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«ТОМСКИЙ ПОЛИТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ»

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LIMITS OF SEQUENCES AND FUNCTIONS

WorkBook

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To the Student

Mathematical tools of engineering science are based on differential calculus and integral calculus. The concept of the limit is essential for calculus. Limits express the concepts of infinite small and infinite large quantities in mathematical terms.

The workbook is prepared for students who study calculus and want to broaden and methodize their knowledge. It will help you to develop problem-solving skills and to focus your attention on important problems of calculus. The topics are presented in the same order as in textbook [1].

The problems concern three content areas: Limits Sequences, Limits of Functions, and Continuity of Functions.

Solving problems try to use graphs that help you with visualization. They also help to explain and to interpret the evaluation of limits.

The tests will reveal your knowledge and skills, your abilities in interpreting symbols, justifying statements and constructing proofs.

After you complete this supplement you should be able to evaluate indeterminate forms, to know the most important limits, to operate with infinitesimal quantities and infinite large variables, and to use standard techniques of taking limits.

To pass the final test with the excellent mark, you should be able to solve basic problems, demonstrate full understanding of all topics, and give a significant portion of the answer successfully. Minor calculation errors are admissible.

Contents

SEQUENCES

$$0! = 1$$
$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$
$$(2n)!! = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n$$
$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$$

Problem 1: Write down a few terms of the below sequences given by their general terms.

a) $a_n = (-1)^{n+1}(2n-1) \quad \Rightarrow \quad \{a_n\} =$

b) $b_n = (-1)^{n+1}\dfrac{1}{2n} \quad \Rightarrow \quad \{b_n\} =$

c) $c_n = \dfrac{2^n}{n!} \quad \Rightarrow \quad \{c_n\} =$

d) $x_n = \dfrac{(-1)^{n+1}3^n}{(2n-1)!} \quad \Rightarrow \quad \{x_n\} =$

e) $y_n = \dfrac{(-1)^{n+1}4^n}{(2n-1)!!} \quad \Rightarrow \quad \{y_n\} =$

f) $z_n = \dfrac{(-1)^n 5^n}{(2n)!!} \quad \Rightarrow \quad \{z_n\} =$

g) $p_n = \dfrac{n+1}{n+3} \quad \Rightarrow \quad \{p_n\} =$

h) $q_n = \dfrac{n}{n^2+2} \quad \Rightarrow \quad \{q_n\} =$

i) $r_n = \dfrac{n!}{n^2} \quad \Rightarrow \quad \{r_n\} =$

j) $s_n = \dfrac{n!}{3^n} \quad \Rightarrow \quad \{s_n\} =$

k) $t_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} =$

$\{t_n\} =$

l) $u_n = \sum_{k=1}^{n} \frac{1}{k(k+2)} =$

$\{u_n\} =$

Problem 2: Find the general terms of the below sequences given by a few first terms.

a) $\{a_n\} = \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8}, \ldots \quad \Rightarrow \quad a_n =$

b) $\{b_n\} = \frac{1}{2}, \frac{2^2}{2^2}, \frac{3^2}{2^3}, \frac{4^2}{2^4}, \ldots \quad \Rightarrow \quad b_n =$

c) $\{c_n\} = \frac{1}{3}, -\frac{2}{4}, \frac{3}{5}, -\frac{4}{6}, \ldots \quad \Rightarrow \quad c_n =$

d) $\{d_n\} = \frac{1}{4}, \frac{3}{8}, \frac{5}{12}, \frac{7}{16}, \frac{9}{20}, \ldots \quad \Rightarrow \quad d_n =$

e) $\{x_n\} = 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}, \ldots \quad \Rightarrow \quad x_n =$

f) $\{y_n\} = 2, 0, 6, 0, 10, 0, 14, \ldots \quad \Rightarrow \quad y_n =$

g) $\{z_n\} = 0, 2, 0, 4, 0, 6, 0, 8, 0, 10, \ldots \quad \Rightarrow \quad z_n =$

h) $\{t_n\} = 1, 0, -1, 0, 1, 0, -1, 0, \ldots \quad \Rightarrow \quad t_n =$

Problem 3: Given a sequence, find the least upper bound and the greatest lower bound, if they exist.

a) $\{(-1)^{n+1} \frac{1}{2n}\}$

b) $\{\frac{(-1)^n 3^n}{(2n-1)!}\}$

c) $\{\frac{n+1}{n+3}\}$

Limits of Sequences

> If for any arbitrary small number $\delta > 0$
> there exists a number N such that
> the condition $n > N$ implies the inequality
> $|x_n - a| < \delta$,
> then a is the **limit of a sequence** $\{x_n\}$,
> $\lim x_n = a$.

