ФЕДЕРАЛЬНОЕ АГЕНТСТВО ПО ОБРАЗОВАНИЮ
Государственное образовательное учреждение высшего профессионального образования
«**ТОМСКИЙ ПОЛИТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ**»

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MATHEMATICS

PREPARATORY COURSE
ALGEBRA

Textbook

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The textbook includes the following sections: The Real Number System; Transformations of Algebraic Expressions; Equations and Inequalities; Elementary Functions and Inverse Functions; Discrete Algebra including The Mathematical Induction Principle.

The course is designed for English speaking students.

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Contents

ALGEBRA

1. The Real Number System

1.1. Basic Notations and Definitions

This part contains definitions for many of the symbols, mathematical notations, and abbreviations used in mathematical and technical literature.

1.1.1. Numbers

- **A positive number** is the number that is greater than zero.
- **A negative number** is the number that is less than zero.
- The number **zero** is a mathematical value intermediate between positive and negative numbers, *i.e.* it is neither positive nor negative.
- **Natural numbers** are the following numbers: $1, 2, 3, 4, \ldots$
- **Integers** are the following numbers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

> All natural numbers and the number zero are integers.

- The numbers, that can be represented as a fraction $\frac{p}{q}$ (where both p and q are integers and q is not equal to zero), are called **rational numbers**.

  All integers are also rational numbers, because any integer can be represented as a fraction $\frac{\text{integer}}{1}$.

  In addition, the fraction $\frac{p}{q}$ can be also represented:

  - either as terminating decimal, *e.g.* $3/4 = 0.75$;
  - or as repeating decimal, *e.g.* $15/11 = 1.3636(36)\ldots$

- **Irrational numbers** are the numbers that can be represented as non-repeating and nonterminating decimals.

  An irrational number cannot be represented as a fraction $\frac{p}{q}$ for any integers p and q.

  Typical examples of irrational numbers are the numbers $\pi \approx 3.14159$ and $\sqrt{2} \approx 1.4142$.

> Irrational numbers cannot be rational numbers, and vice versa.

- **Real** numbers are the numbers that are either rational or irrational.
- An **even** number is an integer that is divisible by the number two.
- An **odd** number is an integer that is not divisible by the number two.

Examples:

- Number 5 is:
  a positive number;
  a natural number;
  an integer;
  a rational number;
  a real number.

- Number (-4.2) is:
  a negative number;
  a rational number;
  a real number.
- Number (-4.2) is not
  an irrational number.

A set of real numbers can be graphically represented by the real number line, that is a straight line, on which an origin (number zero) and a scale are chosen.

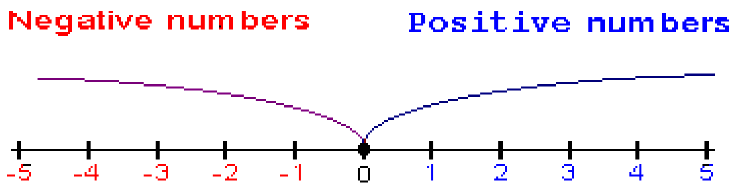


Fig.1

There is one-to-one correspondence between the set of real numbers and points on the real number line: every point on this line corresponds to a real number, and *vice versa*.
All positive real numbers are represented by points, that lie to the right of the number zero, while all negative real numbers are represented by points to the left of the number zero.
All positive numbers are ordered, in ascending order from left to right, to the right side of zero; all negative integers are ordered, in descending order from right to left, to the left side of zero. If the real number is an integer, its point on the number line coincides with one of the notches for an integer; otherwise, its point lies between two successive notches.

1.1.2. Properties of Real Numbers

Most algebraic manipulations are based on the properties of real numbers.
All real numbers have the following properties:

- **Symmetric Property**

The equality $a = b$ implies $b = a$.

Example:
The equality $x + y = z$ implies $z = x + y$.

- **Transitive Property**

Two numbers are equal to each other if each of them is equal to the same number.

In other words, the equalities $a = b$ and $c = b$ imply $a = c$.

Example:
The equalities $x + y = z$ and $z = 4 + c$ imply $x + y = 4 + c$.

- **Substitution Property**

Any number may be substituted for its equal in any expression.

If $a = b$ then a may be replaced by b and b may be replaced by a in any mathematical statement.

Example:
If $x = 2$ and $x + y = c$ then $2 + y = c$.

Addition and Subtraction Properties

If equal numbers are added to equal numbers, then the sums are equal.
If equal numbers are subtracted from equal numbers, then the differences are equal.

If $a = b$ and $c = d$, then $a \pm c = b \pm d$.

Multiplication Property

If equal numbers are multiplied by equal numbers, then the products are equal.

If $a = b$ and $c = d$ then $ac = bd$.

Note: The numbers in a product are called **factors**.

Commutative Laws for Addition and Multiplication

Numbers can be added in any order:
$a + b = b + a$.

Numbers can be multiplied in any order:
$a \cdot b = b \cdot a$

Associative Laws for Addition and Multiplication

Addition items can be combined in any groups:
$a + (b + c) = (a + b) + c$.

Factors can be combined in any groups:
$a(bc) = (ab)c$.

Distributive Law

Parentheses can be expanded; a common factor can be taken out:
$a(b \pm c) = ab \pm ac$
$(a \pm b)c = ac \pm bc$

Identity Axiom of Addition

The sum of any real number and zero is the same real number:
$a + 0 = 0 + a = a$.

Identity Axiom of Multiplication

The product of any real number and number one is the same real number:
$a \cdot 1 = 1 \cdot a = a$.

Additive Inverse Axiom

For any real number a there exists the unique real number $(-a)$ such that

$$a + (-a) = -a + a = 0.$$

The number $(-a)$ is known as the additive inverse of a.

We can say that subtraction is the inverse to addition and addition is the inverse to subtraction.

Addition and subtraction are inverse operations to each other.

- **Multiplicative Inverse Axiom**

For any non-zero real number a there exists the unique real number $(1/a)$ such that

$$a\cdot(1/a)=(1/a)\cdot a=1.$$

The number $(1/a)$ is known as the multiplicative inverse or reciprocal of a.

Multiplication and division are inverse operations to each other.

- The product of zero and any real number is zero.

$$0\cdot a=a\cdot 0=0$$

- For any real numbers a and b one and only one of the following conditions holds:

$a>b$ (a is greater than b)

$a=b$ (a is equal to b)

$a<b$ (a is less than b).

1.2. Absolute Values

The **absolute value** of the real number a is denoted by the symbol $|a|$ and defined as follows:

$$|a|=\begin{cases} a, & \text{if } a\ge 0 \\ -a, & \text{if } a<0 \end{cases} \qquad (1)$$

The absolute value of a non-negative number is the number itself, while the absolute value of a negative number is the negative of the number.

Examples: $|5|=5$

$|-5|=-(-5)=5$

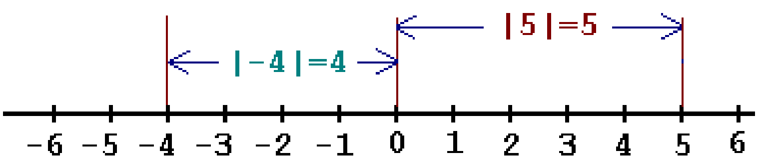
$|0|=0$

Geometric interpretation:

The absolute value of a real number is the distance between the corresponding point
on the number line and zero-point regardless of the direction.

For all numbers a and b, the distance between a and b on the number line is $|a-b|$.

Example: $|-4|=4$ because (-4) is 4 units from 0.



Properties of absolute values

- $|a| \geq 0$
- $|a| = 0$ if and only if $a = 0$
- $|a \cdot b| = |a| \cdot |b|$
- $|\frac{a}{b}| = \frac{|a|}{|b|}$ $(b \neq 0)$
- $|-a| = |a|$
- $|a - b| = |b - a|$
- $|a|^2 = a^2$

1.3. Fractions

A **fraction** is a number written in the form $\frac{a}{b}$, where number a is called a **numerator** and number b is called a **denominator**. Both the numerator and denominator are any real numbers, but the denominator cannot be equal to zero.

The fractions have the following properties:

- The fraction keeps its value when both the numerator and denominator
are multiplied or divided by the same nonzero number:

$$\frac{ac}{bc} = \frac{a}{b}$$

We can use this property to simplify the fraction by factoring the numerator and denominator into prime factors and reducing common factors.

Examples:

- $\frac{30}{45} = \frac{2 \cdot 3 \cdot 5}{3 \cdot 3 \cdot 5} = \frac{2}{3}$
- $\frac{8x-4}{6x-3} = \frac{4(2x-1)}{3(2x-1)} = \frac{4}{3}$

One can also read this property from left to right when it is necessary to reduce a fraction to a different denominator, *e.g.* $\frac{4}{5} = \frac{4 \cdot 2}{5 \cdot 2} = \frac{8}{10}$.

- In order to add (or subtract) fractions with the same denominators, combine the numerators and keep the same denominator:

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \qquad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$$

Two last formulas can be combined into the following uniform expression:

$$\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$$

- In order to add (or subtract) fractions with unlike denominators, reduce the fractions to a common denominator by finding a common multiple of both denominators and then add (or subtract) the fractions with the same denominators:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{bc}{bd} = \frac{ad \pm bc}{bd}$$

Examples:

- $\frac{4}{5} - \frac{2}{3} = \frac{4 \cdot 3}{5 \cdot 3} - \frac{2 \cdot 5}{3 \cdot 5} = \frac{12-10}{15} = \frac{2}{15}$
- $\frac{1}{ab} + \frac{1}{bc} = \frac{c}{abc} + \frac{a}{abc} = \frac{c+a}{abc}$

- The numerator of a product of fractions equals the product of the numerators, and the denominator is equal to the product of the denominators of all the fractions:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

- In order to divide two fractions, invert the second fraction to make the multiplication problem, then multiply:

$$\frac{a}{b} \bigg/ \frac{c}{d} = \frac{a}{b}\frac{d}{c} = \frac{ad}{bc}$$

Example: $\frac{6a}{5b} \bigg/ \frac{3}{b} = \frac{6a}{5b} \cdot \frac{b}{3} = \frac{6ab}{5b \cdot 3} = \frac{2a}{5}$

Equivalent fractions are known as **proportions**.
If two ratios are equal, then their reciprocals are also equal:

$$\frac{a}{b} = \frac{c}{d} \quad \Rightarrow \quad \frac{b}{a} = \frac{d}{c}$$

The proportions may be solved by cross multiplication using the cross product property:

$$\frac{a}{b} = \frac{c}{d} \quad \Rightarrow \quad ad = bc$$

From $\frac{a}{b} = \frac{c}{d}$ it also follows that $\frac{d}{b} = \frac{c}{a}$ and $\frac{a}{c} = \frac{b}{d}$.

One can easily prove the following helpful property of proportions:

For any real numbers t_1 and t_2 that are not equal to zero at the same time, if $\frac{a}{b} = \frac{c}{d} = \lambda$

then $\frac{t_1 a + t_2 c}{t_1 b + t_2 d} = \lambda$.

It looks in a general form as follows:

$$\text{If } \frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n} = \lambda \text{ then } \frac{t_1 a_1 + t_2 a_2 + \ldots + t_n a_n}{t_1 b_1 + t_2 b_2 + \ldots + t_n b_n} = \lambda .$$

Example:

- If $\frac{x}{y} = \frac{z}{w}$, then $\frac{x}{y} = \frac{2x - 7z}{2y - 7w} = \frac{x + 5z}{y + 5w}$.

All the above properties hold true for the quotient of two algebraic expressions. They usually apply for manipulations with rational expressions.

1.4. Sets

A **set** is a finite or infinite collection of objects. The objects are called elements or members of the set. For instance, numbers or words can be considered as elements. Capital letters are usually used as names for sets. The pair of braces, $\{\ \}$, is used to enclose either elements of the set or its description list, using commas to separate the individual elements.

If the set A is defined by the list of its elements, then it can be written in the following format:

$$A = \{\text{list of elements}\}$$

If the element x is an element of a set A, it is written using the symbol $x \in A$. Otherwise, the statement "x is not an element of A" is written symbolically as $x \notin A$.

The set A can be also defined by describing its elements through characterizing properties: "The set A of all elements x such that x has the property P". In this case, the symbol "|" is used instead of the statement "such that", and the set is written in the following format:

$$A = \{x \mid P\}$$

Examples:

- Let A be a set of the elements x, a, b. The set A is defined here by the list of its elements and so it can be denoted as $A = \{a, b, x\}$.
- Let N be the set of all natural numbers: $N = \{1, 2, 3, \ldots\}$

  Then the notation $7 \in N$ means that number seven is a natural number, and the notation $\sqrt{3} \notin N$ means that $\sqrt{3}$ is not a natural number.
- Let B be the set of the natural numbers except number five. Then B may be symbolized as

$$B = \{n \mid n \in N, n \neq 5\}.$$

Note: The set A is a **finite set** whereas N and B are **infinite sets**.

If a set has no elements, it is called a **null set** or an **empty set** and it is denoted by the symbol Ø. Thus, the set of natural numbers $n < 1$ is a null set: $\{\mathrm{n} \mid \mathrm{n} \in \mathrm{N},\ \mathrm{n} < 1\} = \varnothing$.

1.4.1. Some Important Sets

♦ The set of **natural numbers** N:

$$N = \{1, 2, 3, \ldots\} = \{\text{Natural \#s}\} = \{\text{nat. \#s}\}$$

♦ The set of all **integers** is denoted by I:

$$I = \{\ldots, -2, -1, 0, 1, 2, \ldots\} = \{\text{Integer \#s}\}$$

- The set of all **rational numbers** is symbolized as

$$Q = \{p/q \mid q \neq 0, p, q \in I\}$$
$$= \{\text{Rational \#s}\}$$

- The set of all **irrational numbers** is denoted by the symbol H.
- The set of all rational and irrational numbers is the set of **real numbers** that is denoted by the symbol R. The set of real numbers is also called the **continuum**.

1.4.2. Comparison between Sets

The set A is **equal** to the set B if every element of A is an element of B, and *vice versa*.
Notation: $A = B$
Read: A is equal to B
Means: A and B have precisely the same elements.

Example: $\{a,b,c\} = \{c,a,b\}$

The set A is said to be **a proper subset** of the set B if every element of A
is an element of B but $A \neq B$.
Notation: $A \subset B$.
Read: A is a proper subset of B.
Means: Every element of A is also an element of B.

Examples:
- The set of natural numbers is a proper subset of the set real numbers: $N \subset R$.
- The set $\{a,b,c\}$ is a proper subset of the set $\{a,b,c,d\}$: $\{a,b,c\} \subset \{a,b,c,d\}$.

Note: Ø is always considered to be a subset of any set.

The set A is said to be a **subset** of the set B if either A is a proper subset of B or $A = B$.
Notation: $A \subseteq B$
Read: A is a subset of B.
Means: Either $A \subset B$ or $A = B$.

Examples:
- $\{a,b,c\} \subseteq \{a,b,c\}$
- $\{a,b,c\} \subseteq \{a,b,c,d\}$.

The **intersection** of the sets A and B is the set of all elements that are as in A as in B.
Notation: $A \cap B$.
Read: "A intersects B" or "A and B".
Means: The set of all elements that are both in A and in B.

Example: If $A = \{a,b,c\}$ and $B = \{a,c,d,e,f\}$, then $A \cap B = \{a,c\}$.

