ФЕДЕРАЛЬНОЕ АГЕНТСТВО ПО ОБРАЗОВАНИЮ
Государственное образовательное учреждение высшего профессионального образования
«**ТОМСКИЙ ПОЛИТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ**»

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LIMITS OF SEQUENCES AND FUNCTIONS

TextBook

Рекомендовано в качестве учебного пособия
Редакционно-издательским советом
Томского политехнического университета

Издательство
Томского политехнического университета
2009

UDC 517
V.V. Konev. Limits of Sequences and Functions. Textbook. The second edition. Tomsk, TPU Press, 2009, 100 pp.

The textbook consists of two parts devoted to the mathematical concepts of Limits. Different formulations of limits help to understand better the unity of various approaches to this concept. The basic concepts are explained by examples and illustrated by figures.

The textbook is designed for English speaking students.

Reviewed by: V.A. Kilin, Professor of the Higher Mathematics Department, TPU, D.Sc.

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Preface

Every student of a technical university has to be well grounded in mathematics to study engineering science whose mathematical tools are based on Calculus.

The concept of the limit is essential for calculus. It is impossible to overestimate the importance of this concept for modern science. It was a very great advance on all former achievements of mathematics. Limits express the concepts of infinite small and infinite large quantities in mathematical terms. The comprehension of limits creates the necessary prerequisites for understanding other concepts in Differential Calculus and Integral Calculus such as derivatives, definite integrals, series, and solving different problems: calculation of the area of a figure, the length of an arc of a curve, and so on.

This textbook is intended for students studying methods of higher mathematics. It covers such content areas as Limits of Sequences, Basic Elementary Functions, and Limits of Functions.

Each part of the textbook contains basic mathematical conceptions. There are presented different formulations of limits to demonstrate the unity of various approaches to this concept. Intuitive arguments are combined with rigorous proofs of propositions.

Many useful examples and exercises are explained and illustrated graphically.

The book is useful for students specialized in different areas of expertise to broaden and methodize a knowledge of the basic mathematical methods. It can be also used by teachers in the classroom with a group of students.

I thank Professor Victor A. Kilin, who has made useful constructive suggestion about the text. His careful work in checking the text helped me to avoid many inaccuracies.

The author welcomes reader’s suggestions for improvement of future editions of this textbook.

CONTENTS

1. NUMERICAL SEQUENCES

1.1. Basic Definitions

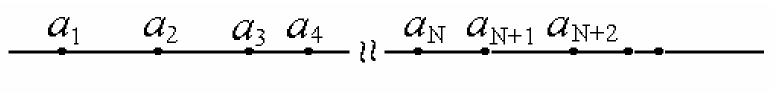
The mathematic concept of a sequence corresponds to our ordinary notion about a sequence of events in a sense of a certain order of events.

A **numerical sequence** is an infinite set of numbers enumerated by a positive integer index in ascending order of values of the index.

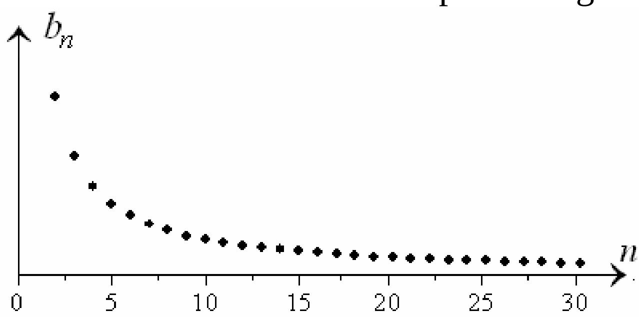
In other words, a sequence is a function $f(n)$ of a discrete variable *n*, whose domain consists of the set of all natural numbers.
The elements of a sequence are called the **terms**. The term $f(n)$ (that is, the *n*-th term) is called the **general term** or **variable** of the sequence.
The general term is denoted by a lower case letter with the subscript *n*:

$$a_n, \quad b_n, \quad x_n, \quad etc.$$

The general term put into braces denotes a sequence: $\{a_n\}$, $\{b_n\}$ $\{x_n\}$, *etc*.
Graphically, a sequence can be represented by points on the number line:



One can also use a two-dimensional chart for presenting a sequence:



A sequence is completely determined by its general term. If a sequence is given by a few first terms, then we need to find the general term.

Examples:

1. The general term $a_n = 2n$ determines the sequence of even numbers:
$$\{a_n\} = 2,\ 4,\ 6, \ldots,\ 2n, \ldots$$
2. The general term $b_n = q^{n-1}$ determines an infinite geometric progression with the common ratio *q*:
$$\{b_n\} = 1,\ q,\ q^2, \ldots,\ q^{n-1}, \ldots$$
3. $\{c_n\} = 2,\ 0,\ 2,\ 0,\ \ldots$ $\quad\Rightarrow\quad$ $c_n = 1 - (-1)^n$.
4. $\{x_n\} = 1,\ 2,\ 6,\ 24,\ 120,\ 720,\ \ldots,$ $\quad\Rightarrow\quad$ $x_n = n!$.

5. $\{y_n\} = 0,\ 1,\ 0,\ \frac{1}{2},\ 0,\ \frac{1}{3},\ 0,\ \frac{1}{4},\ 0,\ \ldots \quad \Rightarrow \quad y_n = \frac{1+(-1)^n}{n}$.

6. $\{z_n\} = 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0,\ \ldots \quad \Rightarrow \quad z_n = \sin\frac{\pi n}{2}$.

7. Let S_n be the sum of the first n elements of a sequence $\{a_n\}$:

$$S_n = \sum_{k=1}^{n} a_k .$$

Then the set $S_1, S_2, \ldots, S_n, \ldots$ is also a sequence $\{S_n\}$ which is called the **sequence of the partial sums** of the sequence $\{a_n\}$.

1.1.1. Bounded Sequences

A sequence $\{x_n\}$ is said to be an **upper-bounded sequence**, if there exists a finite number U such that

$$x_n \le U$$

for all natural numbers n. The number U is said to be an **upper bound** of $\{x_n\}$. Any nonempty upper-bounded sequence has the **least upper bound**.

A sequence $\{x_n\}$ is called a **lower-bounded sequence** if there exists a finite number L such that

$$x_n \ge L$$

for each natural number n. The number L is called a **lower bound** of $\{x_n\}$. Each nonempty lower-bounded sequence has the **greatest lower bound**.
A sequence is called **bounded**, if there exist two finite numbers, L and U, such that

$$L \le x_n \le U$$

for all terms of the sequence. Otherwise, the sequence is **unbounded**.

Examples:

1. The sequence

$$\{\frac{1}{n^2}\} = 1,\quad \frac{1}{4},\quad \frac{1}{9},\quad \frac{1}{16},\quad \frac{1}{25},\quad \ldots,\quad \frac{1}{n^2},\quad \ldots$$

is a bounded, since $0 < \frac{1}{n^2} \le 1$ for all natural numbers n.

2. The sequence

$$\{\frac{n^2}{n+1}\} = \frac{1}{2},\quad \frac{4}{3},\quad \frac{9}{4},\quad \frac{16}{5},\quad \ldots,\quad \frac{n^2}{n+1},\quad \ldots$$

is lower-bounded, since $1/2 \le n^2/(n+1)$ for all natural numbers n. However, the sequence has no a finite upper bound.

3. The sequence

$$\{(-1)^n 2^n\} = -2, \quad 4, \quad -8, \quad 16, \quad \dots$$

is unbounded, since it has no finite bounds.

1.1.2. Monotone Increasing Sequences

A sequence $\{x_n\}$ is called a **monotone increasing** sequence, if

$$x_{n+1} \ge x_n$$

for each natural number n.

A sequence $\{x_n\}$ is called a **monotone decreasing** sequence, if

$$x_{n+1} \le x_n$$

for each natural number n.

Examples of monotone increasing sequences:

$$\{\frac{n}{n+1}\} = \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{4}{5}, \quad \frac{5}{6}, \quad \dots, \quad \frac{n}{n+1}, \quad \dots$$

$$\{n!\} = 1, \; 2, \; 6, \; 24, \; \dots, \; n!, \quad \dots$$

Examples of monotone decreasing sequences:

$$\{\frac{1}{n}\} = 1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \dots, \quad \frac{1}{n}, \quad \dots$$

$$\{2^{-n}\} = \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \frac{1}{16}, \quad \frac{1}{32}, \quad \dots, \quad \frac{1}{2^n}, \quad \dots$$

Examples of non-monotone sequences:

$$\{\frac{(-1)^{n-1}}{n}\} = 1, -\frac{1}{2}, \quad \frac{1}{3}, -\frac{1}{4}, \quad \frac{1}{5}, \quad \dots$$

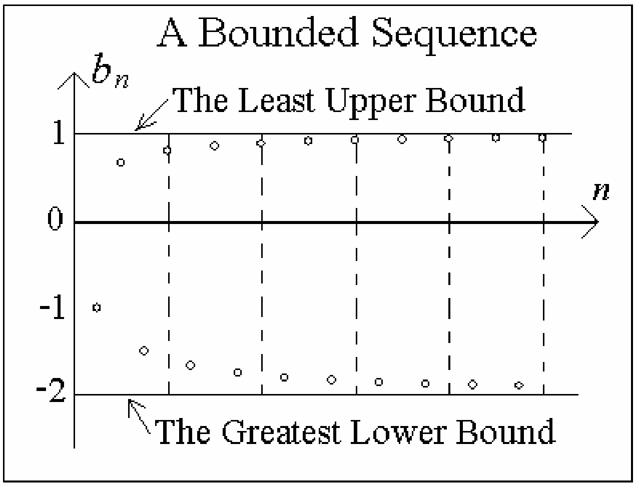
$$\{\sin\frac{\pi n}{2}\} = 1, \; 0, \; -1, \; 0, \; 1, \; 0, \; -1, \; 0, \; \dots$$

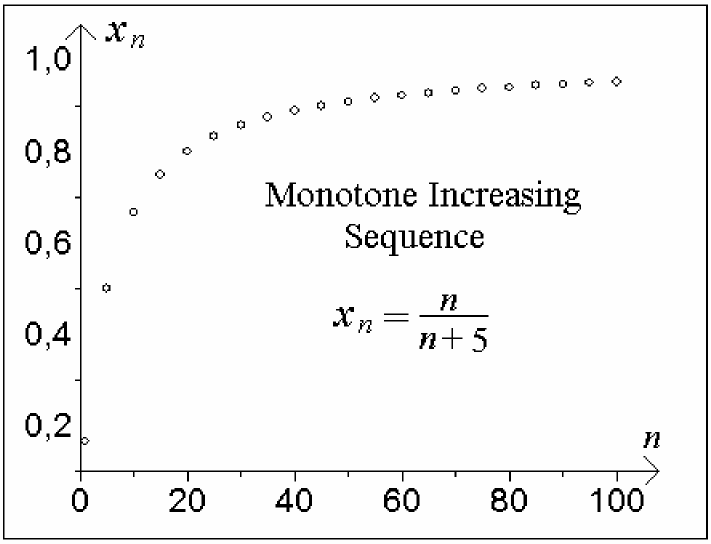
Sequences can also be classified on the basis of a behavior of its terms, if n takes on sufficiently large values. For instance, the variable 2^n increases without any bound with increasing n. In such a case, they say that the variable is infinite large, that is, it approaches infinity as n tends to infinity. Using a symbolical form we write: $2^n \to \infty$ as $n \to \infty$.

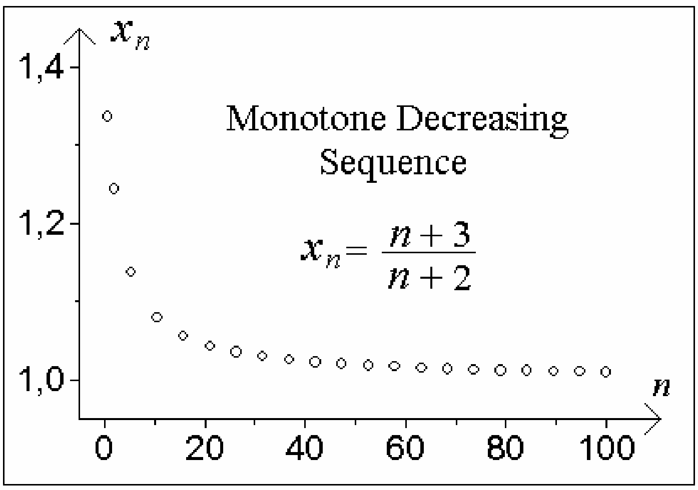
If a variable x_n approaches zero as n tends to infinity, then x_n is said to be an infinite small quantity. In this case we write: $x_n \to 0$ as $n \to \infty$.

At the same time, some variables do not tend to any well-defined number as n tends to infinity, e.g., the terms of the sequence $\{(-1)^n\}$ oscillate between two values, –1 and 1.

1.1.3. Illustrations







1.2. Limits of Numerical Sequences

Intuitive Definition of the Limit:

> The **limit of a sequence** $\{x_n\}$ is a number a such that the terms x_n remain arbitrarily close to a when n is sufficiently large.

This statement is written symbolically in any of the following form:

$$\lim_{n\to\infty} x_n = a\,,$$

$$\lim x_n = a\,,$$

$$x_n \to a \quad \text{as} \quad n \to \infty\,.$$

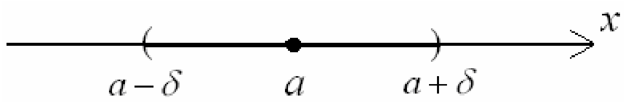
In a mathematical form, the statement "n is sufficiently large" means "starting from some number N"; the statement "the terms x_n remain arbitrarily close to a" means that the absolute value of the difference between x_n and a is getting smaller than any arbitrary small positive number δ.

Translating the above definition to the mathematical language, we obtain the following

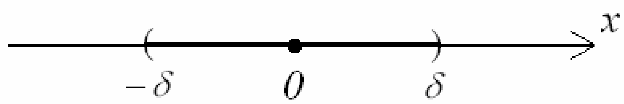
Formal Definition of the Limit:

> Number a is called the **limit of a sequence** $\{x_n\}$ if for any arbitrary small number $\delta > 0$ there exists a number N such that
> $$|x_n - a| < \delta$$
> for each $n > N$.

Geometrically, the inequality $|x_n - a| < \delta$ can be interpreted as the open interval $(a-\delta, a+\delta)$:



In Calculus, the interval $(a-\delta, a+\delta)$ is usually named "**delta neighborhood**" (or **"delta vicinity"**) of the point a. In particular, the delta neighborhood of zero is the open interval $(-\delta, \delta)$:

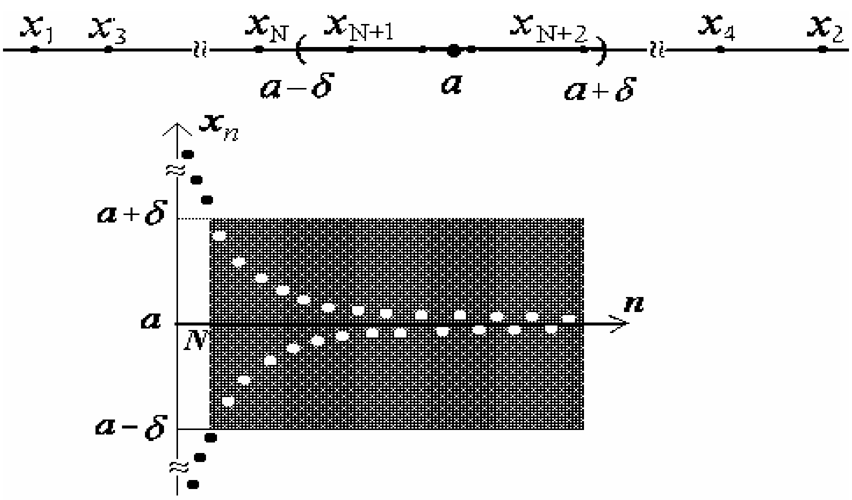


As a rule, we use the term "delta (or epsilon) neighborhood", keeping in mind that δ (or ε) is a small positive number.

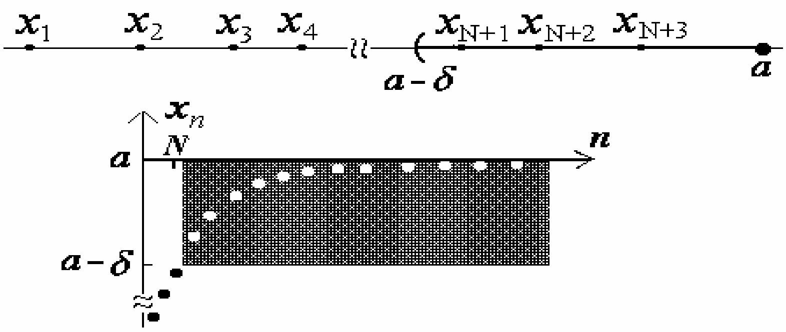
In terms of δ-neighborhood, the limit of a sequence can be defined by the following wording:

> Number a is the **limit of a sequence** $\{x_n\}$
> if any arbitrary small delta neighborhood of the point a
> contains all terms of the sequence, starting from a suitable term.

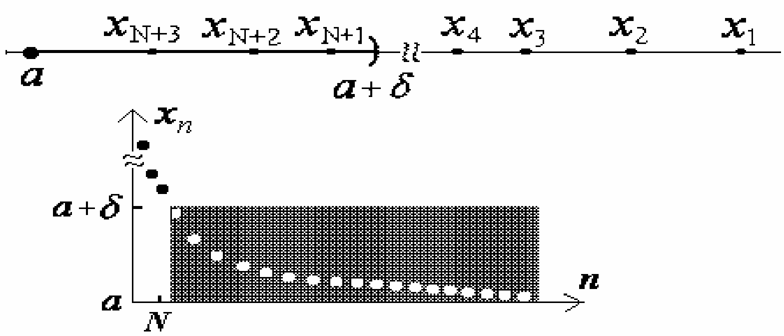
In the figures below, the definition of the limit is illustrated by the number line and using two-dimensional charts for some special cases.



The variable x_n tends to a in an arbitrary way.



The variable x_n tends to a being less than a.



The variable x_n tends to a being greater than a.

If a sequence has a limit a such that a is a finite number, they say that the sequence **converges** to the number a, and the sequence is called **convergent**. Otherwise, the sequence is called **divergent**.

Examples:

1) The sequence $\{(-1)^n\}$ is divergent, since it has no a limit as n tends to infinity.
2) The sequence $\{n^2\}$ is divergent, since it approaches infinity as number n tends to infinity.
3) Prove that the sequence $\{\frac{n}{4n+1}\}$ converges to the number $\frac{1}{4}$.

Intuitive Proof: If n is a sufficiently large number, then number 1 is much less than $4n$ and it can be neglected, that results in

$$\frac{n}{4n+1} \to \frac{1}{4} \quad \text{as} \quad n \to \infty .$$

To prove this statement **rigorously**, we have to show that for any arbitrary small number $\delta > 0$ there exists a number N such that the condition $n > N$ implies the inequality

$$\left| \frac{n}{4n+1} - \frac{1}{4} \right| < \delta .$$

Indeed,

$$\left| \frac{n}{4n+1} - \frac{1}{4} \right| < \delta \quad \Leftrightarrow \quad \left| \frac{4n-(4n+1)}{4(4n+1)} \right| < \delta \quad \Leftrightarrow$$

$$\frac{1}{4(4n+1)} < \delta \quad \Leftrightarrow \quad 16n + 4 > \frac{1}{\delta} \quad \Leftrightarrow$$

$$n > \frac{1}{16}\left(\frac{1}{\delta} - 4\right).$$

Setting $N \geq \frac{1}{16}(\frac{1}{\delta} - 4)$ we obtain that the inequality

$$n > N$$

implies

$$n > \frac{1}{16}\left(\frac{1}{\delta} - 4\right),$$

and hence,

$$\left| \frac{n}{4n+1} - \frac{1}{4} \right| < \delta ,$$

no matter how small positive value of δ is chosen.

1.2.1. Infinitesimal Sequences

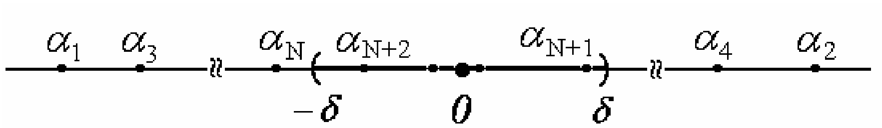
A sequence $\{\alpha_n\}$ is called **infinitesimal**, if it converges to zero:

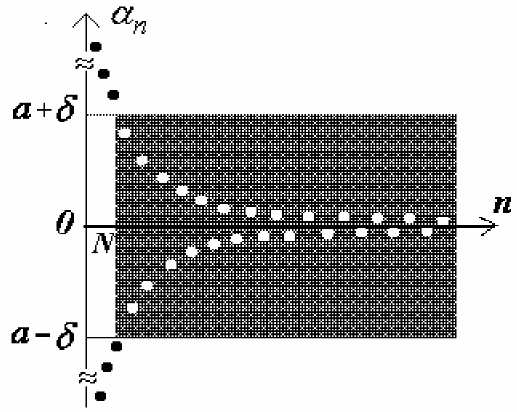
$$\lim \alpha_n = 0.$$

The **Formal Definition** is the following.

> A sequence $\{\alpha_n\}$ is called an **infinitesimal** sequence,
> if for any arbitrary small positive number δ
> there exists a number N such that
> the inequality $n > N$ implies $|\alpha_n| < \delta$.

On the number line, the points α_n of an infinitesimal sequence come arbitrarily close to the zero point as n increases to infinity. It means that the zero point is the accumulation point for any infinitesimal sequence. For $n > N(\delta)$, no matter how small positive δ is chosen, all points α_n remain in the delta neighborhood of zero.





In order to better understand the concept of infinitesimals, try to imagine something divided into millions bits. Then, divided again an obtained bit into millions bits. Repeating this procedure indefinitely many times, we approach to an infinitesimal bit.

Examples:

1. The sequence $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ is an infinitesimal sequence, since $1/n \to 0$ as $n \to \infty$.

Rigorous Proof: We need to show that for any $\delta > 0$ there exists a number N such that the inequality $n > N$ implies $1/n < \delta$.

