ФЕДЕРАЛЬНОЕ АГЕНТСТВО ПО ОБРАЗОВАНИЮ
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LINEAR ALGEBRA, VECTOR ALGEBRA AND ANALYTICAL GEOMETRY

WorkBook

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To the Student

In this workbook, the topics are presented in the same order as in the textbook [1]. The problems concern three content areas: Linear Algebra, Vector Analysis, and Analytical Geometry.

Prerequisites: A student should be able

– to solve linear equations;
– to perform the basic operations on numbers and algebraic expressions;

Linear Algebra topics include the following themes: matrices and determinants; matrix operations; determinant calculations; inverse matrices; systems of linear equations.

The Linear Algebra tests will reveal your knowledge and skills, your abilities in interpreting symbols, justifying statements and constructing proofs. You should be able to apply the properties of determinants and matrix operations and solve linear systems of equations.

The Vector Analysis topics include: linear vector operations; the dot product of vectors; the cross product of vectors; the scalar triple product; geometrical applications of vectors.

After you complete this supplement you should be able to formulate and solve problems basing on vector representation, to use standard techniques of vector analysis.

The Analytical Geometry topics include different forms of equations of straight lines and planes; angles between simple figures; the curves of the second order.

To pass the final test, you should be able to solve basic problems, demonstrate full understanding of all topics, and give a significant portion of the answer successfully. Minor calculation errors are admissible.

You can also use another textbooks [2,3] to understand better the main points of the problems studied.

If you have difficulties with elementary mathematics, turn to textbooks [4,5] and workbook [6-9].

Contents

LINEAR ALGEBRA

Matrix Operations

Problem 1: Which of the below matrices are equal, if any?

$$A = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}, \qquad C = \begin{pmatrix} 4 & 5 & 0 \\ 1 & 2 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 5-1 & \sqrt{25} \\ \sin 90° & 2 \end{pmatrix}, \qquad E = \begin{pmatrix} 4 & 0 & 5 \\ 1 & 0 & 2 \end{pmatrix}.$$

Solution:

Problem 2: Given the matrices

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 0 & 5 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 2 & -2 \\ 4 & 1 \end{pmatrix},$$

find the linear combination $2A - 3B + 4C$.

Solution:

$2A - 3B + 4C =$

Problem 3: Express matrix $A = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ as the linear combination of the matrices

$$X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Solution: $A = \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$

Problem 4: Determine which of the matrix products AB and BA are defined. If the product is appropriate, find the size of the matrix obtained.

1) A is a 3×5 matrix and B is a 5×2 matrix;
2) A is a 3×2 matrix and B is a 2×3 matrix;
3) A is a 4×2 matrix and B is a 4×2 matrix;
4) A is an 1×7 matrix and B is a 7×1 matrix;
5) A and B are square matrices of the fifth order.

Solution:

1)

2)

3)

4)

5)

Problem 5: Let $A=\begin{pmatrix} 5 & -2 & 1 \\ 3 & 4 & 2 \end{pmatrix}$.

Evaluate the matrix products AA^T and $A^T A$

If the difference $AA^T - A^T A$ is defined, find it. If not, explain why.

Solution:

$AA^T =$

$A^T A =$

$AA^T - A^T A$

Problem 6: Find the matrix product AB, if

$$A=\begin{pmatrix} 5 & -2 & 1 \\ 3 & 4 & 2 \end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix} 0 & -2 & 1 \\ -3 & 5 & -1 \\ 4 & -3 & 2 \end{pmatrix}.$$

Solution:

$$AB=\begin{pmatrix} 5 & -2 & 1 \\ 3 & 4 & 2 \end{pmatrix}\begin{pmatrix} 0 & -2 & 1 \\ -3 & 5 & -1 \\ 4 & -3 & 2 \end{pmatrix}=$$

Problem 7: Using the matrices A and B given in Problem 6 find the matrix element $c_{3,2}$ on the third row and second column of the matrix $C = B^T A^T$.

Solution:

$B^T =$

$A^T =$

$c_{3,2} =$

Problem 8: Given three matrices

$$A = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 \\ 6 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} -2 & 8 & 1 \end{pmatrix},$$

find the matrix products $(AB)C$ and $A(BC)$ by straightforward procedure.

Solution:

$AB =$

$(AB)C =$

$BC =$

$A(BC) =$

Problem 9: Let A and B be two diagonal matrices of the order 4. Find the matrix $AB - BA$.

Solution:

$AB - BA =$

Problem 10: Find A^{50}, if $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Solution:

$A^2 =$

$A^3 =$

$A^{50} =$

Problem 11: Let $A=\begin{pmatrix} 0 & -2 & 1 \\ 3 & 4 & -1 \\ 5 & -3 & 7 \end{pmatrix}$. Find a matrix B such that $C = A + B$ is a diagonal matrix.

Solution:

$B =$

Problem 12: Which of the below matrices are symmetric? Which are skew-symmetric?

$$A=\begin{pmatrix} 0 & -2 & 1 \\ 2 & 4 & -1 \\ -1 & 1 & 7 \end{pmatrix},\ B=\begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix},\ \text{and}\ \ C=\begin{pmatrix} 3 & 2 & 5 \\ 2 & -4 & 1 \\ 5 & 1 & 7 \end{pmatrix}.$$

Solution:

Problem 13: Let $AB=\begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$. Find the matrix product $B^T A^T$, where A^T is the transpose of A, and B^T is the transpose of B.

Solution:

$B^T A^T =$

Problem 14: Let $A=\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$ and $f(x) = 3x^2 + 5x - 4$.

Find $f(A)$.