Note: The sets of rational numbers and irrational numbers are mutually exclusive sets and they have nothing in common. Therefore, $H \cap Q = Ø$

The **union** of the sets A and B is the set of all elements that are either in A or B, or both.

Notation: $A \cup B$

Read: "A union B" or "A or B".

Means: The set comprising all elements from A or B.

Example: $\{\text{real } \#s\} = \{\text{irrational } \#s\} \cup \{\text{rational } \#s\}$.

The **Venn diagrams** below illustrate the above definitions graphically.

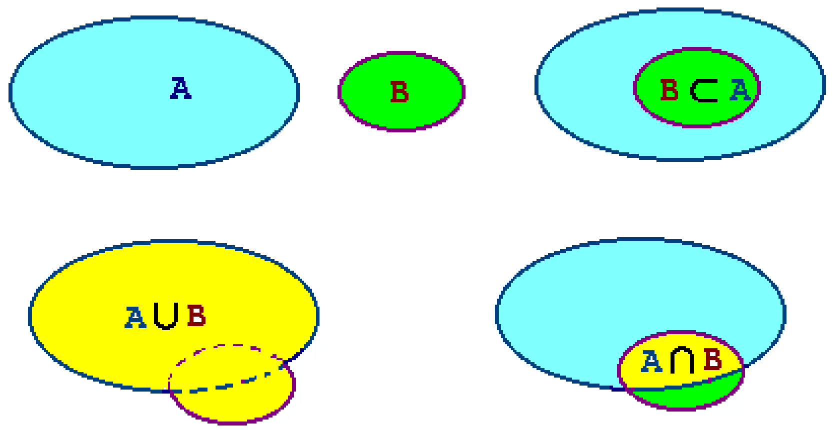


Fig. 2

1.5. Intervals

Intervals are special subsets of real numbers.

An interval may be **finite** or **infinite**. The finite interval of real numbers lies between two real points, a and b. The infinite interval has only one real endpoint and contains all of the other real numbers that lie in the direction of positive or negative infinity from this point.

- If a collection of real numbers lies between a and b, but does not include either of them, the interval is **open**.
  The open interval (a,b) is a set of all real numbers x with $a < x < b$.
- If both endpoints, a and b, are included in the set, the interval is **closed**.
  The closed interval $[a,b]$ is a set of all real numbers x with $a \le x \le b$.
  Open and closed intervals are shown on the number line (Fig. 3a).

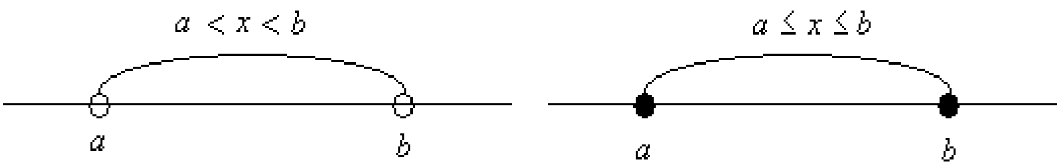


Fig. 3a

- A **half-open** interval contains either a or b.
  The half-open interval $(a,b]$ is a set of real numbers x with $a < x \le b$, while $[a,b)$ is a set of real numbers x with $a \le x < b$.
  Half-open intervals look on the number line as the following (Fig. 3b):

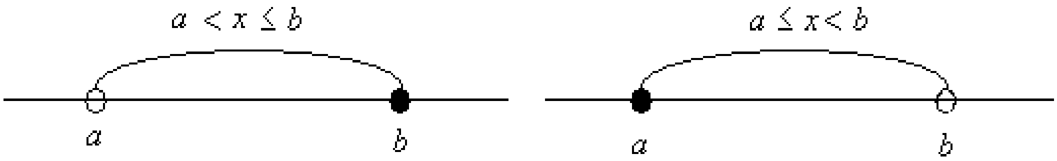


Fig. 3b

- The **infinite** interval is unbounded to the right or to the left, and the infinity symbol is always enclosed by the round bracket to represent it as an open interval. At the same time, the infinite interval may be open or closed at the endpoint.

  Therefore, we have the following cases:
  - the infinite interval $[a, \infty)$ is a set of real numbers x with $a \le x$,
  - the infinite interval (a, ∞) is a set of real numbers x with $a < x$,
  - the infinite interval $(-\infty, b]$ is a set of real numbers x with $x \le b$,
  - the infinite interval $(-\infty, b)$ is a set of real numbers x with $x < b$.

The infinite intervals are shown below on the number line (Fig. 3c).

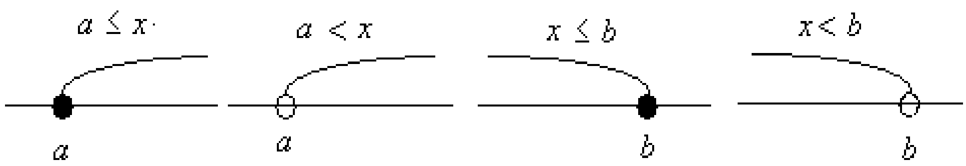


Fig. 3c

The infinite interval $(-\infty, \infty)$ represents the set of all real numbers.

1.6. Exponentiation

In the expression x^a the value x is said to be the base and the value a is called the exponent.

The following mathematical rules are useful in algebraic manipulations involving exponents:

- Any non-zero real number raised to the zeroth power equals 1:

$$x^0 = 1 \qquad (x \ne 0)$$

- A non-zero real number raised to power $(-a)$ is the reciprocal of the same real number raised to power a:

$$x^{-a} = \frac{1}{x^a} \qquad (x \ne 0) \qquad (2)$$

- To multiply powers of a value, add the exponents: $x^a x^b = x^{a+b}$ (3)

- In order to divide powers of a value, subtract the exponent in the denominator from the exponent in the numerator:

$$\frac{x^a}{x^b} = x^{a-b} \qquad (x \ne 0) \qquad (4)$$

- In order to raise powers of a value by a power, multiply the exponents:

$$(x^a)^b = x^{ab} \qquad (5)$$

- A power of a product is equal to the product of powers:

$$(xy)^a = x^a y^a \qquad (6)$$

- The numerator and denominator are raised to the power when raising a fraction to the power.

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \quad (y \neq 0) \qquad (7)$$

Examples:

- $x^3 x^5 / x^6 = x^{3+5-6} = x^2$
- $\dfrac{x^{-7}}{(x^{-2})^3} = x^{-7-(-6)} = x^{-1} = \dfrac{1}{x}$
- $(2^3)^4 = 2^{3 \cdot 4} = 2^{12} = 4096$
- $\dfrac{(x^3)^5 x^{-4}}{x^{11}} = x^{15-4-11} = x^0 = 1$

1.6.1. Rational Exponents

The following is the definition of a radical in which the index n is a natural number greater than one: $n \in N$, $n > 1$.

Number y is said to be the ***n*th root** of a real number x if $y^n = x$.

The *n*th root of x is denoted symbolically by $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

Thus, the relationship between exponents and roots is written as follows:

$$\sqrt[n]{x} = x^{\frac{1}{n}} \qquad (8)$$

In the above expression x is known as the radicand, n is the index of the radical sign $\sqrt{\ }$.

Therefore, both equalities, $y^n = x$ and $y = \sqrt[n]{x}$, express the same statement.

The second root of a number is known as its square root, while its third root is known as its cube root.

If the index n is equal to two, it can be omitted from the expression, *i.e.* the square root of the number x is written as $\sqrt{x} \equiv \sqrt[2]{x}$.

The roots of real numbers may be either real or complex numbers. In particular, the *n*th root of a negative radicand, where n is even, has to be a complex number, since the *n*th power of any real number, where n is even, has to be a positive number.

We will restrict our discussion of exponents and roots to real-number solutions.

No real *n*th root exists when the index n is even and the radicand x is a negative number.

There are two real *n*th roots, y and $(-y)$, when the index n is even and the radicand is positive, because in this case $y^n = (-y)^n$. To avoid confusion, we define the principal *n*th root of a real number, where the *n*th root is a real number, to be the positive *n*th root of the number. When an algebraic expression refers to the *n*th root of a number, and the root is a real number, we generally mean by default the principal (positive) *n*th root of that number.

For instance, the symbol $\sqrt{x}$ is defined for $x \geq 0$ and means the positive square root of the number x.

Therefore,

$$\sqrt{x^2} = |x| \tag{9}$$

Examples:

- Since $5^2 = (-5)^2 = 25$, the square root of 25 has the values 5 and (-5). Its positive square root is 5. Therefore, the principal square root of 25 is number 5.
- $\sqrt{3^2} = |3| = 3$, $\qquad \sqrt{(-3)^2} = |-3| = 3$.

In order to find a square root we can factor an expression under the radical sign to get a perfect square. Then the perfect square is taken out from under the radical sign. What is not a perfect square is left under the radical sign.

Examples: $\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{4^2 \cdot 3} = 4\sqrt{3}$.

The following properties of radicals are based on the rules of exponentiation and the properties of real numbers:

- By setting $a = 1/n$ and $b = m$, we get from rule (5):

  $$x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = (x^m)^{\frac{1}{n}}$$

  Therefore, we have the following formula for radicals:

  $$\sqrt[n]{x^m} = (\sqrt[n]{x})^m \tag{10}$$

- If we set a and b to be equal to $(1/n)$, then from rule (6) it follows that:

  $$(xy)^{\frac{1}{n}} = (x)^{\frac{1}{n}} (y)^{\frac{1}{n}}$$

  Hence, the nth root of a product of numbers is equal to the product of the nth roots:

  $$\sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y} \tag{11}$$

- Similarly to above, the nth root of a quotient of two numbers is equal to the quotient of the nth roots:

  $$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \qquad (y \neq 0) \tag{12}$$

Note: Simplification of algebraic expressions with radicals involves simplification and combination of the quantities within the radical sign. One must ensure that the terms have the same index and the same radicand when radicals are added or subtracted.

Examples:

- $\sqrt[3]{64} = \sqrt[3]{4^3} = 4$
- $\dfrac{\sqrt{6}}{\sqrt{8 \cdot 27}} = \sqrt{\dfrac{6}{8 \cdot 27}} = \dfrac{1}{\sqrt{4 \cdot 9}} = \dfrac{1}{\sqrt{36}} = \dfrac{1}{6}$
- $$\begin{aligned} 9\sqrt{8} - 3\sqrt{32} &= 9\sqrt{2 \cdot 2^2} - 3\sqrt{2 \cdot 4^2} \\ &= 9 \cdot 2\sqrt{2} - 3 \cdot 4\sqrt{2} \\ &= 6\sqrt{2} \end{aligned}$$
- $$\begin{aligned} \sqrt[3]{5} \cdot \sqrt[3]{625} \cdot \sqrt[6]{25} &= \sqrt[3]{5 \cdot 625} \cdot \sqrt[6]{5^2} \\ &= \sqrt[3]{5^5} \cdot \sqrt[3]{5} = \sqrt[3]{5 \cdot 5^5} \\ &= \sqrt[3]{5^6} = 5^2 = 25 \end{aligned}$$

1.6.2. Summary

The most important rules of exponentiation are represented by the following table:

$x^0 = 1$	
$x^{-a} = \frac{1}{x^a}$	$x \neq 0$
$x^a x^b = x^{a+b}$	
$\frac{x^a}{x^b} = x^{a-b}$	$x \neq 0$
$(x^a)^b = x^{ab}$	
$(xy)^a = x^a y^a$	
$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$	$x \neq 0$
$\sqrt[n]{x} = x^{\frac{1}{n}}$	
$\sqrt{x^2} = \|x\|$	
$\sqrt[n]{x^m} = (\sqrt[n]{x})^m$	
$\sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y}$	
$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$	$x \neq 0$

2. Algebraic Expressions

- A **constant** is a symbol that represents a definite mathematical quantity.
- A **variable** is a symbol used to represent an unknown number.
- The number that the variable represents is called its **value**.
- A **term** is a product with an unspecified number of factors, where the factors are variables or constants.
- The variables of a term are said to be **literal factors**, and the product of the constants is called a **coefficient** of the term.
- The term, whose only factors are constants, is called a **constant** term.
- Terms that have the same literal factors but differ only in their numerical coefficients are called **similar terms**.
- The **degree** of a term in one variable is the exponent of that variable.
- An **algebraic expression** is an additive combination of any number of terms. By applying the distributive property, two or more similar terms can be combined into one term. The new term has the same literal factors as the similar terms, but its coefficient is the sum of the coefficients of the similar terms. This process is known as **combining similar terms**.
- The algebraic expression takes on a numerical value when numbers substitute for variables. This process is known as **evaluating algebraic expression**.

Example:

- The algebraic expression

$$4x^3 - 5xy^2 + 3x + 8 - 9x$$

involves the terms $4x^3$, $-5xy^2$, $3x$, $-9x$ and constant 8.

The degree of the term $4x^3$ is 3.
Two terms, $3x$ and $(-9x)$, have the same literal factor, so they are similar terms and can be combined into the single term $(-6x)$.

The given expression is reduced to the following: $4x^3 - 5xy^2 - 6x + 8$ and can be evaluated by setting, *e.g.* $x = 2$ and $y = 3$:

$$4 \cdot 2^3 - 5 \cdot 2 \cdot 3^2 - 6 \cdot 2 + 8 = 32 - 90 - 12 + 8 = -64$$

- A term is called a **monomial** when its every variable has a non-negative integer exponent.
- The **degree** of a monomial is the sum of the exponents of its variables. For example, the degree of the monomial $5x^2y^4$ is (2+4)=6.

2.1. Polynomials

- The finite additive combination of monomials is known as a **polynomial**.
We can say that a monomial is a polynomial with just one term.
- A polynomial with two terms is called a **binomial**; and a polynomial with three terms is a **trinomial**.
- The **degree of a polynomial** is the degree of the monomial with the highest degree.

Examples:

- The polynomial $5x^3z$ is a monomial of degree $(3+1)=4$.
- The polynomial $2x-9y$ is a binomial of degree 1.
- The polynomial $3x+4x^2yz^4-5y^2z^3$ is a trinomial.
  The term $3x$ has degree 1, the term $4x^2yz^4$ has degree $(2+1+4)=7$, and the term $(-5y^2z^3)$ has degree $(2+3)=5$.
  Thus, the above polynomial is the trinomial of degree 7.

A polynomial is one of the most important functions in mathematics and its applications. We can easily manipulate or evaluate the polynomial, but finding its roots is a more difficult task.

There is a worthwhile case when a polynomial has a single variable. We will come to nothing more than polynomials with a single variable that are the most useful.

A **polynomial** with a single variable is an expression that can be written in the following form:

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0 \quad (1)$$

where $n\in N$ and x is a variable.
The numbers a_n are called the coefficients; a_0 is the constant coefficient.

A polynomial is said to be having degree n if $a_n\neq 0$.

A polynomial is said to be a **monic polynomial** if the coefficient of the term of highest degree is one.

Examples:

- The polynomial $P(x)=2$ has degree 0; it is a **constant polynomial**.
- The polynomial $P(x)=5x+7$ has degree 1; it is a **linear polynomial**.
- The polynomial $P(x)=3x^2-x+4$ has degree 2; it is a **quadratic polynomial**.
- The polynomial $P(x)=x^3+2x-1$ has degree 3; it is a **cubic polynomial**.