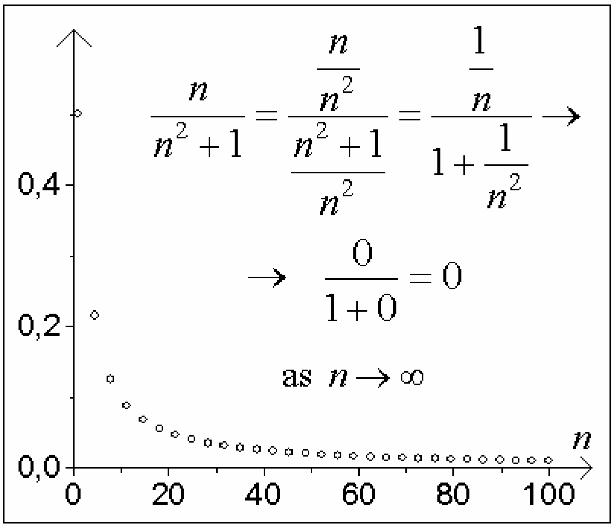
If we set $N \geq \dfrac{1}{\delta}$, then the two-sided inequality $n > N \geq \dfrac{1}{\delta}$ implies the desired inequality $1/n < \delta$ for any arbitrarily small $\delta > 0$.

In particular, setting $\delta = 0.01$ we obtain $\dfrac{1}{n} < 0.01$ for all $n > 100$.

Thus, the given delta neighborhood of the zero point contains all terms of the sequence $\{1/n\}$ except for the first hundred terms.

If $\delta = 0.001$ then $\dfrac{1}{n} < 0.001$ for all $n > 1000$, that is, all points (with $n > 1000$) lie in the delta neighborhood of the zero point.
It does not matter how many terms are beyond a delta vicinity of zero, if only a **finite** number of terms does not belong to that delta vicinity.

2. The variable $\dfrac{n}{n^2+1}$ is infinitesimal as $n \to \infty$.

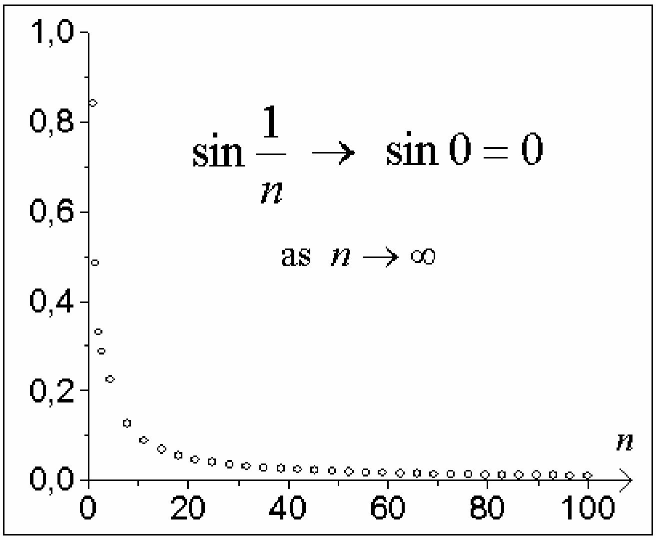


Proof: If we set $N \geq 1/\delta$, then inequality $n > N$ implies $n > 1/\delta$, and so $1/n < \delta$ for any arbitrary small $\delta > 0$.

However, $\dfrac{n}{n^2+1} < \dfrac{n}{n^2} = \dfrac{1}{n}$.

Therefore, $n > N \;\Rightarrow\; \dfrac{1}{n} < \delta \;\Rightarrow$ $\dfrac{n}{n^2+1} < \dfrac{1}{n} < \delta$. Hence, $\lim \dfrac{n}{n^2+1} = 0$.

3. The variable $\sin\dfrac{1}{n}$ is infinitesimal as $n \to \infty$.



Proof: Given any arbitrary small $\delta > 0$, $n > \dfrac{1}{\arcsin\delta} \;\Rightarrow$ $\dfrac{1}{n} < \arcsin\delta \;\Rightarrow\; \sin\dfrac{1}{n} < \delta$.

If we set $N \geq \dfrac{1}{\arcsin\delta}$, then inequality $n > N$ implies $\sin 1/n < \delta$.

The concept of infinitesimals gives a more convenient interpretation of the limit of a sequence.
By the definition, the statement $\lim x_n = a$ means that $|x_n - a| < \delta$ for $n > N$, and hence the difference $(x_n - a)$ is an infinitesimal variable. Therefore, we arrive at the following helpful rule:

> Number a is the **limit of a sequence** $\{x_n\}$,
> if the general term of the sequence can be expressed in the form
> $$x_n = a + \alpha_n,$$
> where α_n is the general term of an infinitesimal sequence.

Example: Evaluate the limit of the sequence $\{\frac{3n}{n+1}\}$ as $n \to \infty$.

Explanation: Number 1 (in the denominator) can be neglected as $n \to \infty$. Then $\frac{3n}{n+1} \to \frac{3n}{n} = 3$, which means

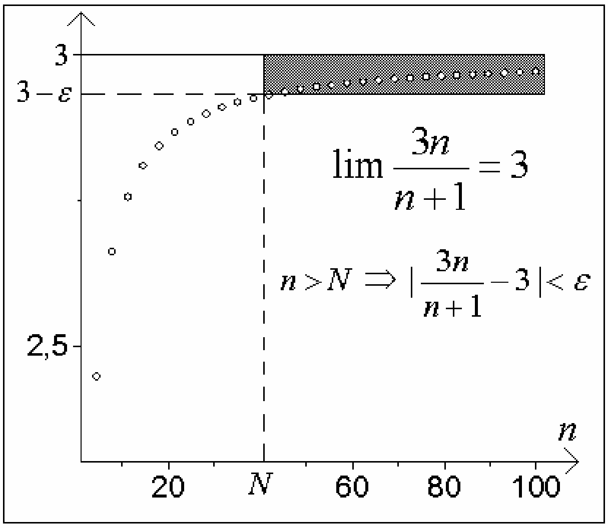
$$\lim \frac{3n}{n+1} = 3.$$

To give a **Formal Proof**, we should express the general term of the sequence as the sum of a constant term and an infinitesimal variable:

$$\frac{3n}{n+1} = \frac{3(n+1)-3}{n+1} = 3 - \frac{3}{n+1}.$$

The expression $(-\frac{3}{n+1})$ is an infinitesimal variable, since $\frac{3}{n+1} \to 0$ as $n \to \infty$. Therefore, the constant term 3 is the limit of the sequence,

$$\lim \frac{3n}{n+1} = 3.$$



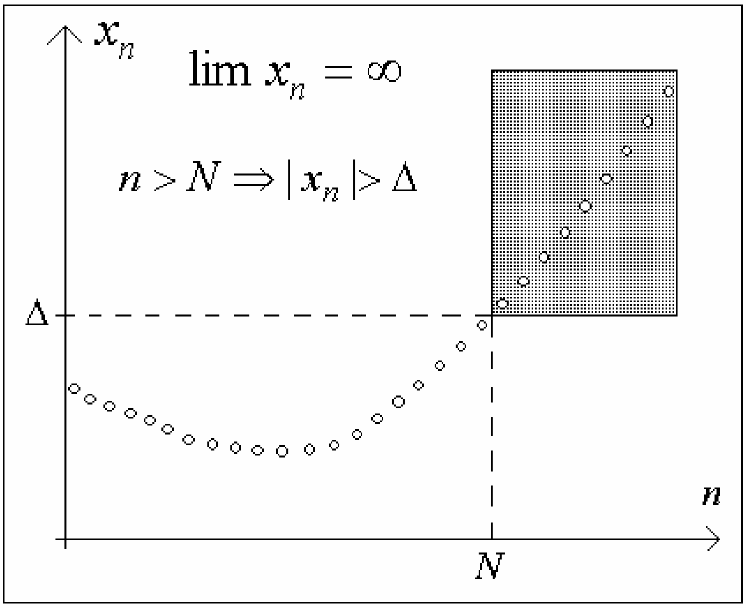
1.2.2. Infinite Large Sequences

A sequence $\{x_n\}$ is called **infinite large** (or **divergent**), if x_n approaches infinity as n tends to infinity.
The formal definition is the following:

A sequence $\{x_n\}$ is called an **infinite large** sequence,
if for any arbitrary large number $\Delta > 0$
there exists a number N such that
$$|x_n| > \Delta$$
for each $n > N$.

Notations:

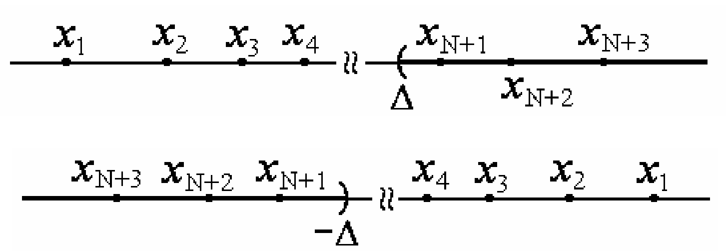
$$\lim_{n\to\infty} x_n = \infty,$$
$$\lim x_n = \infty,$$
$$\text{or} \quad x_n \to \infty \quad \text{as} \quad n \to \infty.$$



If the terms of an infinite large sequence $\{x_n\}$ are, respectively, positive or negative at least starting from a sufficiently large number N, we use the following notations:

$$\lim x_n = +\infty \quad \text{or} \quad \lim x_n = -\infty.$$

Note that a Δ- neighborhood of an infinite point includes either one of the semi-infinite intervals, (Δ, ∞) and $(-\infty, \Delta)$, or both. Terms x_n belong to a Δ- neighborhood of an infinite point, if their absolute values are greater than any arbitrary large number $\Delta > 0$.

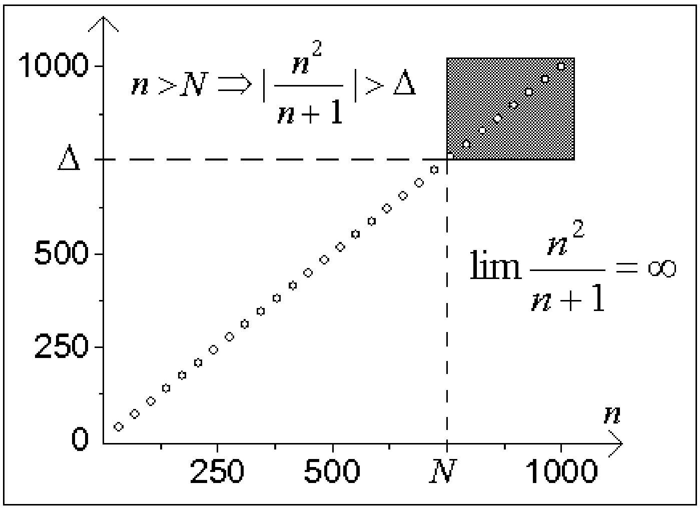


Example:

The sequence $\{\frac{n^2}{n+1}\}$ is an infinite large sequence since

$$\frac{n^2}{n+1} = \frac{n}{1+1/n} \to \frac{\infty}{1+0} = \infty$$

as $n \to \infty$.



Rigorous Proof: We need to show that for any arbitrary large number $\Delta > 0$ there exists a natural number N such that $\frac{n^2}{n+1} > \Delta$ whenever $n > N$.

Note that $\frac{n^2}{n+1} > \frac{n^2}{n+n} = \frac{n}{2}$.

If we set $N \geq 2\Delta$, then the two-sided inequality $n > N \geq 2\Delta$ implies $\frac{n}{2} > \Delta$, and hence

$$\frac{n^2}{n+1} > \frac{n}{2} > \Delta$$

for any arbitrary large number $\Delta > 0$.

1.3. Properties of Infinitesimal Sequences

Property 1

If $\{\alpha_n\}$ is an infinitesimal sequence and $\{b_n\}$ is a bounded sequence, then $\{\alpha_n b_n\}$ is an infinitesimal sequence.

Explanation: First, according to the definition, all terms of the bounded sequence $\{b_n\}$ are restricted by a finite number M, $|b_n| < M$.
Second, an infinitesimal variable α_n approaches zero as $n \to \infty$.
Hence,

$$|\alpha_n b_n| = |\alpha_n| \cdot |b_n| < |\alpha_n| \cdot M \to 0 \cdot M = 0 \quad \text{as} \quad n \to \infty .$$

Rigorous Proof: Since $\{\alpha_n\}$ is an infinitesimal sequence, then for any arbitrary small $\delta > 0$, the positive number δ/M corresponds to a suitable number N such that $|\alpha_n| < \dfrac{\delta}{M}$ for each $n > N$. Therefore,

$$|\alpha_n b_n| = |\alpha_n| \cdot |b_n| < \frac{\delta}{M} M = \delta$$

whenever $n > N$, which required to be proved.

Property 2

If $\{\alpha_n\}$ and $\{\beta_n\}$ are infinitesimal sequences, then $\{\alpha_n \beta_n\}$ is also an infinitesimal sequence.

Explanation: $|\alpha_n + \beta_n| \leq |\alpha_n| + |\beta_n| \to 0 + 0 = 0$ as $n \to \infty$.

Rigorous Proof: For any $\delta > 0$, the number $\delta/2$ corresponds to natural numbers N_1 and N_2 such that

$$|\alpha_n| < \frac{\delta}{2} \qquad \text{for each} \quad n > N_1 \quad \text{and}$$

$$|\beta_n| < \frac{\delta}{2} \qquad \text{for each} \quad n > N_2 .$$

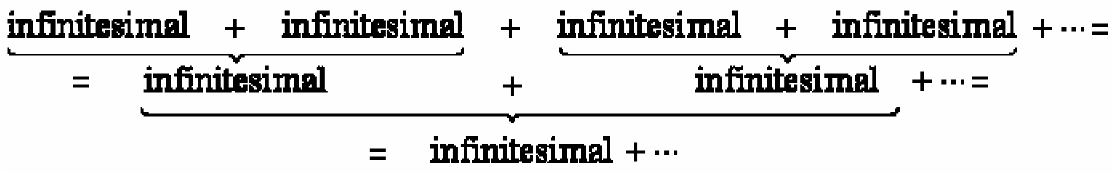
If a number N is not less than each of the numbers N_1 and N_2, then

$$|\alpha_n + \beta_n| \leq |\alpha_n| + |\beta_n| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

for any arbitrary small $\delta > 0$ whenever $n > N$.

Corollary: The sum of any finite number of infinitesimal variables is an infinitesimal variable.

Explanation: The idea of a proof is shown in the drawing below. The sum of any two infinitesimals can be represented by one infinitesimal. Then the sum of two obtained infinitesimal gives also an infinitesimal, *etc*.



Rigorous Proof: Let us apply the mathematical induction principle. Set S_n (with n = 2, 3, 4, …) be the statement "The sum of n infinitesimals is an infinitesimal".

Induction basis: By property 2, the statement S_n is true for $n = 2$.

Induction hypothesis: Assume that the statement S_n holds true for some integer $n \geq 2$.

Induction step: By the hypothesis, the sum of n infinitesimals is an infinitesimal, and so the sum of $(n+1)$ infinitesimals can be considered as the sum consisting of two infinitesimal items. However, the sum of two infinitesimals is an infinitesimal by Property 2. Therefore, the statement S_n implies the statement S_{n+1}, and hence S_n is true for any integer $n \geq 2$.

Property 3

If $\{\alpha_n\}$ is an infinitesimal sequence then $\{\frac{1}{\alpha_n}\}$ is an infinite large sequence and *vice versa*.

Explanation: To verify the proposition, divide number 1 by 1000, 1000000, 1000000000, and so on. Then divide number 1 by 0.001, 0.000001, 0.000000001, and so on. Compare the results.

Rigorous Proof: Let $\{\alpha_n\}$ be an infinitesimal sequence. Then for any $\Delta > 0$ there exists a number N such that $|\alpha_n| < \frac{1}{\Delta}$, which implies $|\frac{1}{\alpha_n}| > \Delta$ for each $n > N$. Hence, $\{\frac{1}{\alpha_n}\}$ is an infinite large sequence.

Likewise, if $\{\alpha_n\}$ is an infinite large sequence, then any $\delta > 0$ corresponds to a number N such that $|\alpha_n| > \frac{1}{\delta}$, and so $|\frac{1}{\alpha_n}| < \delta$ whenever $n > N$.

Hence, $\{\frac{1}{\alpha_n}\}$ is an infinitesimal sequence.

1.4. Properties of Limits of Sequences

Property 1

$$\lim(c \cdot x_n) = c \cdot \lim x_n$$

Proof: Let $\lim x_n = a$ that means

$$x_n = a + \alpha_n,$$

where α_n is an infinitesimal.
Then for any number *c*,

$$c \cdot x_n = c(a + \alpha_n) = ca + c\alpha_n.$$

Since $c\alpha_n$ is an infinitesimal,

$$\lim(c \cdot x_n) = ca,$$

which required to be proved.

Property 2

If there exist finite limits of sequences $\{x_n\}$ and $\{y_n\}$ then $\lim(x_n + y_n) = \lim x_n + \lim y_n$.

Proof: Let $\lim x_n = a$ and $\lim y_n = b$ that means

$$x_n = a + \alpha_n, \quad \text{and} \quad y_n = b + \beta_n,$$

where α_n and β_n are infinitesimals.
Then

$$x_n + y_n = (a + b) + (\alpha_n + \beta_n).$$

By the properties of infinitesimals, the sum $(\alpha_n + \beta_n)$ is an infinitesimal. Therefore,

$$\lim(x_n + y_n) = a + b.$$

Property 3

If there exist finite limits of sequences $\{x_n\}$ and $\{y_n\}$ then $\lim(x_n \cdot y_n) = \lim x_n \cdot \lim y_n$.

Proof: Likewise, the statements

$$\lim x_n = a \quad \text{and} \quad \lim y_n = b$$

imply

$$x_n \cdot y_n = (a + \alpha_n) \cdot (b + \beta_n) = ab + (b\alpha_n + a\beta_n + \alpha_n \beta_n).$$

In view of the properties of infinitesimals, the variable $(b\alpha_n + a\beta_n + \alpha_n \beta_n)$ is an infinitesimal. Therefore,

$$\lim(x_n \cdot y_n) = ab = \lim(x_n) \cdot \lim(y_n).$$

Property 4

If there exist finite limits of sequences $\{x_n\}$ and $\{y_n\}$, and $\lim y_n \neq 0$ then

$$\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n}.$$

Proof: Assume that $\lim x_n = a$ and $\lim y_n = b \neq 0$. To prove the property, we have to represent the quotient $\frac{x_n}{y_n}$ in the form

$$\frac{x_n}{y_n} = \frac{a}{b} + \text{Infinitesimal}.$$

Using simple transformations we obtain

$$\frac{x_n}{y_n} = \frac{a+\alpha_n}{b+\beta_n} = \frac{a}{b} + \left(\frac{a+\alpha_n}{b+\beta_n} - \frac{a}{b}\right)$$

$$= \frac{a}{b} + \frac{ab + b\alpha_n - ab - a\beta_n}{b(b+\beta_n)} = \frac{a}{b} + \frac{b\alpha_n - a\beta_n}{b(b+\beta_n)}.$$

By the properties of infinitesimals,

$$\frac{b\alpha_n - a\beta_n}{b(b+\beta_n)} \to \frac{0}{b^2} = 0 \quad \text{as } n \to \infty.$$

Therefore,

$$\lim \frac{x_n}{y_n} = \frac{a}{b} = \frac{\lim x_n}{\lim y_n}.$$

Property 5

If there exist finite limits of sequences $\{x_n\}$ and $\{x_n{}^p\}$ then

$$\lim x_n{}^p = (\lim x_n)^p$$

Explanation: Let $\lim x_n = a$.

Then

$$x_n = a + \alpha_n,$$

where α_n is an infinitesimal.

Therefore,

$$x_n{}^p = (a+\alpha_n)^p = a^p \cdot \left(1 + \frac{\alpha_n}{a}\right)^p \quad \to$$

$$a^p \cdot (1+0)^p = a^p = (\lim x_n)^p.$$

1.5. Classification of Infinitesimal Sequences

Let $\lim|\frac{\alpha_n}{\beta_n}|=\lambda$, where α_n and β_n are infinitesimal variables as $n\to\infty$.

If $0<\lambda<\infty$, then α_n and β_n are called infinitesimals of **the same order of smallness**.

In particular, if $\lim\frac{\alpha_n}{\beta_n}=1$ then α_n and β_n are called **equivalent** infinitesimals: $\alpha_n\sim\beta_n$.

In that case, they say that the infinitesimals **are equal asymptotically.**

If $\lambda=0$ then α_n is called an infinitesimal of a **higher order of smallness** with respect to β_n, while β_n is an infinitesimal of a **lower order of smallness** with respect to α_n.

If $\lambda=\infty$ then β_n is an infinitesimal of a **higher order of smallness** with respect to α_n, while α_n is an infinitesimal of a **lower order of smallness** with respect to β_n.

If $0<|\lim\frac{\alpha_n}{(\beta_n)^k}|<\infty$, then α_n is called an infinitesimal of **the *k*-th order of smallness** with respect to β_n.

Examples:

1. Infinitesimals $\frac{1}{n^2}$ and $\frac{1}{n^2+5n-2}$ are equal asymptotically as $n\to\infty$, since

$$\frac{1/n^2}{1/(n^2+5n-2)}=\frac{n^2+5n-2}{n^2}=1+\frac{5}{n}-\frac{2}{n^2}\quad\to\quad 1+0-0=1.$$

2. Infinitesimals $\frac{1}{n^2}$ and $\frac{1}{3n^2+n}$ have the same order as $n\to\infty$, since

$$\lim\frac{1/n^2}{1/(3n^2+n)}=\lim\frac{3n^2+n}{n^2}=\lim(3+\frac{1}{n})=3$$

that is a finite number.

3. Show that $\frac{1}{(n+1)^2}$ is an infinitesimal of the second order with respect to $\frac{1}{n}$ as $n\to\infty$.

1.6. Comparison Between Infinitesimal Sequences

1. Let α_n and β_n be two equivalent infinitesimals. Then

$$\beta_n = \alpha_n + \gamma_n,$$

where γ_n – is an infinitesimal of a higher order of smallness with respect to both α_n and β_n.