Problem 4: Give the formal proof of the fact that $\lim\limits_{n\to\infty} \dfrac{n+2}{n+3} = 1$.

Solution:

Problem 5: Prove that $\{\dfrac{1}{\sqrt{n+5}}\}$ is an infinitesimal sequence.

Solution:

Problem 6: Prove that $\{\dfrac{n}{\sqrt{n+1}}\}$ is an infinite large sequence.

Solution:

Problem 7: Show that infinitesimal variables $\dfrac{1}{n^2+5n+6}$ and $\dfrac{1}{n^2-3n+7}$ are equal asymptotically as $n \to \infty$.

Solution:

Problem 8: Show that infinitesimal variables $\dfrac{1}{n^2}$ and $\dfrac{5}{4n^2+n}$ have the same order of smallness as $n \to \infty$.

Solution:

In Problems 9 through 11 find the order of smallness of the given infinitesimal variable with respect to $\frac{1}{n}$ as $n \to \infty$.

Problem 9: $\frac{3}{4n^2} \sim$

Problem 10: $\frac{n}{5n^2 + 2n + 1} \sim$

Problem 11: $\frac{5n + 8}{\sqrt{n^2 + 4n + 7}} \sim$

In Problems 12 through 15 find the increasing order of the given infinite large variable with respect to n as $n \to \infty$.
Which quantities can be neglected in the below expressions?

Problem 12: $n^5 + 3n^4 + 9n + 11 \sim$

Problem 13: $\sqrt[3]{n^9 + 2n^6 + 5n + 7} \sim$

Problem 14: $\frac{n^3 + 6n^2 - 5n + 100}{n^2 + 2n + 9} \sim$

Problem 15: $\frac{n^3 + 6n^2 - 5n + 100}{\sqrt{n^2 + 2n + 9}} \sim$

Number *e*

$$\boxed{\lim_{n\to\infty}(1+\frac{1}{n})^n = e}$$

In Problems 16 through 23 apply the above statement to evaluate the limit of the given variable as $n \to \infty$.

Problem 16: $\lim_{n\to\infty}(1+\frac{3}{n})^n =$

Problem 17: $\lim_{n\to\infty}(1-\frac{1}{3n})^n =$

Problem 18: $\lim_{n\to\infty}(1-\frac{1}{3n})^{n^2+2n} =$

Problem 19: $\lim_{n\to\infty}(1+\frac{1}{4n^2})^{5n} =$

Problem 20: $\lim_{n\to\infty}(1+\frac{2n-3}{n^2+5n+1})^{4n} =$

Problem 21: $\lim_{n\to\infty}(1+\frac{n}{n^2+3n-2})^{\frac{n^2}{n+1}} =$

Problem 22: $\lim_{n\to\infty}(\frac{5n+4}{5n+2})^{-3n+1} =$

Problem 23: $\lim_{n\to\infty}(\sqrt{\frac{2n-1}{2n+7}})^{5n} =$

LIMITS of FUNCTIONS

Let a function $f(x)$ be defined in some neighborhood of a point $x = a$, including or excluding a.
A number A is called the **limit** of $f(x)$ as x tends to a, if for any arbitrary small number $\varepsilon > 0$ there exists a number $\delta > 0$ such that the inequality $|x - a| < \delta$ implies
$|f(x) - A| < \varepsilon$.

$f(x)$ is an **infinitesimal function** as $x \to a$, if $\lim_{x \to a} f(x) = 0$.
$f(x)$ is an **infinite large function** as $x \to a$, if $\lim_{x \to a} f(x) = \infty$.

Problem 1: Give the formal proof that $\lim_{x \to 3} x^2 = 9$. Then, use some values of ε to find $\delta(\varepsilon)$.

Solution:

Problem 2: (i) Prove that $\lim_{x \to 1} \frac{x+4}{x+3} = \frac{5}{4}$.