Solution:

$f(A) =$

Determinants

Problem 1: Apply the Sarrus Rule to calculate the determinants of the matrices

$$A=\begin{pmatrix} 0 & -2 & 1 \\ 2 & 4 & -1 \\ -1 & 1 & 7 \end{pmatrix},\quad B=\begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix},\ \text{and}\quad C=\begin{pmatrix} 3 & 2 & 5 \\ 2 & -4 & 1 \\ 5 & 1 & 7 \end{pmatrix}.$$

Solution:

$$\det A=\begin{vmatrix} 0 & -2 & 1 \\ 2 & 4 & -1 \\ -1 & 1 & 7 \end{vmatrix}=$$

$$\det B=\begin{vmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{vmatrix}=$$

$$\det C=\begin{vmatrix} 3 & 2 & 5 \\ 2 & -4 & 1 \\ 5 & 1 & 7 \end{vmatrix}=$$

Problem 2: Calculate $\det A=\begin{vmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix}$. Using the properties of determinants, express each of the below determinants through $\det A$.

$$\det B=\begin{vmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{vmatrix},\quad \det C=\begin{vmatrix} -5 & 4 & -6 \\ 2 & -1 & 3 \\ 8 & -7 & 9 \end{vmatrix},\quad \det D=\begin{vmatrix} -21 & -24 & -27 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{vmatrix}$$

Solution:

$$\det A=\begin{vmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix}=$$

$$\det B = \begin{vmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{vmatrix} =$$

$$\det C = \begin{vmatrix} -5 & 4 & -6 \\ 2 & -1 & 3 \\ 8 & -7 & 9 \end{vmatrix} =$$

$$\det D = \begin{vmatrix} -21 & -24 & -27 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{vmatrix} =$$

Problem 3: Let $A = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 3 \\ 5 & 2 \end{pmatrix}$. Find the determinant of the matrix product $A^2 B^3$.

Solution:

$$\det A = \begin{vmatrix} 3 & -2 \\ -1 & 4 \end{vmatrix} =$$

$$\det B = \begin{vmatrix} 6 & 3 \\ 5 & 2 \end{vmatrix} =$$

$$\det(A^2 B^3) =$$

Problem 4: Evaluate $\det A^{10}$, if $A = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$.

Solution:

$$\det A = \begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} =$$

$$\det A^{10} =$$

Problem 5: Let A be a square matrix of the third order such that $\det A = 5$. Find

1) $\det(2A)$; 2) $\det A^T$; 3) $\det A^3$, 4) $\det(2A^2 A^T)$.

Solution: $\det(2A) =$

$\det A^T =$

$\det A^3 =$

$\det(2A^2A^T) =$

Problem 6: Let $\begin{vmatrix} 8 & -2 & 1 \\ 2 & 3 & 1 \\ 4 & -5 & 0 \end{vmatrix} = D$. Express the determinants of the below matrices

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 16 & -4 & 2 \\ 4 & -5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 & -2 \\ 3 & 2 & 3 \\ 0 & 2 & -5 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 6 & -5 & 0 \\ 2 & 3 & 1 \\ 4 & -5 & 0 \end{pmatrix}$$

in terms of *D*. The calculation of the given determinant *D* is not required.

Solution:

$$\det A = \begin{vmatrix} 2 & 3 & 1 \\ 16 & -4 & 2 \\ 4 & -5 & 0 \end{vmatrix} =$$

$$\det B = \begin{vmatrix} 3 & 4 & -2 \\ 3 & 2 & 3 \\ 0 & 2 & -5 \end{vmatrix} =$$

$$\det C = \begin{vmatrix} 6 & -5 & 0 \\ 2 & 3 & 1 \\ 4 & -5 & 0 \end{vmatrix} =$$

Problem 7: Let $A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & 1 \\ -1 & 3 & 6 \end{pmatrix}$. Evaluate $\det A$ using expansion according to the second column. Then expand the determinant by the first row and compare the results obtained.

Solution: $\det A = \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & 1 \\ -1 & 3 & 6 \end{vmatrix} = a_{21}A_{21} + \dots$

$$\det A = \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & 1 \\ -1 & 3 & 6 \end{vmatrix} = a_{11}A_{11}\ldots$$

Problem 8: By elementary row and column operations, reduce the matrix $A = \begin{pmatrix} 2 & 3 & -2 \\ 4 & 0 & 7 \\ -1 & 2 & 3 \end{pmatrix}$ to the triangular form and calculate $\det A$.

Solution: $\det A = \begin{vmatrix} 2 & 3 & -2 \\ 4 & 0 & 7 \\ -1 & 2 & 3 \end{vmatrix} =$

Problem 9: Let $A = \begin{pmatrix} 5 & 1 & -2 & 0 \\ 1 & 3 & 6 & -1 \\ 0 & 7 & 1 & 3 \\ 5 & 4 & 2 & 1 \end{pmatrix}$. Calculate $\det A$.

Solution:

$$\det A = \begin{vmatrix} 5 & 1 & -2 & 0 \\ 1 & 3 & 6 & -1 \\ 0 & 7 & 1 & 3 \\ 5 & 4 & 2 & 1 \end{vmatrix} =$$

Inverse Matrices

Problem 1: Find the cofactors of the matrix elements on the second row of the matrix $A = \begin{pmatrix} -3 & 2 & 3 \\ 4 & 1 & 6 \\ 7 & 5 & -1 \end{pmatrix}$.

Solution:

$A_{21} =$

$A_{22} =$

$A_{23} =$

Problem 2: Find the adjoint matrix of $A = \begin{pmatrix} -3 & 2 & 3 \\ 4 & 1 & 6 \\ 7 & 5 & -1 \end{pmatrix}$.

Then find all non-diagonal elements of the matrix product $A \cdot \operatorname{adj} A$. Explain why they are equal to zero.

Solution:

$\operatorname{adj} A =$

$A \cdot \operatorname{adj} A =$

Problem 3: Let A be the matrix given in Problem 2. Evaluate all diagonal elements of the matrix product $A \cdot \operatorname{adj} A$. Explain why they are equal to each other.

Solution:

$A \cdot \operatorname{adj} A =$

Problem 4: Using matrix A given in Problem 2 find the inverse of A. Verify the result by the definition of the inverse matrix.