Proof: By the definition, if $\alpha_n \sim \beta_n$ then $\beta_n / \alpha_n \to 1$ as $n \to \infty$, and so

$$\frac{\beta_n - \alpha_n}{\alpha_n} = (\frac{\beta_n}{\alpha_n} - 1) \to (1-1) = 0.$$

Therefore, the difference $(\beta_n - \alpha_n)$ is an infinitesimal of a higher order of smallness with respect to the given infinitesimals.

2. If β_n is an infinitesimal of a higher order of smallness with respect to α_n, then

$$\alpha_n + \beta_n \sim \alpha_n.$$

It means that β_n is a **negligible quantity** with respect to α_n as $n \to \infty$.

Proof: By the hypothesis, $\beta_n / \alpha_n \to 0$ as $n \to \infty$. Then

$$\alpha_n + \beta_n = \frac{\alpha_n + \beta_n}{\alpha_n}\alpha_n = (1 + \frac{\beta_n}{\alpha_n})\alpha_n \sim (1+0)\cdot\alpha_n = \alpha_n.$$

3. Let α_n and β_n be two infinitesimals of the same order.

If $\lim \frac{\alpha_n}{\beta_n} = \lambda$ then α_n and $\lambda\beta_n$ are equivalent infinitesimals:

$$\alpha_n \sim \lambda\beta_n.$$

In that case, the infinitesimals are said to be **proportional asymptotically.**

Proof: $\frac{\alpha_n}{\lambda\beta_n} = \frac{1}{\lambda}\cdot\frac{\alpha_n}{\beta_n} \to \frac{1}{\lambda}\lambda = 1$ as $n \to \infty$.

Example: Since $\frac{1}{n}$ and $\frac{1}{n+1}$ are two equivalent infinitesimals, then their sum is an infinitesimal of the same order of smallness:

$$\frac{1}{n} + \frac{1}{n+1} \sim \frac{2}{n}.$$

However, their difference is an infinitesimal of the second order with respect to the given infinitesimals: $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$.

1.7. Classification of Infinite Large Sequences

Let $\lim|\frac{\alpha_n}{\beta_n}|=\lambda$, where α_n and β_n are infinite large variables as $n\to\infty$.

If $0<\lambda<\infty$ then α_n and β_n are called infinite large variables of **the same increasing order**.

In particular, if $\lim\frac{\alpha_n}{\beta_n}=1$ then α_n and β_n are called **equivalent** infinite large variables:

$$\alpha_n \sim \beta_n .$$

In that case, infinite large variables are said to be **equal asymptotically**.

If $\lambda=\infty$ then α_n is called an infinite large variable of a **higher order of increase** with respect to β_n, while β_n is an infinite large variable of a **lower increasing order** with respect to α_n.

If $\lambda=0$ then α_n is an infinite large variable of lower order with respect to β_n, while β_n is an infinite large variable of a higher order with respect to α_n.

If $0<|\lim\frac{\alpha_n}{(\beta_n)^k}|<\infty$ then α_n is called an infinite large variable of **the *k*-th order of increase** with respect to β_n.

Examples:

1. Two infinite large variables, n^2 and (n^2+4n+1), are equal asymptotically as $n\to\infty$, since
$$\frac{n^2+4n+1}{n^2}=1+\frac{4}{n}+\frac{1}{n^2} \quad\to\quad 1+0-0=1.$$
2. Both infinite large variables, n^2 and $(5n^2+n)$, have the same increasing order as $n\to\infty$, since
$$\lim\frac{5n^2+n}{n^2}=\lim(5+\frac{1}{n})=5$$
that is a finite number.
3. The variable (n^3+5n^2-1) is an infinite large variable of the third order with respect to n since
$$\lim\frac{4n^3+5n^2-1}{n^3}=\lim(4+\frac{5}{n}-\frac{1}{n^3})=4,$$
and 4 is a finite number.

1.8. Comparison between Infinite Large Sequences

1. Let α_n and β_n be two equivalent infinite large variables. Then

$$\beta_n = \alpha_n + \gamma_n,$$

where γ_n – is an infinitesimal of a lower increasing order with respect to both α_n and β_n.

Proof: By the definition, if $\alpha_n \sim \beta_n$ then $\beta_n / \alpha_n \to 1$ as $n \to \infty$, and so

$$\frac{\beta_n - \alpha_n}{\alpha_n} = (\frac{\beta_n}{\alpha_n} - 1) \to (1-1) = 0.$$

Therefore, the difference $(\beta_n - \alpha_n)$ is an infinite large variable of a lower order of increase with respect to the given variables.

Proof: $\alpha_n \sim \beta_n \quad \Rightarrow \quad \frac{\alpha_n - \beta_n}{\alpha_n} = 1 - \frac{\beta_n}{\alpha_n} \to 1 - 1 = 0.$

2. If β_n is an infinite large variable of a lower increasing order with respect to α_n, then

$$\alpha_n + \beta_n \sim \alpha_n.$$

It means that β_n is a **negligible quantity** with respect to α_n as $n \to \infty$.

Proof: By the hypothesis, $\frac{\beta_n}{\alpha_n} \to 0$ as $n \to \infty$. Then

$$\frac{\alpha_n + \beta_n}{\alpha_n} = 1 + \frac{\beta_n}{\alpha_n} \to 1 + 0 = 1 \quad \Rightarrow \quad \alpha_n + \beta_n \sim \alpha_n.$$

3. Let α_n and β_n be two infinite large variables of the same order.

If $\lim \frac{\alpha_n}{\beta_n} = \lambda$ then α_n and $\lambda \beta_n$ are equivalent infinite large variables:

$$\alpha_n \sim \lambda \beta_n.$$

In that case, the infinite large variables are said to be **proportional asymptotically.**

Proof: $\frac{\alpha_n}{\lambda \beta_n} = \frac{1}{\lambda} \cdot \frac{\alpha_n}{\beta_n} \to \frac{1}{\lambda} \lambda = 1$ as $n \to \infty$.

Examples:

1. In the expression $(n^3 + 5n^2 - 7n + 15)$, the quantity $(5n^2 - 7n + 15)$ is negligible with respect to n^3, since

$$\lim \frac{5n^2 - 7n + 15}{n^3} = \lim(\frac{5}{n} - \frac{7}{n^2} + \frac{15}{n^3}) = 0.$$

2. Two variables, $\sqrt{n^4+n^3}$ and n^2, are equivalent infinite large variables of the second order with respect to *n*.
Their sum is an infinite large variable of the same increasing order:
$$\sqrt{n^4+n^3}+n^2 \sim 2n^2.$$
However, the difference between the given variables is an infinite large variable of the first increasing order with respect to *n*:
$$\sqrt{n^4+n^3}-n^2=\frac{(\sqrt{n^4+n^3}-n^2)(\sqrt{n^4+n^3}+n^2)}{\sqrt{n^4+n^3}+n^2}$$
$$=\frac{n^4+n^3-n^4}{\sqrt{n^4+n^3}+n^2}=\frac{n^3}{\sqrt{n^4+n^3}+n^2}\sim\frac{n^3}{2n^2}=\frac{n}{2}.$$

3. To find the limit of the expression $\frac{n^5+2n^4+9n-3}{n^5+6n^3-4n^2+n+8}$, note that each of the variables, (n^5+2n^4+9n-3) and $(n^5+6n^3-4n^2+8)$, is equivalent to n^5 as $n\to\infty$. Therefore,
$$\lim\frac{n^5+2n^4+9n-3}{n^5+6n^3-4n^2+n+8}=\lim\frac{n^5}{n^5}=1.$$

4. Likewise, $(4n^3+20n^2+53)\sim 4n^3$ and $(2n^3-n^2+3n+15)\sim 2n^3$. Therefore,
$$\lim\frac{4n^3+20n^2+53}{2n^3-n^2+3n+15}=\lim\frac{4n^3}{2n^3}=2.$$

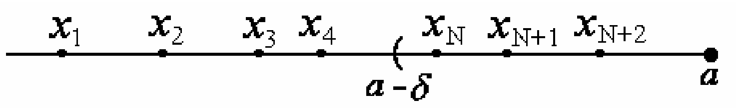
5. To find the limit of the expression $(\sqrt{n^2+5n-3}-n)$ having an indeterminate form $(\infty-\infty)$ as $n\to\infty$, multiply and divide the difference $(\sqrt{n^2+5n-3}-n)$ by the sum $(\sqrt{n^2+5n-3}+n)$ to get the difference between two squares:
$$\sqrt{n^2+5n-3}-n=\frac{(\sqrt{n^2+5n-3}-n)(\sqrt{n^2+5n-3}+n)}{\sqrt{n^2+5n-3}+n}$$
$$=\frac{(\sqrt{n^2+5n-3})^2-n^2}{\sqrt{n^2+5n-3}+n}=\frac{n^2+5n-3-n^2}{\sqrt{n^2+5n-3}+n}=\frac{5n-3}{\sqrt{n^2+5n-3}+n}.$$
Since $(5n-3)\sim 5n$ and $(\sqrt{n^2+5n-3}+n)\sim\sqrt{n^2}+n=2n$, we obtain
$$\lim(\sqrt{n^2+5n-3}-n)=\lim\frac{5n-3}{\sqrt{n^2+5n-3}+n}=\lim\frac{5n}{2n}=\frac{5}{2}.$$

1.9. Theorems of Sequences

Theorem 1

Each monotone increasing upper-bounded sequence has a finite limit.

The below drawing illustrates the theorem.



Proof: Let a be the least upper bound of the sequence $\{x_n\}$.
It means that

i) all the terms of $\{x_n\}$ satisfy the inequality $x_n \le a$;

ii) for any arbitrary small positive δ, the number $(a-\delta)$ is not an upper bound of the sequence.

Therefore, there exists a term x_N, which is greater than $(a-\delta)$:

$$a-\delta < x_N.$$

However, $\{x_n\}$ is a monotone increasing sequence, and so

$$x_N \le x_{N+1} \le x_{N+2} \le \ldots.$$

Thus, all the successors satisfy just the same inequalities, coming arbitrary close to the bound a:

$$a-\delta < x_n \le a$$

for each $n \ge N$. Hence, $\lim x_n = a$.

Theorem 2

Each monotone decreasing lower-bounded sequence has a finite limit.

The theorem can be proved in a similar way.
Proof: Let a be the greatest lower bound a sequence $\{x_n\}$. It means that

i) all the terms of $\{x_n\}$ satisfy the inequality $a \le x_n$;

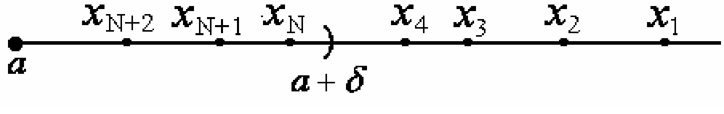
ii) for any arbitrary small positive δ, the number $(a+\delta)$ is not a lower bound of the sequence.

Then there exists a term x_N such that $a \le x_N < a+\delta$.

Since the sequence is monotone decreasing, all the successors remain between the bound a and term x_N:

$$a \le x_n \le x_N < a+\delta$$

for each $n \ge N$, as it is shown in the drawing below.



Hence, $\lim x_n = a$.

Theorem 3

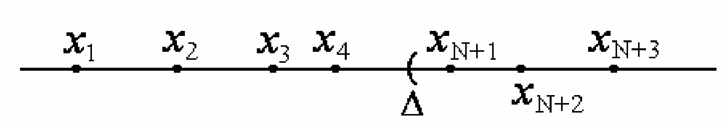
A monotone increasing sequence is divergent, if it has no an upper bound.

Proof: Let Δ be an arbitrary large number. Since Δ is not an upper bound of the sequence $\{x_n\}$, there exists a term x_N, which is greater than Δ. However, $\{x_n\}$ is a monotone increasing sequence, and so each successor is also greater than Δ.

Thus, for any arbitrary large number Δ there exists the corresponding number N such that

$$x_n > \Delta$$

whenever $n > N$.



Hence, the theorem.

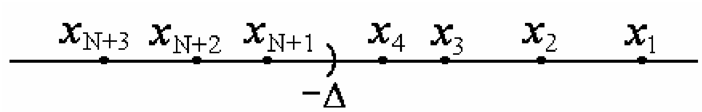
Theorem 4

A monotone decreasing sequence is divergent, if it has no a lower bound.

Proof: By the arguments used in the proof of Theorem 3, we conclude that for any positive number Δ there exists the number N such that

$$x_n < -\Delta,$$

and so $|x_n| > \Delta$ whenever $n > N$.



Hence, the sequence diverges.

1.10. Number *e*

Theorem:

The sequence $\{(1+\frac{1}{n})^n\}$

– is a monotone increasing bounded sequence;

– has the finite limit such that $2 < \lim\limits_{n\to\infty}(1+\frac{1}{n})^n < 3$.

That limit is denoted by the symbol e:

$$e \equiv \lim_{n\to\infty}(1+\frac{1}{n})^n.$$

The number e is an irrational number, $e = 2.71828\ldots$

Proof: First, we prove that $\{a_n\} = \{(1+\frac{1}{n})^n\}$ is a monotone increasing sequence.

By the Binomial Theorem (see [1-3], for example):

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + \frac{n(n-1)(n-2)\cdots 1}{n!}y^n.$$

Setting $x = 1$ and $y = \frac{1}{n}$ and making simple transformations we obtain:

$$a_n = (1+\frac{1}{n})^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \ldots + \frac{n!}{n!}\frac{1}{n^n}$$
$$= 2 + \frac{(n-1)}{n}\frac{1}{2!} + \frac{(n-1)(n-2)}{n^2}\frac{1}{3!} + \ldots + \frac{(n-1)(n-2)\cdots 1}{n^{n-1}}\frac{1}{n!}$$
$$= 2 + (1-\frac{1}{n})\frac{1}{2!} + (1-\frac{1}{n})(1-\frac{2}{n})\frac{1}{3!} + \ldots + (1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})\frac{1}{n!}.$$

Substituting $(n+1)$ for n we get a similar expression for the next term a_{n+1}:

$$a_{n+1} = 2 + (1-\frac{1}{n+1})\frac{1}{2!} + (1-\frac{1}{n+1})(1-\frac{2}{n+1})\frac{1}{3!} + \ldots$$
$$+ (1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{n-1}{n+1})\frac{1}{n!}$$
$$+ (1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{n-1}{n+1})(1-\frac{n}{n+1})\frac{1}{(n+1)!}.$$

Now let us compare the expressions for a_n and a_{n+1} term by term.

First of all, note that in both sums (in the expressions for a_n and a_{n+1}) all the terms are positive, and the number of the terms increases with increasing n.

Starting from the second term, each term of the sum a_{n+1} is greater than the corresponding term of the sum for a_n:

$$(1-\frac{1}{n}) < (1-\frac{1}{n+1}), \quad (1-\frac{1}{n})(1-\frac{2}{n}) < (1-\frac{1}{n+1})(1-\frac{2}{n+1}), \quad \text{and so on.}$$

Therefore, inequality

$$a_{n+1} > a_n$$

proves that $\{a_n\} = \{(1+\frac{1}{n})^n\}$ is a monotone increasing sequence.

Now let us prove that $\{a_n\} = \{(1+\frac{1}{n})^n\}$ is a bounded sequence.

The first term of a monotone increasing sequence is the greatest lower bound of the sequence. Since $a_1 = (1+\frac{1}{1})^1 = 2$, we get the inequality

$$2 \le (1+\frac{1}{n})^n$$

which is valid for all natural numbers *n* that proves the existence of a lower bound.

The existence of an upper bound can be proved by the following simple estimations.

First,

$$a_n = 2 + (1-\frac{1}{n})\frac{1}{2!} + (1-\frac{1}{n})(1-\frac{2}{n})\frac{1}{3!} + \ldots + (1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})\frac{1}{n!}$$

$$< 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$

since $1-\frac{k}{n} < 1$ for all natural numbers *k* and *n*.

Second, $\frac{1}{k!} < \frac{1}{2^k}$ for all integer $k > 2$, and so

$$a_n < 2 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}.$$

The expression on the right side of this inequality includes the sum of *n* terms of the geometric progression with the common ratio 2 that can be easily calculated [2]:

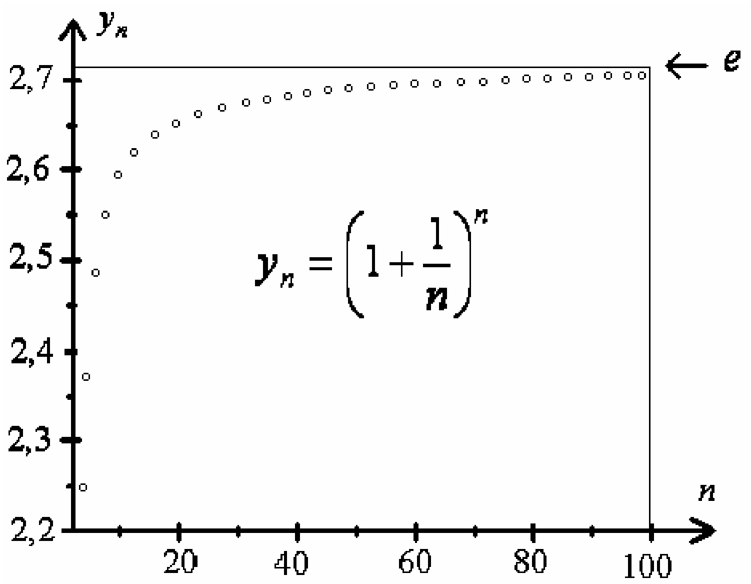
$$\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} = \frac{1}{2}\frac{(1-\frac{1}{2^n})}{1-\frac{1}{2}} = 1 - \frac{1}{2^n} < 1.$$

Thus, $\{a_n\}$ is an upper-bounded sequence, $a_n = (1+\frac{1}{n})^n < 3$ for each natural number *n*.

The sequence $\{(1+\frac{1}{n})^n\}$ satisfies the conditions of Theorem 1, and so it has a finite limit denoted by the symbol *e*,

$$\lim(1+\frac{1}{n})^n = e, \qquad 2 < e < 3.$$

Graphic Illustration



Numerical illustration

n	$(1+\frac{1}{n})^n$
1	2
2	2.25
5	2.48832
10	2.593742460
20	2.653297705
50	2.691588029
100	2.704813829
1000	2.716923932
10 000	2.718145927
100 000	2.718268237
1000 000	2.718280469
…	…
e = 2.71828182845904523536028747135266249775724709…	

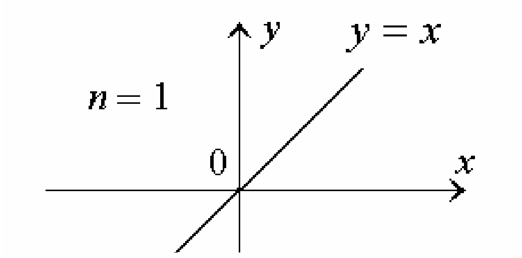
2. FUNCTIONS

2.1. Elementary Functions: Short Review

2.1.1. Power Functions $y = x^n$

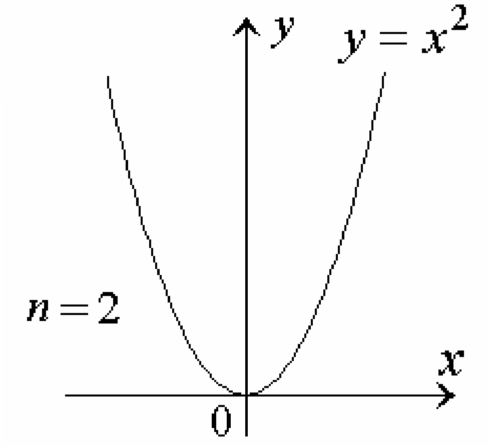
The domain of the power function $y = x^n$ is the set of all real numbers except for $x < 0$, if $n = \frac{1}{2k}$ ($k = 1, 2, \ldots$), and $x \neq 0$, if $n < 0$.

The range of the power function $y = x^n$ depends on the index of the power. If n is an even number then the range contains only non-negative real numbers. For odd numbers n, the range is the set of all real numbers.



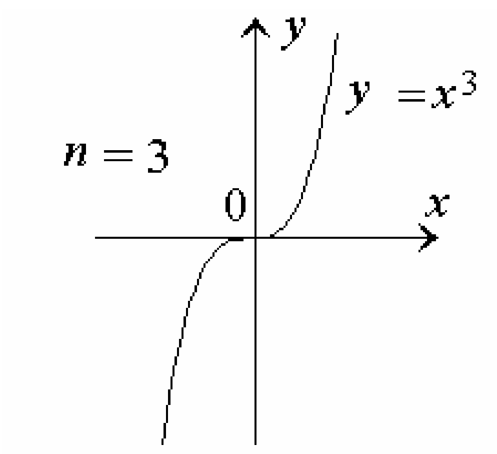
If $n = 1$ then $y = x$ is a linear function, whose graph is a straight line passing through the origin.
Domain: The set of all real numbers.
Range: The set of all real numbers.
Symmetry: An odd function,

$$y(-x) = -y(x).$$



If $n = 2$ then $y = x^2$ is a quadratic function, whose graph is a parabola with the vertex at the origin.
Domain: The set of all real numbers.
Range: The set of all non-negative real numbers.
Symmetry: An even function,

$$y(-x) = y(x).$$

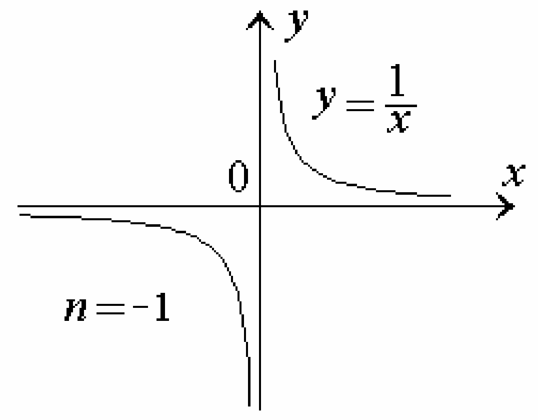


If $n = 3$ then $y = x^3$ is a cubic function, whose graph passes through the origin.