(ii) Setting $\varepsilon = 0.01$ and $\varepsilon = 0.001$ find the corresponding values of $\delta(\varepsilon)$.

Solution:

Problem 3: By contradiction, prove that $\lim_{x \to 2} x^2 \neq 5$.

Solution:

A function $f(x)$ has an infinite limit as x tends to a,
if for any arbitrary large number $E > 0$
there exists a number $\delta = \delta(E) > 0$ such that
the inequality $|x - a| < \delta$ implies
$|f(x)| > E$.

Problem 4: Prove that $f(x) = \dfrac{x-1}{x+1}$ is an infinitesimal function as $x \to 1$.

Solution:

Problem 5: Prove that $f(x) = \dfrac{x-1}{x+1}$ is an infinite large function as $x \to -1$.

Solution:

Problem 6: Prove that $\lim\limits_{x \to -2} \dfrac{x+2}{x^2 - 4} = -\dfrac{1}{4}$.

Solution:

Problem 7: Prove that $\lim\limits_{x \to 2} \dfrac{x+2}{x^2 - 4} = \infty$.

Solution:

A number A is the **limit** of $f(x)$ as $x \to \infty$,
if for any arbitrary small number $\varepsilon > 0$
there exists the corresponding number $\Delta = \Delta(\varepsilon) > 0$
such that the inequality $|x| > \Delta$ implies
$|f(x) - A| < \varepsilon$.

Problem 8: Prove that $f(x) = \dfrac{1}{x+1}$ is an infinitesimal function as $x \to \infty$.

Solution:

Problem 9: Prove that $\lim\limits_{x \to \infty} \dfrac{x}{x+1} = 1$.

Solution:

Problem 10: Prove that $f(x) = \dfrac{x^2}{x+1}$ is an infinite large function as $x \to \infty$.

Solution:

Problem 11: Prove that:

(i) $y = 2^x$ is an infinite large function as $x \to +\infty$, $\lim\limits_{x \to +\infty} 2^x = +\infty$;

(ii) $y = 2^x$ is an infinitesimal function as $x \to -\infty$, $\lim\limits_{x \to -\infty} 2^x = 0$.

Solution:

1. $\lim\limits_{x\to a} c\,f(x) = c\lim\limits_{x\to a} f(x)$.

If there exist both limits, $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$, then there exist the limits of the sum, product and quotient of the functions:

2. $\lim\limits_{x\to a}(f(x)\pm g(x)) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$.

3. $\lim\limits_{x\to a}(f(x)\cdot g(x)) = \lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x)$.

4. $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ (if $\lim\limits_{x\to a} g(x)\neq 0$).

In Problems 12 through 28 evaluate the limit of the given function.
Hints. Use the following algebraic transformations:

- factor the numerator and denominator of a fraction to cancel the common factors;
- reduce the sum of (or difference between) fractions to the common denominator; then apply the previous hint.

Problem 12: $\lim\limits_{x\to 2}\dfrac{4-x^2}{x-2} =$

Problem 13: $\lim\limits_{x\to 1}\dfrac{x-1}{\sqrt{x}-1} =$

Problem 14: $\lim\limits_{x\to 3}\dfrac{x^3-27}{9-x^2} =$

Problem 15: $\lim_{x\to -1}\dfrac{x^2+7x+6}{x^2-2x-3}=$

Problem 16: $\lim_{x\to 2}\dfrac{x^2-4x+4}{(x^2-2x)(x^2-5x+6)}=$

Problem 17: $\lim_{x\to 3}\dfrac{x^2-6x+9}{(x^2-3x)(x+1)}=$

Problem 18: $\lim_{x\to 3}\dfrac{(x^2-3x)(x+1)}{x^2-6x+9}=$

Problem 19: $\lim_{x\to 1}\dfrac{\sqrt{x+3}-2}{x-1}=$

Problem 20: $\lim_{x\to 3}\dfrac{\sqrt{12-x}-3}{9-x^2}=$

Problem 21: $\lim_{x\to 2}\dfrac{\sqrt{6-x}-2}{\sqrt{x+7}-3}=$

Problem 22: $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x+8}-2}{x-1}=$

Problem 23: $\lim\limits_{x\to 2}\left(\dfrac{1}{x^2-x-2}-\dfrac{1}{x^2+x-6}\right)=$