Solution:

$A^{-1} =$

Verification: $A^{-1}A =$

$AA^{-1} =$

Problem 5: Find the inverse of $A = \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix}$. Check the result by the definition.

Solution:

$\det A =$ $\qquad$ $\operatorname{adj} A =$

$A^{-1} =$ $\qquad$ $AA^{-1} =$

Problem 6: Let $A=\begin{pmatrix}5 & 4\\ 3 & 2\end{pmatrix}$ and $B=\begin{pmatrix}2 & -6\\ 1 & 7\end{pmatrix}$. Solve for X the matrix equation $AX=B$. Verify the solution.

Solution: $X=$

$A^{-1}=$

Verification:

Problem 7: Let A and B be the matrices given in Problem 6. Solve for X the matrix equation $XA^2=B$. Verify the solution.

Solution: $X=$

$A^{-1}=$

Verification:

Problem 8: Evaluate the inverse of $A=\begin{pmatrix}1 & 0 & 5\\ -2 & 4 & 0\\ 3 & -1 & 1\end{pmatrix}$. Find the product $A^{-1}A$ to check up the result.

Solution:

$\det A=\begin{vmatrix}1 & 0 & 5\\ -2 & 4 & 0\\ 3 & -1 & 1\end{vmatrix}=$

$\operatorname{adj} A=$ $A^{-1}=$

Check up:

Problem 9: Evaluate the inverse of $A=\begin{pmatrix}1 & 5\\ -2 & -7\end{pmatrix}$ by means of the elementary transformations of the extended matrix $(A\,|\,I)$.

Solution: $(A\,|\,I)=\left(\begin{array}{cc|cc}1 & 5 & 1 & 0\\ -2 & -7 & 0 & 1\end{array}\right)\to$

$A^{-1}=$

LINEAR ALGEBRA

Systems of Linear Equations

Problem 1: Reduce the matrix

$$A = \begin{pmatrix} 3 & -4 & 1 & 5 & -2 \\ 2 & 1 & -3 & 0 & 4 \\ 3 & 7 & -10 & -5 & 14 \end{pmatrix}$$

to the row echelon form and find the rank of A.

Solution:

$$\begin{pmatrix} 3 & -4 & 1 & 5 & -2 \\ 2 & 1 & -3 & 0 & 4 \\ 3 & 7 & -10 & -5 & 14 \end{pmatrix} \to$$

Problem 2: Find the rank of $A = \begin{pmatrix} 11 & -3 & 18 & 3 \\ -2 & -1 & -11 & 5 \\ 7 & -2 & 13 & 1 \\ 3 & -1 & 8 & -2 \end{pmatrix}$ by elementary transformations.

Solution:

$$\begin{pmatrix} 11 & -3 & 18 & 3 \\ -2 & -1 & -11 & 5 \\ 7 & -2 & 13 & 1 \\ 3 & -1 & 8 & -2 \end{pmatrix} \to$$

rank A =

Hint: You can interchange two rows or columns, multiply a row or column by a nonzero number and multiply a row (column) by a number to add the result obtained to another row (column).

Problem 3: Solve the system of linear equations

$$\begin{cases} 5x_1 + 2x_2 - 4x_3 = 4 \\ x_1 + 3x_2 + 4x_3 = 3 \\ 2x_1 + 4x_2 + x_3 = 1 \end{cases}$$

via Gaussian elimination. Check whether the solution satisfies all the given equations.

Solution:

$$\left(\begin{array}{ccc|c} 5 & 2 & -4 & 4 \\ 1 & 3 & 4 & 3 \\ 2 & 4 & 1 & 1 \end{array}\right) \to$$

Solution check:

Problem 4: Use Gaussian elimination to solve the system of equations

$$\begin{cases} 7x_1 + x_2 - 5x_3 = 3 \\ 4x_1 + x_2 - 3x_3 = 5 \\ -2x_1 + x_2 + x_3 = 2 \end{cases}$$

Solution:

$$\left(\begin{array}{ccc|c} 7 & 1 & -5 & 3 \\ 4 & 1 & -3 & 5 \\ -2 & 1 & 1 & 2 \end{array}\right) \to$$

Solution check:

Problem 5: Find the general solution and a particular solution of the linear system, which is given by the augmented matrix

$$\left(\begin{array}{cccc|c} -3 & -2 & 2 & 1 & 2 \\ 2 & 1 & -1 & 1 & -2 \\ -1 & 1 & -3 & 1 & 4 \end{array}\right).$$

Check the solution by substituting the values of the unknowns.

Solution:

$$\left(\begin{array}{cccc|c} -3 & -2 & 2 & 1 & 2 \\ 2 & 1 & -1 & 1 & -2 \\ -1 & 1 & -3 & 1 & 4 \end{array}\right) \rightarrow$$

The general solution is

A particular solution is

Solution check:

Problem 6: Let $A = \begin{pmatrix} -2 & 2 & 5 & 3 \\ 2 & -2 & 1 & 1 \\ 5 & -5 & -2 & 4 \end{pmatrix}$ be the coefficient matrix of the homogeneous system of linear equations $AX = 0$. Find the general solution.

Solution:

$$\begin{pmatrix} -2 & 2 & 5 & 3 \\ 2 & -2 & 1 & 1 \\ 5 & -5 & -2 & 4 \end{pmatrix} \rightarrow$$

The general solution is

Solution check:

Problem 7: The homogeneous system of linear equations is given the coefficient matrix $A = \begin{pmatrix} 2 & 1 & -5 \\ -3 & -1 & 2 \\ 4 & 1 & 0 \end{pmatrix}$. Determine the number of solutions.