Polynomials are not always given in an expanded form as above. For instance, the expression $(x-4)(x^2+1)$ is also a polynomial of degree 3, as it can be easily checked.

2.2. Algebraic Transformations

It is a general situation when one needs to write a particular algebraic expression in the simplest possible form. Although it is difficult to say exactly what one means in all cases by the "simplest form", a worthwhile practical procedure is to look at many different forms of an expression, and pick out the one that involves the smallest number of parts. There are many different ways to write the same algebraic expression. In most cases, it is best simply to experiment, trying different transformations until we get a suitable form.
It is impossible to formulate any general-purpose method of getting expressions into the simplest form. For instance, if one has an expression with a single variable, one can choose to write it as a sum of terms, a product, and so on.

When one has an expression with several variables, there is an even wider selection of possible forms. One can, for example, group terms in the expression in such a way that one or another of the variables is chosen as major.

Even when we deal with polynomials and rational expressions, there are many different ways to write any particular expression. If we consider more complicated expressions, involving, for example, trigonometric functions, the variety of possible forms becomes still greater.

However, we can try to formulate some simple principles that are suitable for solving some specific tasks. First of all let us define the following common rules:

- Perform a sequence of algebraic transformations on the expression and return to the simplest form you found.
- Simplify the expression making use of factoring or expanding of some parts of expression.
- Collect together terms that involve the same powers or radicals, *etc*.
- Put all terms over a common denominator or separate into terms with simple denominators.
- Cancel common factors between the numerator and denominator, *etc*.

Here we consider the following common and useful methods of manipulating and simplifying algebraic expressions, equations and inequalities: factoring, expanding and rationalizing the denominator.

2.2.1. Factoring

"Factor expression" is the same as "write expression as a product of factors".

This procedure often gives simpler expressions. As one example, the following expression

$$(x+y^2)(2x-y)^3(x^2+3y)^4$$

is the polynomial with two variables and 36 terms.

As is known, the process of factoring a real number involves expressing the number as a product of prime numbers that are irreducible factors, *i.e*. each of which has only two factors, the number one and the prime number itself.

Similarly, we can factor a polynomial expression by representing it as a product of irreducible polynomials, *i.e*. the polynomials each of which cannot be further reduced to other factors aside from the number one and itself.

Transformations of expressions by expanding or factoring are always correct, whatever values the symbolic variables in the expressions may have.

Examples: Factor the following expressions to irreducible factors:

- $60 = 2\cdot 2\cdot 3\cdot 5$
- $15a - 6a^2 = 3a(5-2a)$
- $a^2 + 2ab - 3a - 6b = a(a+2b) - 3(a+2b) = (a+2b)(a-3)$

Problem 1:

Reduce to a product of factors the **difference between two squares:** $a^2 - b^2$.

Solution: First, we subtract and add the product ab:

$$a^2 - b^2 = a^2 - ab + ab - b^2$$

Then, we combine the terms by pairs and take out the common factors:

$$\begin{aligned} a^2 - ab + ab - b^2 &= a(a-b) + b(a-b) \\ &= (a-b)(a+b) \end{aligned}$$

Finally, we get the following helpful formula:

$$\boxed{a^2 - b^2 = (a-b)(a+b)} \qquad (2)$$

Examples:

- $9x^2 - 25 = (3x)^2 - 5^2 = (3x-5)(3x+5)$
- $$\begin{aligned}(5x+3)^2 - 16x^2 &= (5x+3)^2 - (4x)^2 \\ &= (5x+3-4x)(5x+3+4x) \\ &= (x+3)(9x+3) \\ &= 3((x+3)(3x+1)\end{aligned}$$
- $x^4 - 1 = (x^2)^2 - 1^2 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$
- $$\begin{aligned}x^4 + 1 &= x^4 + 2x^2 + 1 - 2x^2 \\ &= ((x^2)^2 + 2x^2 + 1) - 2x^2 \\ &= (x^2+1)^2 - (\sqrt{2}x)^2 \\ &= (x^2 + 1 - \sqrt{2}x)(x^2 + 1 + \sqrt{2}x)\end{aligned}$$

Problem 2:

Reduce to a product of factors the following quadratic polynomial: $a^2 + 2ab + b^2$

Solution: First, we rewrite the term $2ab$ as $ab + ab$; next, we combine the terms by pairs; then, we take out the common factors:

$$\begin{aligned}a^2 + 2ab + b^2 &= a^2 + ab + ab + b^2 \\ &= (a^2 + ab) + (ab + b^2) \\ &= a(a+b) + b(a+b) = (a+b)(a+b)\end{aligned}$$

Finally, we obtain the following formula for the perfect square:

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2} \qquad (3)$$

Corollary: From the last formula one can easily get another formula for the perfect square by substituting $(-b)$ for b:

$$(a + (-b))^2 = a^2 + 2a(-b) + (-b)^2 \quad \Rightarrow$$

$$\boxed{(a-b)^2 = a^2 - 2ab + b^2} \qquad (4)$$

Formulas (3)-(4) can now be combined into a uniform formula:

$$\boxed{(a \pm b)^2 = a^2 \pm 2ab + b^2} \qquad (5)$$

Problem 3:

Reduce to a product of factors the **difference between two cubes:** $a^3 - b^3$

Solution: We can use a similar way as above, but now we first add and subtract the terms a^2b and ab^2; then we combine the terms by pairs and take out the common factors:

$$\begin{aligned} a^3 - b^3 &= a^3 - a^2 b + a^2 b - ab^2 + ab^2 - b^3 \\ &= a^2(a-b) + ab(a-b) + b^2(a-b) \\ &= (a-b)(a^2 + ab + b^2) \end{aligned}$$

We have one more helpful formula:

$$\boxed{a^3 - b^3 = (a-b)(a^2 + ab + b^2)} \qquad (6)$$

Corollary: From the last formula one can easily get a formula for the **sum of two cubes** by substituting $(-b)$ for b:

$$a^3 - (-b)^3 = (a - (-b))(a^2 + a\cdot(-b) + (-b)^2) \quad \Rightarrow$$

$$\boxed{a^3 + b^3 = (a+b)(a^2 - ab + b^2)} \qquad (7)$$

Examples:

- $$\begin{aligned} 125 - 8x^3 &= 5^3 - (2x)^3 \\ &= (5-2x)(5^2 + 5\cdot 2x + (2x)^2) \\ &= (5-2x)(25 + 10x + 4x^2) \end{aligned}$$
- $125 + 8x^3 = 5^3 + (2x)^3 = (5+2x)(25 - 10x + 4x^2)$
- $$\begin{aligned} \frac{x^3-1}{x^2-1} &= \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} \\ &= \frac{x^2+x+1}{x+1} = \frac{(x^2+2x+1)-x}{x+1} \\ &= \frac{(x+1)^2 - x}{x+1} = \frac{(x+1)^2}{x+1} - \frac{x}{x+1} = x+1-\frac{x}{x+1} \end{aligned}$$

2.2.2. Expanding

Here we consider one more method of transformation between different forms of algebraic expressions.

Expanding is the inverse operation to factoring. We can get another form of an algebraic expression if we multiply out products and powers, writing the result as a sum of terms. We give below some examples.

Examples: We can also read formulas (3)-(5) from right to left and expand the following expression:

- $(5x-1)^2 = (5x)^2 - 2\cdot 5x + 1 = 25x^2 - 10x + 1$
- $(2x + 3y^2)^2 = (2x)^2 + 2\cdot 2x \cdot 3y^2 + (3y^2)^2 = 4x^2 + 12xy^2 + 9y^4$
- $(3 \pm 4x)^2 = 3^2 \pm 2\cdot 3\cdot 4x + (4x)^2 = 9 \pm 24x + 16x^2$

Problem 4: Prove the following formulas:

$$\boxed{(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3} \qquad (8)$$

$$\boxed{(a-b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3} \qquad (9)$$

Proof: Let us start from the cube of sum (8) and expand the expression on the left-hand side:

$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= a^3+a^2b+2ab^2+b^2a+b^3\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

Then, as above, we get a formula for the cube of the difference by substituting $(-b)$ for b:

$$\begin{aligned}(a-b)^3 &= a^3+3a^2(-b)+3a(-b)^2+(-b)^3\\ &= a^3-3a^2b+3ab^2-b^3\end{aligned}$$

Excellent advice: Memorize and use the above formulas.

Examples:

- $$\begin{aligned}(2+5x^2)^3 &= 2^3+3\cdot 2^2\cdot 5x^2+3\cdot 2\cdot\cdot(5x)^2+(5x)^3\\ &= 8+60x^2+150x^4+125x^4\end{aligned}$$
- $$\begin{aligned}(2a-5b)^3 &= (2a)^3-3\cdot(2a)^2 5b+3\cdot 2a(5b)^2-(5b)^2\\ &= 8a^3-60a^2b+150ab^2-125b^2\end{aligned}$$

2.2.3. Rationalizing Denominators

Since a rational expression is the quotient of two algebraic expressions, it can be represented in a fractional form. In doing so, it is often desirable to eliminate all the terms involving radicals from the denominator of the fraction. This is known as rationalizing the denominator.

Examples:

- $\sqrt{\dfrac{a}{b}}=\sqrt{\dfrac{ab}{b^2}}=\dfrac{\sqrt{ab}}{\sqrt{b^2}}=\dfrac{\sqrt{ab}}{|b|}$
- $\dfrac{2}{\sqrt{5}}=\dfrac{2\sqrt{5}}{\sqrt{5}\sqrt{5}}=\dfrac{2\sqrt{5}}{5}$
- $\sqrt{\dfrac{3}{11}}=\sqrt{\dfrac{33}{11^2}}=\dfrac{\sqrt{33}}{\sqrt{11^2}}=\dfrac{\sqrt{33}}{11}$

There are tree cases when the denominator can be easily rationalized.

1. Let the denominator be a binomial radical expression $(a+b)$, where each item, a and b, can contain radicals, but the difference between two squares, a^2-b^2, cannot do so. Then one can multiply both the numerator and denominator of the fraction by factor $(a-b)$ to rationalize the denominator (in view of formula for the difference between two squares):

$$\frac{1}{a+b}=\frac{a-b}{(a+b)(a-b)}=\frac{a-b}{a^2-b^2} \qquad (10)$$

Example: Rationalize the denominator of the given expression: $\dfrac{1}{(2-\sqrt{3})}$.

Solution: First, we multiply both the numerator and the denominator by the factor $(2+\sqrt{3})$. Then we use the formula (2):

$$\frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})}$$
$$= \frac{2+\sqrt{3}}{2^2-(\sqrt{3})^2} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}$$

Example: Rationalize the denominator of the given expression: $1/(\sqrt{2}+1)$.

Solution: Now we multiply both the numerator and the denominator by the factor $(\sqrt{2}-1)$ and end up by using formula (2):

$$\frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{\sqrt{2}-1}{(\sqrt{2})^2-1} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1$$

Example: Rationalize the denominator of the given expression:

$$\frac{1}{\sqrt{15}-\sqrt{3}+\sqrt{5}-1}.$$

Solution: First of all we have to factor the denominator. Making use of the identity $\sqrt{15}=\sqrt{5}\sqrt{3}$, one can combine the terms by pairs and take out the common factor:

$$\sqrt{15}-\sqrt{3}+\sqrt{5}-1 = \sqrt{5}\sqrt{3}-\sqrt{3}+\sqrt{5}-1$$
$$= \sqrt{3}(\sqrt{5}-1)+(\sqrt{5}-1) = (\sqrt{5}-1)(\sqrt{3}+1)$$

Now the fraction can be represented as a product of the fractions:

$$\frac{1}{\sqrt{15}-\sqrt{3}+\sqrt{5}-1} = \frac{1}{\sqrt{5}-1}\cdot\frac{1}{\sqrt{3}+1}$$

so that each of them, taken separately, can be rationalized as above:

$$\frac{1}{\sqrt{5}-1}\cdot\frac{1}{\sqrt{3}+1} = \frac{\sqrt{5}+1}{(\sqrt{5}-1)(\sqrt{5}+1)}\cdot\frac{\sqrt{3}-1}{(\sqrt{3}+1)(\sqrt{3}-1)}$$
$$= \frac{1}{4}(\sqrt{5}+1)\frac{1}{2}(\sqrt{3}-1) = \frac{1}{8}(\sqrt{5}+1)(\sqrt{3}-1)$$

Example: Rationalize the denominator of the given expression: $\dfrac{1}{(\sqrt{7}+\sqrt{5})^2}$

Solution: First, we rationalize the denominator of the fraction $1/(\sqrt{7}+\sqrt{5})$:

$$\frac{1}{\sqrt{7}+\sqrt{5}} = \frac{\sqrt{7}-\sqrt{5}}{(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})}$$
$$= \frac{\sqrt{7}-\sqrt{5}}{(\sqrt{7})^2-(\sqrt{5})^2} = \frac{\sqrt{7}-\sqrt{5}}{7-5} = \frac{1}{2}(\sqrt{7}-\sqrt{5})$$

which being squared gives:

$$\frac{1}{(\sqrt{7}+\sqrt{5})^2} = \frac{1}{4}((\sqrt{7})^2-2\sqrt{7}\sqrt{5}+(\sqrt{5})^2)$$
$$= \frac{1}{4}(7-2\sqrt{35}+5) = 3-\frac{\sqrt{35}}{2}$$

Example: Rationalize the denominator of the given expression: $\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}$

Solution: As above, we first rationalize the denominator of the fraction $1/(\sqrt{2}-1)$:

$$\frac{1}{\sqrt{2}-1}=\frac{(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}=\frac{\sqrt{2}+1}{2-1}=\sqrt{2}+1$$

Then, we multiply the above by the term ($\sqrt{2}$ + 1) and take a square root:

$$\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}=\sqrt{(\sqrt{2}+1)^2}=\sqrt{2}+1$$

2. Now let the denominator be a trinomial radical expression with the structure (a^2+ab+b^2), where each item can contain radicals, but the difference between cubes, a^3-b^3, cannot do so. Then the problem of rationalizing the denominator is solved in a similar way: we multiply both the numerator and denominator of the fraction by the factor $(a-b)$ and use the formula for the difference between two cubes:

$$\frac{1}{a^2+ab+b^2}=\frac{a-b}{(a^2+ab+b^2)(a-b)}=\frac{a-b}{a^3-b^3} \qquad (11)$$

3. Let the denominator be a trinomial radical expression with the structure (a^2-ab+b^2), where each item, but not the sum of cubes, a^3+b^3, can contain radicals. Then we have a similar problem, so it can be solved as above:

$$\frac{1}{a^2-ab+b^2}=\frac{a+b}{(a^2-ab+b^2)(a+b)}=\frac{a+b}{a^3+b^3} \qquad (12)$$

The below examples involve the cases when a denominator can be rationalized by using the formulas for the sum and difference of two cubes in view of the properties of radicals.