Problem 24: $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2+x}-\dfrac{1}{x^2-3x}\right)=$

Problem 25: $\lim\limits_{x\to -1}\left(\dfrac{1}{x+1}-\dfrac{2}{x^2-1}\right)=$

Problem 26: $\lim\limits_{x\to 0}\dfrac{\tan^2 3x}{1-\cos^2 3x}=$

Problem 27: $\lim\limits_{x\to \infty}\dfrac{4x^2+5x+1}{3x^2-x}=$

Problem 28: $\lim\limits_{x\to \infty}\dfrac{x^3-2x^2+4x+1}{5x^3+7x^2-x}=$

1. If $f(x)$ is a bounded function and $\alpha(x)$ is an infinitesimal function as x tends to a, then $f(x)\alpha(x)$ is an infinitesimal function.
2. The sum of any finite number of infinitesimal functions is an infinitesimal function.

Helpful Rule:

$$f(x) = A + \text{infinitesimal} \quad \text{as } x \to a$$
$$\Leftrightarrow$$
$$\lim_{x \to a} f(x) = A$$

In Problems 29 through 34 apply the above rule to evaluate the limits of the given functions.

Problem 29: $\lim\limits_{x \to \infty} \dfrac{5x+1}{x} =$

Problem 30: $\lim\limits_{x \to \infty} \dfrac{2x-1}{x+3} =$

Problem 31: $\lim\limits_{x \to \infty} \dfrac{1}{2x^2} =$

Problem 32: $\lim\limits_{x \to \infty} \dfrac{4x-1}{2x^2} =$

Problem 33: $\lim\limits_{x \to \infty} \dfrac{3x^2+4x-1}{2x^2} =$

Problem 34: $\lim\limits_{x \to \infty} \dfrac{x^3+3x^2+4x-1}{2x^2} =$

$\alpha(x)$ and $\beta(x)$ are **infinitesimal functions of the same order of smallness** as *x* tends to *a*,

if $\lim\limits_{x\to a}\alpha(x)=0$, $\lim\limits_{x\to a}\beta(x)=0$, and

$$0<\left|\lim_{x\to a}\frac{\alpha(x)}{\beta(x)}\right|<\infty.$$

$$\alpha(x)\sim\beta(x)$$

Infinitesimal functions $\alpha(x)$ and $\beta(x)$ are **equivalent** as *x* tends to *a*,

if $\lim\limits_{x\to a}\dfrac{\alpha(x)}{\beta(x)}=1$.

An infinitesimal function $\alpha(x)$ has a **higher order of smallness** with respect to $\beta(x)$ as *x* tends to *a*,

if $\lim\limits_{x\to a}\dfrac{\alpha(x)}{\beta(x)}=0$.

$\alpha(x)$ is an infinitesimal of the ***n*-th order** with respect to $\beta(x)$ as *x* tends to *a*,

if $\alpha(x)$ and $(\beta(x))^n$ are infinitesimal functions of the same order:

$$0<\left|\lim_{x\to a}\frac{\alpha(x)}{(\beta(x))^n}\right|<\infty.$$

Problem 35: Prove that

$$\alpha(x)=\frac{1}{x+5}\quad\text{and}\quad\beta(x)=\frac{1}{3x+8}$$

are infinitesimal functions of the same order as $x\to\infty$.

Solution:

Problem 36: Prove that

$$\alpha(x)=\frac{1}{x^2+2}\quad\text{and}\quad\beta(x)=\frac{1}{4x^2-7}$$

are infinitesimal functions of the same order as $x\to\infty$.

Solution:

Problem 37: Prove that

$$\alpha(x)=\frac{1}{x^2+2x+5} \quad \text{and} \quad \beta(x)=\frac{1}{3x^2-x+8}$$

are infinitesimal functions of the same order as $x\to\infty$.

Solution:

Problem 38: Prove that

$$\alpha(x)=x^2-9 \quad \text{and} \quad \beta(x)=6x-18$$

are equivalent infinitesimal functions as $x\to 3$.

Solution:

Problem 39: Prove that

$$\alpha(x)=2\sqrt{x^2-5}-4 \quad \text{and} \quad \beta(x)=3x-9$$

are equivalent infinitesimal functions as $x\to 3$.

Solution:

Problem 40: Prove that

$$\alpha(x)=x^2-9 \quad \text{and} \quad \beta(x)=6x-18$$

are equivalent infinitesimal functions as $x\to 3$.