Solution:

$$\det A = \begin{vmatrix} 2 & 1 & -5 \\ -3 & -1 & 2 \\ 4 & 1 & 0 \end{vmatrix} =$$

Problems 8: Use Cramer's Rule to solve the following system of linear equations.

$$\begin{cases} 5x_1 + 2x_2 - 4x_3 = 4 \\ x_1 + 3x_2 + 4x_3 = 3 \\ 2x_1 + 4x_2 + x_3 = 1 \end{cases}$$

Solution:

$$D = \det A = \begin{vmatrix} 5 & 2 & -4 \\ 1 & 3 & 4 \\ 2 & 4 & 1 \end{vmatrix} =$$

$$D_1 = \begin{vmatrix} 4 & 2 & -4 \\ 3 & 3 & 4 \\ 1 & 4 & 1 \end{vmatrix} =$$

$$D_2 = \begin{vmatrix} 5 & 4 & -4 \\ 1 & 3 & 4 \\ 2 & 1 & 1 \end{vmatrix} =$$

$$D_3 = \begin{vmatrix} 5 & 2 & 4 \\ 1 & 3 & 3 \\ 2 & 4 & 1 \end{vmatrix} =$$

$x_1 = \dfrac{D_1}{D} =$ $\qquad$ $x_2 =$ $\qquad$ $x_3 =$

Solution check:

Problems 9: Solve the following system of linear equations by Cramer's Rule.

$$\begin{cases} 2x_1 - x_2 + 5x_3 = 19 \\ x_1 + x_2 - 3x_3 = -3 \\ 2x_1 + 4x_2 + x_3 = 6 \end{cases}$$

Solution:

$$D = \det A = \begin{vmatrix} 2 & -1 & 5 \\ 1 & 1 & -3 \\ 2 & 4 & 1 \end{vmatrix} =$$

$$D_1 = \begin{vmatrix} 19 & -1 & 5 \\ -3 & 1 & -3 \\ 6 & 4 & 1 \end{vmatrix} =$$

$$D_2 = \begin{vmatrix} 2 & 19 & 5 \\ 1 & -3 & -3 \\ 2 & 6 & 1 \end{vmatrix} =$$

$$D_3 = \begin{vmatrix} 2 & -1 & 19 \\ 1 & 1 & -3 \\ 2 & 4 & 6 \end{vmatrix} =$$

$$x_1 = \frac{D_1}{D} = \qquad x_2 = \frac{D_2}{D} = \qquad x_3 = \frac{D_3}{D} =$$

Solution check:

Problem 10: Given the system of linear equations

$$\begin{cases} x_1 + 2x_2 + 3x_3 = a \\ 4x_1 + 5x_2 + 6x_3 = b \\ 7x_1 + 8x_2 + 9x_3 = c \end{cases}$$

For which values of a, b, and c the system is consistent?

Hint: Transform the augmented matrix to the reduced row echelon form and then apply Cramer's General Rule.

Solution:

Problem 11: Given the system of linear equations

$$\begin{cases} x_1 + 2x_2 + 3x_3 = a \\ 4x_1 + 5x_2 + 6x_3 = b \\ 7x_1 + 8x_2 + 9x_3 = 0 \end{cases}$$

For which values of a and b the system is inconsistent?

Solution:

Problem 12: Find the values of a for which the following system of linear equations

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 3 \\ -x_1 + x_2 = 2 \\ x_1 + \alpha x_2 - x_3 = -2. \end{cases}$$

has the unique solution.

Solution:

Problem 13: Given the reduced row echelon form of the augmented matrix,

$$\overline{A} = \left(\begin{array}{cccc|c} 1 & 3 & -1 & 5 & 2 \\ 0 & 7 & 0 & 2 & 4 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right),$$

Find the number of solutions of the corresponding linear system. It is not necessary to solve the system.

Solution:

Problem 14: Given the reduced row echelon form of the augmented matrix,

$$\overline{A} = \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 5 & 6 & 7 & 2 \\ 0 & 0 & 8 & 9 & 3 \\ 0 & 0 & 0 & 0 & a \end{array}\right).$$

How many solutions has the corresponding linear system?

Solution:

VECTOR ALGEBRA

Linear Vector Operations

$$\lambda\, \boldsymbol{a} = \{\lambda\, a_1, \lambda\, a_2, \lambda\, a_3\}$$
$$\boldsymbol{a} + \boldsymbol{b} = \{a_1 + b_1, a_2 + b_2, a_3 + b_3\}.$$

Problem 1: Given two vectors $\boldsymbol{a} = \{4,1,-2\}$ and $\boldsymbol{b} = \{1,5,-2\}$ in Cartesian coordinate system, find the lengths of the vectors $\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{q} = \boldsymbol{a} - \boldsymbol{b}$.

Solution:

$\boldsymbol{p} =$ $\quad\quad\quad\quad$ $|\,\boldsymbol{p}\,| =$

$\boldsymbol{q} =$ $\quad\quad\quad\quad$ $|\,\boldsymbol{q}\,| =$

Problem 2: Let $\boldsymbol{a} = \{4,1,-2\}$ and $\boldsymbol{b} = \{1,5,-2\}$. Solve for $\boldsymbol{p}$ the following vector equation:

$$3\,\boldsymbol{a} - 2\,\boldsymbol{p} = 4\,\boldsymbol{b}.$$

Solution:

$\boldsymbol{p}$ =

Problem 3: Let $A(1,2,3)$ be the common origin of two vectors $\boldsymbol{a} = \overrightarrow{AB} = \{-2,4,7\}$ and $\boldsymbol{b} = \overrightarrow{AC} = \{3,-1,5\}$. Find the coordinates of the points B and C.

Solution:

Problem 4: Find the unit vector $\boldsymbol{u}$ in the direction of $\overrightarrow{AB}$, if $A(5,3,-1)$ and $B(2,7,-1)$.

Solution:

$\boldsymbol{u}$ =

Problem 5: Let $\boldsymbol{a} = \{-3,4,0\}$. Find the unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{a}$.

Solution:

$\boldsymbol{u}$ =

In Problems 6 through 8 determine whether the given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

Problem 6: $\boldsymbol{a} = \{-2, 4, 5\}$ and $\boldsymbol{b} = \{8, 16, -20\}$.