Examples:

- $$\frac{1}{\sqrt[3]{25}-4\sqrt[3]{5}+16}=\frac{1}{(\sqrt[3]{5})^2-4\sqrt[3]{5}+4^2}=\frac{\sqrt[3]{5}+4}{((\sqrt[3]{5})^2-4\sqrt[3]{5}+4^2)(\sqrt[3]{5}+4)}=\frac{\sqrt[3]{5}+4}{(\sqrt[3]{5})^3+4^3}=\frac{\sqrt[3]{5}+4}{5+64}=\frac{\sqrt[3]{5}+4}{69}$$

- $$\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2}}=\frac{\sqrt[3]{x}-\sqrt[3]{y}}{((\sqrt[3]{x})^2+\sqrt[3]{x}\cdot\sqrt[3]{y}+(\sqrt[3]{y})^2)(\sqrt[3]{x}-\sqrt[3]{y})}=\frac{\sqrt[3]{x}-\sqrt[3]{y}}{(\sqrt[3]{x})^3-(\sqrt[3]{y})^3}=\frac{\sqrt[3]{x}-\sqrt[3]{y}}{x-y}$$

We can also use formulas (11)-(12) to solve similar problems.

Example: Rationalize the denominator of the given expression: $1/(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)$.

Solution: Setting $a=\sqrt[3]{x}$ and $b=2$ we get from formula (11) the following result:

$$\frac{1}{\sqrt[3]{x^2}+2\sqrt[3]{x}+4}=\frac{\sqrt[3]{x}-2}{x-8}.$$

3. Algebraic Equations and Inequalities

- An **algebraic equation** is a mathematical statement of the equivalence (in a certain well-defined sense) of two algebraic expressions, *i.e.* it states that one algebraic expression is equal to another algebraic expression.
- An **algebraic inequality** is a mathematical statement comparing two algebraic expressions. One algebraic expression can be
  - greater than (>) another;
  - less than (<) another;
  - greater than or equal to $(\geq)$ another;
  - less than or equal to $(\leq)$ another algebraic expression.

The symbols $a << b$ and $a >> b$ are used to denote "a is much less than b" and "a is much greater than b", respectively.

- The **solution** of equations and inequalities involves finding the values of the variables that make the mathematical statements true.
  The addition and multiplication properties of equalities and inequalities as well as the properties of real numbers, are used to simplify the equation or the inequality as much as possible, prior to formulating the solution set for the variable in question.

3.1. Properties of Equations and Inequalities

The following properties of equalities apply to equations.

- The **addition property of equalities** states that

  $a = b$ if and only if $a + c = b + c$ for any c.

  This property applies to equations, but it is better to say:

  > Any number or expression can be added to both sides of an equation to produce an equivalent equation.

- The **multiplication property of equalities** states that

  $a = b$ if and only if $ac = bc$ for any $c \neq 0$.

  A more suitable wording for equations is the following:

  > Both sides of an equation can be multiplied by the same non-zero quantity to produce an equivalent equation.

- The **addition property of inequalities** states that

  $a > b$ if and only if $a + c > b + c$ for any c.

  In other words:

  > Any number or expression can be added to both sides of an inequality to produce an equivalent inequality.

- The **multiplication property of inequalities** is given in two cases.
  The first case states that

  if $a > b$ and $c > 0$ then $ac > bc$.

The second case states that

if $a > b$ and $c < 0$ then $ac < bc$.

> Both sides of an inequality can be multiplied by the same positive quantity to produce an equivalent inequality:
>
> If both sides of an inequality are multiplied by the same negative quantity, then the inequality symbol must be reversed:

3.2. Linear Equations

A **linear equation** in one variable is that equation which can be put into the following form:

$$ax + b = 0 \tag{1}$$

where a and b are constants ($a \neq 0$), and x is a variable.
Let us note that the expression on the left-hand side (1) is a linear polynomial.
Here is the solution of the equation (1):

$$x = -\frac{b}{a} \tag{2}$$

Example: Solve the equation $a_1x + b_1 = a_2x + b_2$ for the variable x.

Solution: In order to solve the given equation, first we have to subtract a_2x and b_1 from both sides.

$$a_1x - a_2x = b_2 - b_1$$

Next, we combine similar terms: $(a_1 - a_2)x = b_2 - b_1$

If $a_1 \neq a_2$, then the solution is $x = \dfrac{b_2 - b_1}{a_1 - a_2}$.

If $a_1 = a_2$ and $b_1 = b_2$, then the solution set is any $x \in R$.
If $a_1 = a_2$ but $b_1 \neq b_2$, then the given equation has no solution.

3.3. Linear Inequalities

A **linear inequality** in one variable is that inequality which can be put into one of the following forms:

$$ax + b \geq 0 \tag{3a}$$
$$ax + b > 0 \tag{3b}$$

where a and b are constants ($a \neq 0$), and x is a variable.

- If $a > 0$, then the solution set is respectively

$$x \geq -b/a \tag{4a}$$
$$x > -b/a \tag{4b}$$

- Otherwise, if $a < 0$, then the solution set is

$$x \leq -b/a \tag{5a}$$
$$x < -b/a \tag{5b}$$

The endpoint of the solution's interval may be included or not. It depends on whether the inequality contains the inequality symbol "$\geq$" ("$\leq$") or "$>$" ("$<$").

The process of solving inequalities is similar to that of equations, except that properties of inequalities apply.

Example: Solve the following inequality: $-5x+3 \geq 2x+17$

Solution: First, we have to simplify this inequality by subtracting the term $2x$ and the number 3 from both sides:

$$-7x \geq 14.$$

Then we divide both sides of the above inequality by the negative number (-7) to get the solution set: $x \leq -2$

3.4. Linear Equations Involving Absolute Values

1. Let a linear equation involve some absolute value $|\ ax + b\ |$.

Following the definition of an absolute value, we can drop the absolute symbol, provided that the correct sign is chosen. There are two possible cases:

1) if $ax+b \geq 0$, then $|ax+b| = ax+b$;

2) if $ax+b<0$, then $|ax+b| = -(ax+b)$.

Therefore, the initial equation is split into two equations such that each of them does not contain the absolute value bars. Hence, we must solve two ordinary linear equations and choose solutions to satisfy the above conditions.

Example 1: Solve the equation $|2x+3|=5$

Solution:

Case 1: If $2x+3 \geq 0$, that means $x \geq -3/2$, then the absolute symbol can be simply dropped:

$|2x+3|=5 \quad \Rightarrow \quad 2x+3=5 \quad \Rightarrow \quad x=1$, provided that $x \geq -3/2$.

That is a true statement.

Case 2: If $2x+3<0$, that means $x<-3/2$, then we have to change the sign in front of the expression $(2x+5)$ when the absolute symbol is dropped:

$|2x+3|=5 \quad \Rightarrow \quad -(2x+3)=5 \quad \Rightarrow \quad x=-4$, provided that $x<-3/2$.

That is true.

The solution set is the union of the solutions involving case 1 and case 2:

$$\{x \mid x=-4, x=1\}.$$

Example 2: Solve the equation $|2x+15|=1$

Solution:

Case 1: If $2x+15 \geq 0$, that means $x \geq -\frac{15}{2}$, then

$|2x+15|=1 \quad \Rightarrow \quad 2x+15=1 \quad \Rightarrow \quad x=-7$, provided that $x \geq -\frac{15}{2}$.

That is certainly true.

Case 2: If $2x+15<0$, then $-(2x+15)=1 \quad \Rightarrow \quad x=-4$, provided that $x<-\frac{15}{2}$.

That is a contradiction, since $(-4)>(-15/2)$.

Therefore, the value $x=-4$ is not the solution for the considered equation, *i.e.* the solution set for case 2 is the empty set Ø.

Hence, the solution set involves the singular value $x=-7$.

2. When an equation involves absolute values $|ax+b|$ and $|cx+d|$, then we have to make a few steps to solve the equation:
   - Solve each intermediate equation below for x to find out where the expressions change their signs:

$$ax+b=0 \quad \Rightarrow \quad x=x_1$$
$$cx+d=0 \quad \Rightarrow \quad x=x_2$$

Let x_1 be less than x_2. We use this statement for the sake of determinacy only because the opposite case can be considered in a similar way.

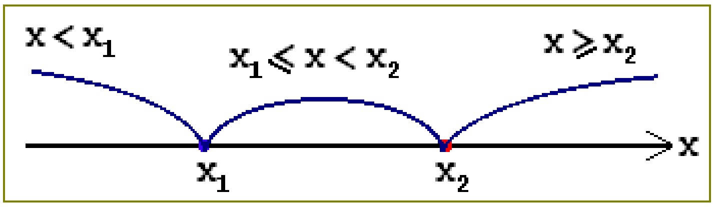


Fig. 1

We get three intervals: $(-\infty, x_1)$, $[x_1, x_2)$ and $[x_2, +\infty)$; hence, we have to solve three ordinary equations when absolute bars are dropped and correct signs of the expressions are applied. The positive or negative value of each expression must be evaluated separately according to the definition of the absolute value.

- It is necessary to select consistent solutions by testing whether each solution lies in the corresponding interval. The solution set is the union of the solutions involving all cases.

Example 3: Solve the given equation

$$|3x+4| = |7x-2| - 4x \qquad (6)$$

Solution:

1) We solve two intermediate linear equations:

$$3x+4=0 \quad \Rightarrow \quad x=-\frac{4}{3}$$
$$7x-2=0 \quad \Rightarrow \quad x=\frac{2}{7}$$

Thus, we have obtained the following three intervals: $(-\infty, -4/3)$, $[-4/3, 2/7)$ and $[2/7, +\infty)$.

2) Now we consider three cases:

Case 1: If $x < -\frac{4}{3}$, then $|3x+4| = -(3x+4)$ and $|7x-2| = -(7x-2)$.

Hence, equation (6) can be transformed in the following way:

$$|3x+4| = |7x-2| - 4x \quad \Rightarrow \quad -3x-4 = -7x+2-4x \quad \Rightarrow$$
$$8x=6 \quad \Rightarrow \quad x=\frac{3}{4}, \quad \text{provided that} \quad x < -\frac{4}{3}.$$

That is a contradiction. Hence, the equation (6) has no solution when $x < -\frac{4}{3}$.

Case 2: If $-\frac{4}{3} \le x < \frac{2}{7}$, then $|3x+4| = 3x+4$ and $|7x-2| = -(7x-2)$.

As above, we get the following result:

$$|3x+4| = |7x-2| - 4x \quad \Rightarrow \quad 3x+4 = -7x+2-4x \quad \Rightarrow$$
$$14x=-2 \quad \Rightarrow \quad x=-\frac{1}{7}, \quad \text{provided that} \quad -\frac{4}{3} \le x < \frac{2}{7}.$$

That is true.

Case 3: If $x > \frac{2}{7}$, then $|3x+4| = 3x+4$ and $|7x-2| = 7x-2$.

Hence, the equation (6) implies $3x+4 = 7x-2-4x \quad \Rightarrow \quad 4 = -2$.

That is a contradiction and so equation (6) has no solution when $x > \frac{2}{7}$.

3) Therefore, we get finally that the singular $x = -1/7$ is the solution of equation (6).

4) The above can be illustrated by the following drawing:

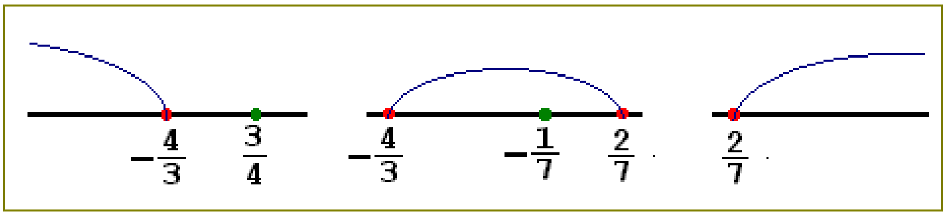


Fig. 2

Thus, we can make the following conclusion:

> If an equation involves absolute values, then its solution involves solving two or more equations.

3.5. Linear Inequalities Involving Absolute Values

Let a linear inequality involves some absolute value $|ax+b|$.

As in case of equation we have two possible cases when the absolute symbol is dropped. Then, we can solve the problem in a usual way due to the initial inequality is split into two inequalities, such that each of them does not contain absolute value bars. Hence, we must solve two ordinary linear inequalities and choose solutions to satisfy the corresponding conditions.

In some cases we can easily write the solution set basing in view of the following statements:

- If $|x-b| < a$ $(a > 0,\ b \in R)$, then $b-a < x < b+a$
  If $|x| < a$ $(a > 0)$, then $-a < x < a$.
- If $|x-b| \le a$ $(a \ge 0, b \in R)$, then $b-a \le x \le b+a$
  If $|x| \le a$ $(a \ge 0)$, then $-a \le x \le a$
- If $|x-b| > a$ $(a > 0, b \in R)$, then $x \in (-\infty, b-a) \cup (a, b+\infty)$
  If $|x| > a$ $(a > 0)$, then $x \in (-\infty, -a) \cup (a, +\infty)$
- If $|x-b| \ge a$ $(a \ge 0, b \in R)$, then $x \in (-\infty, b-a] \cup [a, b+\infty)$
  If $|x| \ge a$ $(a \ge 0)$, then $x \in \{-\infty, -a] \cup [a, +\infty)$
- If the values a and b are both positive or negative and $a < b$, then $a^{-1} > b^{-1}$.

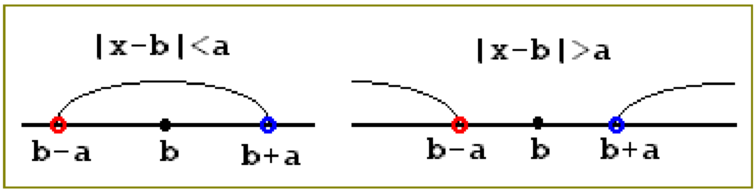


Fig. 3

Example 1: Solve the inequality $|3x-1|<5$.

Solution: According to the above statement it follows from the given inequality that

$$1-5<3x<1+5 \qquad \Rightarrow \qquad -\frac{4}{3}<x<2$$

Example 2: Solve the inequality $|4x+5|\geq 3$.

Solution: As above $|4x+5|\geq 3 \quad \Rightarrow \quad 4x\leq(-3-5)$ or $4x\geq 3-5 \quad \Rightarrow$ $x\leq -2$ or $x\geq(-1/2)$

Therefore, the solution set is $(x\,|\,(x\leq -2)\bigcup(x\geq -\frac{1}{2}))$

Example 3: Solve the inequality $|3x-1|<2x$.

Solution:

Case 1: If $3x-1\geq 0$, that means $x\geq\frac{1}{3}$, then

$$|3x-1|<2x \quad \Rightarrow \quad 3x-1<2x \quad \Rightarrow \quad x<1, \text{ provided that } x\geq\frac{1}{3}.$$

So the solution set in this case is $x\in[\frac{1}{3},1)$.

Case 2: If $3x-1<0$, that means $x<\frac{1}{3}$, then

$$|3x-1|<2x \quad \Rightarrow \quad -(3x-1)<2x \quad \Rightarrow \quad x>\frac{1}{5}, \text{ provided that } x<\frac{1}{3}.$$

Now the solution set is $x\in(\frac{1}{5},\frac{1}{3})$.

One can easily find the union of the of the solutions involving cases 1 and 2:

$$(\frac{1}{5},\frac{1}{3})\bigcup[\frac{1}{3},1\}=(\frac{1}{5},1)$$

Therefore, $|3x-1|<2x$ for any $x\in=(\frac{1}{5},1)$.

Example 4: Solve the inequality $|x+2|\leq 5x-10$.

Solution:

Case 1: If $x+2\geq 0$, that means $x\geq -2$, then

$$|x+2|\leq 5x-10 \quad \Rightarrow \quad x+2\leq 5x-10 \quad \Rightarrow \quad x\geq 3, \text{ provided that } x\geq -2.$$

Hence, the solution set in this case includes any $x\in[3,+\infty)$.