Solution:

Problem 41: Prove that

$$\alpha(x)=x^2-4x+4$$

is an infinitesimal function of the second order of smallness with respect to

$$\alpha(x)=x-2$$

as $x\to 2$.

Solution:

Problem 42: Find the order of smallness of the infinitesimal function

$$\alpha(x) = 6x^2 + 5x + \sqrt{x}$$

with respect to

$$\beta(x) = \sqrt{x} \text{ as } x \to 0.$$

Solution:

Problem 43: Find the order of smallness of the infinitesimal function

$$\alpha(x) = 6x^2 + 5x + \sqrt{x}$$

with respect to

$$\beta(x) = x \text{ as } x \to 0.$$

Solution:

Problem 44: Find the order of smallness of the infinitesimal function

$$\alpha(x) = 6x^2 + 5x + \sqrt{x}$$

with respect to

$$\beta(x) = x - \sqrt{x} \text{ as } x \to 0.$$

Solution:

Problem 45: Let $\alpha(x) = 3x^2 + 7x + \sqrt{x}$.
Which terms of $\alpha(x)$ are negligible quantities with respect to x as $x \to 0$?
Which terms of $\alpha(x)$ are negligible quantities with respect to x as $x \to +\infty$?
Solution:

Problem 46: Compare with each other the orders of smallness of the infinitesimal functions

$$\alpha(x) = 2x^3 + 5x + \sqrt[3]{x}$$

and

$$\beta(x) = x^3 - 4x + \sqrt{x}.$$

Which terms of $\alpha(x)$ and $\beta(x)$ are negligible quantities with respect to x as $x \to 0$?
Solution:

$\alpha(x)$ and $\beta(x)$ are infinite large functions of the **same increasing order** as $x \to a$,

if $\lim\limits_{x\to a} \alpha(x) = \infty$, $\lim\limits_{x\to a} \beta(x) = \infty$, and

$$0 < \left| \lim_{x\to a} \frac{\alpha(x)}{\beta(x)} \right| < \infty .$$

$$\alpha(x) \sim \beta(x)$$

Infinite large functions $\alpha(x)$ and $\beta(x)$ are **equivalent** as $x \to a$,

if $\lim\limits_{x\to a} \dfrac{\alpha(x)}{\beta(x)} = 1$.

An infinite large function $\alpha(x)$ has a **higher increasing order** with respect to $\beta(x)$ as x tends to a,

if $\lim\limits_{x\to a} \dfrac{\alpha(x)}{\beta(x)} = \infty$.

$\alpha(x)$ is an infinite large function of the n-th increasing order with respect to $\beta(x)$ as x tends to a,

if $\alpha(x)$ and $(\beta(x))^n$ are infinite large functions of the same order:

$$0 < \left| \lim_{x\to a} \frac{\alpha(x)}{(\beta(x))^n} \right| < \infty .$$

Problem 47: Prove that

$$\alpha(x) = x^2 + 2x + 5$$

and

$$\beta(x) = 3x^2 - x + 8$$

are infinite large functions of the same order as $x \to \infty$.

Solution:

Problem 48: Prove that

$$\alpha(x)=\frac{1}{x^2-9} \quad \text{and} \quad \beta(x)=\frac{1}{6x-18}$$

are equivalent infinite large functions as $x \to 3$.
Solution:

Problem 49: Prove that

$$\alpha(x)=\frac{1}{x^2-4x+4}$$

is an infinite large function of the second increasing order with respect to

$$\alpha(x)=\frac{1}{x-2}$$

as $x \to 2$.
Solution:

Problem 50: Determine whether or not

$$\alpha(x)=\frac{1}{6x^2+5x+\sqrt{x}}$$

is an infinite large function of a higher increasing order with respect to

$$\beta(x)=\frac{1}{x-\sqrt{x}}$$

as $x \to 0$.
Solution:

Problem 51: Compare with each other the increasing orders of the infinite large functions

$$\alpha(x)=\frac{1}{2x^3+5x+\sqrt[3]{x}} \quad \text{and} \quad \beta(x)=\frac{1}{x^3-4x+\sqrt{x}}.$$

Which terms in the denominators of $\alpha(x)$ and $\beta(x)$ are negligible quantities with respect to x as $x \to 0$?
Solution:

The Most Important Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$. $\sin x \sim x$, $\tan x \sim x$, $\arcsin x \sim x$, $\arctan x \sim x$, $\cos x \sim 1 - \frac{x^2}{2}$ as $x \to 0$

In Problems 1 through 16 evaluate the limit of the given function.