Solution:

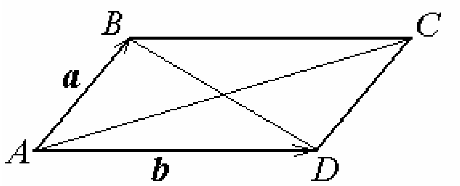
Problem 7: $\boldsymbol{a} = \{3, 4, 5\}$ and $\boldsymbol{b} = \{6, 7, 8\}$.

Solution:

Problem 8: $\boldsymbol{a} = \{3, -5, 1\}$ and $\boldsymbol{b} = \{6, -10, 2\}$.

Solution:

Problem 9: Let A, B, C, and D be the vertices of a parallelogram. Express vectors $\boldsymbol{d}_1 = \overrightarrow{AC}$ and $\boldsymbol{d}_2 = \overrightarrow{BD}$ as linear combinations of two adjacent vectors $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AD}$.



Solution:

$\boldsymbol{d}_1 =$

$\boldsymbol{d}_2 =$

Problem 10: Assume that the vectors $\boldsymbol{d}_1 = \overrightarrow{AC}$ and $\boldsymbol{d}_2 = \overrightarrow{BD}$ join the opposite vertices of a parallelogram. Find the adjacent vectors $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AD}$.

Solution:

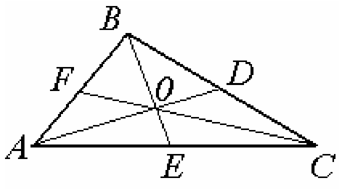
$\boldsymbol{a} = \overrightarrow{AB} =$

$\boldsymbol{b} = \overrightarrow{AD} =$

Problem 11: Given a triangle with the vertices at the points $A(2,1,-1)$, $B(3,4,4)$, and $C(5,2,3)$, find

(i) the medians *AD*, *BE*, and *CF*;

(ii) the coordinates of the medians interception point.



Solution:

$\overrightarrow{AD} =$ $\qquad$ $|AD| =$

$\overrightarrow{BE} =$ $\qquad$ $|BE| =$

$\overrightarrow{CF} =$ $\qquad$ $|CF| =$

Problem 12: Let $A(2,1,-1)$ and $B(8,-2,11)$ be the endpoints of a linear segment *AB*.

Find the coordinates of the point C dividing *AB* in the ratio 2:1.

Solution:

Problem 13: Consider two vectors $\boldsymbol{a} = \{-2,4,5\}$ and $\boldsymbol{b} = \{x,-3,z\}$.

For which values of z the equation

$$|\boldsymbol{a}| = |\mathbf{b}|$$

has exactly one solution with respect to *x*?

For which values of *z* the equation has exactly two solutions with respect to *x*?

For which values of *z* the equation for *x* has no solutions?

Solution:

Scalar Product

$$\boldsymbol{a}\cdot\boldsymbol{b}=a_x b_x+a_y b_y+a_z b_z$$
$$\boldsymbol{a}\cdot\boldsymbol{b}=a\,b\cos\theta$$
$$\mathrm{Proj}_{\boldsymbol{b}}\boldsymbol{a}=a\cos\theta$$

Problem 1: Find the scalar product of two vectors $\boldsymbol{a}=\{2,-3,4\}$ and $\boldsymbol{b}=\{7,5,1\}$.

Solution:

$\boldsymbol{a}\cdot\boldsymbol{b}=$

Problem 2: Let $|\boldsymbol{a}|=3$, $|\boldsymbol{b}|=4$, and the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ be equal to $60°$. Find the scalar product $\boldsymbol{a}\cdot\boldsymbol{b}$.

Solution:

$\boldsymbol{a}\cdot\boldsymbol{b}=$

Problem 3: Let $\boldsymbol{a}=4\boldsymbol{p}-3\boldsymbol{q}$ and $\boldsymbol{b}=\boldsymbol{p}+2\boldsymbol{q}$. Find the scalar product $\boldsymbol{a}\cdot\boldsymbol{b}$, if $|\boldsymbol{p}|=5$, $|\boldsymbol{q}|=2$, and vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ form the angle of $30°$.

Solution:

In Problems 4 through 6 find the angle between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Problem 4: $\boldsymbol{a}=\{2,-3,4\}$ and $\boldsymbol{b}=\{-1,5,-4\}$.

Solution:

$|\boldsymbol{a}|=$ $|\boldsymbol{b}|=$

$\boldsymbol{a}\cdot\boldsymbol{b}=$

$\cos\theta=$

Problem 5: $\boldsymbol{a}=\{0,3,4\}$ and $\boldsymbol{b}=\{2,4,-4\}$.

Solution:

$|\boldsymbol{a}|=$ $|\boldsymbol{b}|=$

$\boldsymbol{a}\cdot\boldsymbol{b}=$

$\cos\theta=$

Problem 6: $\boldsymbol{a} = \{4,1,-1\}$ and $\boldsymbol{b} = \{0,3,7\}$.

Solution:

$|\boldsymbol{a}| =$ $|\boldsymbol{b}| =$

$\boldsymbol{a} \cdot \boldsymbol{b} =$

$\cos\theta =$

In Problems 7 through 9 find the projection of ***a*** on ***b***.

Problem 7: $\boldsymbol{a} = \{6,5,-1\}$ and $\boldsymbol{b} = \{2,-3,4\}$.

Solution:

$\boldsymbol{a} \cdot \boldsymbol{b} =$

$|\boldsymbol{b}| =$

$\mathrm{Proj}_{\boldsymbol{b}}\boldsymbol{a} =$

Problem 8: $\boldsymbol{a} = \{1,-1,5\}$ and $\boldsymbol{b} = \{4,3,0\}$.

Solution:

$\boldsymbol{a} \cdot \boldsymbol{b} =$

$|\boldsymbol{b}| =$

$\mathrm{Proj}_{\boldsymbol{b}}\boldsymbol{a} = a\cos\theta$

Problem 9: $\boldsymbol{a} = \{-2,-1,6\}$ and $\boldsymbol{b} = \{2,5,1\}$.