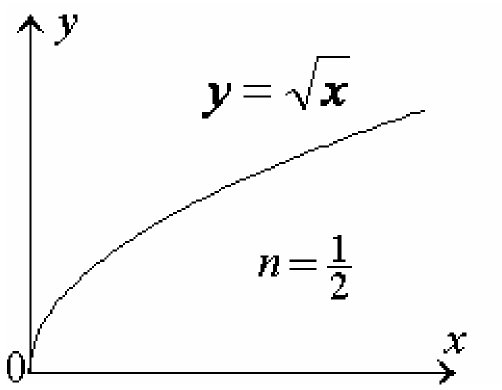
Domain: The set of all real numbers.

Range: The set of all real numbers.

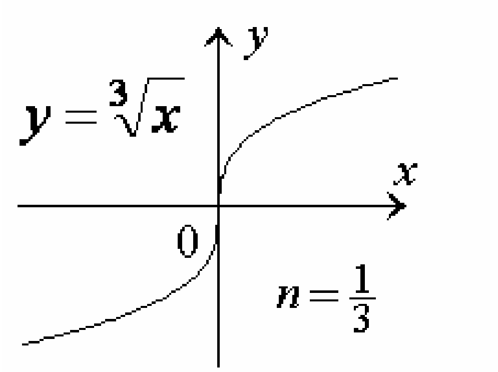
Symmetry: An odd function.



If $n = -1$ then the equation $y = \frac{1}{x}$ describes the hyperbola.
Domain: The set of all positive and negative real numbers.
Range: The set of all positive and negative real numbers.
Symmetry: An odd function.



If $n = \frac{1}{2}$ then $y = \sqrt{x}$ is the inverse function of $y = x^2$ provided that $x \geq 0$.
Domain: The set of all non-negative real numbers.
Range: The set of all non-negative real numbers.

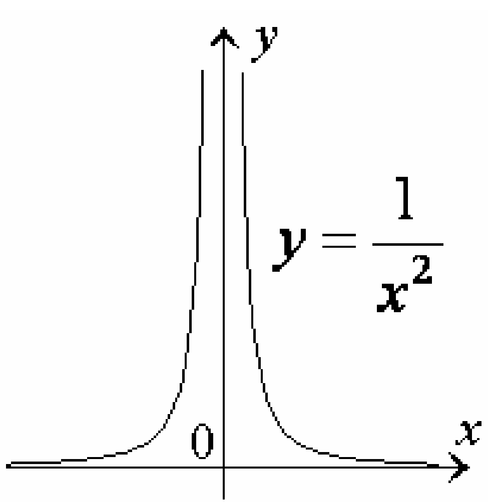


If $n = \frac{1}{3}$ then $y = \sqrt[3]{x}$ is the inverse function of $y = x^3$.

Domain: The set of all real numbers.

Range: The set of all real numbers

Symmetry: An odd function.



If $n = -2$ then $y = \frac{1}{x^2}$.
Domain: The set of all real numbers.
Range: The set of all positive real numbers
Symmetry: An even function.

2.1.2. Exponential Functions $y = a^x$

Requirements: $a > 0$ and $a \neq 1$.
Domain: The set of all real numbers.
Range: The set of all positive real numbers.
Properties:
If $a > 1$ then

– $y = a^x$ is a monotone increasing function, that is,

$$x_2 > x_1 \quad \Leftrightarrow \quad a^{x_2} > a^{x_1};$$

– graphs of the function $y = a^x$ tends to the *x*-axis asymptotically as $x \to -\infty$, and tends to infinity as $x \to +\infty$.

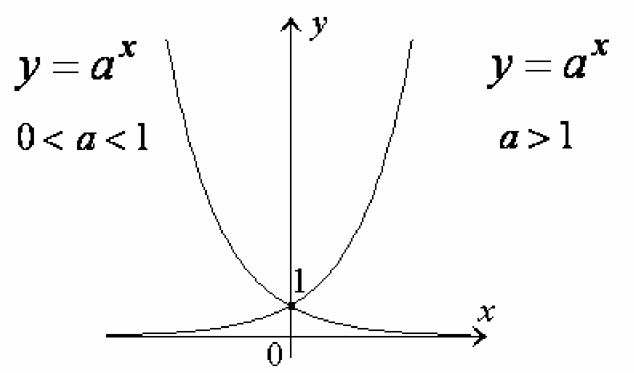
If $0 < a < 1$ then

– $y = a^x$ is a monotone decreasing function, that is,

$$x_2 > x_1 \quad \Leftrightarrow \quad a^{x_2} < a^{x_1};$$

– graphs of the function $y = a^x$ tends to the *x*-axis asymptotically as $x \to +\infty$, and tends to infinity as $x \to -\infty$.

Graphs:



Basic Formulas:

$a^0 = 1$
$a^{-x} = \dfrac{1}{a^x}$
$a^{x_1} \cdot a^{x_2} = a^{x_1 + x_2}$
$\dfrac{a^{x_1}}{a^{x_2}} = a^{x_1 - x_2}$
$(a^x)^n = a^{nx}$

The reader can find more detail discussion of the properties of elementary functions, for example, in [1-3].

2.1.3. Logarithmic Functions $y = \log_a x$

Note: $a > 0$ and $a \neq 1$.
Domain: The set of all positive numbers.
Range: The set of all real numbers.

The logarithmic function is defined as the inverse of the exponential function:

$$y = \log_a x \quad \Leftrightarrow \quad x = a^y .$$

Properties:
If $a > 1$ then

- $y = \log_a x$ is a monotone increasing function, that is,

$$x_2 > x_1 \quad \Leftrightarrow \quad \log_a x_2 > \log_a x_1 ;$$

- $\log_a x \to -\infty$ and graphs of the function $y = \log_a x$ tends to the y-axis asymptotically as $x \to +0$;
- $\log_a x \to +\infty$ as $x \to +\infty$.

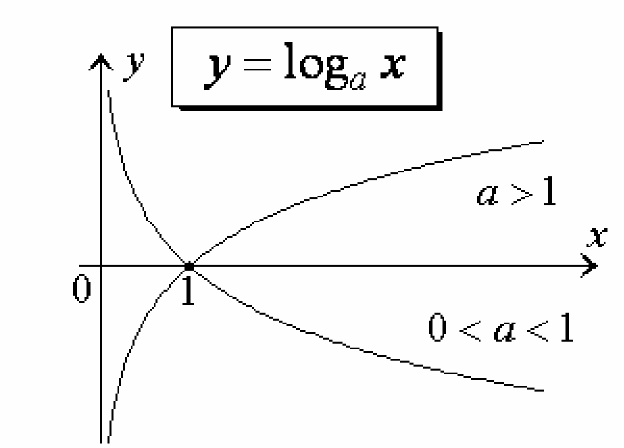
If $0 < a < 1$ then

- $\log_a x$ is a monotone decreasing function, that is,

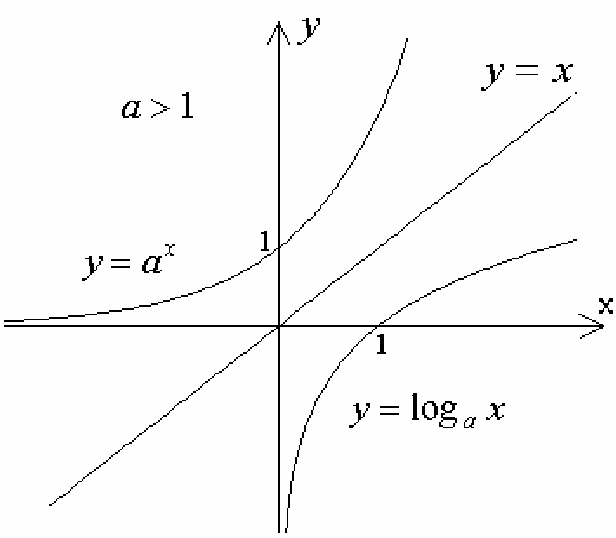
$$x_2 > x_1 \quad \Leftrightarrow \quad \log_a x_2 < \log_a x_1 ;$$

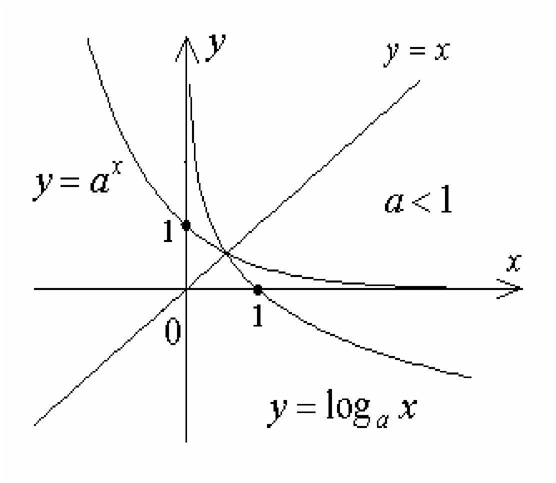
- $\log_a x \to +\infty$ and graphs of the function $y = \log_a x$ tends to the y-axis asymptotically as $x \to +0$;
- $\log_a x \to -\infty$ as $x \to +\infty$.

Graphs:



Since $y = a^x$ and $y = \log_a x$ are inverse functions of each other, their graphs look as the mirror images of each other across the bisector of the first and third quadrants, see the figures below.





Basic Formulas:

$\log_a 1 = 0$
$\log_a a = 1$
$\log_a x_1 + \log_a x_2 = \log_a (x_1 \cdot x_2)$
$\log_a x_1 - \log_a x_2 = \dfrac{\log_a x_1}{\log_a x_2}$
$n \log_a x = \log_a x^n$
$\log_a x = \dfrac{\log_c x}{\log_c a}$

The function $\log_{10} x$ is referred to as $\log x$ ($\lg x$ in Russian books).
The function $\log_e x$ is denoted by $\ln x$ and it is called the **natural logarithm**.

2.1.4. Trigonometric Functions

Sine Function $y = \sin x$
Cosine Function $y = \cos x$

The reader can find more detail discussion of the properties of trigonometric functions, for example, in [1-3].

Domains: The set of all real numbers.

Ranges: $|\sin x| \le 1$, $|\cos x| \le 1$.

Properties:

- $\sin x$ and $\cos x$ are periodic functions with period 2π:

$$\sin(x + 2\pi) = \sin x, \quad \cos(x + 2\pi) = \cos x;$$

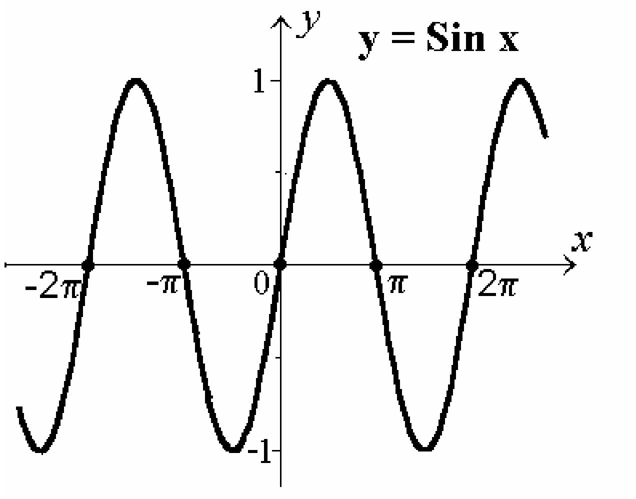
- $\sin x$ is an odd function:

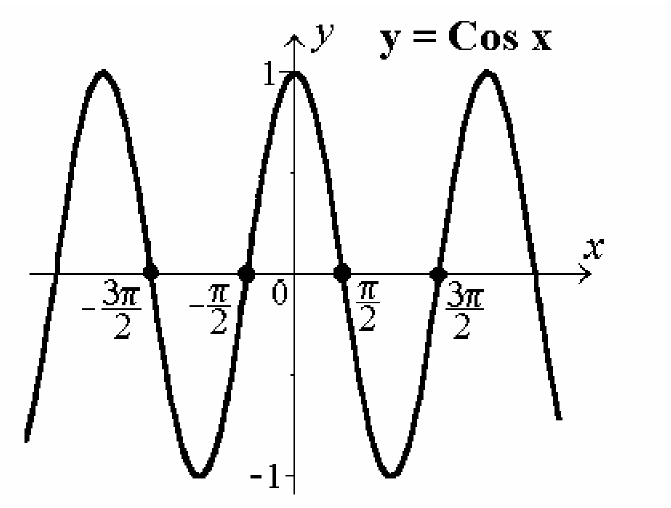
$$\sin(-x) = -\sin x;$$

- $\cos x$ is an even function:

$$\cos(-x) = \cos x.$$

Graphs:





Basic Formulas:

Addition Formulas for Sines and Cosines

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double-Angle Formulas for Sines and Cosines

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

Half-Angle Formulas for Sines and Cosines

$$2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha$$

$$2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha$$

Relationships between Sines and Cosines

$$\cos^2\alpha + \sin^2\alpha = 1$$

$$\sin\alpha = \cos(\alpha - \frac{\pi}{2}) = \cos(\frac{\pi}{2} - \alpha)$$

$$\cos\alpha = \sin(\alpha + \frac{\pi}{2}) = \sin(\frac{\pi}{2} - \alpha)$$

Other Formulas

$$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha \pm \beta}{2}\cos\frac{\alpha \mp \beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha + \beta}{2}\sin\frac{\alpha - \beta}{2}$$

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha - \beta) + \sin(\alpha + \beta))$$

Tangent Function $y = \tan x$

Domain: The set of all real numbers except for $x = \frac{\pi}{2} + \pi n$, where n is any integer.

Range: The set of all real numbers.

Properties:

– $\tan x$ is a periodic function with period π:

$$\tan(x + \pi) = \tan x;$$

– $\tan x$ is an odd function:

$$\tan(-x) = -\tan x.$$

Cotangent Function
$y = \cot x$

Domain: The set of real numbers except for $x = \pi n$, where n is any integer.

Range: The set of all real numbers.

Properties:

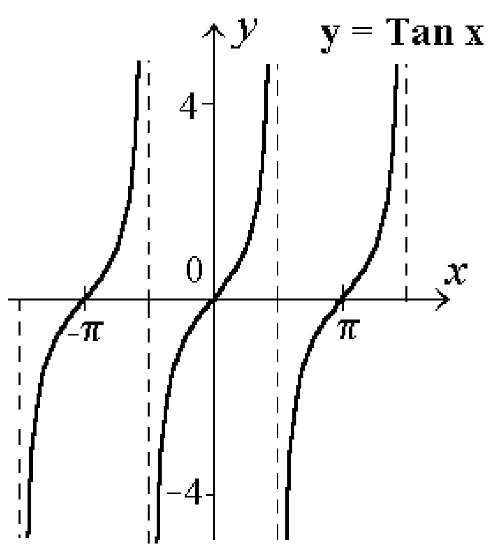
– $\cot x$ is a periodic function with period π:

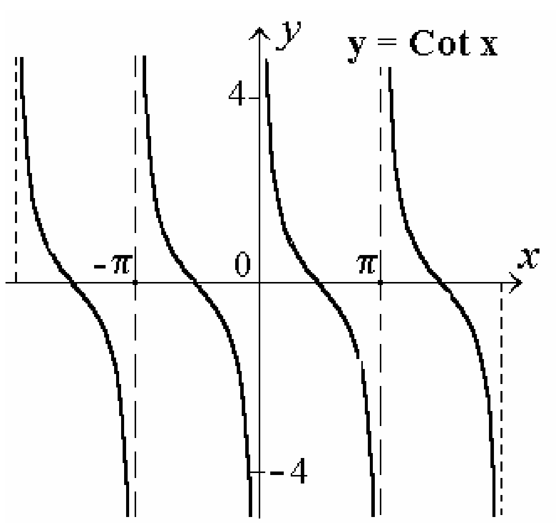
$$\cot(x + \pi) = \cot x;$$

– $\cot x$ is an odd function:

$$\cot(-x) = -\cot x.$$

Graphs:





Relationships between Trigonometric Functions

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\tan x = \frac{1}{\cot x}$$

$$1 + \tan^2 x = \frac{1}{\cos^2 x}$$

$$1 + \cot^2 x = \frac{1}{\sin^2 x}$$

$$\tan x = \cot(\frac{\pi}{2} - x)$$

$$\cot x = \tan(\frac{\pi}{2} - x)$$

Addition Formulas for Tangents and Cotangents

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}$$

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha \cot\beta \mp 1}{\cot\alpha \pm co\,t\beta}$$

Double-Angle Formulas for Tangents and Cotangents

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$$\cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha}$$

Half-Angle Formulas for Tangents and Cotangents

$$\tan\frac{\alpha}{2} = \frac{1 - \cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1 + \cos\alpha}$$

$$\cot\frac{\alpha}{2} = \frac{1 + \cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1 - \cos\alpha}$$

Other Formulas

$$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha \pm \beta)}{\cos\alpha \cos\beta}$$

$$\cot\alpha \pm \cot\beta = \frac{\sin(\beta \pm \alpha)}{\sin\alpha \sin\beta}$$

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

Values of Trigonometric Functions for Special Angles:

Angle x		sin x	cos x	tan x	cot x
Degrees	Radians				
0	0	0	1	0	undefined
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
90	$\frac{\pi}{2}$	1	0	undefined	0

2.1.5. Inverse Trigonometric Functions

Inverse Sine Function is referred as

$$y = \arcsin x \quad \text{or as} \quad y = \sin^{-1} x .$$

Domain: $-1 \le x \le 1$. **Range**: $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$.

Properties:

– $\arcsin x$ is a monotone increasing function;
– $\sin^{-1}(\sin x) = x$;
– $\sin(\sin^{-1} x) = x$.

The solution set of the equation $\sin x = a$:

$$x = (-1)^n \arcsin a + \pi n , \quad (n = 0, \pm 1, \pm 2, \ldots) .$$

Inverse Cosine Function is referred as

$$y = \arccos x \quad \text{or as} \quad y = \cos^{-1} x .$$

Domain: $-1 \le x \le 1$. **Range**: $0 \le \arccos x \le \pi$.

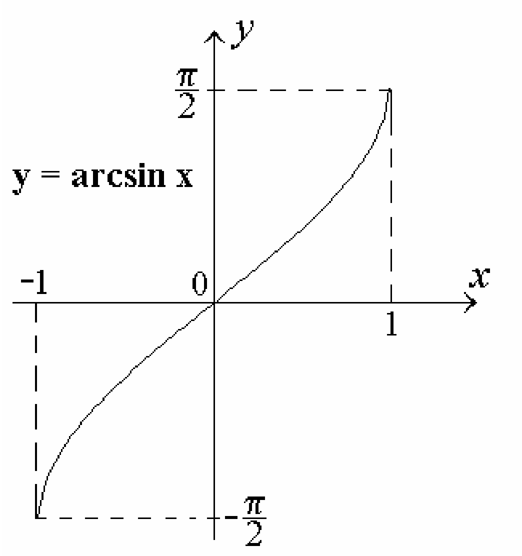
Properties:

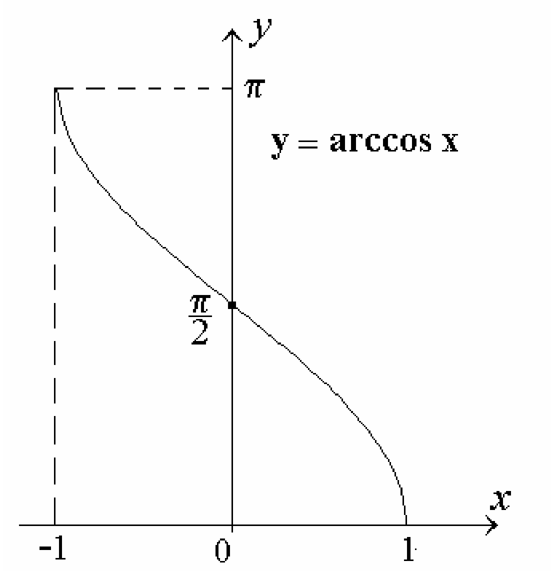
– $\arccos x$ is a monotone decreasing function;
– $\cos^{-1}(\cos x) = x$;
– $\cos(\cos^{-1} x) = x$.

The solution set of the equation $\cos x = a$:

$$x = \pm \arccos a + 2\pi n , \quad (n = 0, \pm 1, \pm 2, \ldots) .$$

Graphs:





Inverse Tangent Function $y = \arctan x$ (or $y = \tan^{-1} x$).

Domain: The set of all real numbers. **Range**: $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$.

Properties:
- $\arctan x$ is a monotone increasing function;
- $\tan^{-1}(\tan x) = x$;
- $\tan(\tan^{-1} x) = x$.

The solution set of the equation $\tan x = a$:

$$x = \arctan a + \pi n, \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Inverse Cotangent Function $y = \cot^{-1} x$.

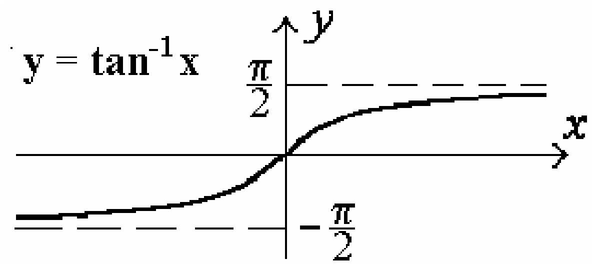
Domain: The set of all real numbers. **Range**: $0 < \cot^{-1} x < \pi$.

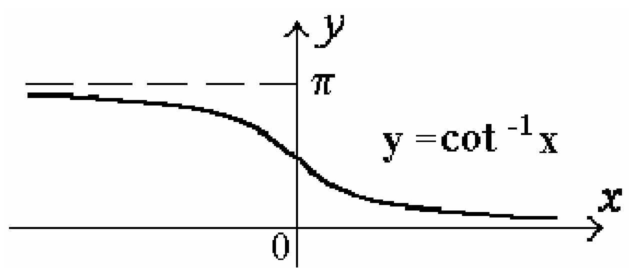
Properties:
- $\cot^{-1} x$ is a monotone decreasing function;
- $\cot^{-1}(\cot x) = x$;
- $\cot(\cot^{-1} x) = x$.

