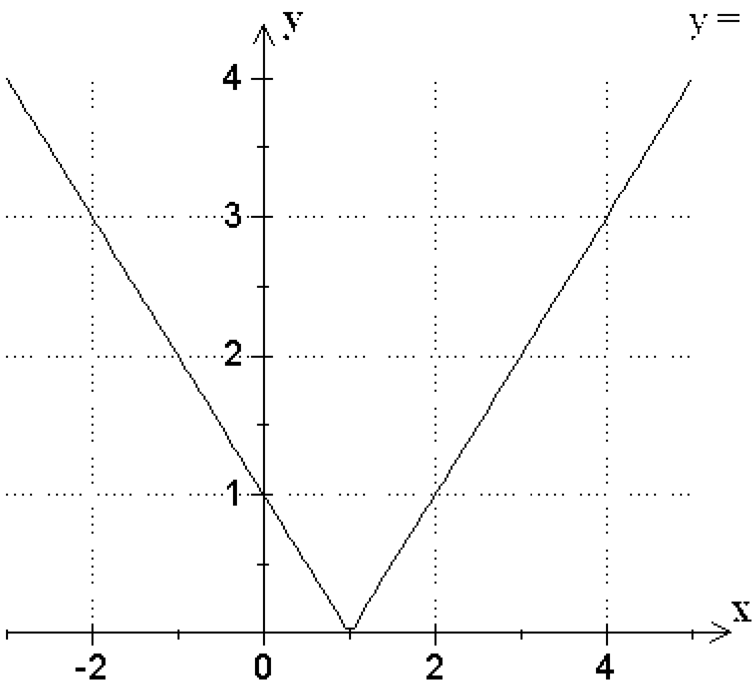
7.6. Match the following functions with their graphs.