Solution:

$\boldsymbol{a} \cdot \boldsymbol{b} =$

$|\boldsymbol{b}| =$

$\mathrm{Proj}_{\boldsymbol{b}}\boldsymbol{a} = a\cos\theta$

In Problems 10 through 14 find the value of a parameter p to satisfy the required condition.

Problem 10: $\boldsymbol{a} = \{6,5,-1\}$, $\boldsymbol{b} = \{p,-3,4\}$, and $\boldsymbol{a} \perp \boldsymbol{b}$.

Solution:

$\boldsymbol{a} \cdot \boldsymbol{b} =$

$p =$

Problem 11: $\boldsymbol{a} = \{2, p, -3\}$, $\boldsymbol{b} = \{3, 2, -5\}$, and $\boldsymbol{a} \perp \boldsymbol{b}$.

Solution:

$\boldsymbol{a} \cdot \boldsymbol{b} =$

$p =$

Problem 12: $\boldsymbol{a} = \{-6, 2, 5\}$, $\boldsymbol{b} = \{3, -1, p\}$, and $\boldsymbol{a} \parallel \boldsymbol{b}$.

Solution:

$\frac{a_x}{b_x} =$

$p =$

Problem 13: $\boldsymbol{a} = \{2, p, -3\}$, $\boldsymbol{b} = \{10, -5, -15\}$, and $\boldsymbol{a} \parallel \boldsymbol{b}$.

Solution:

$\frac{a_x}{b_x} =$

$p =$

Problem 14: $\boldsymbol{a} = \{1, 2, -3\}$, $\boldsymbol{b} = \{4, 5, p\}$, and $\boldsymbol{a} \parallel \boldsymbol{b}$.

Solution:

$\frac{a_x}{b_x} =$

$p =$

Problem 15: Let $\boldsymbol{a} = \{4, -2, 1\}$. Fund the direction cosines.

Solution:

$|\boldsymbol{a}| =$

$\cos\alpha =$

$\cos\beta =$

$\cos\chi =$

Vector Product

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

$$|\boldsymbol{a}\times\boldsymbol{b}|=ab\sin\theta,\qquad \boldsymbol{a}\times\boldsymbol{b}\perp\boldsymbol{a},\qquad \boldsymbol{a}\times\boldsymbol{b}\perp\boldsymbol{b}$$

Problem 1: Find the vector product of two vectors $\boldsymbol{a}=\{2,-3,4\}$ and $\boldsymbol{b}=\{7,5,1\}$.

Solution:

$\boldsymbol{a}\times\boldsymbol{b}=$

Problem 2: Let $|\boldsymbol{a}|=2$, $|\boldsymbol{b}|=7$, and the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ be equal to $30°$. Find the absolute value of the vector $\boldsymbol{a}\times\boldsymbol{b}$.

Solution:

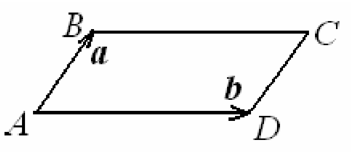
$|\boldsymbol{a}\times\boldsymbol{b}|=$

Problem 3: Simplify the vector product $(2\boldsymbol{a}-3\boldsymbol{b})\times(\boldsymbol{a}+4\boldsymbol{b})$.

Solution:

$(2\boldsymbol{a}-3\boldsymbol{b})\times(\boldsymbol{a}+4\boldsymbol{b})=$

Problem 4: Let *A*(2, 3, –2), *B*(1, 7, -1), and *C*(3, 4, 5) be three adjacent vertices of a parallelogram. Find the area *S* of the parallelogram.



Solution:

$\boldsymbol{a}=\overrightarrow{AB}=$

$\boldsymbol{b}=\overrightarrow{AC}=$

$\boldsymbol{a}\times\boldsymbol{b}=$

$S=$

Problem 5: Find the area of the triangle with the vertices at the points $A(5,6,2)$, $B(4,8,7)$, and $C(5,3,4)$.

Solution:

$\boldsymbol{a} = \overrightarrow{AB} =$

$\boldsymbol{b} = \overrightarrow{AC} =$

$\boldsymbol{a} \times \boldsymbol{b} =$

Area =

Problem 6: Given two vectors, $\boldsymbol{a} = \{1, -5, 4\}$ and $\boldsymbol{b} = \{2, -2, 1\}$, find a vector $\boldsymbol{c}$ such that $\boldsymbol{c} \perp \boldsymbol{a}$ and $\boldsymbol{c} \perp \boldsymbol{b}$.

Solution:

Problem 7: Given the parallelogram with the adjacent vertices at the points $A(-1,3,2)$, $B(4,1,2)$, and $C(3,4,4)$, find the length of the height from the vertex B to the base AC.

Solution:

$\overrightarrow{AB} =$

$\overrightarrow{AC} =$

$\overrightarrow{AB} \times \overrightarrow{AC} =$

$|\overrightarrow{AB}| =$

$h =$

Problem 7: Given the triangle with the vertices at the points $A(-1,3,2)$, $B(4,1,2)$, and $C(3,4,4)$, find the length of the height from the vertex B to the base AC.

Solution:

$\overrightarrow{AB} =$ $\qquad$ $\overrightarrow{AC} =$

$\overrightarrow{AB} \times \overrightarrow{AC} =$

$|\overrightarrow{AB}| =$ $\qquad$ $h =$

Scalar Triple Product

$$([\boldsymbol{a}, \boldsymbol{b}], \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Problem 1: Determine whether the vectors $\boldsymbol{a} = \{-3, 4, -2\}$, $\boldsymbol{b} = \{0, -3, 2\}$, and $\boldsymbol{c} = \{2, 1, -1\}$ are linear independent.

Solution:

$\boldsymbol{abc} =$

Conclusion:

Problem 2: Determine whether the vectors $\boldsymbol{a} = \{1, 5, -2\}$, $\boldsymbol{b} = \{4, -1, 3\}$, and $\boldsymbol{c} = \{2, -2, 1\}$ form a basis.