Problem 1: $\lim_{x \to 0} \frac{\sin 2x}{5x} =$

Problem 2: $\lim_{x \to 0} \frac{\sin 4x}{\sin 3x} =$

Problem 3: $\lim_{x \to 0} \frac{\tan 4x}{\tan 3x} =$

Problem 4: $\lim_{x \to 0} \frac{\sin^5 8x}{3x^2 \tan^3 4x} =$

Problem 5: $\lim_{x \to 0} \frac{3 \arcsin 5x}{4 \sin 3x} =$

Problem 6: $\lim_{x \to 0} \frac{2 \arctan 6x}{3 \arcsin 2x} =$

Problem 7: $\lim_{x \to \frac{\pi}{4}} \frac{\sin^2 x - \cos^2 x}{\tan^2 x - 1} =$

Problem 8: $\lim\limits_{x\to 0}\dfrac{1-\cos 6x}{\sin^2 5x}=$

Problem 9: $\lim\limits_{x\to\infty} x^3\sin\dfrac{1}{4x^2}=$

Problem 10: $\lim\limits_{x\to 0}\dfrac{1-\sqrt{\cos 2x}}{\tan x(1-\cos\sqrt{x})}=$

Problem 11: $\lim\limits_{x\to 1/2}\dfrac{\sin(\pi x)-1}{2x-1}=$

Problem 12: $\lim\limits_{x\to 3}\dfrac{\tan(\pi x)}{x-3}=$

Problem 13: $\lim\limits_{x\to \pi/4}\dfrac{\tan x-1}{x-\pi/4}=$

Problem 14: $\lim\limits_{x\to 0}\dfrac{\cos 2x-\cos 6x}{x}=$

Problem 15: $\lim\limits_{x\to 0}\dfrac{\cos^2 3x-\cos^2 x}{x^2}=$

Problem 16: $\lim\limits_{x\to 0}\dfrac{\sin 5x-\sin 3x}{2x}=$

Problem 17: Calculate approximately $\sin 15°$. Compare your result with the exact value

$$\sin 15° = 0{,}25881904510252076234889883762 4...$$

Do not forget to transform the degrees to the radian measure of angle before calculating!

Solution:

Problem 18: Calculate approximately $\cos 80°$. Compare your result with the exact value

$$\cos 80° = 0{,}17364817766693034885171662676 93...$$

Solution:

Problem 19: Calculate approximately $\sin 60°$. Compare your result with the exact value

$$\cos 60° = 0{,}86602540378443864676372317075294...$$

Solution:

Problem 20: Calculate approximately $\tan 15°$. Compare your result with the exact value

$$\tan 60° = 0{,}26794919243112270647255365849413...$$

Solution:

Problem 21: Calculate approximately $\cot 80°$. Compare your result with the exact value

$$\cot 80° = 0{,}17632698070846497347109038686862...$$

Solution:

2. $$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e,$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e,$$

$$(e = 2.71828\ldots).$$

If $\alpha(x)$ and $\beta(x)$ are infinitesimal functions as $x \to a$, then

$$\lim_{x\to a}(1+\alpha(x))^{\frac{1}{\beta(x)}} = e^{\lim\limits_{x\to a}\frac{\ln(1+\alpha(x))}{\beta(x)}}.$$

In Problems 22 through 32, apply the above formulas to evaluate the limit of the given function.