Case 2: If $x+2<0$, that means $x<-2$, then

$$|x+2|\leq 5x-10 \quad \Rightarrow \quad -(x+2)\leq 5x-10 \quad \Rightarrow \quad x\geq\frac{4}{3}, \text{ provided that } x<-2$$

. This case is impossible.

Therefore, the solution set contains case 1 only: $x\in[3,+\infty)$.

If an inequality involves two or more absolute values, then its solution involves solving three or more inequalities. Then, it is necessary to select the solutions obtained by testing whether each of them lies in the corresponding interval. The solution set is the union of the solutions involving all cases.

3.6. Quadratic Equations

A **quadratic equation** in one variable x is that equation which can be written in the following form:

$$\boxed{ax^2 + bx + c = 0} \qquad (7)$$

where a, b and c are constants ($a \neq 0$).

Equation (7) is also said to be a second-degree equation.

We can see that the expression on the left-hand side (7) is a quadratic polynomial. There is nothing special about the symbol x in these equations; any other letter could be used. The equation

$$az^2 + bz + c = 0$$

is also a quadratic equation that is quadratic in one variable, namely, z.

An expression is said to be a monic quadratic in a single variable x if it can be written as $x^2 + \frac{b}{a}x + \frac{c}{a}$. Therefore, any quadratic equation in only one variable can be rewritten so that one side is a monic quadratic by dividing both sides by the numerical coefficient of the quadratic term:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (8)$$

There are a few methods of solving the second-degree equations:

- completing the square,
- using the quadratic formula,
- factoring.

3.6.1. Completing the Square

Let us transform the quadratic polynomial on the left-hand side of equation (8) by adding and subtracting the constant to complete the perfect square:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a}$$

$$= (x + \frac{b}{2a})^2 - ((\frac{b}{2a})^2 - \frac{c}{a})$$

We get the equation that is equivalent to the original one:

$$(x + \frac{b}{2a})^2 = ((\frac{b}{2a})^2 - \frac{c}{a}).$$

Now we reduce the right side to a common denominator:

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} \qquad (9)$$

The value

$$D = b^2 - 4ac$$

is said to be a discriminant of the quadratic equation. The sign of the discriminant is an important characteristic of the quadratic equation.

There are tree possible cases: $D < 0$, $D = 0$ and $D > 0$.

Case 1: If $D < 0$, then in view of equality (9) we get $(x + \frac{b}{2a})^2 < 0$.

This is a contradiction. Therefore, equation (8) has no real roots, *i.e.* the solution set for case 1 is the empty set Ø.

Case 2: If $D = 0$, then from (9) we get

$$(x+\frac{b}{2a})^2=0$$

Therefore, equation (8) has one real root or rather two real roots that are equal to each other:

$$x=-\frac{b}{2a} \qquad (10)$$

Case 3: If $D>0$, then by taking the square root of the both sides, the equation (2) can be transformed into the following form::

$$|x+\frac{b}{2a}|=\frac{\sqrt{b^2-4ac}}{2a} \qquad \Rightarrow$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad (11a)$$

Formula (11a) is known as **quadratic formula.** It gives the complete solution of the quadratic equation (7) and it is usually written as follows:

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{D}}{2a} \qquad (11b)$$

Example 1: The equation $3x^2-x+4=0$ has no real roots because the discriminant $D=(-1)^2-4\cdot3\cdot5=-59<0$

Example 2: The equation $x^2-6x+9=0$ has the solution $x=6/2=3$ in view of formula (10) because $D=36-36=0$.

Example 3: The equation $x^2+6x+5=0$ has the solution set $x_1=-5,\quad x_2=-1$ in view of formula (11).

3.6.2. Factoring a Polynomial Expression

Another way to solve a quadratic equation is based on factoring a polynomial expression, *i.e.* by representing it as a product of irreducible polynomials. This worthwhile method is suitable for solving another kind of equations too.

- If $D<0$, then equation (8) has no real roots, and polynomial $x^2+\frac{b}{a}x+\frac{c}{a}$ cannot be reduced to other factors aside from the number one and itself.
- If $D=0$, then roots for equation (8) coincide with each other: $x_1=x_2$. So the considered polynomial can be represented as

$$x^2+\frac{b}{a}x+\frac{c}{a}=(x-x_1)^2 \qquad (12)$$

- If $D>0$, then equation (8) has two real roots x_1 and x_2 ($x_1\neq x_2$), *i.e.* the polynomial $x^2+\frac{b}{a}x+\frac{c}{a}$ is equal to zero, if and only if either $x=x_1$ or $x=x_1$. Among the second-degree polynomials there is only one, namely $(x-x_1)(x-x_2)$, that has the same properties. Consequently,

$$x^2+\frac{b}{a}x+\frac{c}{a}=(x-x_1)(x-x_2) \qquad (13)$$

Let us remove the parentheses on the right side of equation (13) and combine similar terms:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - x(x_1 + x_2) + x_1x_2 \qquad \Rightarrow$$

$$(\frac{b}{a} + x_1 + x_2)x = x_1x_2 - \frac{c}{a}$$

This is a true statement when

$$\frac{b}{a} + x_1 + x_2 = 0 \text{ and } x_1x_2 - \frac{c}{a} = 0.$$

Hence, the sum of the roots for the quadratic equation (8) produces the relationship

$$\boxed{x_1 + x_2 = -\frac{b}{a}} \qquad (14)$$

and the product of the roots produces the relationship

$$\boxed{x_1x_2 = \frac{c}{a}} \qquad (15)$$

These helpful statements may be used to find the roots or check if the found roots are correct.

Example 3: Solve the quadratic equation $x^2 - 4x - 12 = 0$.

Solution: First, we add and subtract $2x$ to the left side of the equation, next group the terms by pairs, then take out the common factor:

$$x^2 - 4x - 12 = 0 \quad \Rightarrow \quad (x^2 + 2x) - 6x - 12 = 0 \quad \Rightarrow$$

$$x(x+2) - 6(x+2) = 0 \quad \Rightarrow \quad (x+2)(x-6) = 0$$

A product of terms is equal to zero if only any of them equals zero. Hence, the solution set is $x = -2$ and $x = 6$.

Example 4: Solve the quadratic equation $x^2 + 4x - 5 = 0$.

Solution: One can easily see that

$$-4 = 1 + (-5) \qquad \text{and} \qquad -5 = 1 \cdot (-5)$$

Hence, in view of relationships (7)-(8) we get the solution set $x = -5$ and $x = 1$.

Example 5: Solve the quadratic equation $x^2 - 11x + 24 = 0$.

Solution: It is evident that

$$11 = 3 + 8 \qquad \text{and} \qquad 24 = 3 \cdot 8$$

Hence, in view of relationships (7)-(8) we get the solution set $x = 3$ and $x = 8$.

Check:

If $x = 3$, then $x^2 - 11x + 24 = 0 \Rightarrow 3^2 - 33 + 24 \equiv 0$. That is true.

If $x = 8$, then $x^2 - 11x + 24 = 0 \Rightarrow 8^2 - 88 + 24 \equiv 0$. That is true.

Example 6: Solve the **cubic equation** $x^3 + 4x^2 + x - 6 = 0$.

Solution: Let us transform the cubic polynomial on the left-hand side of the equation. First, we subtract and add the term $2x^2$. Next, we combine the terms by pairs. Then, we factor the obtained expression:

$$x^3+4x^2+x-6=(x^3-x^2)-(x^2-x)+(6x^2-6)$$
$$=x^2(x-1)-x(x-1)+6(x^2-1)$$
$$=x^2(x-1)-x(x-1)+6(x-1)(x+1)$$
$$=(x-1)(x^2-x+6x+6)$$
$$=(x-1)(x^2+5x+6)=(x-1)(x+2)(x+3)$$

Thus, $(x-1)(x+2)(x+3)=0$. A product of terms is equal to zero if only any of the terms equals zero. Hence, the solution set is $x=-3,\quad x=-2$ and $x=1$.

3.7. Quadratic Inequalities

A quadratic inequality in one variable is that inequality which can be put into one of the following forms:

$$ax^2+bx+c>0 \qquad (16a)$$
$$ax^2+bx+c\geq 0 \qquad (16b)$$

where a, b and c are constants ($a\neq 0$), and x is a variable.

In order to solve the quadratic inequality it is necessary first to solve the corresponding quadratic equation (7): $ax^2+bx+c=0$.

There are three possible cases:

1. If $D<0$, then equation (7) has no real roots, and the expression ax^2+bx+c has the same sign as the coefficient a for each value of x.
2. If $D=0$, then roots for equation (7) coincide with each other: $x_1=x_2$. So the expression ax^2+bx+c has the same sign as the coefficient a for each value x except $x=x_1=x_2$ when it is equal to zero.
3. If $D>0$, then equation (7) has two real roots x_1 and x_2 ($x_1<x_2$), so the polynomial ax^2+bx+c changes its sign when the variable x jumps over x_1 or x_2. Therefore, there are three intervals: $(-\infty,x_1)$, (x_1,x_2) and (x_2,∞).

- If $a>0$, then the solution set for inequalities (16a)-(16b) is respectively $\{x\,|\,x<x_1\cup x>x_2\}$ or $\{x\,|\,x\leq x_1\cup x\geq x_2\}$.
- If $a<0$, then the solution set for inequalities (16) is respectively $\{x\,|\,x_1<x<x_2\}$ or $\{x\,|\,x_1\leq x\leq x_2\}$.

We can use the number line to get the solution set for the inequality.

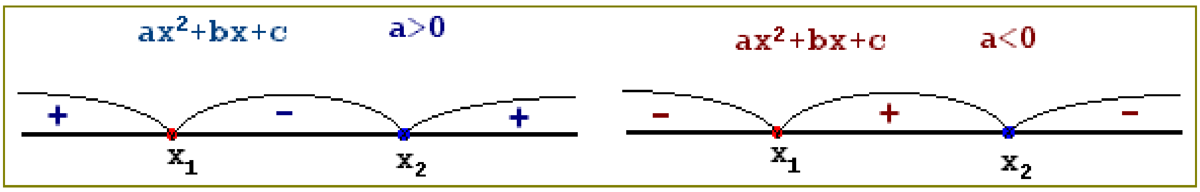


Fig. 1

Example 1: Solve the following inequality:

$$x^2+2x-8>0 \qquad (17)$$

Solution: The equation $x^2+2x-8=0$ has two real roots: $x_1=-4$ and $x_2=2$.
Therefore, the solution set for inequality (17) is

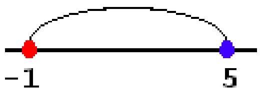
$$\{x \mid x<-4 \cup x>2\}$$

Example 2: Solve the following inequality:

$$x^2-4x-5\le 0 \qquad (18)$$

Solution: The equation $x^2-4x-5=0$ has two real roots: $x_1=-1$ and $x_2=5$. One can see from Fig. 1 that the expression x^2-4x-5 is less than zero when $x_1<x<x_2$. Therefore, the solution set for inequality (18) is

$$\{x \mid -1\le x\le 5\}$$



Example 3: Solve the following inequality:

$$x^2+6x+9>0 \qquad (19)$$

Solution: The roots for the equation $x^2+6x+9=0$ coincide with each other: $x_1=x_2=-3$.
Therefore, the solution set for inequality (19) is any $x\in R$ except $x=-3$.

Example 4: Solve the following inequality:

$$x^2+3x+5>0 \qquad (20)$$

Solution: The equation $x^2+3x+5>0$ has no real roots, so the expression x^2+3x+5 being positive does not change its sign. Therefore, the solution set for inequality (20) is any $x\in R$.

Example 5: Solve the following inequality: $x^2-4x+4\le 0$
Solution: The roots for the equation $x^2-4x+4=0$ coincide with each other: $x_1=x_2=2$.
Hence, the expression x^2-4x+4 is either positive for $x\ne 2$ or equals zero when $x=2$.
Therefore, the solution set for the inequality is $x=2$.

Example 6: Solve the following inequality: $x^2-4x+5\le 0$
Solution: Since the equation $x^2-4x+5=0$ has no real roots, so the quadratic polynomial x^2-4x+5 being positive does not change its sign for any $x\in R$.
Therefore, the solution set is the empty set Ø.

4. Functions

4.1. Introduction to Cartesian Coordinate System

Let us consider two number lines in the plane, one horizontal and one vertical. The horizontal line is called x-axis and the vertical line is called y-axis. These two perpendicular lines intercross at some point that is called an origin.

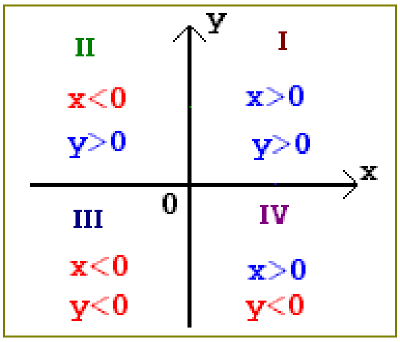


Fig.1

The number lines make up the axes of a coordinate system. The horizontal line is called the x-axis, which is positive to the right and negative to the left from the origin. The vertical line is called the y-axis, which is positive going up and negative going down from the origin.

The x-axis and y-axis divide the x, y-plane into four parts called quadrants. The quadrants are numbered counter-clockwise form one to four. We can see that a point has a positive x-coordinate when it lies in the first or fourth quadrant, while its x-coordinate is negative if a point lies in the second or third quadrant. The y- coordinates are positive for points from the first and second quadrants, and they are negative when points are in the third or fourth quadrant.

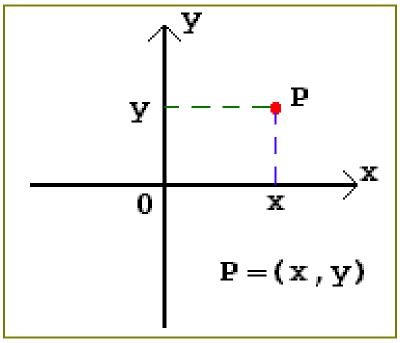


Fig.2

Any point in the plane can be described by an ordered pair of real numbers (x, y) that are called the $x-$and $y-$ coordinates of the point. The first number, x, is called an **abscissa**; it describes the displacement of the point from the origin along the x-axis. The second number, y, is called an **ordinate** and describes the displacement of the point away from the origin along the y-axis. The ordered pair is always listed (x-coordinate, y-coordinate). We assign the number pair (0,0) to the origin. The point 0 often refers to the origin (0,0).

4.2. Basic Definitions

A set of ordered pairs (x, y) is called a **relation**. The set of the first components in the ordered pairs is called the **domain**, and the set of the second components is called the **range**.

A **function** is such a relation that each element of the domain specifies one and only one element of the range; then y is said to be function of the argument x. Functions are usually represented using the function notation by the equation $y = f(x)$, but they can be also determined by means of tables or graphically.

A function f is said to map X onto Y if for every y in Y, there is some x in X such that $f(x) = y$.

A function f is said to be one to one if $f(x) = f(y)$ implies that $x = y$.