Solution:

$\boldsymbol{abc} =$

Conclusion:

Problem 4: Determine whether four points $A(2, -3, 1)$, $B(3, 3, 0)$, $C(3, 1, 4)$, and $D(5, -4, 1)$ lie in the same plane.

Solution:

$\boldsymbol{a} = \overrightarrow{AB} =$

$\boldsymbol{b} = \overrightarrow{AC} =$

$\boldsymbol{c} = \overrightarrow{AD} =$

$\boldsymbol{abc} =$

Conclusion:

Problem 5: Find the volume V of the tetrahedron with the vertices at the points $A(3,-1,4)$, $B(4,4,4)$, $C(1,0,2)$, and $D(1,5,2)$.

Solution: $\boldsymbol{a} = \overrightarrow{AB} =$

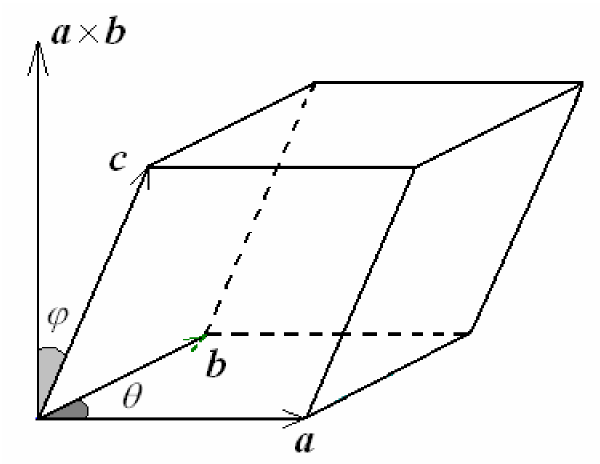
$\boldsymbol{b} = \overrightarrow{AC} =$

$\boldsymbol{c} = \overrightarrow{AD} =$

$\boldsymbol{abc} =$

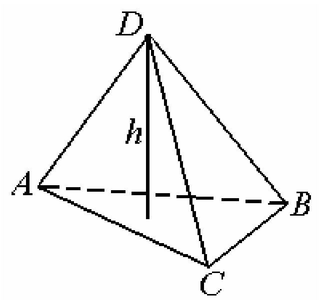
$V =$

Problem 6: Find the volume of the parallelepiped constructed on the vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ as it is shown in the figure.



Solution:

Problem 7: The tetrahedron is given by the vertices $A(0,-1,1)$, $B(2,0,3)$, $C(0,4,1)$, and $D(3,3,3)$.
Find the height from the point D to the base ABC.



Solution:

ANALYTICAL GEOMETRY

Straight Lines

$$\frac{x-x_0}{q_x}=\frac{y-y_0}{q_y}=\frac{z-z_0}{q_z}$$

$$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}=\frac{z-z_0}{z_1-z_0}$$

$$\cos\theta=\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}|\cdot|\boldsymbol{q}|}=\frac{p_xq_x+p_yq_y+p_zq_z}{\sqrt{p_x^2+p_y^2+p_z^2}\sqrt{q_x^2+q_y^2+q_z^2}}$$

Problem 1: Find the canonical equations of the line passing through the point $M(2,–3, 4)$ and being parallel to the vector $\boldsymbol{a}=\{5,–1, 7\}$.

Solution:

Problem 2: Find equations of the line passing through the point $M(4, 0, 8)$ and being parallel to the vector $\overrightarrow{AB}$, if $A(-3,2,6)$ and $B(1,4,-4)$.

Solution:

Problem 3: Find equations of the line passing through the point $M(0,–1,3)$ and being parallel to the line $\frac{x-3}{2}=\frac{y+5}{1}=\frac{z-7}{-4}$.

Solution:

Problem 4: Let L be a line passing through the points $M_1(4,-1,3)$ and $M_2(3,5,-2)$.

Determine whether the point $A(1,3,6)$ lies on the line L.

Solution:

Problem 5: Let L be a line passing through the points $M_1(4,-1,3)$ and $M_2(3,5,-2)$. Find a few other points on the line L.

Solution:

Problem 6: In the x, y–plane a line is given by the equation

$$2x-3y+24=0.$$

Find

(i) any two points on the line;
(ii) the slope of the line;
(iii) the x-intercept and y- intercept.

Solution:

(i)

(ii)

(iii)

Problem 7: In the x, y–plane, find the equation of the line passing through the point $M_1(2,-4)$ and being perpendicular to the vector $\boldsymbol{n}=\{3,1\}$.

Solution:

Problem 8: Let $M_1(1,5)$ and $M_2(-2,3)$ be the points on a line.
Which of the following points, $A(6,4)$, $B(2,7)$ and $C(2,10)$, are the points on the line?

Solution:

Problem 9: Let $3x-2y+8=0$ and $x+4y-5=0$ be two lines in the x, y–plane. Find the cosine of the angle between the lines.

Solution:

$\boldsymbol{n}_1=$ $\boldsymbol{n}_2=$

$\cos\theta=$

Problem 10: Transform the equation of the line $x+4y-5=0$ in *x*, *y*–plane to the intercept form.

Solution:

Problem 11: Let *A* = {2, –1}, *B* = {4, 4} and *C* = {9, 7} be the vertices of a triangle. Find the equation of the altitude from the vertex *A*. Write down the equation in the intercept form.

Solution:

Problem 12: Find the distance from the point $M(-2,5)$ to the line $4x-3y+1=0$.

Solution:

Problem 13: Let *ABC* be a triangle with the vertices at the points $A=\{3,5\}$, $B=\{2,-1\}$ and $C=\{6,7\}$ in the *x*, *y*–plane.

Find the length of the altitude from the vertex *A*.