Problem 22: $\lim\limits_{x\to 0}\left(1+\dfrac{x}{2}\right)^{5x} =$

Problem 23: $\lim\limits_{x\to 0}\left(1-\dfrac{4x}{3}\right)^{2x} =$

Problem 24: $\lim\limits_{x\to 0}\left(1+\dfrac{x}{4}\right)^{-2x+3} =$

Problem 25: $\lim\limits_{x\to\infty}\left(\dfrac{x-2}{x}\right)^{3x} =$

Problem 26: $\lim\limits_{x\to\infty}\left(\frac{x+2}{x}\right)^{3x+5} =$

Problem 27: $\lim\limits_{x\to\infty}\left(\frac{x-4}{x+3}\right)^{x} =$

Problem 28: $\lim\limits_{x\to\infty}\left(\frac{5x-2}{5x+2}\right)^{7x} =$

Problem 29: $\lim\limits_{x\to\infty}\left(\frac{5x+2}{5x-2}\right)^{7x+4} =$

Problem 30: $\lim\limits_{x\to 2}(2x-3)^{\frac{3x}{x-2}} =$

Problem 31: $\lim\limits_{x\to 3}(x^2-8)^{\frac{x}{x-3}} =$

Problem 32: $\lim\limits_{x\to 0}(\sqrt{\cos 6x})^{\frac{-1}{x^2}} =$

3. $$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1,$$

$\ln(1+x)\sim x$ as $x\to 0$.

IN PROBLEMS 33 THROUGH 46, APPLY THE ABOVE FORMULAS TO EVALUATE THE LIMITS OF THE GIVEN FUNCTIONS.

Problem 33: $\lim\limits_{x\to 0}\dfrac{\ln(1+4x)}{6x}=$

Problem 34: $\lim\limits_{x\to 0}\dfrac{\ln\sqrt{1-7x}}{5x}=$

Problem 35: $\lim\limits_{x\to 0}\dfrac{\ln(1+2\sin 3x)}{4x}=$

Problem 36: $\lim\limits_{x\to 0}\dfrac{\ln(1+2\tan 3x)}{4x}=$

Problem 37: $\lim\limits_{x\to 0}\dfrac{\ln(1-3\arcsin 5x)}{8\tan 3x}=$

Problem 38: $\lim\limits_{x\to 4}\dfrac{\ln(x-3)}{x-4}=$

Problem 39: $\lim\limits_{x\to +\infty} x\cdot(\ln(x-4)-\ln x)=$

Problem 40: $\lim\limits_{x\to+\infty} x\cdot(\ln(x-4)-\ln x)=$

Problem 41: $\lim\limits_{x\to+\infty} x\cdot(\ln(x+3)-\ln(x+1))=$

Problem 42: $\lim\limits_{x\to+\infty} x\cdot(\ln\sqrt{x+5}-\ln\sqrt{x-2})=$

Problem 43: $\lim\limits_{x\to\infty}(x\cdot\ln\sqrt{\dfrac{x+5}{x-2}})=$

Problem 44: $\lim\limits_{x\to\infty}(x^2\cdot\ln\sqrt{\dfrac{3x^2+5}{3x^2-2}})=$

Problem 45: $\lim\limits_{x\to0}\dfrac{\ln(\cos 4x)}{2\arctan 6x}=$

Problem 46: $\lim\limits_{x\to0}\dfrac{\ln(2-\cos 4x+3x-x^2+4x\tan 2x)}{5x-4x^2+x^3\arcsin 3x}=$

4. $\lim_{x\to 0}\dfrac{e^x-1}{x}=1$,
$e^x-1\sim x$ as $x\to 0$.

In Problems 47 through 52, apply the above formulas to evaluate the limit of the given function.

Problem 47: $\lim_{x\to 0}\dfrac{e^{5x}-1}{3x}=$

Problem 48: $\lim_{x\to 0}\dfrac{e^{\sqrt{x}}-1}{2x-\sqrt{x}}=$

Problem 49: $\lim_{x\to 0}\dfrac{e^{\arcsin 4x}-1}{3x}=$

Problem 50: $\lim_{x\to 2}\dfrac{e^x-e^2}{x-2}=$

Problem 51: $\lim_{x\to 0}\dfrac{e^{3x}-\cos 6x}{x+x^2+2\sin x}=$

Problem 52: $\lim_{x\to 0}\dfrac{2\tan 5x-4\ln(1+7x)-4\arcsin x}{3x^2-6\arctan 3x+e^{4x}-1}=$

5. $$\lim_{x\to 0}\frac{(1+x)^n-1}{x}=n,$$

$$(1+x)^n-1\sim nx \quad \text{as} \quad x\to 0.$$

In Problems 53 through 58, apply the above formulas to evaluate the limits of the given functions.