The solution set of the equation $\cot x = a$:

$$x = \cot^{-1} a + \pi n, \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Graphs:





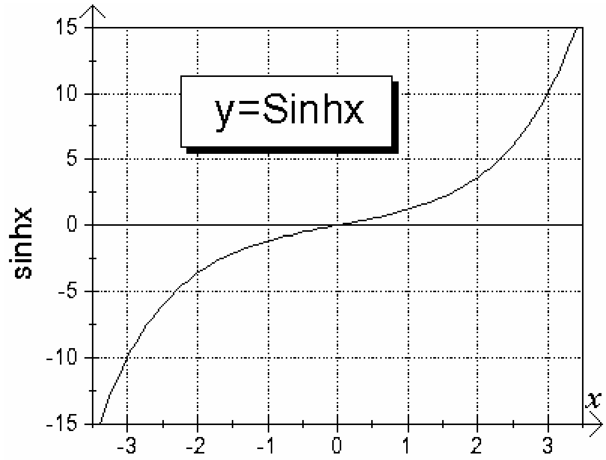
2.1.6. Hyperbolic Functions

1. The **hyperbolic sine** $\sinh x$ is defined by the following formula:

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Domain: The set of all real numbers.
Range: The set of all real numbers.
The hyperbolic sine is an odd function because

$$\sinh(-x) = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} = -\sinh x.$$

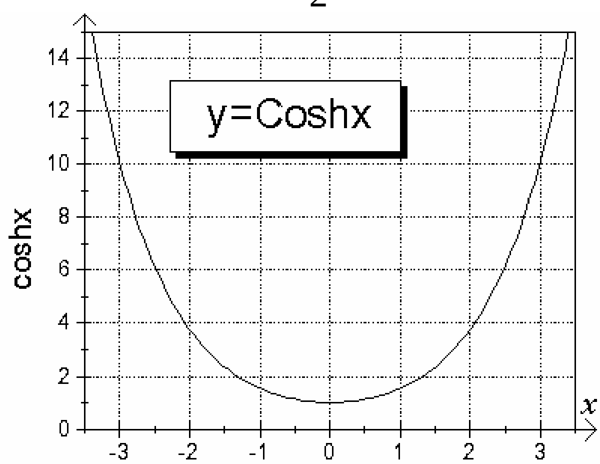


2. The **hyperbolic cosine** $\cosh x$ is defined as

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Domain: The set of all real numbers.
Range: The set of all non-negative real numbers.
The hyperbolic cosine is an even function, since

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh x.$$



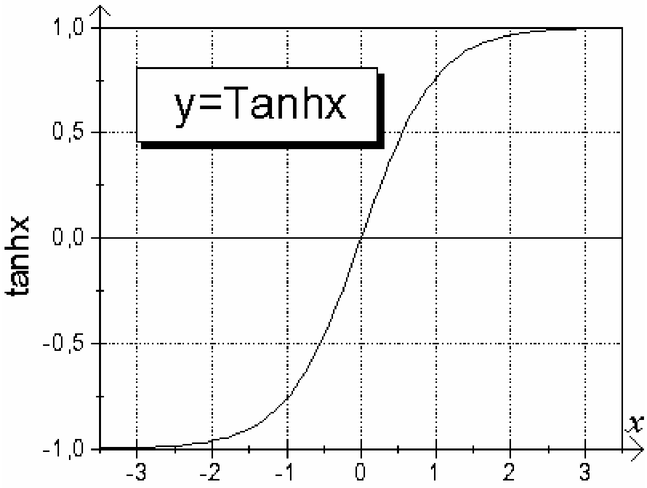
3. The **hyperbolic tangent** $\tanh x$ is defined as the ratio between $\sinh x$ and $\cosh x$:

$$\tanh x = \frac{\sin x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Domain: The set of all real numbers.

Range: The set of real numbers $|x| < 1$.

The hyperbolic tangent is an odd function due to the symmetry properties of $\sinh x$ and $\cosh x$.



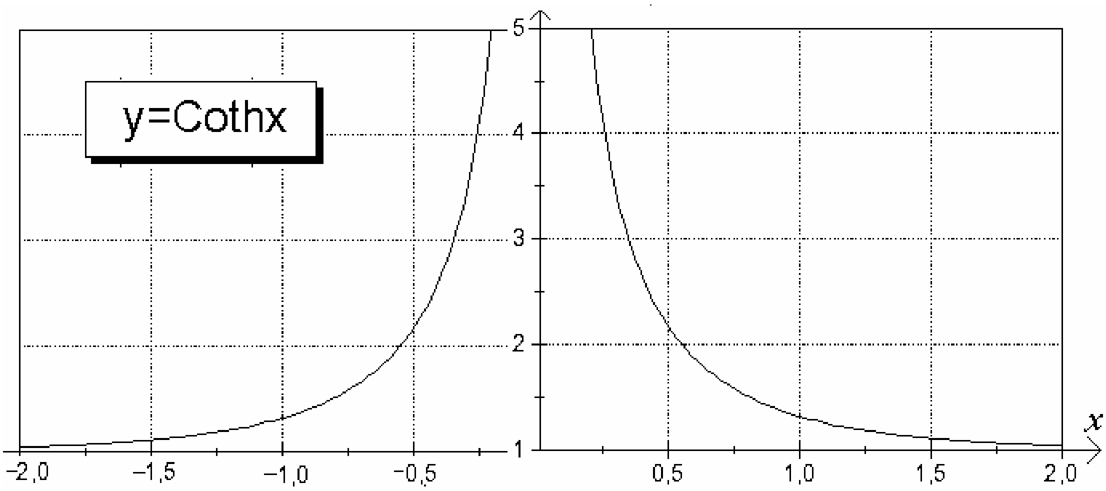
4. The **hyperbolic cotangent** $\coth x$ is the ratio of $\cosh x$ to $\sinh x$:

$$\coth x = \frac{\cos x}{\sinh x} = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

Domain: The set of all real numbers except for $x = 0$.

Range: The set of all real numbers.

The hyperbolic cotangent is an odd function due to the symmetry properties of $\sinh x$ and $\cosh x$.



Basic Formulas:

Addition Formulas for $\sinh x$ and $\cosh x$

$$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \sinh\beta\cosh\alpha$$

$$\cosh(\alpha \pm \beta) = \cosh\alpha\cosh\beta \pm \sinh\alpha\sinh\beta$$

Double-Angle Formulas for $\sinh x$ and $\cosh x$

$$\sinh 2\alpha = 2\sinh\alpha\cosh\alpha$$

$$\cosh 2\alpha = \cosh^2\alpha + \sinh^2\alpha$$

Half-Angle Formulas for $\sinh x$ and $\cosh x$

$$2\sinh^2\frac{\alpha}{2} = \cosh\alpha - 1$$

$$2\cosh^2\frac{\alpha}{2} = 1 + \cosh\alpha$$

Other Formulas

$$\sinh\alpha \pm \sinh\beta = 2\sinh\frac{\alpha \pm \beta}{2}\cosh\frac{\alpha \mp \beta}{2}$$

$$\cosh\alpha + \cosh\beta = 2\cosh\frac{\alpha + \beta}{2}\cosh\frac{\alpha - \beta}{2}$$

$$\cosh\alpha - \cosh\beta = 2\sinh\frac{\alpha + \beta}{2}\sinh\frac{\alpha - \beta}{2}$$

$$\sinh\alpha\sinh\beta = \frac{1}{2}(\cosh(\alpha+\beta) - \cosh(\alpha-\beta))$$

$$\cosh\alpha\cosh\beta = \frac{1}{2}(\cosh(\alpha+\beta) + \cosh(\alpha-\beta))$$

$$\sinh\alpha\cosh\beta = \frac{1}{2}(\sinh(\alpha+\beta) + \sinh(\alpha-\beta))$$

Relationships between Hyperbolic Functions

$$\cosh^2\alpha - \sinh^2\alpha = 1,$$

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}, \quad \tanh x = \frac{1}{\coth x},$$

$$1 - \tanh^2 x = \frac{1}{\cosh^2 x}, \quad \coth^2 x - 1 = \frac{1}{\sinh^2 x}.$$

2.2. Limits of Functions

2.2.1. Preliminary Discussion

Let $f(x) = x^2$ and the values of the variable x belong to a small vicinity of the point $x = 2$. Then it looks like evident that the values of the function $f(x)$ lie in a small vicinity of 4, that is, $x^2 \to 4$ as $x \to 2$. In this example we can directly substitute $x = 2$ to get the limit value of $f(x) = x^2$ as $x \to 2$.

However, if a function is not defined at some point $x = a$, we need to use another way of looking to find the limit value of $f(x)$ as $x \to a$.

Sometimes, similar problems can be solved algebraically, for instance,

$$f(x) = \frac{x^2 - a^2}{x - a} = \frac{(x-a)(x+a)}{x-a} = x + a \to 2a, \quad \text{as } x \to a.$$

We see that $f(x)$ approaches $2a$ as x tends to a. Therefore, by the supplementary condition

$$f(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & \text{if} \quad x \neq a \\ 2, & \text{if} \quad x = a \end{cases}$$

the domain of $f(x) = \dfrac{x^2 - a^2}{x - a}$ can be extended to include the point $x = a$.

Practically, we have found the limit of the given function as $x \to a$. In many other cases the evaluation of the limits is more complicated.

The above example shows that it is possible to operate with expressions of the form $\frac{0}{0}$ by the limit process. Some other indeterminate forms can be reduced to the form $\frac{0}{0}$ by algebraic transformations. For example, if $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then the fraction $\frac{f}{g}$ is an indeterminate form $\frac{\infty}{\infty}$, which can be reduced to the form $\frac{0}{0}$ by dividing both the numerator and denominator by the product $(f \cdot g)$: $\dfrac{f}{g} = \dfrac{1/g}{1/f}$, where $1/g \to 0$ and $1/f \to 0$ as $x \to a$.

Using limits one can also investigate the asymptotic behavior of functions at infinity. The comprehension of limits creates the necessary prerequisites for understanding all other concepts in Calculus.

2.2.2. Basic Conceptions and Definitions

Here we will give different formulations of limits in order to demonstrate the unity of various approaches to this concept. Intuitive arguments will be combined with rigorous proofs of propositions.

Intuitive Definition:

> The **limit** of a function $f(x)$ is a number *A* such that
> the values of $f(x)$ remain arbitrarily close to *A* when
> the independent variable x is sufficiently close to a specified point *a*:
> $$\lim_{x\to a} f(x) = A.$$

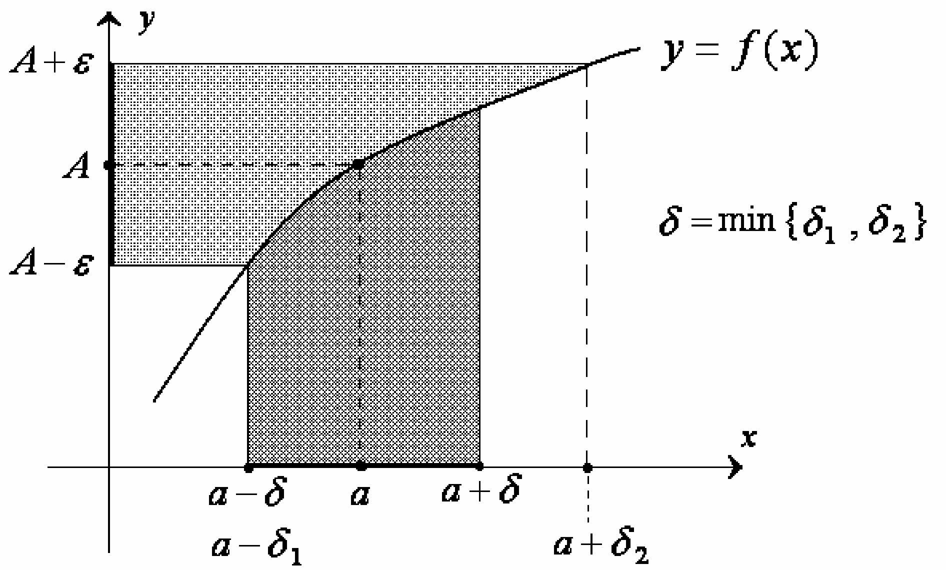
One can also say that the values of the function $f(x)$ approach the number *A* as the variable *x* tends to the point *a*.

Other Notation: $f(x) \to A$ as $x \to a$.

The above definition gives the general idea of limits. It can be easily translated into the rigorous mathematical language.

The words "the values of a given function $f(x)$ remains arbitrarily close to *A*" mean that $| f(x) - A |$ is less than any number $\varepsilon > 0$, no matter how small ε is chosen. The only thing that matters is how the function is defined in a small neighborhood of the limit point.

By a value of ε we set an acceptable deviation of $f(x)$ from the limit value *A*, that is, ε means a variation of $f(x)$ from *A*, which can be disregarded. The bound values $A-\varepsilon$ and $A+\varepsilon$ determine the corresponding interval $(a-\delta_1, a+\delta_2)$ of the values of the independent variable *x* around its limit point $x = a$.



The above drawing illustrates that for any points x in the interval

$$a - \delta_1 < x < a + \delta_2, \qquad (1)$$

the corresponding values of $f(x)$ lie in the epsilon vicinity of the point *A*,

$$A - \varepsilon < f(x) < A + \varepsilon. \qquad (2)$$

Setting $\delta = \min\{\delta_1, \delta_2\}$, we can change inequality (1) by the inequality

$$a - \delta < x < a + \delta. \qquad (3)$$

If condition (1) implies inequality (2), then inequality (3) implies inequality (2) even more. It is more convenient to operate with a symmetric delta neighborhood of the point *a*, and nothing more in this change.

Formally, the limit of a function is defined as follows:

> Let a function $f(x)$ be defined in some neighborhood of a point *a*, including or excluding $x = a$.
> A number *A* is called the **limit** of $f(x)$ as *x* tends to *a*,
> if for any arbitrary small number $\varepsilon > 0$
> there exists the corresponding number $\delta = \delta(\varepsilon) > 0$ such that
> the inequality $|x - a| < \delta$ implies
> $|f(x) - A| < \varepsilon$.

The inequality $|x - a| < \delta$ expresses the condition that values of the variable *x* are in an immediate vicinity of the limit point *a*.

If *a* is an infinite point then any neighborhood of *a* consists of sufficiently large values of *x*, and so it is necessary to modify the above definition for case of $x \to \infty$.

If $x \to \infty$, the limit of a function is defined by the following wording:

> Number *A* is called the **limit** of $f(x)$ as $x \to \infty$,
> if for any arbitrary small number $\varepsilon > 0$
> there exists the corresponding number $\Delta = \Delta(\varepsilon) > 0$
> such that the inequality $|x| > \Delta$ implies
> $|f(x) - A| < \varepsilon$.

There are two special cases of great importance:

1. $f(x) \to 0$ as $x \to a$, and
2. $f(x) \to \infty$ as $x \to a$.

In the first case, the limit of the function equals zero,

$$\lim_{x \to a} f(x) = 0,$$

and $f(x)$ is called an **infinitesimal function**.

If $f(x) \to A$ as x tends to a, then the difference between $f(x)$ and its limit value A approaches zero as $x \to a$.
It means that $f(x) - A = \alpha(x)$ is an infinitesimal function as $x \to a$. Therefore, if a number A is the limit of a function $f(x)$ as $x \to a$, then $f(x)$ can be expressed as

$$f(x) = A + \alpha(x),$$

where $\alpha(x)$ is an infinitesimal function as x tends to a.
Thus, we have obtained the following **helpful rule** of finding the limit of a function:

$$f(x) = A + \text{infinitesimal} \quad \text{as } x \to a$$
$$\Leftrightarrow$$
$$\lim_{x \to a} f(x) = A$$

Examples of infinitesimal functions:

$x^2 \to 0$ as $x \to 0$	$x^3 - 8$ as $x \to 2$
$\sin x \to 0$ as $x \to 0$	$\sin x \to 0$ as $x \to \pi$
$e^x - 1 \to 0$ as $x \to 0$	$1/x \to 0$ as $x \to \infty$
$\ln x \to 0$ as $x \to 1$	$\ln(1 + x) \to 0$ as $x \to 0$

In the second case, the statement "$f(x) \to \infty$ as $x \to a$" has the following mathematical wording:

> If for any arbitrary large number $E > 0$ there exists the corresponding number $\delta = \delta(E) > 0$ such that the inequality
> $|x - a| < \delta$ implies $|f(x)| > E$, then
> the function $f(x)$ has an infinite limit as x tends to the point a.

If $f(x) \to \infty$ as $x \to a$, the function is called an **infinite large function** that is written symbolically as

$$\lim_{x \to a} f(x) = \infty$$

or

$$f(x) \to \infty \quad \text{as} \quad x \to a.$$

The symbolical notations

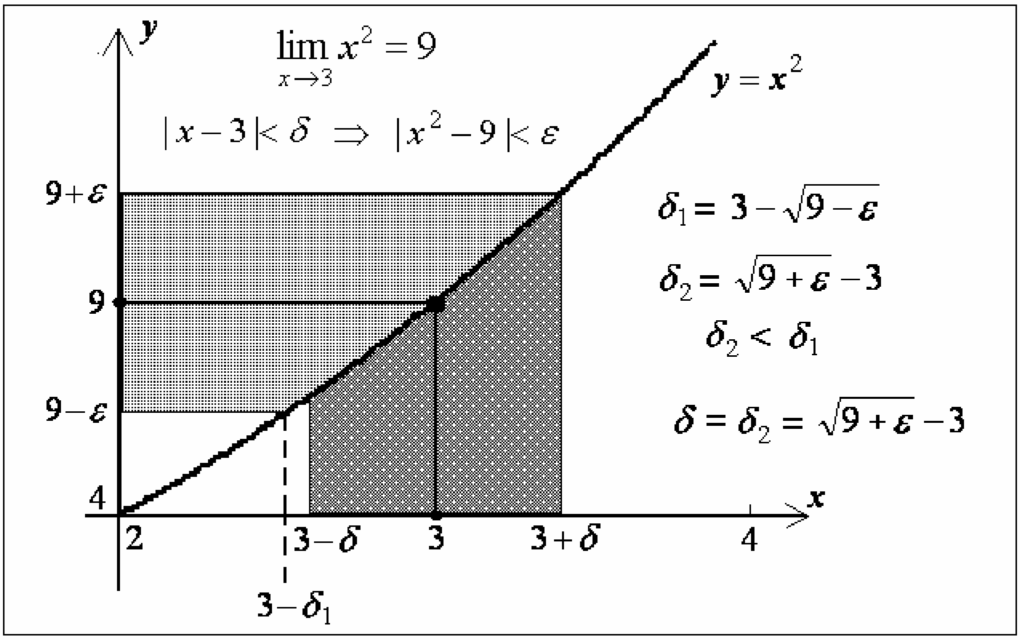
$$\lim_{x \to a} f(x) = +\infty \quad \text{and} \quad \lim_{x \to a} f(x) = -\infty$$

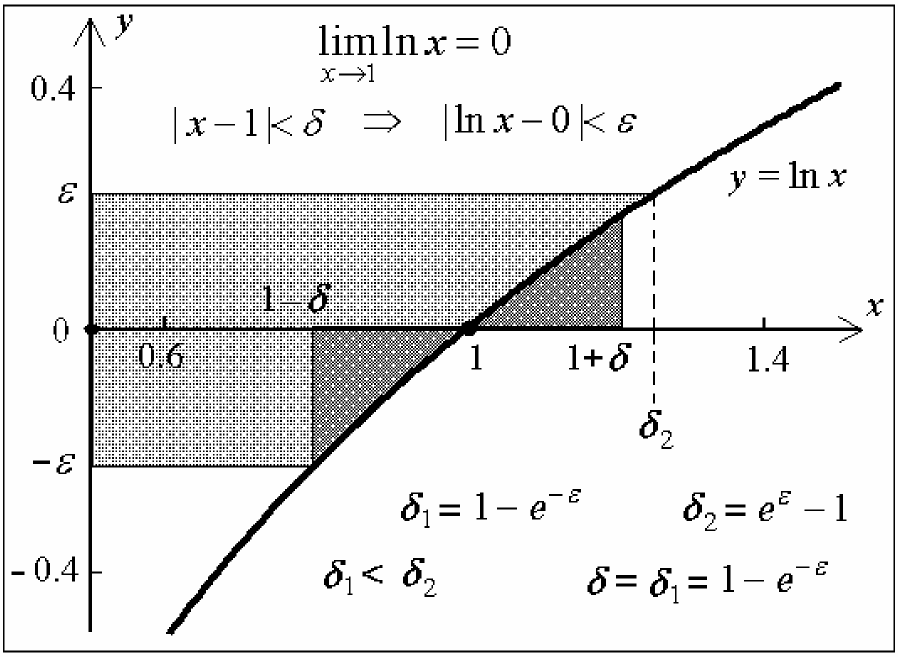
mean that infinite function $f(x)$ is positive defined or negative defined, respectively, at least in some sufficiently small vicinity of the point $x = a$.