Solution:

Problem 14: Find the point of intersection of the lines

$$3x-2y+8=0 \quad \text{and} \quad x+4y-5=0.$$

Solution:

$$\begin{cases} 3x-2y+8=0 \\ x+4y-5=0 \end{cases} \quad \Rightarrow$$

Problem 15: Find the point of intersection of the lines

$$x-2y-3=0 \quad \text{and} \quad -3x+6y+5=0.$$

Solution:

$$\begin{cases} 3x-2y+8=0 \\ x+4y-5=0 \end{cases} \quad \Rightarrow$$

Planes

$$Ax + By + Cz + D = 0,$$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0,$$

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

Problem 1: Find the general equation of the plane passing through the point $M(5,-2,1)$ and being parallel to the vectors $\boldsymbol{a} = \{-4,3,0\}$ and $\boldsymbol{b} = \{1,-6,2\}$.

Solution:

Problem 2: Find the general equation of the plane passing through the points $M_1(-1,4,2)$, $M_2(3,4,-1)$, and $M_3(0,5,6)$.

Solution:

Problem 3: Find the general equation of the plane passing through the point $M(1,-2,-3)$ and being perpendicular to the vector $\boldsymbol{n} = \{7,-4,-1\}$.

Solution:

Problem 4: A plane is given by the equation $x - 2y + 3z - 6 = 0$. Find a unit normal vector $\boldsymbol{u}$ to the plane and any two points in the plane.

Solution:

Problem 5: Transform the equation of the plane $3x + 2y - 4z - 24 = 0$ to the intercept form.

Solution:

Problem 6: Find the point of intersection of the line

$$\frac{x-1}{2}=\frac{y+3}{-2}=\frac{z}{5}$$

and the plane $3x-y+2z=4$.

Solution:

Problem 7: Let L be the line of interception of two planes $2x-y+4z=5$ and $x+y-2z=6$. Find the canonical equations of the line L.

Solution:

Problem 8: Find the angle between two planes $-3x+4y-z-5=0$ and $2x+3y-1=0$.

Solution:

$\cos\theta=$

Problem 9: Find the angle between the plane $4x+3y-5z+2=0$ and the line $\frac{x+3}{7}=\frac{y-2}{4}=\frac{z+5}{-1}$.

Solution:

$\cos\theta=$

Problem 10: Find the distance from the point $M(-2,7,-1)$ to the plane $4x-3y+5=0$.

Solution:

Problem 11: Find the point of intersection of the planes $x-2y+z+5=0$, $3x+y-z+3=0$, and $x-2z+1=0$.

Solution:

Quadratic Curves

$$(x-x_0)^2+(y-y_0)^2=R^2,$$

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_2)^2}{b^2}=1,$$

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_2)^2}{b^2}=\pm1,$$

$$(y-y_0)^2=\pm2p(x-x_0),$$

$$(x-x_0)^2=\pm2p(y-y_0).$$

Problem 1: Sketch the graph of $x^2-4x+y^2+6y-3=0$.

Solution:

Problem 2: Given the circle $x^2-4x+y^2+6y-3=0$, find the radius and coordinates of the center.

Solution:

Problem 3: Write down the equations of the circle with center at the point *M*(–3, 5) and radius 2.

Solution:

Problems 4–9: Reduce each of the following equations to the canonical form:

4) $3x^2-12x+2y^2+4y=11$;
5) $3x^2-12x+2y^2+4y=-14$;
6) $2x^2-4x+3y^2+12y=-15$;
7) $x^2-6x-2y^2+8y=3$;
8) $2x^2+4x-3y^2-12y=0$;
9) $x^2+2x-4y^2=-1$.

Give the detailed description of the curves.

Solution:

4) $3x^2 - 12x + 2y^2 + 4y = 11 \quad \Rightarrow$

5) $3x^2 - 12x + 2y^2 + 4y = -14 \quad \Rightarrow$

6) $2x^2 - 4x + 3y^2 + 12y = -15 \quad \Rightarrow$

7) $x^2 - 6x - 2y^2 + 8y = 3 \quad \Rightarrow$

8) $2x^2 + 4x - 3y^2 - 12y = 0 \quad \Rightarrow$

9) $x^2 + 2x - 4y^2 = -1 \quad \Rightarrow$

Problem 10: Using the equation $\frac{(x-5)^2}{25} + \frac{(y+3)^2}{9} = 1$ recognize the curve and find

i) the location of the center;
ii) the major axis and minor axis;
iii) the coordinates of the focuses;
iv) the eccentricity;
v) the equations of the axes of symmetry.

Solution:

Problem 11: Using the equation $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{16} = 1$ find

i) the location of the center;
ii) the coordinates of the focuses;
iii) the eccentricity of the hyperbola;
iv) the equations of the asymptotes.

Solution:

References

1. V.V. Konev. Linear Algebra, Vector Algebra and Analytical Geometry. Textbook. Tomsk. TPU Press, 2009, 112p.
2. K.P. Arefiev, O.V. Boev, A.I. Nagornova, G.P. Stoljarova, A.N. Harlova. Higher Mathematics, Part 1. Textbook. Tomsk, TPU Press, 1998, 97p.
3. Murray H. Plotter, Charles B. Morrey. Intermediate Calculus, Springer, 1985.
4. V.V. Konev. Mathematics, Preparatory Course. Textbook. Tomsk. TPU Press, 2009, 104p.
5. V.V. Konev. The Elements of Mathematics. Textbook. Tomsk. TPU Press, 2009, 140p.
6. V.V. Konev. Mathematics, Preparatory Course: Algebra, Workbook. TPU Press, 2009, 60p.
7. V.V. Konev. Mathematics, Preparatory Course: Geometry and Trigonometry, Workbook. Tomsk. TPU Press, 2009, 34p.
8. V.V. Konev. The Elements of Mathematics. Workbook, Part 1. Tomsk. TPU Press, 2009, 54p.
9. V.V. Konev. The Elements of Mathematics. Workbook, Part 2. Tomsk. TPU Press, 2009, 40p.

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Linear Algebra, Vector Algebra and Analytical Geometry

Workbook

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