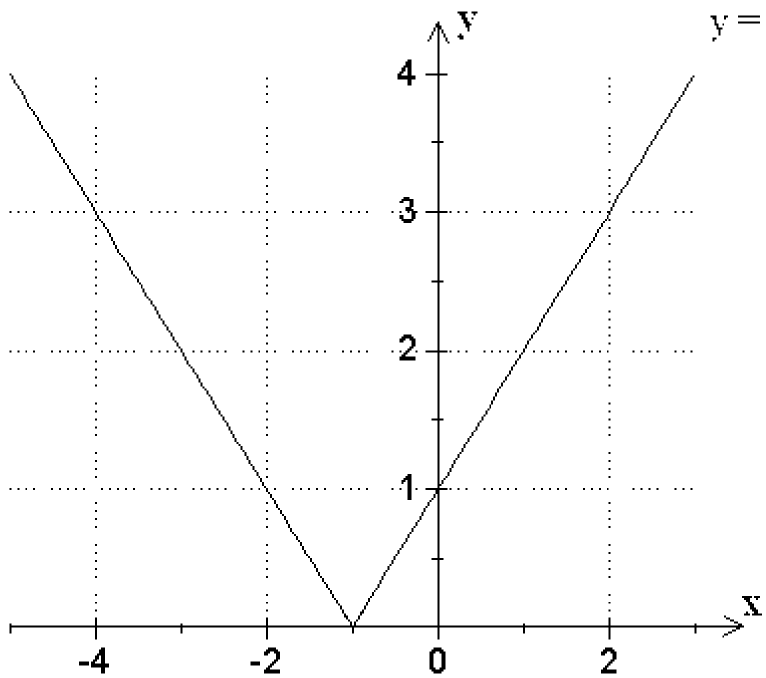
a) $y = |x+1|$

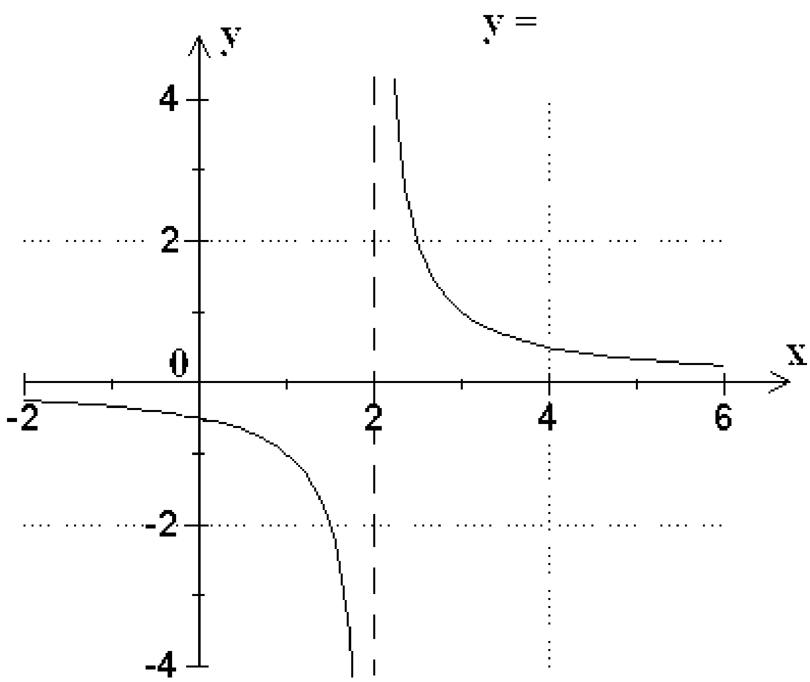
b) $y = \dfrac{1}{(x-2)}$

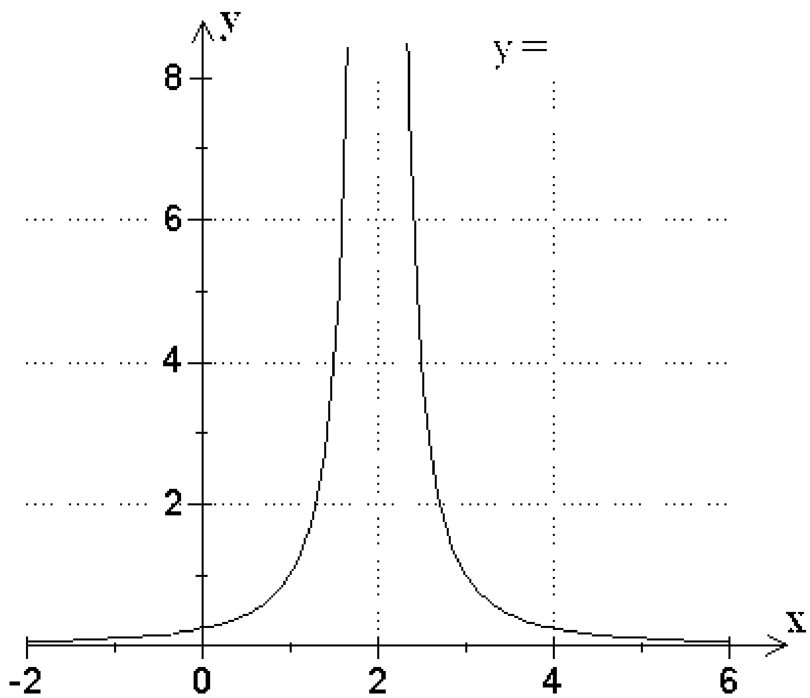
c) $y = \dfrac{1}{(x-2)^2}$

c) $y = |x-1|$






y =
y
x
4
2
0
-2
-4
-2
2
4
6


y
y =
x
8
6
4
2
-2
0
2
4
6

7.7. Find the equation and the slope of the straight line going through the points:

a) (-2; 1) and (3; 5).

b) (1; 4) and (5; -1).

c) (2; -3) and (7; -3).

d) (-4; -2) and (-4; 5).

Sketch each graph, showing the intercepts.

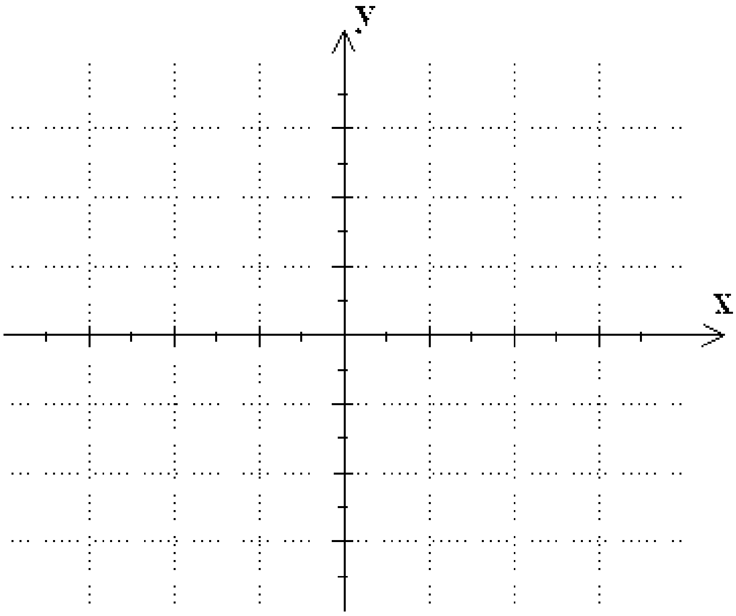
Hints:

- Recall that
  - the equation of a line can be written as follows:
  $$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}.;$$
  - the slope of a line between two different points, $(x_1;\,y_1)$ and $(x_2;\,y_2)$, is $k=\dfrac{y_2-y_1}{x_2-x_1}$.
- To find the x-intercept of the line $y = f(x)$, set $y=0$ in its equation and solve the equation $f(x)=0$ for x.
- To find the y-intercept, set $x=0$ and solve for y.

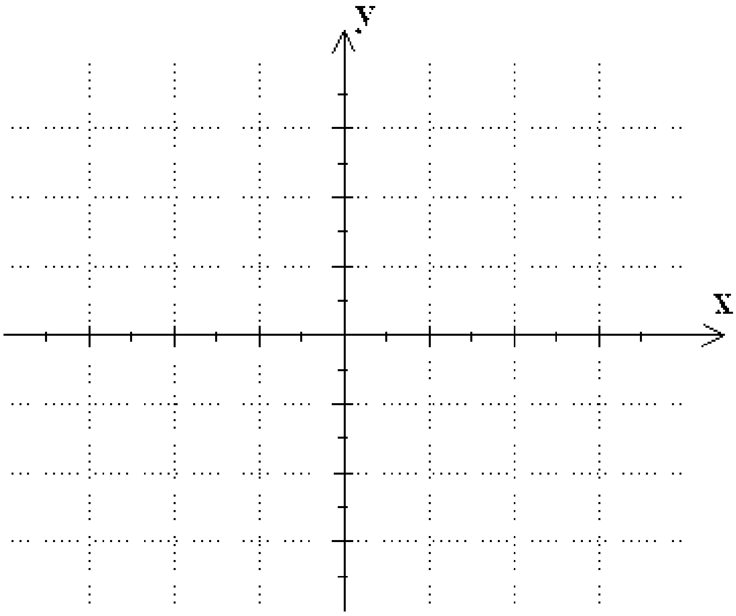
  If the equation is written in the slope-intercept form
  $$f(x)=kx+b$$
  then the y-intercept is equal to the constant b.

Solution:

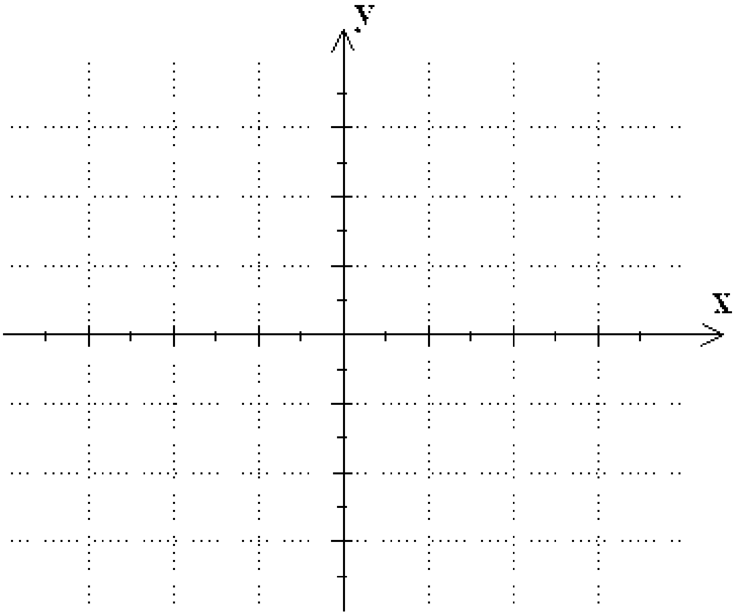
a)