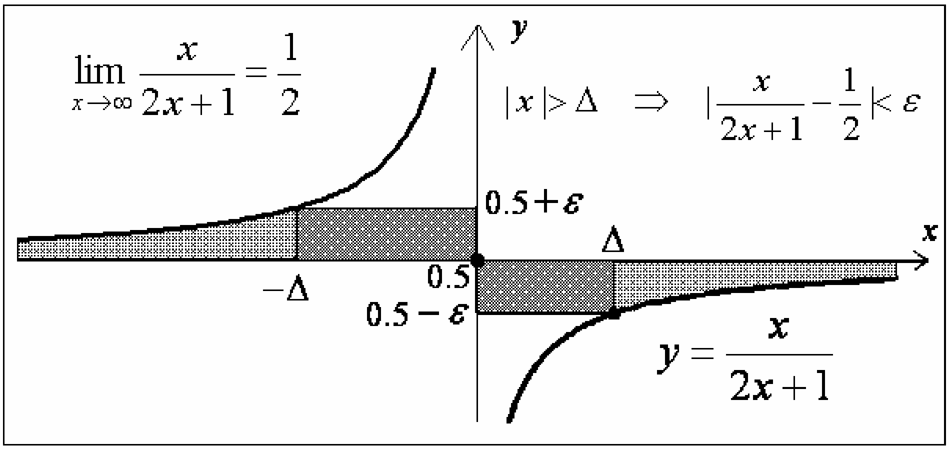
Examples of infinite large functions:

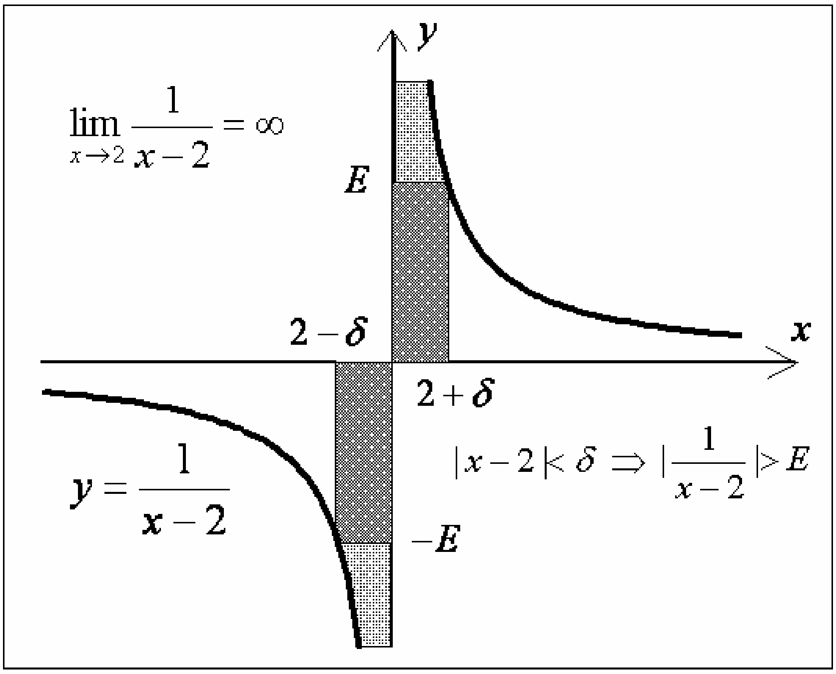
$1/x^2 \to +\infty$ as $x \to 0$	$x^2 \to +\infty$ as $x \to +\infty$
$e^x \to +\infty$ as $x \to +\infty$	$e^{-x} \to +\infty$ as $x \to -\infty$
$\tan x \to \pm\infty$ as $x \to \pi/2$	$\cot^2 x \to +\infty$ as $x \to 0$
$\ln x \to -\infty$ as $x \to 0$	$\ln x \to +\infty$ as $x \to +\infty$

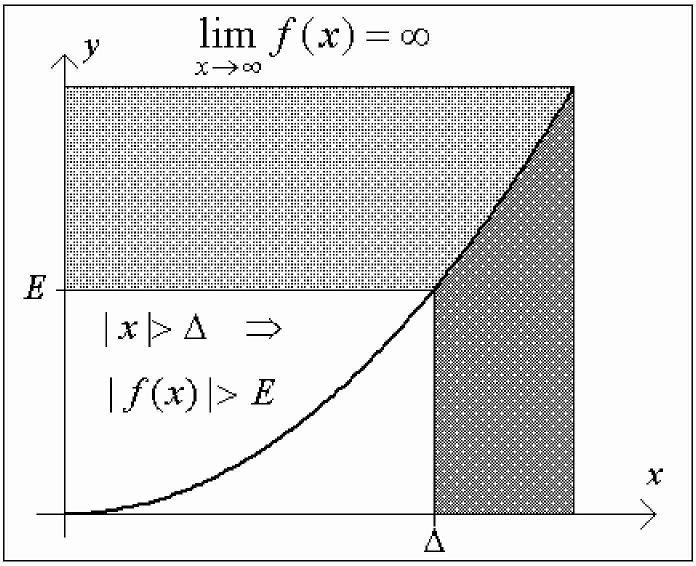
2.2.2.1. Illustrations






$\lim_{x\to\infty} \frac{x}{2x+1} = \frac{1}{2}$
$|x| > \Delta \;\Rightarrow\; |\frac{x}{2x+1} - \frac{1}{2}| < \varepsilon$
y
x
0.5+ε
0.5
0.5−ε
−Δ
Δ
$y = \frac{x}{2x+1}$


$\lim_{x\to 2} \frac{1}{x-2} = \infty$
y
x
E
−E
2−δ
2+δ
$|x-2| < \delta \Rightarrow |\frac{1}{x-2}| > E$
$y = \frac{1}{x-2}$


$\lim_{x\to\infty} f(x) = \infty$
y
x
E
Δ
$|x| > \Delta \;\Rightarrow$
$|f(x)| > E$

2.2.2.2. One-Sided Limits

Now suppose that $f(x) \to A_1$ as $x \to a$ provided that x belongs to a right-sided neighborhood of the point a ($x > a$). Then the number A_1 is called the **right-sided limit** of $f(x)$,

$$A_1 = \lim_{x \to a+0} f(x).$$

One can also use the following symbolic form to express this statement:

$$f(x) \to A_1 \quad \text{as} \quad x \to a+0 \quad \text{or simply} \quad f(a+0).$$

The left-sided limit has a similar meaning. If $f(x) \to A_2$ as $x \to a-0$ (that is, $x < a$), then A_2 is **the left-sided limit** of $f(x)$:

$$A_2 = \lim_{x \to a-0} f(x) = f(a-0).$$

If x tends to zero being less than zero, we white $x \to -0$, while the direction of approaching x to zero from the side of positive values is denoted by the symbolical form $x \to +0$.

In terms of $\varepsilon - \delta$, one-sided limits are defined as follows:

The number A_1 is called the **right-sided limit** of $f(x)$ as x tends to a, if for any arbitrary small $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that the inequality $a < x < a + \delta$ implies

$$|f(x) - A_1| < \varepsilon.$$

Likewise, if $|f(x) - A_2| < \varepsilon$ whenever $a - \delta < x < a$, then A_2 is the **left-sided limit** of $f(x)$ as x tends to a.

By the definition of the limit, $f(x) \to A$ as $x \to a$, no matter what sequence of values of x, converging to a, is chosen. Therefore, the following theorem holds true:

> $f(x)$ has the limit at the point a if and only if
> there exist one-sided limits as $x \to a$, which are equal to each other:
> $$\lim_{x \to a-0} f(x) = \lim_{x \to a+0} f(x) = A \Leftrightarrow \lim_{x \to a} f(x) = A.$$

Example: Find the limit of the function $f(x) = \dfrac{1}{1 + 5^{\frac{1}{x-2}}}$ at the point $x = 2$.

Solution: Consider one-sided limits of $f(x)$ as $x \to 2 \pm 0$:

$$\lim_{x \to 2-0} f(x) = \frac{1}{1 + 5^{-\infty}} = \frac{1}{1+0} = 1 \quad \text{and} \quad \lim_{x \to 2+0} f(x) = \frac{1}{1 + 5^{+\infty}} = \frac{1}{1+\infty} = 0.$$

They differ from each other, and so $f(x)$ has no a limit at $x = 2$.

2.2.3. Properties of Infinitesimal Functions

Property 1:

> Let $f(x)$ be a function bounded at least in some neighborhood of a point *a* and $\alpha(x)$ be an infinitesimal function as *x* tends to *a*.
> Then the product $f(x)\alpha(x)$ is an infinitesimal function.

Explanation: The absolute values of the bounded function $f(x)$ are restricted by a finite positive number *M*, $|f(x)| < M$, for any *x* in some neighborhood of the point *a*.
Since $\alpha(x)$ is an infinitesimal function as *x* tends to *a*, then

$$|f(x)\,\alpha(x)| < M\,|\alpha(x)| \quad \to \quad M \cdot 0 = 0$$

Rigorous Proof: Since $f(x)$ is a function bounded in some neighborhood of the point *a*, then there exists a finite positive number *M* such that

$$|f(x)| < M \tag{4}$$

whenever

$$|x - a| < \delta_1 . \tag{5}$$

Since $\alpha(x)$ an infinitesimal function in the same neighborhood of the point *a*, any positive number ε / M corresponds to a positive number δ_2 such that the inequality

$$|x - a| < \delta_2 \tag{6}$$

implies

$$|\alpha(x)| < \frac{\varepsilon}{M} . \tag{7}$$

Let us set $\delta = \min\{\delta_1, \delta_2\}$. Then condition $|x - a| < \delta$ implies inequalities (5) and (6), that results in inequalities (4) and (7).
Therefore, for any arbitrary small number ε, we obtain that

$$|\alpha(x) \cdot f(x)| = |\alpha(x)| \cdot |f(x)| < \frac{\varepsilon}{M} M = \varepsilon$$

whenever the values of x are in the delta neighborhood of the point a.
This proves that $\alpha(x)\,f(x)$ is an infinitesimal function.

Property 2:

> The sum of two infinitesimal functions is an infinitesimal function.

Explanation: If $\alpha(x) \to 0$ and $\beta(x) \to 0$ as $x \to a$, then

$$\alpha(x) + \beta(x) \to 0 + 0 = 0 \quad \text{as} \quad x \to a .$$

Rigorous Proof: Let $\alpha(x)$ and $\beta(x)$ be infinitesimal functions as $x \to a$.

Then for any arbitrary small positive number $\varepsilon/2$ there exist the corresponding numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|x-a|<\delta_1 \quad \Rightarrow \quad |\alpha(x)|<\frac{\varepsilon}{2}$$

and

$$|x-a|<\delta_2 \quad \Rightarrow \quad |\beta(x)|<\frac{\varepsilon}{2}.$$

If $\delta = \min\{\delta_1, \delta_2\}$ then inequality $|x-a|<\delta$ implies both $|x-a|<\delta_1$ and $|x-a|<\delta_2$, and hence

$$|\alpha(x)+\beta(x)| \le |\alpha(x)|+|\beta(x)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

for any arbitrary small number ε.

Corollary:

> The sum of any finite number of infinitesimal functions is **an infinitesimal function.**

Explanation: The sum of two infinitesimals is an infinitesimal, the sum of which with a third infinitesimal is also an infinitesimal, and so on.
The statement can be proved rigorously by the mathematical induction principle just in the same manner that was used in a case of sequences. (See Chapter 1, p. 18-19.)

Example: $5x + 2\sin x + 4x\tan x - 3\ln(1+x)$ is an infinitesimal function as $x \to 0$, since each term of the sum is an infinitesimal function.

2.2.4. Properties of Limits of Functions

Property 1:

> A constant factor can be taken out the sign of the limit,
> $$\lim_{x\to a} c\, f(x) = c \lim_{x\to a} f(x).$$

Proof: By the rule formulated on page 47,

$$\lim_{x\to a} f(x) = A \quad \Rightarrow \quad f(x) = A + \alpha(x),$$

where $\alpha(x)$ is an infinitesimal function as $x \to a$.

Therefore, $\quad c\, f(x) = c\,A + c\,\alpha(x) \quad \Rightarrow \quad \lim_{x\to a} c\, f(x) = c\,A.$

Properties 2-4:

If there exist both limits, $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$,
then there exist the limits of the sum, product and quotient of the functions:

2. $\lim\limits_{x\to a}(f(x)\pm g(x)) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$.

3. $\lim\limits_{x\to a}(f(x)\cdot g(x)) = \lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x)$.

4. $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ (if $\lim\limits_{x\to a} g(x) \neq 0$).

Let us prove, for example, Property 3.
The statements

$$\lim_{x\to a} f(x) = A \quad \text{and} \quad \lim_{x\to a} g(x) = B$$

mean that

$$f(x) = A + \alpha(x) \quad \text{and} \quad g(x) = B + \beta(x),$$

where $\alpha(x)$ and $\beta(x)$ are infinitesimal functions as $x \to a$.
Therefore,

$$f(x)\,g(x) = (A+\alpha(x))(B+\beta(x))$$
$$= AB + (A\beta(x) + B\alpha(x) + \alpha(x)\beta(x)).$$

Since $A\beta(x) + B\alpha(x) + \alpha(x)\beta(x)$ is also an infinitesimal function as $x \to a$, then

$$\lim_{x\to a} f(x)\,g(x) = AB = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x).$$

2.2.5. Examples

1. Evaluate $\lim\limits_{x\to 0}\dfrac{\tan x}{\sin x}$.

Solution: In order to evaluate an indeterminate form $0/0$, we need to select and cancel common infinitesimal factors in the numerator and denominator of the given expression:

$$\frac{\tan x}{\sin x} = \frac{\sin x}{\sin x \cos x} = \frac{1}{\cos x}.$$

By the properties of limits,

$$\lim_{x\to 0}\frac{1}{\cos x} = \frac{1}{\lim\limits_{x\to 0}\cos x} = \frac{1}{1} = 1.$$

2. Evaluate $\lim\limits_{x\to 1}\left(\dfrac{1}{x^2+x-2}-\dfrac{1}{x^2-4x+3}\right)$.

Solution: Here we deal with an indeterminate form $\infty-\infty$.
Transform the expression under the sign of the limit:

$$\frac{1}{x^2+x-2}-\frac{1}{x^2-4x+3}=\frac{-5x+5}{(x^2+x-2)(x^2-4x+3)}$$

$$=-5\frac{x-1}{(x+2)(x-1)(x-3)(x-1)}=-5\frac{1}{(x+2)(x-1)(x-3)}.$$

We have obtained the expression of the form $\dfrac{\text{constant}}{0}$, and so

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x^2+5x}\right)=\infty .$$

3. Find $\lim\limits_{x\to 1}\dfrac{x^2-1}{x^2+x-2}$ (an indeterminate form $\dfrac{0}{0}$).

Solution: Present the fraction in factored form; then cancel the common infinitesimal factors:

$$\lim_{x\to 1}\frac{x^2-1}{x^2+x-2}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x+2)}=\lim_{x\to 1}\frac{x+1}{x+2}=\frac{2}{3}.$$

4. Evaluate $\lim\limits_{x\to 3}\dfrac{x^2-4x+3}{x^3-2x^2-x-6}$ (an indeterminate form $\dfrac{0}{0}$).

Solution: Using the idea of reducing the common factors, we obtain

$$\lim_{x\to 3}\frac{x^2-4x+3}{x^3-2x^2-x-6}=\lim_{x\to 3}\frac{(x-3)(x-1)}{(x-3)(x^2+x+2)}=\lim_{x\to 3}\frac{x-1}{x^2+x+2}=\frac{1}{7}.$$

5. Find $\lim\limits_{x\to\infty}\dfrac{4x^2+5x+1}{3x^2-x}$ (an indeterminate form $\dfrac{\infty}{\infty}$).

Solution: In order to evaluate the indeterminate form $\dfrac{\infty}{\infty}$, divide the numerator and denominator x^2 and then apply the properties of limits:

$$\lim_{x\to\infty}\frac{4x^2+5x+1}{3x^2-x}=\lim_{x\to\infty}\frac{(4x^2+5x+1)/x^2}{(3x^2-x)/x^2}=\lim_{x\to\infty}\frac{4+5/x+1/x^2}{3-1/x}$$

$$=\frac{\lim\limits_{x\to\infty}(4+5/x+1/x^2)}{\lim\limits_{x\to\infty}(3-1/x)}=\frac{4+5\lim\limits_{x\to\infty}1/x+\lim\limits_{x\to\infty}1/x^2}{3-\lim\limits_{x\to\infty}1/x}=\frac{4}{3}.$$

6. Evaluate $\lim_{x\to 2}\frac{\sqrt{7+x}-3}{x-2}$ (an indeterminate form $\frac{0}{0}$).

Solution: Multiply the numerator and denominator by the sum $(\sqrt{7+x}+3)$ to complete the difference between two squares. Then cancel the like infinitesimal factors:

$$\lim_{x\to 2}\frac{\sqrt{7+x}-3}{x-2}=\lim_{x\to 2}\frac{(\sqrt{7+x}-3)(\sqrt{7+x}+3)}{(x-2)(\sqrt{7+x}+3)}$$

$$=\lim_{x\to 2}\frac{(\sqrt{7+x})^2-9}{(x-2)(\sqrt{7+x}+3)}=\lim_{x\to 2}\frac{7+x-9}{(x-2)(\sqrt{7+x}+3)}$$

$$=\lim_{x\to 2}\frac{x-2}{(x-2)(\sqrt{7+x}+3)}=\lim_{x\to 2}\frac{1}{\sqrt{7+x}+3}=\frac{1}{6}.$$

7. Evaluate $\lim_{x\to 1}\frac{\sqrt{3+x}-2}{\sqrt{2-x}-1}$ (an indeterminate form $\frac{0}{0}$).

Solution: Likewise, complete the difference between squares to select and cancel the common infinitesimal factors. Note that

$$\sqrt{3+x}-2=\frac{(\sqrt{3+x}-2)(\sqrt{3+x}+2)}{\sqrt{3+x}+2}=\frac{3+x-4}{\sqrt{3+x}+2}=\frac{x-1}{\sqrt{3+x}+2}$$

and

$$\sqrt{2-x}-1=\frac{(\sqrt{2-x}-1)(\sqrt{2-x}+1)}{\sqrt{2-x}+1}=\frac{2-x-1}{\sqrt{2-x}+1}=-\frac{x-1}{\sqrt{2-x}+1}.$$

Therefore,

$$\lim_{x\to 1}\frac{\sqrt{3+x}-2}{\sqrt{2-x}-1}=-\lim_{x\to 1}\frac{\sqrt{2-x}+1}{\sqrt{3+x}+2}=-\frac{2}{4}=-\frac{1}{2}.$$

8. Evaluate $\lim_{x\to 2}\frac{x^3-8}{x-2}$ (an indeterminate form $\frac{0}{0}$).

Solution: In view of the formula of the difference between two cubes,

$$\lim_{x\to 2}\frac{x^3-8}{x-2}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}=\lim_{x\to 2}(x^2+2x+4)=12.$$

9. $$\lim_{x\to+\infty}\frac{5^x-3^x}{5^x+3^x}=\lim_{x\to+\infty}\frac{1-(\frac{3}{5})^x}{1+(\frac{3}{5})^x}=\frac{1-0}{1+0}=1.$$

10. $$\lim_{x\to-\infty}\frac{5^x-3^x}{5^x+3^x}=\lim_{x\to-\infty}\frac{(5/3)^x-1}{(5/3)^x+1}=\frac{0-1}{0+1}=-1.$$

11. $$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)=\lim_{x\to 1}\frac{x+1-2}{x^2-1}=\lim_{x\to 1}\frac{x-1}{x^2-1}$$
$$=\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)}=\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}.$$

12. $$\lim_{x\to\infty}\frac{(5x+4)^{70}(5x-3)^{30}}{(5x+7)^{100}}=\lim_{x\to\infty}\frac{(5+4/x)^{70}(5-3/x)^{30}}{(5+7/x)^{100}}$$
$$=\lim_{x\to\infty}\frac{5^{70}5^{30}}{5^{100}}=\lim_{x\to\infty}\frac{5^{100}}{5^{100}}=1.$$

13. $$\lim_{x\to\infty}\frac{(5x+4)^{70}(5x-3)^{30}}{(5x+7)^{100}}=\lim_{x\to\infty}\frac{(5x)^{70}(5x)^{30}}{(5x)^{100}}$$
$$=\lim_{x\to\infty}\frac{(5x)^{70}(5x)^{30}}{(5x)^{100}}=\lim_{x\to\infty}\frac{(5x)^{100}}{(5x)^{100}}=1.$$

14. $$\lim_{x\to 3}\frac{\sqrt{3x}-x}{x-3}=\lim_{x\to 3}\frac{\sqrt{x}(\sqrt{3}-\sqrt{x})}{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}=-\lim_{x\to 3}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{3}}=-\frac{1}{2}.$$

15. $$\lim_{x\to 1}\frac{x-1}{\sqrt[3]{x}-1}=\lim_{x\to 1}\frac{(\sqrt[3]{x})^3-1^3}{\sqrt[3]{x}-1}$$
$$=\lim_{x\to 1}\frac{(\sqrt[3]{x}-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{\sqrt[3]{x}-1}=\lim_{x\to 1}(\sqrt[3]{x^2}+\sqrt[3]{x}+1)=3.$$

2.2.6. Classification of Infinitesimal Functions

Infinitesimal functions can be classified in the same manner that was used in the corresponding section devoted to the sequences.

Two infinitesimal functions, $\alpha(x)$ and $\beta(x)$, are called the **infinitesimal functions of the same order of smallness** as *x* tends to *a*, if their ratio has a finite non-zero limit:

$$0<\left|\lim_{x\to a}\frac{\alpha(x)}{\beta(x)}\right|<\infty.$$

In that case, infinitesimal functions $\alpha(x)$ and $\beta(x)$ are said to be **proportional** to each other in some vicinity of the point *a*.

In particular, if $\lim_{x\to a}\frac{\alpha(x)}{\beta(x)}=1$, infinitesimal functions $\alpha(x)$ and $\beta(x)$ are called **equivalent** as *x* tends to *a* that is denoted symbolically as

$$\alpha(x)\sim\beta(x).$$

An infinitesimal function $\alpha(x)$ has a **higher order of smallness** with respect to $\beta(x)$ as x tends to a, if

$$\lim_{x\to a}\frac{\alpha(x)}{\beta(x)}=0.$$

In this case, $\beta(x)$ is called an infinitesimal function of a **lower order of smallness** with respect to $\alpha(x)$.

An infinitesimal function $\alpha(x)$ is called an **infinitesimal of the *n-th* order** with respect to $\beta(x)$ as x tends to a, if $\alpha(x)$ and $(\beta(x))^n$ are infinitesimal functions of the same order:

$$0<\left|\lim_{x\to a}\frac{\alpha(x)}{(\beta(x))^n}\right|<\infty.$$

Examples

1. Infinitesimal functions $\alpha(x)=x^2-4$ and $\beta(x)=x-2$ as $x\to 2$ have the same order, since

$$\lim_{x\to 2}\frac{\alpha(x)}{\beta(x)}=\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4,$$

   and 4 is a finite non-zero number.

2. If $x\to\infty$ then $\alpha(x)=\dfrac{1}{4x^2+3x+8}$ and $\beta(x)=\dfrac{1}{x^2-x+20}$ are infinitesimal functions of the same order, since their ratio tends to a finite non-zero number:

$$\frac{\alpha(x)}{\beta(x)}=\frac{1/(4x^2+3x+8)}{1/(x^2-x+20)}=\frac{x^2-x+20}{4x^2+3x+8}$$

$$=\frac{\dfrac{x^2-x+20}{x^2}}{\dfrac{4x^2+3x+8}{x^2}}=\frac{1-\dfrac{1}{x}+\dfrac{20}{x^2}}{4+\dfrac{3}{x}+\dfrac{8}{x^2}}\to\frac{1-0+0}{4+0+0}=\frac{1}{4}.$$

3. The limit of the ratio of the infinitesimal functions $\alpha(x)=x\sin x$ and $\beta(x)=\tan x$ as $x\to 0$ equals zero:

$$\lim_{x\to 0}\frac{\alpha(x)}{\beta(x)}=\lim_{x\to 0}\frac{x\sin x}{\tan x}=\lim_{x\to 0}\frac{x\sin x}{\sin x}\cos x=\lim_{x\to 0}x\cos x=0.$$

   Therefore, $\alpha(x)$ is an infinitesimal function of a higher order of smallness with respect to $\beta(x)$.