Examples:

- If $f(x) = x^2$, then $f(3) = 3^2 = 9$

- The domain of the function $f(x) = 4x + 1$ is $\{= \{x \mid \text{any } x \in R\}$ and its range is $\{f(x) \mid \text{any } x \in R\}$.
- The domain of the function $f(x) = \dfrac{x}{x-2}$ is $D = \{x \mid x \neq 2\}$ because a denominator cannot be equal to zero. However, the function $f(x)$ can have any values, so its range is $\{f(x) \mid \text{any } x \in R\}$.

Here is an example of the function that is determined by means of a table:

x	-3	-2	-1	0	1	2	3
$f(x)$	5	2	0	-1	3	4	5

Table 1.

We can also use a graphic representation of the dependence between x and y coordinates. Thus, let us plot the ordered pairs above and connect the points with a smooth curve. Then we get a graph that is made up by all of the ordered pairs.

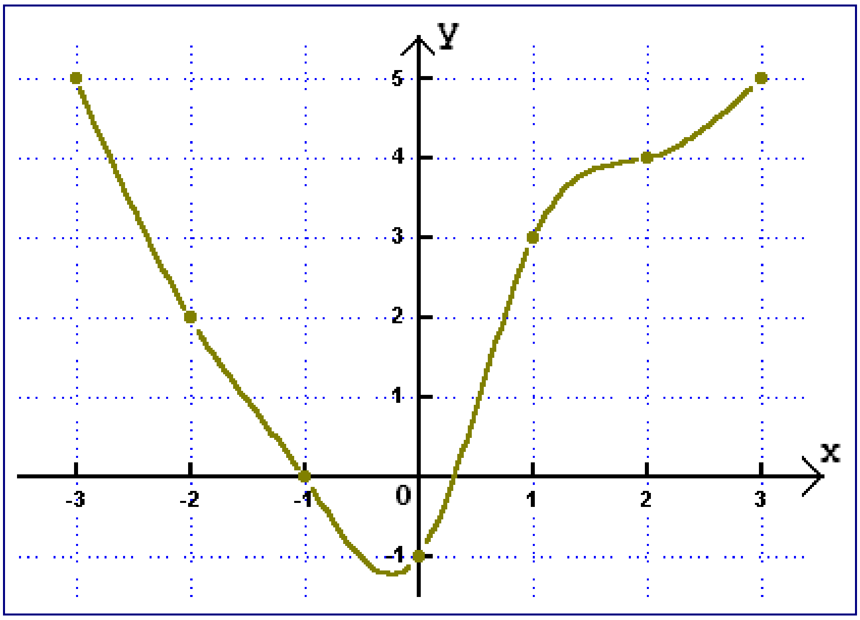


Fig.3

The graphs can be helpful as illustrations of equations and inequalities, *i.e.* we can see the equation through the graph. Sometimes the graphical representation is used to find the solution of equations.

Consider some relation, *i.e.* a set of ordered pairs (x, y). This relation determines some function $y = f(x)$. The inverse of (x, y), that is (y, x), determines the inverse function $y = g(x)$.

> The functions, $f(x)$ and $g(x)$, are said to be **inverse** of each other if
>
> $$f(g(x)) = g(f(x)) = x$$

The inverse function is often denoted by the symbol $f^{-1}(x)$ so

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x \qquad (1)$$

In order to find the inverse function of $f(x)$ we have to replace $f(x)$ with y, next replace x with y and y with x, and then solve the equality for y.

Example: Find inverse functions of $f(x) = 7x - 2$.

- $f(x) = 7x - 2 \quad \Rightarrow \quad y = 7x - 2 \quad \Rightarrow \quad x = 7y - 2 \quad \Rightarrow \quad y = (x+2)/7$

  Thus, $f^{-1}(x) = (x+2)/7$.

  Let us check whether this function is inverse of $f(x)$:

$$f(f^{-1}(x)) = f(\frac{x+2}{7}) = 7\frac{x+2}{7} - 2 = (x+2) - 2 = x$$

$$f^{-1}(f(x)) = f^{-1}(7x-2) == \frac{(7x-2)+2}{7} = x.$$

  The inverse function test is correct.

4.3. Graphs of Some Algebraic Functions

1. **Linear function** in the slope-intercept form: $f(x) = kx + b$.

   Slope of a line between two different points, (x_1, y_1) and (x_2, y_2), is

$$k = \frac{y_2 - y_1}{x_2 - x_1}.$$

   It does not matter which two points are selected on a line; the slope is always the same.
   The slope of a horizontal line is equal to zero because in this case $y_1 = y_2$
   The slope of a vertical line is undefined because in this case $x_1 = x_2$ but one never divides by zero.
   The point where a line crosses or touches the x-axis or y-axis is called an **intercept**.
   In order to find the x-intercepts for the graph of a function $y = f(x)$ we have to set $y = 0$ and solve the equation $f(x) = 0$. The y-intercept is found from the expression $y = f(0)$.

The graphs of some linear functions are shown in the drawing below.
We can see the line $y = 4$ with a zero-slope, the lines with positive and negative slopes, and the vertical line whose slope is undefined. The intercepts are also shown.

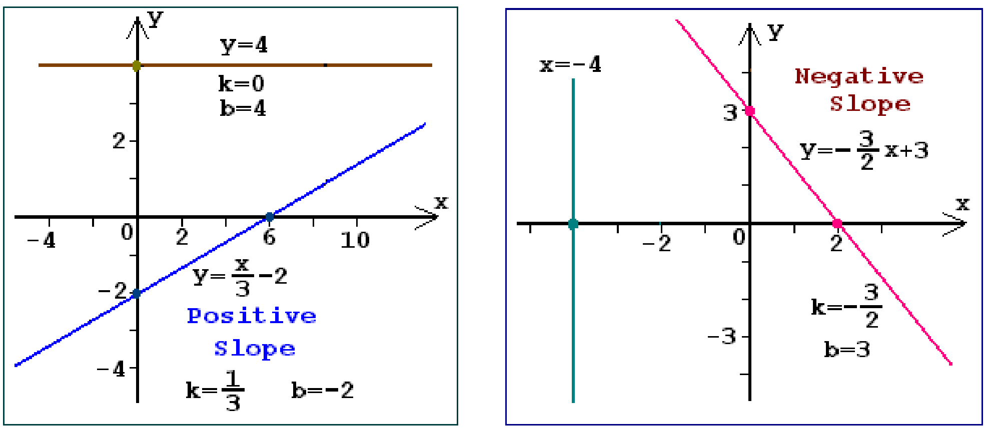


Fig.4

2. Quadratic function: $f(x) = ax^2 + bx + c$ $(a \neq 0)$
   The graph of this function is a parabola.
   - If $b = 0$ and $c = 0$, then the graph of the function $f(x) = ax^2$ is a parabola with a vertex at the origin and it is symmetric with respect to y-axis.
   - If $b = 0$, then the graph of the function $f(x) = ax^2 + c$ is a parabola with a vertex in the y-axis and symmetric with respect to the y-axis.
   - If $b \neq 0$, then a parabola is shifted along the $x-axis$.

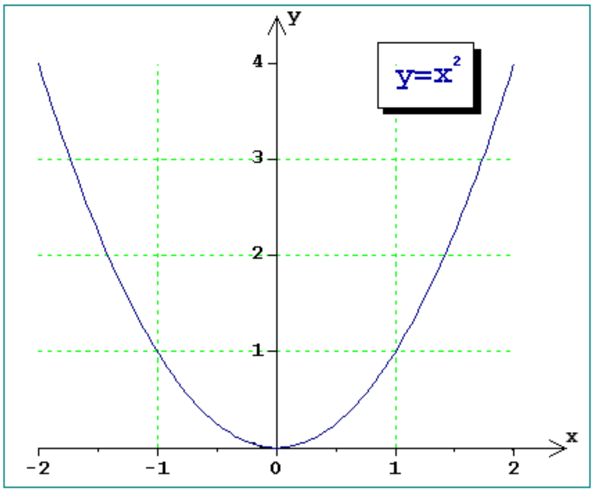


Fig.5a

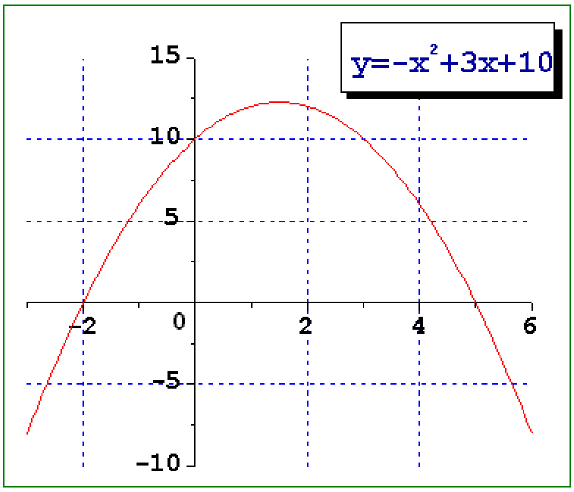


Fig.5b

3. The graph of cubic parabola $f(x) = ax^3 + bx^2 + cx + d$ is shown the drawings below:

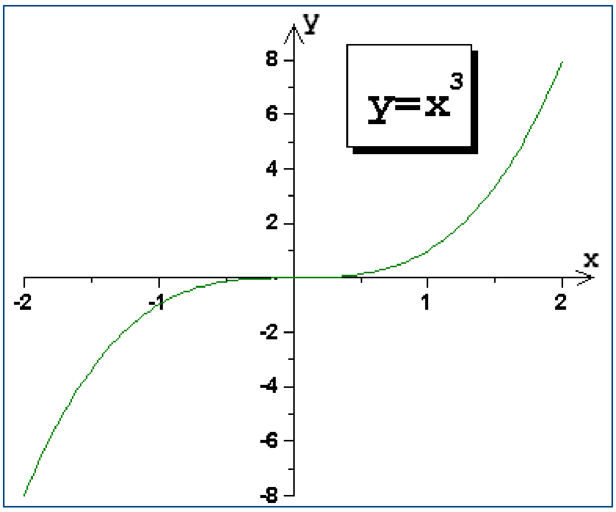


Fig.6a

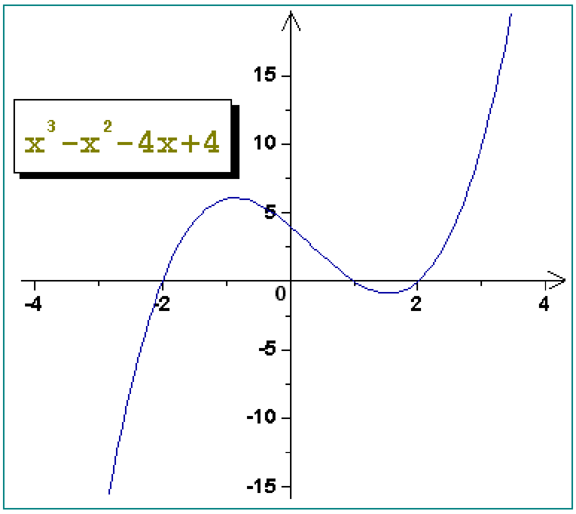


Fig.6b

4. You can also see a few more examples of graphs of frequently used functions.
   - Hyperbola: $f(x) = 1/x$
   - $f(x) = |x|$

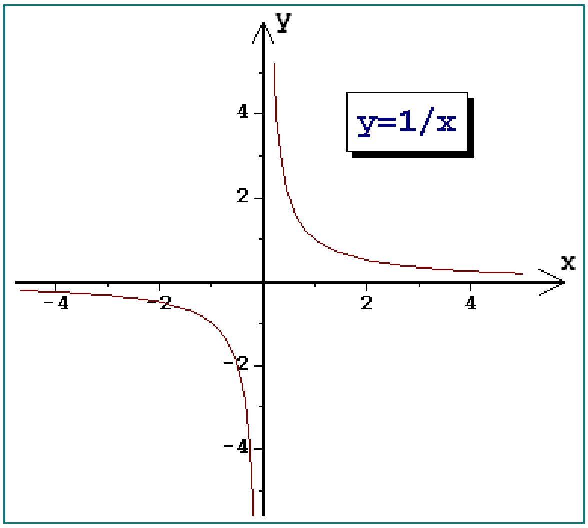


Fig.7a

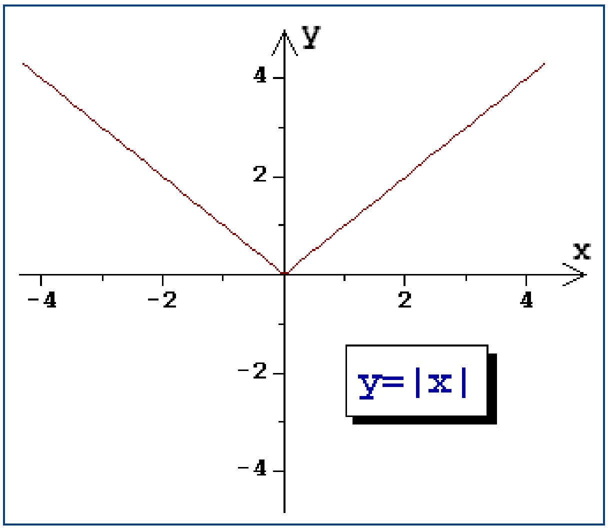


Fig.7b

4.4. Symmetry of Functions

- A function $f(x)$ is said to be an **even** function if for any x in its domain

$$f(-x) = f(x) \qquad (2)$$

> The graph of the even function is symmetric with respect to the y-axis.

Examples of even functions: x^2, x^4, $|x|$.

- A function $f(x)$ is an **odd** function if

$$f(-x) = -f(x) \qquad (3)$$

for any x in its domain.

> The graph of the odd function is symmetric with respect to the origin.

Examples of odd functions: x, x^3, $1/x$.

- A function $f(x)$ is said to be **periodic** if there exists a positive number T such that for all x in its domain

$$f(x+T) = f(x) \qquad (4)$$

The smallest number T is called a **period**.

4.5. Exponential Functions

The exponential function to base a has the following form:

$$f(x) = a^x, \qquad x \in R \qquad (5)$$

where the constant a is called the base ($a > 0$ and $a \neq 1$).
The domain of any exponential function consists of all real numbers while its range consists of positive real numbers only.

Here are some useful properties of exponential functions:

- $a^x = a^y$ if and only if $x = y$.
- If $a > 1$, then from $x < y$ it follows that $a^x < a^y$.
- If $0 < a < 1$, then from $x < y$ it follows that $a^x > a^y$.

When $a > 1$, the function a^x increases towards infinity as x approaches infinity, while it decreases to zero as x approaches negative infinity.
When $0 < a < 1$, the function a^x decreases to zero as $x \to \infty$, while it increases towards infinity as $x \to -\infty$.
A graph of the exponential function lies above the x-axis and has no x-intercepts, because the value of a^x is positive for all x and can never be equal to zero.
Since $f(0) = a^0 = 1$, the graph of the exponential function $f(x) = a^x$ includes the point $(0,1)$.
We can see that the graphs of $f(x) = a^{-x}$ and $f(x) = a^x$ are reflections of each other through the y-axis.