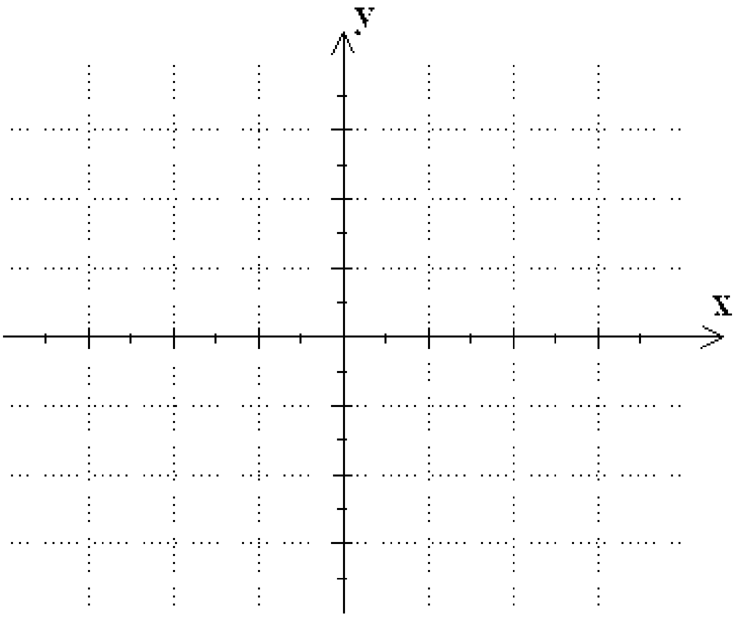
b)



c)



d)



7.8. How many intersection points have the graphs of the following functions?

a) $y = |\log|x+1||$ and $y = 1$

b) $y = x^2 - 4|x| + 3$ and $y = \frac{1}{2}$

c) $y = 2^{\frac{|x|}{x}}$ and $y = |x+1|$

Solution:

a) **Hint**:

- An effective stepwise procedure for plotting the graph of the function $y = |\log|x+1||$ is the following.

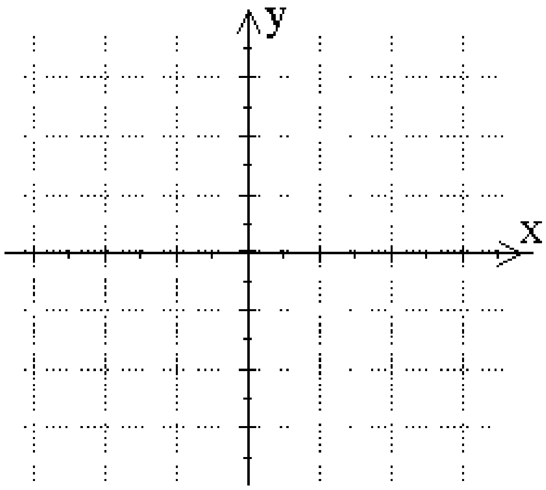
  1) Plot the graph of the function $y = \log x$ and its mirror image through the y-axis to get the graph of the function $y = \log|x|$.

  2) Translate it along the x-axis on one unit to the left to get the graph of the function $y = \log|x+1|$.

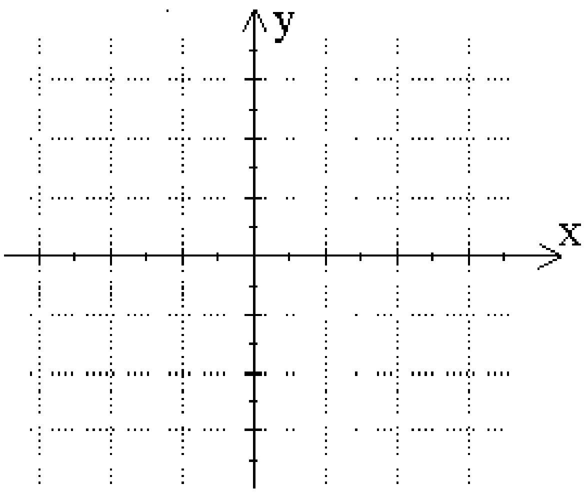
  3) Replace those parts of the graph $y = \log|x+1|$ which lie below the y-axis by their mirror image through the x-axis to get the graph of the function $y = |\log|x+1||$.

- Then plot the graph of the function $y = 1$ and count the number of intersection points.