4. Let $\alpha(x)=x$ and $\beta(x)=\sqrt[3]{x}$. Since

$$\lim_{x\to 0}\frac{\alpha}{\beta^3}=\lim_{x\to 0}\frac{x}{(\sqrt[3]{x})^3}=\lim_{x\to 0}\frac{x}{x}=1,$$

that is, the limit is a finite number, then $\alpha(x)$ is an infinitesimal function of the third order with respect to $\beta(x)$ as $x\to 0$.

5. Given two infinitesimal functions $\alpha(x)=\dfrac{1}{x+2}$ and $\beta(x)=\dfrac{1}{x+5}$ as $x\to\infty$,

$$\lim_{x\to\infty}\frac{\alpha}{\beta}=\lim_{x\to\infty}\frac{\frac{1}{x+2}}{\frac{1}{x+5}}=\lim_{x\to\infty}\frac{x+5}{x+2}=\lim_{x\to\infty}\frac{\frac{x+5}{x}}{\frac{x+2}{x}}=\lim_{x\to\infty}\frac{1+\frac{5}{x}}{1+\frac{2}{x}}=1.$$

Therefore, $\alpha(x)$ and $\beta(x)$ are equivalent infinitesimal functions as $x\to\infty$.

6. Expressions $\alpha(x)=\dfrac{1}{x^2-3x+7}$ and $\beta(x)=\dfrac{1}{x^2+2x+10}$ determine equivalent infinitesimal functions as $x\to\infty$, since the limit of their ratio equals unity:

$$\lim_{x\to\infty}\frac{\alpha(x)}{\beta(x)}=\lim_{x\to\infty}\frac{x^2+2x+10}{x^2-3x+7}=\lim_{x\to\infty}\frac{1+\frac{2}{x}+\frac{10}{x^2}}{1-\frac{3}{x}+\frac{7}{x^2}}=1.$$

2.2.7. Comparison Between Infinitesimal Functions

Rule 1: Let $\alpha(x)$ and $\beta(x)$ be two equivalent infinitesimal functions as $x\to a$. Then

$$\beta(x)=\alpha(x)+\gamma(x),$$

where $\gamma(x)$ is an infinitesimal function of a higher order of smallness as $x\to a$.

Proof: By the definition, if $\alpha(x)\sim\beta(x)$ then $\dfrac{\beta(x)}{\alpha(x)}\to 1$ as $x\to a$, and so

$$\frac{\beta(x)-\alpha(x)}{\alpha(x)}=\frac{\beta(x)}{\alpha(x)}-1 \quad\to\quad 1-1=0.$$

Therefore, the difference $(\beta(x)-\alpha(x))$ is an infinitesimal function of a higher order of smallness with respect to the given infinitesimals as $x\to a$.

Rule 2: If $\beta(x)$ is an infinitesimal function of a higher order of smallness with respect to $\alpha(x)$ as $x \to a$, then

$$\alpha(x)+\beta(x) \sim \alpha(x).$$

It means that $\beta(x)$ is a **negligible quantity** with respect to $\alpha(x)$ as $x \to a$.

Proof: By the hypothesis, $\dfrac{\beta(x)}{\alpha(x)} \to 0$ as $x \to a$. Then

$$\alpha(x)+\beta(x)=\frac{\alpha(x)+\beta(x)}{\alpha(x)}\alpha(x)=(1+\frac{\beta(x)}{\alpha(x)})\alpha(x) \sim (1+0)\cdot\alpha(x)=\alpha(x).$$

Rule 3: If $\alpha(x)$ and $\beta(x)$ are infinitesimal functions of the same order and

$$0<\lim\frac{\alpha(x)}{\beta(x)}=\lambda<\infty,$$

then $\alpha(x)$ and $\lambda\beta(x)$ are equivalent infinitesimal functions,

$$\alpha(x) \sim \lambda\beta(x).$$

Examples:

1. Since $\dfrac{1}{7x^2-x+4} \overset{x\to\infty}{\sim} \dfrac{1}{7x^2}$ and $\dfrac{1}{8x^2+6x+1} \overset{x\to\infty}{\sim} \dfrac{1}{8x^2}$,

   then $\displaystyle \lim_{x\to\infty}\frac{1/(7x^2-x+4)}{1/(8x^2+6x+1)}=\lim_{x\to\infty}\frac{1/(7x^2)}{1/(8x^2)}=\lim_{x\to\infty}\frac{8x^2}{7x^2}=\frac{8}{7}.$

2. Infinitesimal functions x^2 and x^3 have higher orders of smallness with respect to x as $x \to 0$. Therefore, $x+x^2+x^3 \sim x$, that is, (x^2+x^3) is a negligible quantity with respect to x as x tends to zero.

3. Since $\tan x$ and $\sin x$ are equivalent infinitesimal functions as $x \to 0$, the difference between them is an infinitesimal function of a higher order of smallness with respect to each of them. Really,

$$\tan x-\sin x=\frac{\sin x}{\cos x}-\sin x=\sin x(\frac{1}{\cos x}-1)$$
$$=\sin x\frac{1-\cos x}{\cos x}=\sin x\frac{(1-\cos x)(1+\cos x)}{\cos x(1+\cos x)}$$
$$=\frac{\sin x(1-\cos^2 x)}{\cos x(1+\cos x)}=\frac{\sin^3 x}{\cos x(1+\cos x)} \to \text{const}\cdot\sin^3 x$$

   as $x \to 0$.

   Thus, $\tan x-\sin x$ is an infinitesimal function of the third order of smallness with respect to both functions, $\sin x$ and $\tan x$, as $x \to 0$.

2.2.8. Classification of Infinite Large Functions

Two infinite large functions, $\alpha(x)$ and $\beta(x)$, have the **same increasing order** as $x \to a$, if their ratio has a finite nonzero limit,

$$0 < \left| \lim_{x \to a} \frac{\alpha(x)}{\beta(x)} \right| < \infty .$$

In particular, if

$$\lim_{x \to a} \frac{\alpha(x)}{\beta(x)} = 1$$

then $\alpha(x)$ and $\beta(x)$ are called **equivalent** infinite large functions as $x \to a$ that is denoted symbolically as

$$\alpha(x) \sim \beta(x) .$$

An infinite large function $\alpha(x)$ has a **higher increasing order** with respect to $\beta(x)$ as *x* tends to *a*, if

$$\lim_{x \to a} \frac{\alpha(x)}{\beta(x)} = \infty .$$

Correspondingly, $\beta(x)$ is an infinite large function of a **lower increasing order** with respect to $\alpha(x)$.

Let $\alpha(x)$ and $(\beta(x))^n$ be two infinite large functions of the same order,

$$0 < \left| \lim_{x \to a} \frac{\alpha(x)}{(\beta(x))^n} \right| < \infty .$$

Then $\alpha(x)$ is called an infinite large function **of the *n*-th order** with respect to $\beta(x)$ as *x* tends to *a*.

2.2.9. Comparison Between Infinite Large Functions

Rule 1: Let $\alpha(x)$ and $\beta(x)$ be two infinite large functions as $x \to a$. Then

$$\beta(x) = \alpha(x) + \gamma(x) ,$$

where $\gamma(x)$ is an infinite large function of a lower increasing order as $x \to a$.

Rule 2: The difference between two equivalent infinite large functions is a quantity of a lower increasing order:

$$\alpha(x) \sim \beta(x) \quad \Rightarrow \quad \frac{\alpha(x) - \beta(x)}{\alpha(x)} = 1 - \frac{\beta(x)}{\alpha(x)} \to 1 - 1 = 0 .$$

Rule 3: If $\alpha(x)$ is an infinite large function of a higher increasing order with respect to $\beta(x)$ as $x \to a$ then the sum $\alpha(x) + \beta(x)$ is an infinite large function equivalent to $\alpha(x)$:

$$\frac{\alpha(x)+\beta(x)}{\alpha(x)} = 1+\frac{\beta(x)}{\alpha(x)} \;\to\; 1+0=1 \quad \Rightarrow \quad \alpha(x)+\beta(x) \sim \alpha(x).$$

In this case, $\beta(x)$ is said to be a **negligible quantity** with respect to $\alpha(x)$ as $x \to a$.

If $\alpha(x)$ and $\beta(x)$ are infinite large functions of the same increasing order as $x \to a$ and $0 < \lim\limits_{x\to a} \dfrac{\alpha(x)}{\beta(x)} = \lambda < \infty$, then $\alpha(x)$ and $\lambda\,\beta(x)$ are equivalent infinite large functions,

$$\alpha(x) \sim \lambda\,\beta(x).$$

In this case, they say that infinite large functions are **proportional asymptotically** as $x \to a$.

Examples

1. Infinite large functions $f(x) = x^2$ and $g(x) = x^2 + 5x$ as $x \to \infty$ have the same increasing order, since $\lim\limits_{x\to\infty} f/g$ is a finite number:

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \lim_{x\to\infty} \frac{x^2+5x}{x^2} = \lim_{x\to\infty}(1+\frac{5}{x}) = 1.$$

Moreover, the limit equals 1, and so $f(x)$ and $g(x)$ are equivalent infinite large functions as $x \to \infty$.

One can see that $5x$ (in the numerator) is a negligible quantity with respect to x^2 as $x \to \infty$.

Generally, if $k < n$ and a is a finite number, then any power function $a\,x^k$ is a negligible quantity with respect to x^n as $x \to \infty$.

For instance,

$$4x^5 + 7x + 50 \sim 4x^5 \qquad \text{and} \qquad 6x^5 - 3x + 8 \sim 6x^5.$$

Hence,

$$\lim_{x\to\infty} \frac{4x^5+7x+50}{6x^5-3x+8} = \lim_{x\to\infty} \frac{4x^5}{6x^5} = \frac{2}{3}.$$

2. The infinite large function $\sqrt{4x^2+3x+7}$ is equal asymptotically to $2x$ as $x \to \infty$, since $3x+7$ is a negligible quantity with respect to $4x^2$.

Likewise, $\sqrt[3]{x^3+5x^2-8x+2} \sim x$, since $5x^2-8x+2$ is a negligible quantity with respect to x^3. Therefore,

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+3x+7}}{\sqrt[3]{x^3+5x^2-8x+2}} = \lim_{x\to\infty} \frac{2x}{x} = 2.$$

3. Evaluate $\lim\limits_{x\to\infty} \dfrac{3x^2+5x+1}{\sqrt{16x^4+x^3-9x^2+7x+2}}$.

Solution: Since

$$3x^2+5x+1 \sim 3x^2$$

and

$$\sqrt{16x^4+x^3-9x^2+7x+2} \sim \sqrt{16x^4} = 4x^2,$$

as $x\to\infty$, then

$$\lim_{x\to\infty} \frac{3x^2+5x+1}{\sqrt{16x^4+x^3-9x^2+7x+2}} = \lim_{x\to\infty} \frac{3x^2}{4x^2} = \frac{3}{4}.$$

4. Both functions, $f(x)=\sqrt{x^4+x^3}$ and $g(x)=x^2+x$, are infinite large functions as $x\to\infty$.
Prove that:
   1) $f(x)\sim g(x)$;
   2) both functions, $f(x)$ and $g(x)$, are infinite large functions of the second increasing order with respect to *x* as $x\to\infty$;
   3) the difference $f(x)-g(x)$ is a quantity of a lower increasing order with respect to the given functions $f(x)$ and $g(x)$.

Solution:

1) $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^4+x^3}}{x^2+x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x^4+x^3}/x^2}{(x^2+x)/x^2} = \lim\limits_{x\to\infty} \dfrac{\sqrt{1+1/x}}{1+1/x} = 1.$

2) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^4+x^3}}{x^2} = \lim\limits_{x\to\infty} \sqrt{1+1/x} = 1$ and $\lim\limits_{x\to\infty} \dfrac{x^2+x}{x^2} = \lim\limits_{x\to\infty} (1+\dfrac{1}{x^2}) = 1.$

3) By the formula of difference between two squares,

$$f-g = \frac{(f-g)(f+g)}{f+g} = \frac{f^2-g^2}{f+g} = \frac{(\sqrt{x^4+x^3})^2-(x^2+x)^2}{\sqrt{x^4+x^3}+x^2+x}$$

$$= \frac{x^4+x^3-x^4-2x^3-x^2}{\sqrt{x^4+x^3}+x^2+x} = \frac{-x^3-x^2}{\sqrt{x^4+x^3}+x^2+x}.$$

If $x\to\infty$ then

$$x^3+x^2 \sim x^3$$

and

$$\sqrt{x^4+x^3}+x^2+x \sim \sqrt{x^4}+x^2 = 2x^2.$$

Hence, $f-g \sim -x/2$, that is, the difference is an infinite large function of the first increasing order with respect to *x*.

2.3. The Most Important Limits

2.3.1. Theorem 1

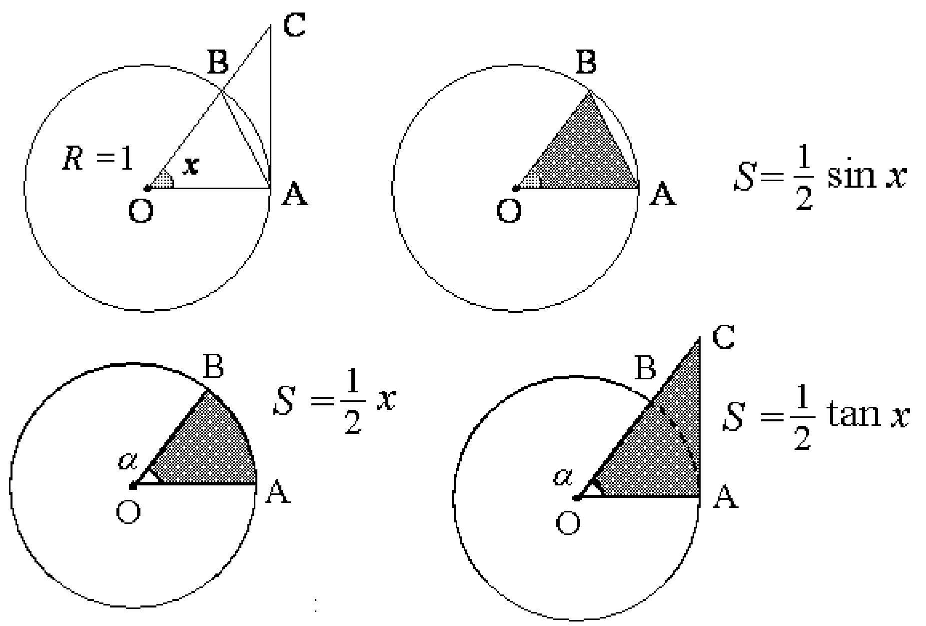
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

This statement can be also expressed as

$$\sin x \sim x \quad \text{as} \quad x \to 0.$$

Proof: Note that $\frac{\sin x}{x}$ is an even function. Therefore, we may consider only the case of positive values of variable *x* in a vicinity of zero.

Let *x* be a central angle (in radians) of the unit circle. Compare the areas of the figures shown in the drawings below.



The area of the triangle *OAB* is

$$S_{\Delta OAB} = \frac{1}{2} \sin x.$$

The area of the circular sector *OAB* is

$$S_{OAB} = \frac{1}{2} x.$$

The area of the triangle *OAC* is

$$S_{\Delta OAC} = \frac{1}{2} \tan x.$$

Evidently,

$$\sin x < x < \tan x$$

for any $0 < x < \pi/2$.

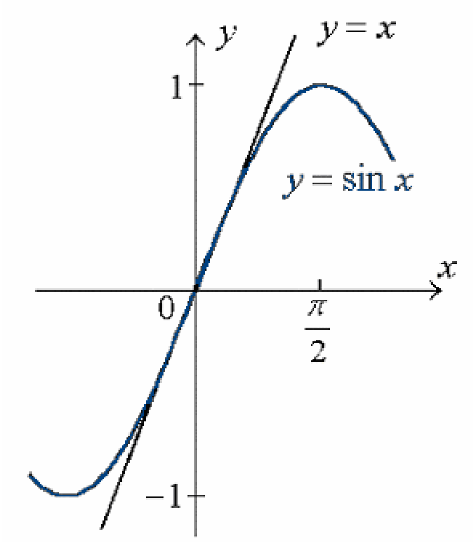
Recall that $\tan x = \dfrac{\sin x}{\cos x}$ and perform simple algebraic transformations:

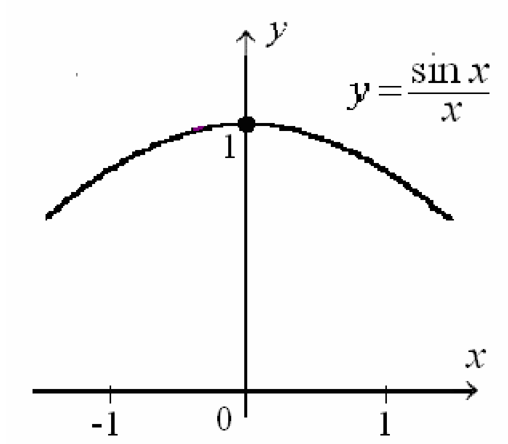
$$\sin x < x < \tan x \quad \Rightarrow \quad 1 < \frac{x}{\sin x} < \frac{1}{\cos x} \quad \Rightarrow$$

$$1 > \frac{\sin x}{x} > \cos x .$$

If $x \to 0$ then $\cos x \to 1$, and hence $\dfrac{\sin x}{x} \to 1$.

Graphic Illustrations:





Note: If $\alpha(x)$ is an infinitesimal function as $x \to a$, then

$$\sin \alpha(x) \sim \alpha(x) \qquad (\text{as } x \to a),$$

$$\lim_{x \to a} \frac{\sin \alpha(x)}{\alpha(x)} = 1,$$

independently of a type of the function $\alpha(x)$ and value of *a*. The only thing that matters is a smallness of $\alpha(x)$ as *x* tends to *a*.

For instance,

$$\sin(x^3 - 8) \sim (x^3 - 8) \qquad \text{as} \quad x \to 2,$$

$$\sin \frac{1}{x} \sim \frac{1}{x} \qquad \text{as} \quad x \to \infty .$$

Examples:

1. $\displaystyle \lim_{x \to 0} \frac{\sin 7x}{\sin 3x} = \lim_{x \to 0} \frac{7x}{3x} = \frac{7}{3}.$

Another **Solution**: $\displaystyle \lim_{x \to 0} \frac{\sin 7x}{\sin 3x} = \frac{7}{3} \lim_{x \to 0} \frac{\sin 7x}{7x} \frac{3x}{\sin 3x} = \frac{7}{3}.$

2. $$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sin x}{x\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}=1.$$

Therefore,

$$\boxed{\tan x \sim x \quad \text{as } x\to 0.}$$

3. Evaluate $\lim_{x\to 0}\frac{\arcsin x}{x}$.

Solution: By changing the variable $x=\sin t$ we obtain $t=\arcsin x$ and $t\to 0$ as $x\to 0$. Therefore,

$$\lim_{x\to 0}\frac{\arcsin x}{x}=\lim_{t\to 0}\frac{t}{\sin t}=1.$$

Thus, infinitesimal function $\arcsin x$ is equivalent to x as $x\to 0$,

$$\boxed{\arcsin x \sim x \quad \text{as } x\to 0}$$

4. Evaluate $\lim_{x\to 0}\frac{\arctan x}{x}$.

Solution: Likewise, substitution $t=\arctan x$ implies $x=\tan t$ and $t\to 0$ as $x\to 0$. Therefore,

$$\lim_{x\to 0}\frac{\arctan x}{x}=\lim_{x\to 0}\frac{x}{\tan x}=1,$$

$$\boxed{\arctan x \sim x \quad \text{as } x\to 0.}$$

5. Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Solution: By making use of the half-angle identity the numerator can be expressed through a sine function:

$$1-\cos x=2\sin^2\frac{x}{2}.$$

In view of Theorem 1, $\sin x/2 \sim x/2$. Therefore,

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim_{x\to 0}\frac{2(\frac{x}{2})^2}{x^2}=\lim_{x\to 0}\frac{x^2}{2x^2}=\frac{1}{2}.$$

It means that $1-\cos x\sim\frac{x^2}{2}$, or

$$\boxed{\cos x \sim 1-\frac{x^2}{2} \quad \text{as } x\to 0.}$$

6. Evaluate $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\sin x - \cos x}{\tan x - 1}$.

Solution: Using the identity $\tan x - 1 = \dfrac{\sin x}{\cos x} - 1 = \dfrac{\sin x - \cos x}{\cos x}$ we get

$$\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{\tan x - 1} = \lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{\sin x - \cos x}\cos x = \lim_{x\to\frac{\pi}{4}} \cos x = \frac{\sqrt{2}}{2}.$$

7. Evaluate $\lim\limits_{x\to 0} \dfrac{1-\cos 8x}{1-\cos 2x}$.

Solution: By the trigonometric identity $1-\cos t = 2\sin^2 \dfrac{t}{2}$,

$$\lim_{x\to 0} \frac{1-\cos 8x}{1-\cos 2x} = \lim_{x\to 0} \frac{2\sin^2 4x}{2\sin^2 x} = \lim_{x\to 0} \frac{(4x)^2}{x^2} = 16.$$

8. Evaluate $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{x^2}$.

Solution: Since $1-\cos^2 t = \sin^2 t$ and $\sin t \sim t$, then

$$\lim_{x\to 0} \frac{1-\cos^2 x}{x^2} = \lim_{x\to 0} \frac{\sin^2 x}{x^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

9. Evaluate $\lim\limits_{x\to 0} \dfrac{5\sin 2x}{4\tan 3x}$.

Solution: If $x \to 0$ then $\sin 2x \sim 2x$ and $\tan 3x \sim 3x$. Therefore,

$$\lim_{x\to 0} \frac{5\sin 2x}{4\tan 3x} = \frac{5}{4}\lim_{x\to 0} \frac{2x}{3x} = \frac{5}{6}.$$

10. Evaluate $\lim\limits_{x\to\infty} x^2 \sin \dfrac{1}{x^2}$.

Solution: Since $1/x^2$ is an infinitesimal function as $x \to \infty$, then $\sin 1/x^2 \sim 1/x^2$, and so

$$\lim_{x\to\infty} x^2 \sin\frac{1}{x^2} = \lim_{x\to\infty} x^2 \frac{1}{x^2} = 1.$$

11. Evaluate $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\tan 7x}{\tan 3x}$.

Solution: In order to evaluate an indeterminate form $\dfrac{\infty}{\infty}$, set $t = x - \dfrac{\pi}{2}$:

$$\tan 7x = \tan 7(t + \frac{\pi}{2}) = \tan(7t + \frac{7\pi}{2}) = -\cot 7t = -\frac{1}{\tan 7t}$$

and

$$\tan 3x = \tan 3(t + \frac{\pi}{2}) = -\frac{1}{\tan 3t}.$$

Since $x \to \frac{\pi}{2}$, then $t = x - \frac{\pi}{2} \to 0$. Therefore,

$$\lim_{x \to \frac{\pi}{2}} \frac{\tan 7x}{\tan 3x} = \lim_{t \to 0} \frac{\tan 3t}{\tan 7t} = \lim_{t \to 0} \frac{3t}{7t} = \frac{3}{7}.$$

12. Evaluate $\lim_{x \to 0} \frac{1 - \sqrt{\cos x}}{1 - \cos \sqrt{x}}$.

Solution: Transform the numerator:

$$1 - \sqrt{\cos x} = \frac{(1 - \sqrt{\cos x})(1 + \sqrt{\cos x})}{1 + \sqrt{\cos x}} = \frac{1 - (\sqrt{\cos x})^2}{1 + \sqrt{\cos x}}$$

$$= \frac{1 - \cos x}{1 + \sqrt{\cos x}} = \frac{2\sin^2 \frac{x}{2}}{1 + \sqrt{\cos x}} \overset{x \to 0}{\sim} \frac{2(\frac{x}{2})^2}{1 + 1} = \frac{x^2}{4}.$$

Now transform the denominator:

$$1 - \cos \sqrt{x} = 2\sin^2 \frac{\sqrt{x}}{2} \overset{x \to 0}{\sim} 2\left(\frac{\sqrt{x}}{2}\right)^2 = \frac{x}{2}.$$

Then we get

$$\lim_{x \to 0} \frac{1 - \sqrt{\cos x}}{1 - \cos \sqrt{x}} = \lim_{x \to 0} \frac{x^2/4}{x/2} = \lim_{x \to 0} \frac{x}{2} = 0.$$

13. Evaluate $\lim_{x \to 1} \frac{\tan(\pi x)}{x - 1}$.

Solution: Substitution $t = x - 1$ implies

$$\tan(\pi x) = \tan(\pi t + \pi) = \tan \pi t.$$

Since $x \to 1$, then $t = x - 1 \to 0$.