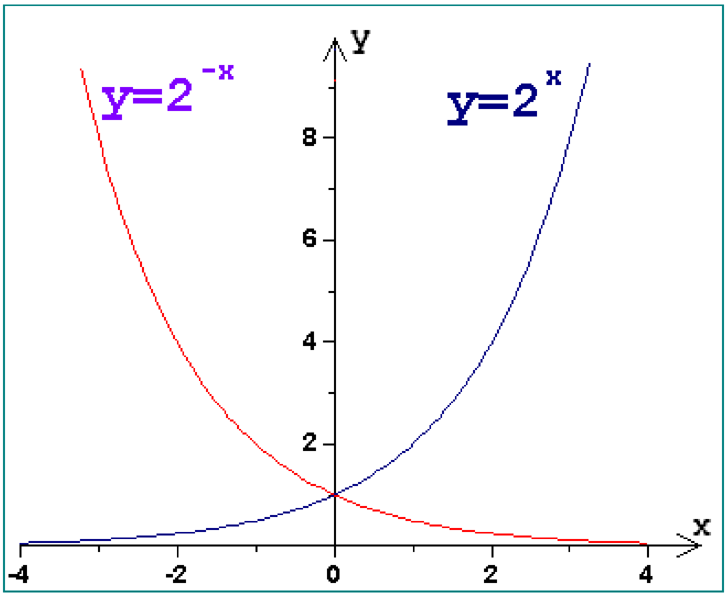


Fig. 8

4.6. Logarithmic Functions

If $a > 0$ and $a \neq 1$, then a **logarithm** to the base a of a positive real number x is the real number y such that the yth power of a is equal to x, *i.e.*

$$y = \log_a x \qquad \text{whenever} \qquad x = a^y \qquad (6)$$

We can see that the logarithm function $f(x) = y = \log_a x$ is suitable for solving the exponential equation $y = a^x$ for x in terms of the variable y.
The logarithmic function has a domain that consists only of positive real numbers while its range consists of all real numbers.

> The base must be positive and different from 1.
>
> The expression that you are taking the logarithm of must also be positive.

The function $\log_{10} x$ is referred to as simply $\log x$.

Problem 1: Prove that the functions $f(x) = \log_a x$ and $g(x) = a^x$ are inverse of each other.

Proof: From the definition of the logarithmic function by combining of equalities (6) there follows the important identity:

$$\boxed{x = a^{\log_a x}} \tag{7}$$

We can also combine equalities (6) in another order:

$$\boxed{y = \log_a a^y} \tag{8}$$

Hence,

$$f(g(x)) = f(a^x) = \log_a a^x \equiv x,$$

$$g(f(x)) = g(\log_a x) = a^{\log_a x} \equiv x.$$

Thus, $f(g(x)) = g(f(x)) = x$, that proves the given statement.

Corollary: Since the functions $f(x) = \log_a x$ and $g(x) = a^x$ are inverse of each other, their graphs are mirror images of each other across the line $y = x$.

Problem 2: Prove the following identities:

$$\log_a 1 = 0 \tag{9}$$

$$\log_a a = 1 \tag{10}$$

Proof: Since $a^0 = 1$ and $a^1 = a$ so in view of definition (6) we get formulas (9)-(10).

Problem 3: Prove the following identities:

$$\log_a |xy| = \log_a |x| + \log_a |y| \tag{11}$$

$$\log_a \left|\frac{x}{y}\right| = \log_a |x| - \log_a |y| \tag{12}$$

$$\log_a |x|^y = y \log_a |x| \tag{13}$$

Proof: 1) In view of identity (7) we have

$$a^{\log_a |x|} = |x| \tag{14}$$

$$a^{\log_a |y|} = |y| \tag{15}$$

$$a^{\log_a |xy|} = |xy| \tag{16}$$

Next, we multiply both sides of equalities (14) and (15) and then transform the products making use of the properties of exponents:

$$a^{\log_a |x|} \cdot a^{\log_a |x|} = |x||y| \quad \Rightarrow \quad a^{\log_a |x| + \log_a |y|} = |xy| \tag{17}$$

Finally, we compare the last formula with identity (16) and conclude that

$$a^{\log_a |x| + \log_a |y|} = a^{\log_a |xy|}$$

In view of the properties of the exponential function noted above, we can make conclusion about validity of formula (11).

2) Formula (12) can be proved in a similar way, but now we have to divide equalities (14) and (15) one by another. One can easily get the following relationship:

$$a^{\log_a|x|-\log_a|y|}=\left|\frac{x}{y}\right| \qquad (18)$$

Then, as above, we can write down the identity $a^{\log_a|x/y|}=|x/y|$ and compare it with (18) to complete the proof.

3) In order to prove formula (13) we can make the following transformations:

$$a^{y\cdot\log_a|x|}=(a^{\log_a|x|})^y=|x|^y$$

$$a^{\log_a|x|^y}=|x|^y$$

$$a^{y\cdot\log_a|x|}=a^{\log_a|x|^y} \quad \Rightarrow \quad \log_a|x|^y=y\log_a|x|.$$

Identity (13) is proved.

When it is necessary to change in the base of a logarithmic function, the following equality can be used:

$$\boxed{\log_a x=\frac{\log_c x}{\log_c a}} \qquad (19)$$

Corollary 1: By changing in the base of a logarithmic function we get a new identity:

$$\log_{1/a} x=\frac{\log_a x}{\log_a(1/a)}=\frac{\log_a x}{\log_a a^{-1}}=\frac{\log_a x}{-\log_a a}=-\log_a x$$

Therefore, a logarithmic function in the base a differs from a logarithmic function in the base $\frac{1}{a}$ in the sign only:

$$\boxed{\log_{1/a} x=-\log_a x} \qquad (20)$$

Hence, their graphs are mirror images of each other across the x-axis.

Corollary 2: As above, making use of formulas (19), (13) and (10) we can also get a more general formula:

$$\log_{(a^b)} x=\frac{\log_a x}{\log_a a^b}=\frac{\log_a x}{b\log_a a}=-\frac{\log_a x}{b}$$

$$\boxed{\log_{a^b} x=-\frac{\log_a x}{b}} \qquad (16)$$

Examples:

- $\log_5 x=3 \quad \Rightarrow \quad x=5^3=125$
- $\log\sqrt{x}=2 \quad \Rightarrow \quad \sqrt{x}=10^2=100 \quad \Rightarrow \quad x=10000$
- $\log_2 16=\log_2 2^4=4\log_2 2=4$
- $\log_6 2+\log_6 3=\log_6(2\cdot 3)=\log_6 6=1$
- $\log_3\sqrt[5]{81}=\log_3(81)^{1/5}=\frac{1}{5}\log_3(3^4)=\frac{4}{5}\log_3 3=\frac{4}{5}$
- $\log_5 400-\log_5 16=\log_5(20/4)^2=2\log_5 5=2$

- $\log_{1/7} 49 = \log_{7^{-1}} 7^2 = -2\log_7 7 = -2$

4.6.1. Graphs of Logarithmic Functions

When $a > 1$, the function $y = \log_a x$ increases towards infinity as $x \to \infty$, while it will approach the y-axis asymptotically as $x \to 0$.

When $0 < a < 1$, the function $y = \log_a x$ decreases continuously as x grows.

Since $\log_a 1 = 0$, the graph of each logarithmic function $f(x) = \log_a x$ includes the point $(1,0)$. The graphs of $f(x) = \log_a x$ and $f(x) = \log_{1/a} x$ are reflections of each other through the x-axis.

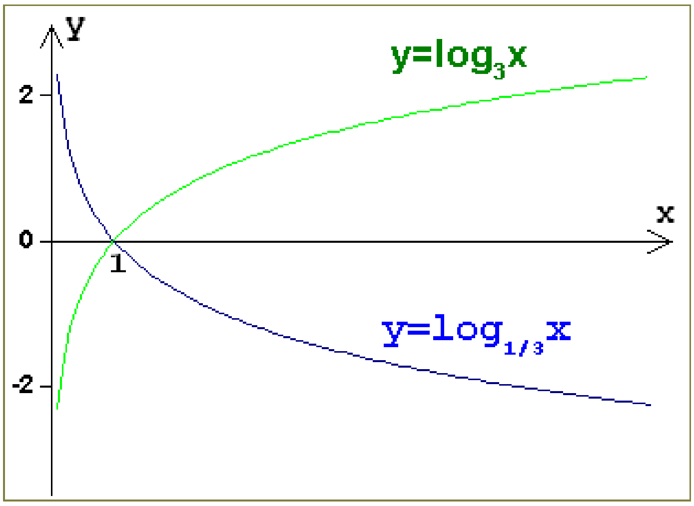


Fig. 9

The below drawings illustrate the general property of inverse functions: the graphs of functions inverse of each other, $y = \log_a x$ and $y = a^x$, are mirror images of each other across the line $y = x$.

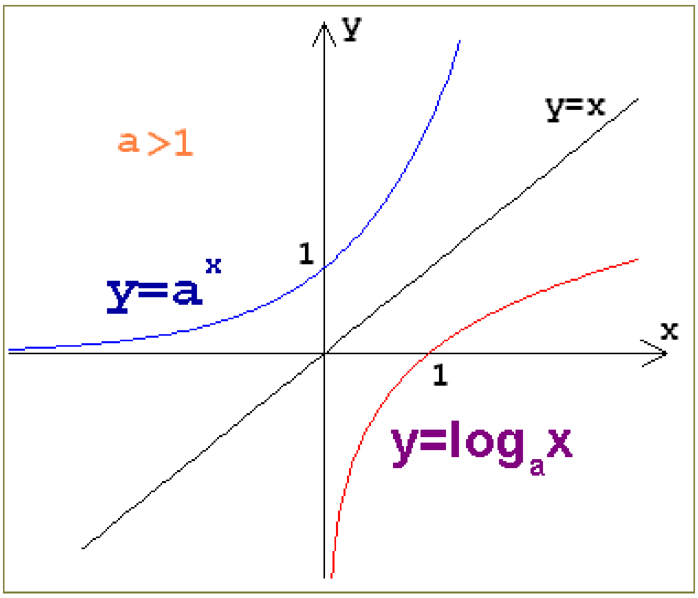


Fig. 10a

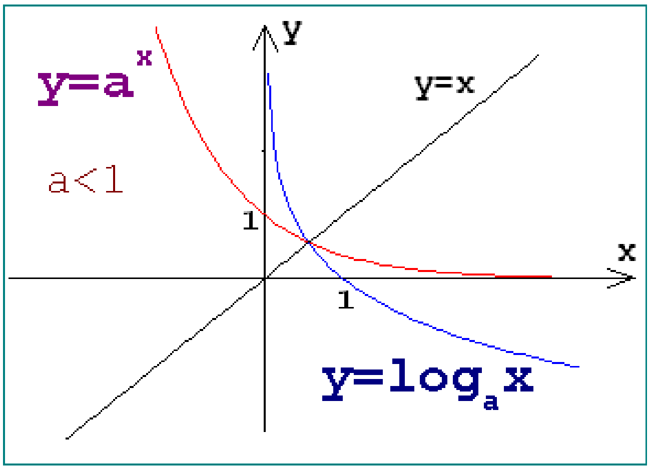


Fig.10b

4.6.2. Natural Logarithm

One of the most important numbers used as the base for exponential and logarithmic functions is denoted as e. It is an irrational number, with its value approximated as

$$e = 2.718281828459045235360287 4.$$

Exponential and logarithmic functions with the base e occur in many practical applications, including those involving growth and decay.

The natural logarithm of a positive real number x is defined as the logarithm to the base e of the number x. The natural logarithm $\log_e x$ is denoted as $\ln x$:

$$\ln x \equiv \log_e x .$$

By definition, the equality $\ln x = y$ implies that $e^y = x$.

In order to convert logarithm from base 10 to base e, one can proceed from on formula (14):

$$\log x \equiv \log_{10} x = \frac{\log_e x}{\log_e 10} = \frac{\ln x}{\ln 10} \approx \frac{\ln x}{2.3} \approx 0.43 \ln x$$

Since the functions $f(x) = e^x$ and $f(x) = \ln x$ are inverse of each other, then

$$\ln e^x = x$$
$$e^{\ln x} = x$$

The natural logarithm possesses the same properties as common logarithms. For instance,

$$\ln e = 1,$$
$$\ln 1 = 0..$$

Examples:

- $\ln \sqrt[5]{e} = \ln(e)^{1/5} = \frac{1}{5} \ln e = \frac{1}{5}$
- $\ln \frac{\sqrt{e}}{3} = \ln \sqrt{e} - \ln 3 = \frac{1}{2} - \ln 3$

5. Discrete Algebra

5.1. Mathematical Induction Principle

Induction is a mathematical method suitable for proving an infinite sequence of statements. The statements can be represented, for example, by mathematical equations or inequalities involving the variable n.

The main idea of this method is the following.
Let S_n be an infinite sequence of statements for $n = 0, 1, 2, ...$

If the statement S_n is true for $n = 0$, and if the truth of S_n implies that S_{n+1} is true, then S_n is true for every non-negative integer n.

The induction principle is based on a quite clear self-intuitive premise. Indeed, if a particular statement S_n is true for $n = 0$ and S_n implies that S_{n+1} is also true, then the statement S_n is true for $n = 0 + 1 = 1$. Similarly, S_n is true for $n = 1 + 1 = 2$, $n = 2 + 1 = 3$, and so on for all non-negative integers.

> The mathematical induction principle includes three components:
> - the induction basis,
> - the induction hypothesis,
> - the induction step.

The **induction basis** is such a statement that being true gives a starting point for the induction. Therefore, in order to form the induction basis one has to prove (or check) that the statement S_n is true for some integer $n = k$. Usually, one takes $k = 0$ or $k = 1$.
When we try to prove the truth of some general statement, it is quite naturally to <u>check</u> whether it is valid in a particular case.

The **induction hypothesis** is an assumption of the truth of the statement S_n for some integer $n \ge k$. In other words, we are ready to believe that the statement S_n holds true for some integer $n \ge k$. At this stage of induction we <u>suppose</u> the truth of the statement S_n but prove nothing.

The **induction step** is the main stage of induction. If the statement S_n implies S_{n+1}, provided $n \ge k$, then S_n must be true for all integers $n \ge k$. Here, we proceed from verifications and assumptions to direct proving of the statement. So that in order to conclude that S_n is true for any integer $n \ge k$, we <u>must prove</u> the statement S_{n+1} being based on the assumption S_n.

The above discussion shows that the mathematical induction method can be represented by the following pattern.

> - **The induction basis:** The statement S_k is true for some integer k.
> - **The induction hypothesis:** The statement S_n holds true for some integer $n \ge k$.
> - **The induction step**: If the statement S_n implies S_{n+1}, provided $n \ge k$, then S_n must be true for all integers $n \ge k$.

According to this scheme, the procedure of proving the validity of some statement S_n for all integers $n \geq k$ also includes three stages:
First, we need to originate a basis of induction.
Second, we have to formulate an induction hypothesis.
Finally, we must prove that the statement S_n implies S_{n+1} that in view of a man-made assumption allows to complete the proof.

Note: If $S_n \Rightarrow S_{n+1}$ but the statement S_k is false, then we can at least conclude that S_n is false for every $n \leq k$, but we cannot say anything about S_n for $n > k$.

The method of mathematical induction is very helpful in proving many statements about integers. The following examples illustrate the technique of it using in practice.

Example 1: Prove that $2^n > n$ for all positive integers n.

Proof: Let S_n be the statement: $2^n > n$.

Induction basis: The statement S_1 is true: $2^1 = 2 > 1$.

Induction hypothesis: Suppose that the statement S_n (*i.e.* inequality $2^n > n$) holds for some integer $n \geq 1$.