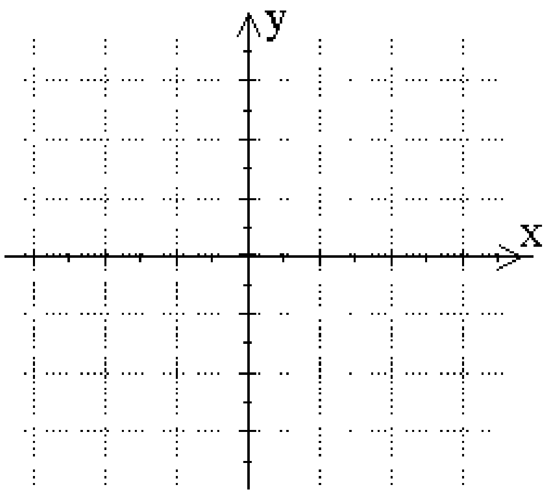
$y = \log x$



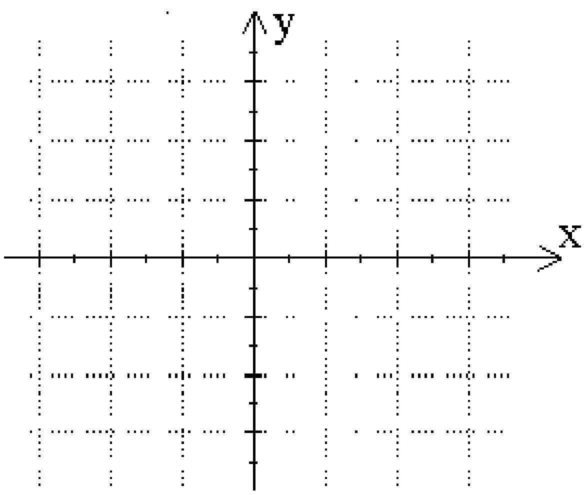
$y = \log |x|$



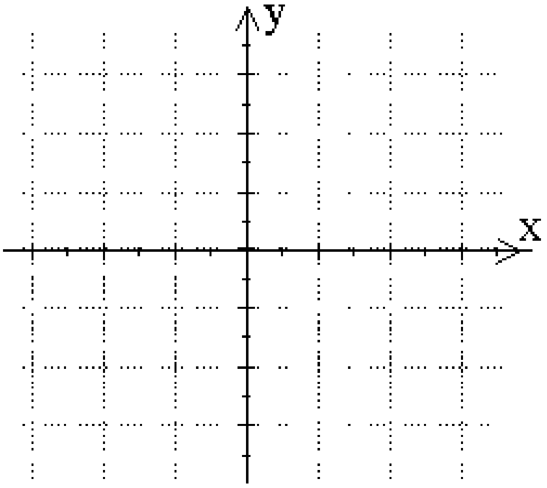
$y = \log |x+1|$



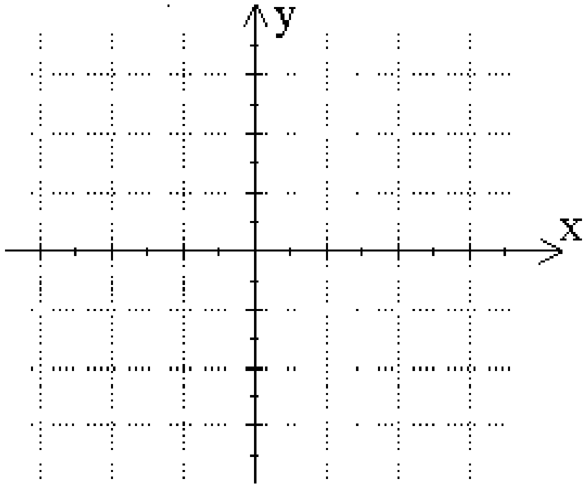
$y = |\log |x+1||$ and $y = 1$



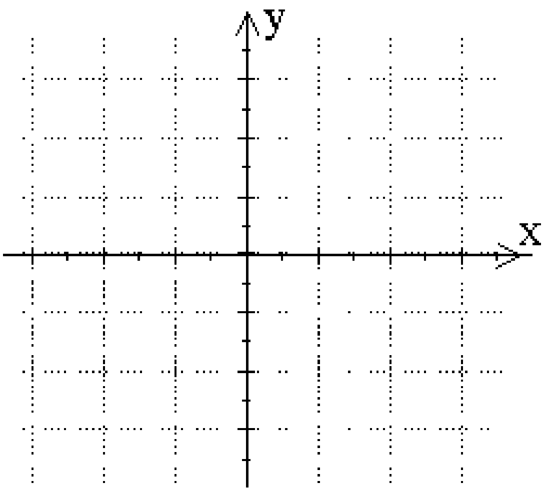
b) $y = x^2 - 4x + 3$



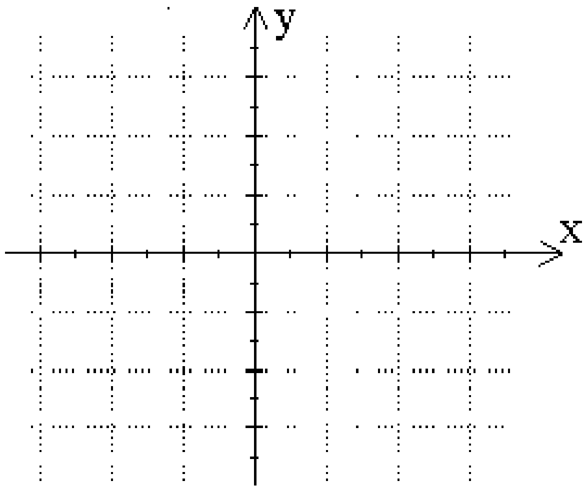
$y = x^2 - 4|x| + 3$ and $y = \frac{1}{2}$



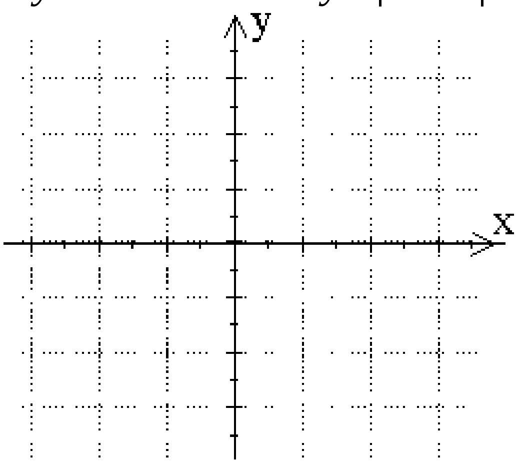
c) $y = x + 1$



$y = |x + 1|$



$y = 2^{\frac{|x|}{x}}$ and $y = |x + 1|$



7.9. Simplify the following expressions:

a) $\log_{\frac{1}{5}}(\sqrt{125}\frac{1}{625}) =$

b) $2^{3-\log_2 5} =$

c) $36^{\log_6 5} + 10\log^{1-\log 2} - 3^{\log_9 36} =$

7.10. Solve the following equations:

a) $(\frac{2}{5})^{3-2x} = \frac{125}{8}$