Problem 53: $\lim\limits_{x\to 0}\dfrac{\sqrt{1+7x}-1}{2x}=$

Problem 54: $\lim\limits_{x\to 0}\dfrac{\sqrt{1-3\sin 4x}-1}{5x}=$

Problem 55: $\lim\limits_{x\to 0}\dfrac{\sqrt[4]{1+2\tan 6x}-1}{3\sin 2x}=$

Problem 56: $\lim\limits_{x\to 0}\dfrac{\sqrt[5]{1-3x}-1}{\sqrt{1+5x}-1}=$

Problem 57: $\lim\limits_{x\to 0}\dfrac{(1-4x)^{50}-1}{8x}=$

Problem 58: $\lim\limits_{x\to 0}\dfrac{\sqrt[4]{1+2x}+3x-\cos 6x}{\sin 5x-x^2}=$

CONTINUITY OF FUNCTIONS

A function $f(x)$ is **continuous** at a point *a*, if

$$\lim_{x\to a-0} f(x) = \lim_{x\to a+0} f(x) = f(a).$$

A point *a* is the **point of discontinuity of the first kind**, if the jump

$$\left|\lim_{x\to a-0} f(x) - \lim_{x\to a+0} f(x)\right|$$

is a finite non-zero number.

Otherwise, the point *a* is **point of discontinuity of the second kind**.

A point *a* is the **point of removable discontinuity**, if

$$\lim_{x\to a-0} f(x) = \lim_{x\to a+0} f(x) \neq f(a).$$

Properties of Continuous Functions

1. The sum of a finite number of continuous functions is a continuous function.
2. The product of a finite number of continuous functions is a continuous function.
3. The quotient of two continuous functions is a continuous function except for the points where the denominator is equal to zero.

All elementary functions are continuous in their domains.

In Problems 1 through 34 test the given functions for continuity.
If there exist points of discontinuity, determine their kind.
If some point is a point of removable discontinuity, redefine the function at that point by the supplementary condition to include that point into the domain of the function.

Problem 1: $f(x) = x^2 - 3x + 5$

Conclusion and Explanation:

Problem 2: $f(x) = 2\sin x + e^{3x}$

Conclusion and Explanation:

Problem 3: $f(x) = 5\arctan 3x - 4\sqrt{x^2+1}$

Conclusion and Explanation:

Problem 4: $f(x) = \dfrac{1}{x-2}$

Conclusion and Explanation:

Problem 5: $f(x) = \sin(\dfrac{1}{x-2})$

Conclusion and Explanation:

Problem 6: $f(x) = e^{\frac{1}{x-2}}$

Conclusion and Explanation:

Problem 7: $f(x) = \dfrac{1}{1+e^{3-x}}$

Conclusion and Explanation:

Problem 8: $f(x) = \dfrac{1}{e^x - e}$

Conclusion and Explanation:

Problem 9: $f(x) = \dfrac{1}{3 + 2^{\frac{1}{x-4}}}$

Conclusion and Explanation:

Problem 10: $f(x) = \dfrac{1}{3 + 2^{\frac{5x+1}{x-4}}}$

Conclusion and Explanation:

Problem 11: $f(x) = \dfrac{1}{4^{\frac{x}{x+3}} - 1}$

Conclusion and Explanation:

Problem 12: $f(x) = \dfrac{x}{4^{\frac{x}{x+3}} - 1}$

Conclusion and Explanation:

Problem 13: $f(x)=\begin{cases} 4x, & \text{if } x \le 2 \\ x^2, & \text{if } x > 2 \end{cases}$

Conclusion and Explanation:

Problem 14: $f(x)=\begin{cases} x+3, & \text{if } x < 0 \\ 3-x, & \text{if } 0 \le x \le 3 \\ x^2-8, & \text{if } x > 3 \end{cases}$

Conclusion and Explanation:

Problem 16: $f(x)=\begin{cases} \sin x, & \text{if } x < 0 \\ 2, & \text{if } 0 \le x < 4 \\ \sqrt{x}, & \text{if } x \ge 4 \end{cases}$

Conclusion and Explanation:

Problem 17: $f(x)=\begin{cases} 5^x, & \text{if } x < 0 \\ \tan x, & \text{if } 0 \le x \le \frac{\pi}{4} \\ \left|\frac{\pi-4}{4}-x\right|, & \text{if } x > \frac{\pi}{4} \end{cases}$

Conclusion and Explanation:

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