Induction step: $2^{n+1} = 2 \cdot 2^n > 2 \cdot n = n + n \geq n + 1$.

We can see that the inequality $2^n > n$ implies the inequality $2^{n+1} > (n+1)$. Therefore, the proof of the above inequality by mathematical induction is complete.

Example 2: For all positive integers n we have the following formula:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad (1)$$

Note: The last formula is written using the Σ-notation: $\sum_{i=1}^{n} i = 1 + 2 + \ldots + n$. We will employ similar notations hereinafter.

Proof: Let S_n be statement (1).

Induction basis: The statement S_1 is certainly true because $1 = \frac{1 \cdot 2}{2}$.

Induction hypothesis: Let equality (1) hold true for some integer $n \geq 1$.

Induction step: We verify the statement S_{n+1} which can be read as: $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$.

If S_n is true, then

$$\begin{aligned}\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \qquad \text{(by induction hypothesis)} \\ &= (n+1)\left(\frac{n}{2} + 1\right) = \frac{(n+1)(n+2)}{2}\end{aligned}$$

That is true.
Therefore, S_n is true for $n \geq 1$ by induction.

Example 3: For all positive integers n the following formula is valid:

$$\sum_{i=1}^{n}(2i-1)=n^2 \qquad (2)$$

Proof: Let S_n be statement (2).

Induction basis: The statement S_1 is read as: $\sum_{i=1}^{1}(2i-1)\equiv 1=1$. That is true.

Induction hypothesis: Suppose that equality (2) holds true for some integer $n\geq 1$.
Induction step: We verify that $S_n \Rightarrow S_{n+1}$

$$\begin{aligned}\sum_{i=1}^{n+1}(2i-1)&=\sum_{i=1}^{n}(2i-1)+(2(n+1)-1)\\&=n^2+(2n+1)\quad \text{(by induction hypothesis)}\\&=(n+1)^2\end{aligned}$$

If S_n is true, then S_{n+1} must be true. Formula (2) is proved by induction.

Example 4: For all positive integers n we have the following formula:

$$\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6} \qquad (3)$$

Proof: Let S_n be statement (3).

Induction basis: The statement S_1 is read as: $1^2=\frac{1(1+1)(2+1)}{6}=1$. That is true.

Induction hypothesis: Let equality (3) be valid for some integer $n\geq 1$.

Induction step: The statement S_{n+1} is read as: $\sum_{i=1}^{n+1}i^2=\frac{(n+1)(n+2)(2n+3)}{6}$.

We verify that $S_n \Rightarrow S_{n+1}$

$$\begin{aligned}\sum_{i=1}^{n+1}i^2&=\sum_{i=1}^{n}i^2+(n+1)^2\\&=\frac{n(n+1)(2n+1)}{6}+(n+1)^2\quad \text{(by induction hypothesis)}\\&=\frac{n+1}{6}(n(2n+1)+6(n+1))\quad \text{(common term is taken out)}\\&=\frac{n+1}{6}(2n^2+7n+6)\\&=\frac{n+1}{6}((2n^2+4n)+(3n+6))\quad \text{(by grouping terms)}\\&=\frac{n+1}{6}(2n(n+2)+3(n+2))\\&=\frac{(n+1)(n+2)(2n+3)}{6}\end{aligned}$$

Thus if S_n is true, so is S_{n+1}.

Example 5: Use mathematical induction to prove that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 \qquad (4)$$

is true for all positive integers n.

Proof: Let S_n be statement (4).

We check S_1 which is read as: $1^3 = \left(\frac{1(2)}{2}\right)^2 = 1$. That is true.

Assume that equality (4) is true. Then

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \qquad \text{(by induction hypothesis)}$$

$$= \frac{(n+1)^2}{4}(n^2 + 4(n+1)) \qquad \text{(common term is taken out)}$$

$$= \frac{(n+1)^2}{4}(n^2 + 4n + 4)$$

$$= \frac{(n+1)^2}{4}(n+2)^2 \qquad \text{(by factoring)}$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2 = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$$

Thus if S_n is true, so is S_{n+1}. Therefore, S_n is true for all positive n by induction.

Note: We can write down one more surprising formula by comparing formulas (1) and (4) with each other:

$$(1+2+\ldots+n)^2 = 1^3 + 2^3 + \ldots + n^3 \qquad (5a)$$

That looks in the Σ-notation as follows:

$$\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3 \qquad (5b)$$

Example 6: Let n be any non-negative integer and $q \neq 1$, then the following formula is valid:

$$\sum_{i=0}^{n} q^i = \frac{q^{n+1} - 1}{q - 1} \qquad (6)$$

Proof: Let S_n be statement (6).

Let us check the above formula for $n = 0$: $q^0 = 1 = (q-1)/(q-1)$

The statement S_0 is true.

Suppose S_n is true. Then we get

$$\sum_{i=0}^{n+1} q^i = \sum_{i=0}^{n} q^i + q^{n+1} = \frac{q^{n+1}-1}{q-1} + q^{n+1}$$
$$= \frac{q^{n+1}-1+q^{n+1}(q-1)}{q-1}$$
$$= \frac{q^{n+2}-1}{q-1} = \frac{q^{(n+1)+1}-1}{q-1}$$

by expanding the parentheses and combining similar terms.

Thus, the statement S_n implies S_{n+1}. Therefore, by the induction principle we can conclude that S_n is true for all integers $n \geq 0$. Formula (6) is proved.

Example 7: For all positive integers n the following formula is valid:

$$\sum_{i=1}^{n} (2i-1)^2 = \frac{n(4n^2-1)}{3} \qquad (7)$$

Proof: Let S_n be statement (7).

Let us check S_1: $1 = \frac{1\cdot(4-1)}{3}$. That is true.

Let us denote that $(4n^2-1) = (2n-1)(2n+1)$. Then the statement S_{n+1} is read as:

$$\sum_{i=1}^{n+1} (2i-1)^2 = \frac{(n+1)(2(n+1)-1)(2(n+1)+1)}{3}$$
$$= \frac{(n+1)(2n+1)(2n+3)}{3}$$

Suppose that S_n is true. Then we pass to the induction step:

$$\sum_{i=1}^{n+1} (2i-1)^2 = \sum_{i=1}^{n} (2i-1)^2 + (2(n+1)-1)^2$$
$$= \frac{n(4n^2-1)}{3} + (2n+1)^2 = \frac{n(2n-1)(2n+1)}{3} + (2n+1)^2$$
$$= \frac{(2n+1)}{3}(n(2n-1)+3(2n+1))$$
$$= \frac{(2n+1)}{3}(2n^2+5n+3) = \frac{(2n+1)}{3}((2n^2+2n)+(3n+3))$$
$$= \frac{(2n+1)}{3}(2n(n+1)+3(n+1)) = \frac{(2n+1)(2n+3)(n+1)}{3}$$

We can see that S_n implies S_{n+1}. Therefore, by induction principle we come to the conclusion that S_n is true for all positive integers n.

Example 8: Use mathematical induction to check the validity of the assumption

$$2^n > n^2 \qquad (8)$$

for positive integers n.

Solution: Let S_n be statement (8).

Let us check S_1: $2 > 1$. That is true.

Suppose that S_n is true. Then $2^{n+1} = 2 \cdot 2^n > 2n^2$.

So we must test whether $2n^2 > (n+1)^2$.

$$2n^2 > (n+1)^2 \quad \Rightarrow \quad n^2 - 2n - 1 > 0 \quad \Rightarrow \quad n > 1 + \sqrt{2} \quad \Rightarrow \quad n \geq 3.$$

Therefore, the statement S_1 cannot be taken as the induction basis.

Let us check S_n for $n \geq 3$:

$n = 3$: $2^3 = 8 > 3^2 = 9$. That is a false statement.

$n = 4$: $2^4 = 16 > 4^2 = 16$. That is false.

$n = 5$: $2^5 = 32 > 5^2 = 25$. That is true.

Thus, the starting point is $n = 5$.

We can see that the statement S_5 is true and S_n implies S_{n+1} for all integers $n \geq 5$.

Hence, $2^n > n^2$ for all integers $n \geq 5$.

5.2. Arithmetic Progression

An **arithmetic progression** is a sequence in which each term (after the first) is determined by adding a constant to the preceding term. This constant is said to be the **common difference** of the arithmetic progression. The following equations express this sentence mathematically:

$$a_{n+1} = a_n + d \tag{9}$$

$$a_{n+1} = a_1 + nd \tag{10}$$

Here a_1 is the first term of the arithmetic progression;

a_n is its nth term;

d is the common difference of the arithmetic progression;

$n \in N$.

Consider the sum of the first n terms of an arithmetic progression:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_{n-1} + a_n \tag{11}$$

Let us note that sum (11) consists of equal pairs:

$$\begin{aligned} a_1 + a_n &= (a_1 + d) + (a_n - d) = a_2 + a_{n-1} \\ &= (a_2 + d) + (a_{n-1} - d) = a_3 + a_{n-2} \\ &= \ldots \end{aligned}$$

Therefore, the sum S_n holds its value if each term in (11) is replaced by $(a_1 + a_n)/2$. Since sum (11) contains n terms, so we get the following formula:

$$S_n = \sum_{k=1}^{n} a_k = \frac{(a_1 + a_n)n}{2} \tag{12}$$

The last formula can be also written as

$$S_n = (a_1 + \frac{(n-1)d}{2})n \tag{13}$$

making use of the equality $a_n = a_1 + (n-1)d$.

Example: Calculate the sum of the first 10 terms of the arithmetic progression if $a_2 = 4$ and $a_5 = 22$.

Solution: From equality (2) it follows that

$$a_2 = a_1 + d = 4 \qquad (14)$$

$$a_5 = a_1 + 4d = 22 \qquad (15)$$

We can find the common difference of the arithmetic progression by subtracting equality (14) from (15):

$$3d = 18 \qquad \Rightarrow \qquad d = 6.$$

Then from (14) we calculate the first term of the arithmetic progression:

$$a_1 = 4 - d = 4 - 6 = -2$$

The sum of the first 10 terms of the arithmetic progression in view of formula (13) is

$$S_{10} = (-2 + \frac{9 \cdot 6}{2}) \cdot 10 = 250$$

5.3. Geometric Progression

A **geometric progression** is a sequence in which each term (after the first) is determined by multiplying the preceding term by a constant. This constant is called the **common ratio** of the arithmetic progression. The following equations express this statement mathematically:

$$a_{n+1} = a_n \, q \qquad (16)$$

$$a_{n+1} = a_1 \, q^n \qquad (17)$$

Here a_1 is the first term of the geometric progression;
a_n is the nth term of the geometric progression;
q is the common ratio of the geometric progression;
$n \in N$.

Let us calculate the sum of the first n terms of the geometric progression:

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \ldots + a_n = a_1 + a_1 q + a_1 q^2 + \ldots + a_1 q^{n-1} \\ &= a_1(1 + q + q^2 + \ldots + q^{n-1}) = a_1 \sum_{k=1}^{n-1} q^k. \end{aligned} \qquad (18)$$

At fist, let us multiply both sides of the above equality by the factor q:

$$qS_n = a_1 \sum_{k=2}^{n} q^k, \qquad (19)$$

then subtract equality (18) from equality (19) and simplify both sides:

$$qS_n - S_n = a_1 \sum_{k=2}^{n} q^k - a_1 \sum_{k=1}^{n-1} q^k \qquad \Rightarrow$$

$$S_n(q-1) = a_1(\sum_{k=2}^{n-1} q^k + q^n) - a_1(\sum_{k=2}^{n-1} q^k + q) \qquad \Rightarrow$$

$$S_n(q-1) = a_1(q^n - 1)$$

The sum S_n can be obtained by dividing both sides of the latter by the factor $(q-1)$, if $q \neq 1$
:

$$S_n = \frac{a_1(q^n - 1)}{q - 1} \qquad (20)$$

Note: If $|q| < 1$, then $q^n \to 0$ when $n \to \infty$. Hence, the sum of an infinite number of terms of decreasing geometric progression is equal to

$$S_\infty = \frac{a_1}{1 - q} \qquad (21)$$

Example: Calculate the sum of the first ten terms of the geometric progression and find the seventh term if the sum of the first five terms $S_5 = 31$ and the common ratio $q = 2$.

Solution:

1) From formula (20) we get $a_1 = \dfrac{S_5(q-1)}{q^5 - 1} = \dfrac{31}{32 - 1} = 1$.
2) Making use of formula (20) for $n = 10$ we get $S_{10} = 2^{10} - 1 = 1023$.
3) From formula (17) we get $a_7 = a_1 q^6 = 2^6 = 64$.

5.4. Binomial Theorem

At first let us introduce binomial coefficients defined by the following formula:

$$C_n^k = \frac{n!}{k!(n-k)!} \qquad (22)$$

The symbol "$n!$" has to be read as "n factorial" and means the product of all natural numbers from 1 to n:

$$n! = 1 \cdot 2 \cdot \ldots \cdot n$$

A zero factorial is equal to one unit by definition: $0! = 1$.
It is clear that $(n+1)! = (n+1) \cdot n!$

The binomial coefficient C_n^k gives a number of ways for choosing k objects from a set of n objects, regardless of the order in which the k objects are chosen.
Making use of definition (22) we get the following relationships:

$$C_n^{n-k} = C_n^k = \frac{n!}{k!(n-k)!}$$

$$C_n^0 = C_n^n = \frac{n!}{0!(n-0)!} = 1$$

$$C_n^1 = \frac{n!}{1!(n-1)!} = n$$

One can also check the validity of the recursion relation between the binomial coefficients

$$C_n^{k-1} + C_n^k = C_{n+1}^k \qquad (23)$$

which allows by means of constructing the Pascal's triangle to evaluate C_n^k in a simple way.

Pascal's triangle is a triangular array of binomial coefficients. Its structure is evident from the table below.

C_0^0						1						k=0
C_1^k					1		1					k=0, 1
C_2^k				1		2		1				k=0, 1, 2
C_3^k			1		3		3		1			k=0, 1, 2, 3
C_4^k		1		4		6		4		1		k=0, 1,2,3,4
C_5^k	1		5		10		10		5		1	k=0, 1,2,3,4,5
...		...		...		...		...		...		...

Examples:

- $C_3^1 = C_2^0 + C_2^1 = 1 + 2 = 3$
- $C_4^1 = C_3^0 + C_3^1 = 1 + 3 = 4$
- $C_4^2 = C_3^1 + C_3^2 = 3 + 3 = 6$
- $C_5^4 = C_4^3 + C_4^4 = 4 + 1 = 5$

The Binomial Theorem: For any positive integer n and real a and b the following formula is valid:

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k . \qquad (24)$$

Setting $a = 1$ and $b = x$ we get from the last formula:

$$(1+x)^n = \sum_{k=0}^{n} C_n^k x^k$$

One can see that binomial coefficients are the coefficients of x in the expansion of $(1+x)^n$.

Examples:

- $(a+b)^2 = C_2^0 a^2 + C_2^1 ab + C_2^2 b^2 = a^2 + 2ab + b^2$.
- $(a+b)^3 = C_3^0 a^3 + C_3^1 a^2 b + C_3^2 a b^2 + C_3^3 b^3 = a^3 + 3a^2 b + 3ab^2 + b^3$
- $(a+b)^5 = a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5$

The binomial theorem is employed in calculus, combinatorial analysis, statistics, *etc*.

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