b) $\log_2(2^x - 7) = 3 - x$

c) $10^{x^3+1} - 10^{x^3-1} = 99$

d) $\log(64\sqrt[24]{2^{x^2-40x}}) = 0$

e) $x^{\log_{x^2}(x^2-1)} = \sqrt{8}$

Hint:

Recall the properties of the exponential and logarithm functions.

Solution:

a) $(\frac{2}{5})^{3-2x} = \frac{125}{8}$

b) $\log_2(2^x - 7) = 3 - x$

c) $10^{x^3+1} - 10^{x^3-1} = 99$

d) $\log(64\sqrt[24]{2^{x^2-40x}}) = 0$

e) $x^{\log_{x^2}(x^2-1)} = \sqrt{8}$

7.11. Solve the following inequalities:

a) $\log_2 \log_3 \frac{3x}{x-2} > 0$; b) $\log_x (x+2) > 2$;

c) $3^{2x} - 34 \cdot 2^{-x} - 72 \cdot 4^{-x} > 0$; d) $\frac{1}{\log_2 x - 1} - \frac{1}{\log_2 x} > 0$.

Solution:

a) $\log_2 \log_3 \frac{3x}{x-2} > 0$

b) $\log_x (x+2) > 2$

c) $3^{2x} - 34 \cdot 2^{-x} - 72 \cdot 4^{-x} > 0$

d) $\frac{1}{\log_2 x - 1} - \frac{1}{\log_2 x} > 0$.

Discrete Algebra

Notations:

- The sum of numbers indexed by integers is written using $\sum$-notation as follows:

$$a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

The naming of the summation variable is irrelevant, *i.e.*

$$\sum_{i=1}^{n} a_i = \sum_{k=1}^{n} a_k$$

Check the following formulae for $n = 3$:

a) $$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$$

b) $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

c) $$\sum_{i=1}^{n} 1 = n$$

d) $$\sum_{i=1}^{n} 0 = 0$$

e) $$\sum_{i=1}^{n} (a_i - a_{i-1}) = a_n - a_0$$

f) $$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i ,$$ where $1 \leq m \leq n$

g) $$\sum_{i=1+j}^{n+j} a_i = \sum_{i=1}^{n} a_{i+j}$$

Solution:

a) $\sum_{i=1}^{3} ca_i =$

b) $\sum_{i=1}^{3} (a_i + b_i) =$

c) $\sum_{i=1}^{3} 1 =$

d) $\sum_{i=1}^{3} 0 =$

e) $\sum_{i=1}^{3} (a_i - a_{i-1}) =$

f) $\sum_{i=1}^{2} a_i + \sum_{i=3}^{3} a_i =$

g) $\sum_{i=3}^{5} a_i =$

h) $\sum_{i=1}^{3} a_{i+2} =$

The product of all natural numbers from 1 to n is denoted by the symbol "!" (factorial): $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

There is a convention that $0! = 1$.

Check the following identities:

a) $n! = n \cdot (n-1)!$

b) $(n+1)! = (n+1) \cdot n \cdot (n-1)!$

c) $1! = 1$

d) $2! = 2$

e) $3! = 6$

f) $4! = 24$

g) $5! = 120$

h) $6! = 720$

---

8. Mathematical Induction Principle

Recall the main idea of this method:

Let S_n be an infinite sequence of statements for $n = 0,1,2,\ldots$

If the statement S_n is true for $n = 0$, and if the truth of S_n implies that S_{n+1} is true, then S_n is true for every non-negative integer n.

The procedure of proving the validity of some statement S_n for all integers $n \geq k$ includes the following three stages:

- You have to originate a basis of induction.
- You have to formulate an induction hypothesis.
- You must prove that the statement S_n implies S_{n+1}.

Note: If $S_n \Rightarrow S_{n+1}$ but the statement S_k is false, then one cannot conclude anything about S_n for $n > k$.

8.1. Prove that $3^n > n$ for all positive integers n.

Proof:

Let S_n be the statement $3^n > n$

Hint:

- In order to form the induction basis you have to check that the statement S_n is true for $n = 1$.

- Formulate the induction hypothesis. At this stage of induction you suppose the truth of the statement S_n but prove nothing.

- The induction step is the main stage of induction. If the statement S_n implies S_{n+1}, provided $n \geq 1$, then S_n must be true for all integers $n \geq 1$. Here, you proceed from verifications and assumptions to direct proving of the statement.

8.2. Prove that for all positive integers n the following formula is valid:

$$\sum_{i=1}^{n}(i-\frac{1}{2})=\frac{1}{2}n^2$$

Proof:

Induction basis:

Induction hypothesis:

Induction step:

8.3. Prove the following formula for all positive integers n:

$$\sum_{i=1}^{n}q^i=\frac{q(1-q^n)}{1-q}$$

Proof:

Induction basis:

Induction hypothesis:

Induction step:

8.4. Prove the following formula for all positive integers n:

$$\sum_{i=1}^{n}(i-\frac{1}{2})^2=\frac{n}{3}(n^2-\frac{1}{4})$$

Proof:

Induction basis:

Induction hypothesis:

Induction step:

9. Arithmetic Progression

9.1. Calculate the sum of the first 15 terms of the arithmetic progression if $a_4=5$ and $a_7=14$. Find the tenth term.

Solution:

9.2. Find the second term of the arithmetic progression if $a_5 = 3$ and the sum of the first ten terms $S_{10} = 35$.

Solution:

10. Geometric Progression

10.1. Calculate the sum of the first ten terms of the geometric progression and find the seventh term if the common ratio $q = 0.5$ and $a_{10} = 4$.

Solution:

10.2. Find the eighth term of the geometric progression and the common ratio if $a_3 = 2$ and $a_5 = 4$.

Solution:

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Workbook

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