Therefore, $\tan \pi t \sim \pi t$ that results in

$$\lim_{x \to 1} \frac{\tan(\pi x)}{x - 1} = \lim_{t \to 0} \frac{\tan(\pi t)}{t} = \lim_{t \to 0} \frac{\pi t}{t} = \pi.$$

2.3.1.1. Calculus of Approximations

Here and below we use the radian measure of angles unless the contrary is allowed.

Theorem 1 states the approximation formula

$$\sin x \approx x,$$

which is valid for any x in some small vicinity of zero.

Other trigonometric functions can be expressed through the sine function. For instance,

$$\cos x = 1 - 2\sin^2 \frac{x}{2} \approx 1 - 2\cdot(\frac{x}{2})^2 = 1 - \frac{x^2}{2},$$

$$\tan x = \frac{\sin x}{\cos x} \approx \frac{x}{1 - x^2/2} \approx x.$$

The below drawings illustrate the error range of the above approximation formulas. A measure of inaccuracy $\frac{\cos x - (1 - x^2/2)}{\cos x}\cdot 100\,\%$ is shown in the additional window.

We can hardly ever see any differences between graphs of functions $y = \cos x$ and $y = 1 - \frac{x^2}{2}$ for $|x| < 0.8\,\text{rad} \approx 45°$.

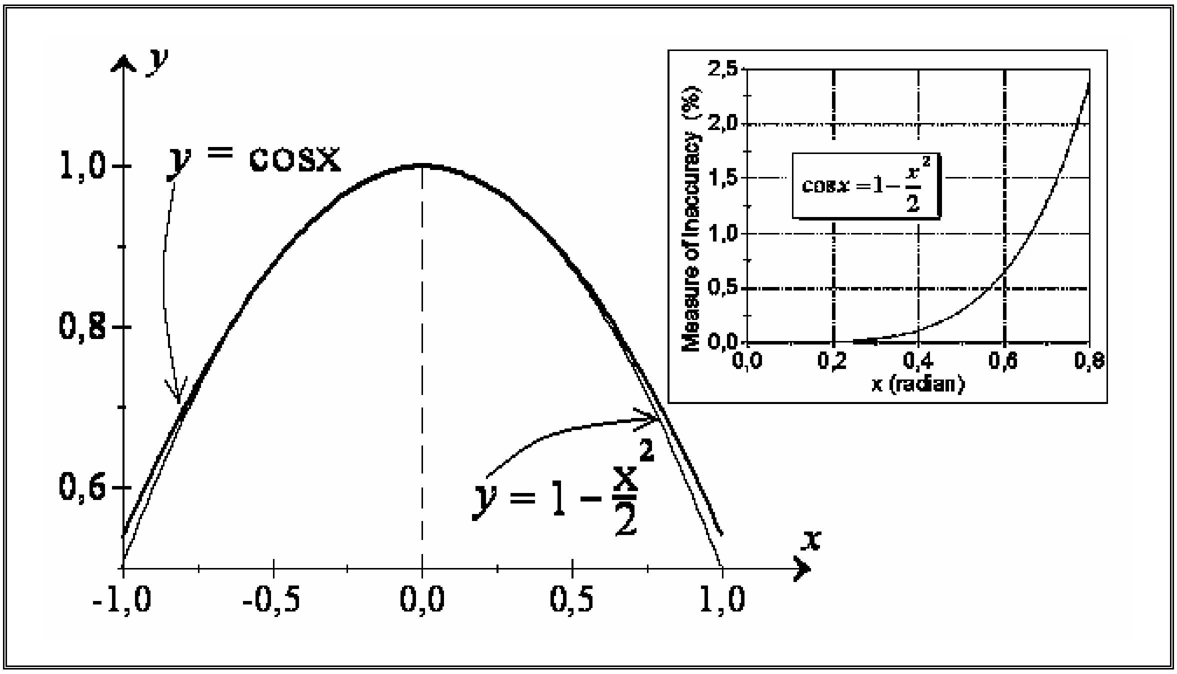
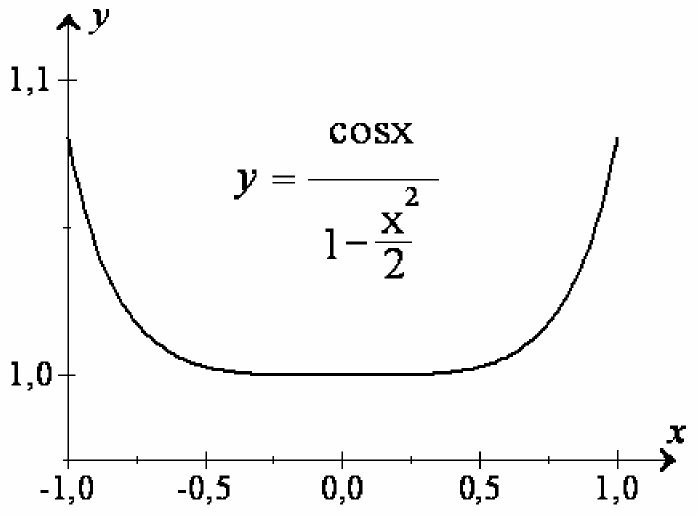


Fig. 1. Graphs of functions $y = \cos x$ (upper curve) and $y = 1 - x^2/2$ (lower curve).

The ratio of the cosine function to its polynomial approximation is shown in Fig.2. One can see that the quadratic polynomial $1-\frac{x^2}{2}$ fits well the cosine function in a wide range of values of *x*.



In Fig. 2, the graphs of the functions $y = \tan x$ and $y = x$ are presented. A measure of inaccuracy $\frac{\tan x - x}{\tan x} \cdot 100\%$ is shown in the additional window.

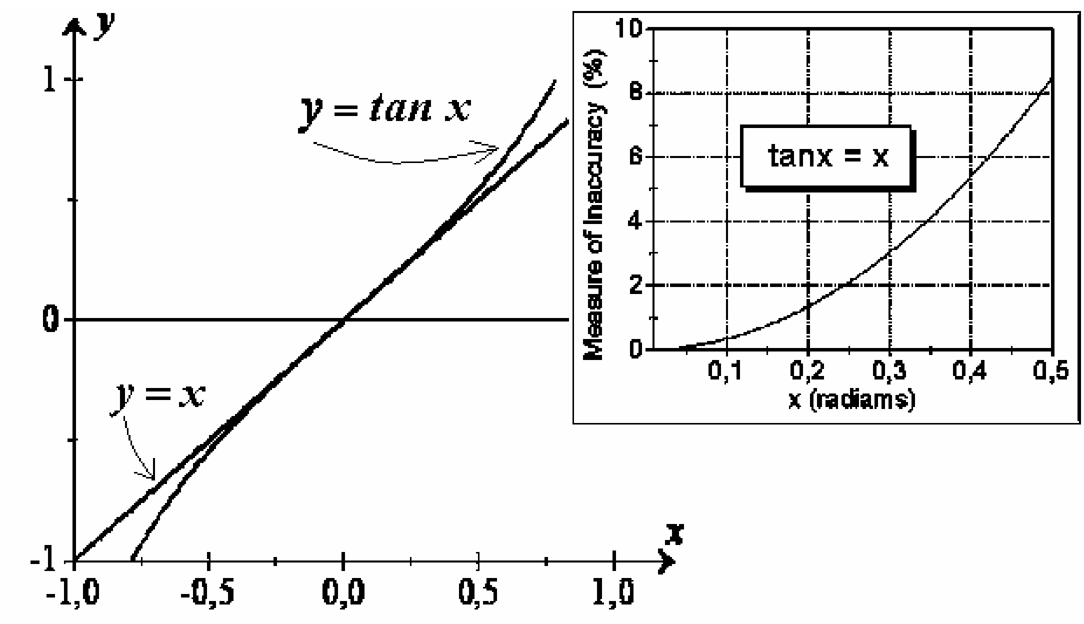


Fig. 2. The graph of the functions $y = \tan x$ and $y = x$.

Consider a few numerical examples.

Approximation formulas for $f(x)$	Approximate values of $f(x)$	Exact values of $f(x)$
$\sin x \approx x$	$\sin 10° = \sin\frac{\pi}{18} \approx \frac{\pi}{18} \approx 0.1745$	0.1736…
	$\sin 30° = \sin\frac{\pi}{6} \approx \frac{\pi}{6} \approx 0.52$	0.5
$\cos x \approx 1 - \frac{x^2}{2}$	$\cos 20° = \cos\frac{\pi}{9} \approx 1 - \frac{1}{2}(\frac{\pi}{9})^2 \approx 0.9391$	0.9397…
	$\cos 30° = \cos\frac{\pi}{6} \approx 1 - \frac{\pi^2}{72} \approx 0.8629$	$\frac{\sqrt{3}}{2} = 0.8660...$
$\tan x \approx x$	$\tan 10° = \tan\frac{\pi}{18} \approx \frac{\pi}{18} \approx 0.1745$	0.1763…
	$\tan 20° = \tan\frac{\pi}{9} \approx \frac{\pi}{9} \approx 0.3491$	0.36397…

2.3.2. Theorem 2

$$\lim_{x \to 0}(1+x)^{\frac{1}{x}} = e$$

$(e = 2.71828\ldots)$

Using substitution $t = \frac{1}{x}$ and then returning to the symbol *x*, we can express Theorem 2 in the other form:

$$\lim_{x \to \infty}(1+\frac{1}{x})^x = e\,.$$

(See detailed discussion of the theorem in Chapter 1, pp. 28-31).

Note: If $\alpha(x)$ is an infinitesimal function as $x \to a$, then

$$\lim_{x \to a}(1+\alpha(x))^{\frac{1}{\alpha(x)}} = e\,.$$

For instance,

$$\lim_{x \to 0}(1+\sin x)^{1/\sin x} = e\,,$$

$$\lim_{x \to \infty}\left(1+\frac{1}{5x}\right)^{5x} = e\,.$$

The well-known logarithmic identity

$$a^b = e^{b\ln a}$$

can be generalized by the limit process that results in the following **Important Rule:**

If $\alpha(x)$ and $\beta(x)$ are infinitesimal functions as $x \to a$, then

$$\lim_{x\to a}(1+\alpha(x))^{\frac{1}{\beta(x)}} = e^{\lim\limits_{x\to a}\frac{\ln(1+\alpha(x))}{\beta(x)}}. \qquad (8)$$

Let $f(x) \to 1$ and $g(x) \to \infty$ as $x \to a$.

Then $(f(x))^{g(x)}$ is an indeterminate form 1^∞ as $x \to a$.

To apply Theorem 2, it is necessary to reduce $(f(x))^{g(x)}$ to the standard form $(1+\alpha(x))^{\frac{1}{\alpha(x)}}$, where $\alpha(x)$ is an infinitesimal function as x tends to infinity.

The general procedure of the reducing is the following:

$$f^g = [1^\infty] = (1+(f-1))^g$$

$$= (1+(f-1))^{\frac{1}{f-1}\cdot g\cdot(f-1)} = [1+\alpha]^{\frac{1}{\alpha}\cdot g\cdot(f-1)},$$

where $\alpha = f-1$. Thus,

$$\lim_{x\to a}(f(x))^{g(x)} = \lim_{x\to a}(1+(f(x)-1))^{\frac{1}{f(x)-1}\cdot g(x)\cdot(f(x)-1)}$$
$$= e^{\lim\limits_{x\to a} g(x)\cdot(f(x)-1)}, \qquad (9)$$

that is the given problem is reduced to evaluation of $\lim\limits_{x\to a} g(x)(f(x)-1)$.

Examples:

1. Evaluate $\lim\limits_{x\to\infty}\left(\frac{x+4}{x}\right)^x$.

Solution: Since $(1+4/x)^{\frac{x}{4}} \to e$ as $x\to\infty$, then

$$\lim_{x\to\infty}\left(\frac{x+4}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x = \lim_{x\to\infty}\left(\left(1+\frac{4}{x}\right)^{\frac{x}{4}}\right)^4 = \left(\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^{\frac{x}{4}}\right)^4 = e^4.$$

2. Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x}\right)^{x+2}$.

Solution: Note that $\left(\dfrac{x+3}{x}\right)^{x+2}=\left(1+\dfrac{3}{x}\right)^{x}\left(1+\dfrac{3}{x}\right)^{2}$, where $\left(1+\dfrac{3}{x}\right)^{x}$ is an indeterminate form 1^{∞} as $x\to\infty$, while the expression $(1+3/x)^2$ tends to 1 as $x\to\infty$.
Therefore,

$$\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^{x+2}=\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x}\left(1+\frac{3}{x}\right)^{2}=\lim_{x\to\infty}\left(\left(1+\frac{3}{x}\right)^{\frac{x}{3}}\right)^{3}=e^{3}.$$

3. Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{x-4}{x}\right)^{x}$.

Solution:

$$\lim_{x\to\infty}\left(\frac{x-4}{x}\right)^{x}=\lim_{x\to\infty}\left(1-\frac{4}{x}\right)^{x}=\lim_{x\to\infty}\left(1-\frac{4}{x}\right)^{(-\frac{x}{4})\cdot(-4)}=e^{-4}=\frac{1}{e^{4}}.$$

4. Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x+5}\right)^{x}$.

Solution:

$$\lim_{x\to\infty}\left(\frac{x+3}{x+5}\right)^{x}=\lim_{x\to\infty}\left(\frac{(x+3)/x}{(x+5)/x}\right)^{x}=\lim_{x\to\infty}\left(\frac{1+3/x}{1+5/x}\right)^{x}$$

$$=\frac{\lim\limits_{x\to\infty}(1+3/x)^{x}}{\lim\limits_{x\to\infty}(1+5/x)^{x}}=\frac{\lim\limits_{x\to\infty}(1+3/x)^{\frac{x}{3}\cdot 3}}{\lim\limits_{x\to\infty}(1+5/x)^{\frac{x}{5}\cdot 5}}=\frac{e^{3}}{e^{5}}=\frac{1}{e^{2}}.$$

5. Evaluate $\lim\limits_{x\to 1}(2x-1)^{\frac{4x}{x-1}}$.

Solution:

$$\lim_{x\to 1}(2x-1)^{\frac{4x}{x-1}}=\lim_{x\to 1}(1+2(x-1))^{\frac{8x}{2(x-1)}}$$

$$=\lim_{x\to 1}\left((1+2(x-1))^{\frac{1}{2(x-1)}}\right)^{8x}=\lim_{x\to 1}e^{8x}=e^{8}.$$

2.3.3. Theorem 3

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$$

This statement can also be expressed in the form

$$\ln(1+x)\sim x \quad \text{as} \quad x\to 0.$$

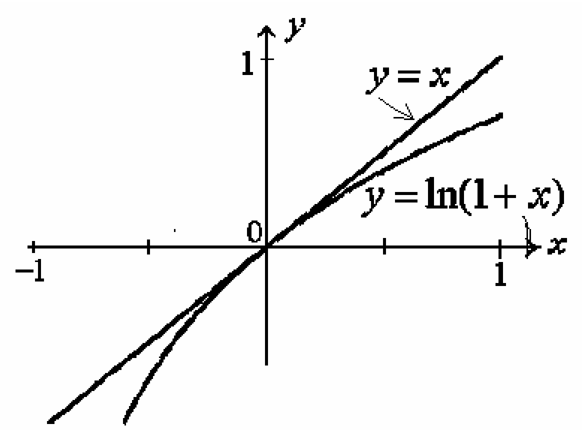
Proof: By the properties of logarithms,

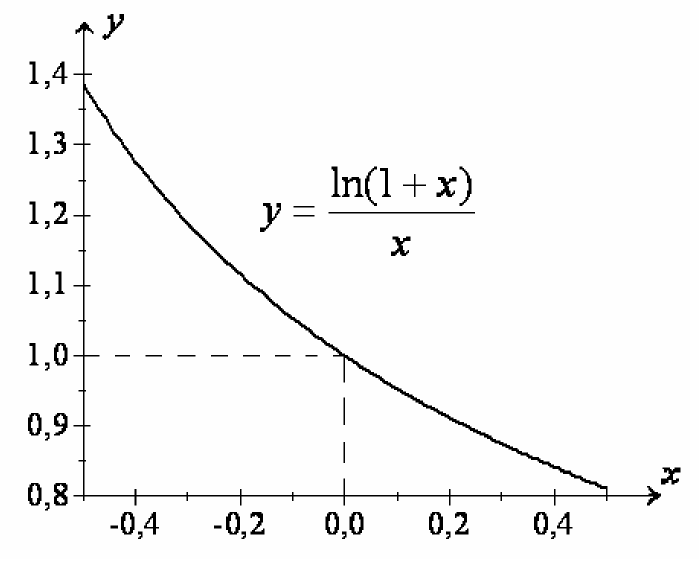
$$\frac{1}{x}\ln(1+x)=\ln(1+x)^{\frac{1}{x}}.$$

Recall that $(1+x)^{\frac{1}{x}}\to e$ as $x\to 0$. However, if the quantity $(1+x)^{\frac{1}{x}}$ approaches number *e*, then its natural logarithm tends to $\ln e$,

$$\ln(1+x)^{\frac{1}{x}}\to \ln e=1 \quad \text{as} \quad x\to 0.$$

Graphic Illustrations:





Note: If $\alpha(x)$ is an infinitesimal function as $x \to a$, then

$$\ln(1+\alpha(x)) \sim \alpha(x) \quad \text{as } x \to a,$$

$$\lim_{x\to a} \frac{\ln(1+\alpha(x))}{\alpha(x)} = 1.$$

For instance,

$$\ln(1+\sqrt[3]{2-x}) \sim \sqrt[3]{2-x} \qquad \text{as } x \to 2,$$

$$\ln(1-\sin^2 x) \sim -\sin^2 x \qquad \text{as } x \to 0,$$

$$\lim_{x\to 0} \frac{\ln(1+2x^4-5x)}{2x^4-5x} = 1.$$

Examples:

1. Evaluate $\lim_{x\to 0} \frac{\ln(1-2x)}{7x}$.

Solution: In view of the above note,

$$\lim_{x\to 0} \frac{\ln(1-2x)}{7x} = -\frac{2}{7} \lim_{x\to 0} \frac{\ln(1-2x)}{-2x} = -\frac{2}{7}.$$

2. Evaluate $\lim_{x\to 0} \frac{\ln(1+\sin 4x)}{\tan 2x}$.

Solution: By Theorem 1, $\sin 4x \sim 4x$ and $\tan 2x \sim 2x$ as $x \to 0$.
In addition, $\ln(1+\sin 4x) \sim \sin 4x \sim 4x$ as $x \to 0$.
Therefore,

$$\lim_{x\to 0} \frac{\ln(1+\sin 4x)}{\tan 2x} = \lim_{x\to 0} \frac{4x}{2x} = 2.$$

3. Evaluate $\lim_{x\to 0} \frac{\ln(1+9x-\arcsin^2 x)}{\tan 3x}$.

Solution: First, $(9x-\arcsin^2 x)$ is infinitesimal function as $x \to 0$, and so

$$\ln(1+9x-\arcsin^2 x) \sim (9x-\arcsin^2 x).$$

Second, $\arcsin^2 x \sim x^2$ is a negligible quantity with respect to $9x$ as $x \to 0$, and so

$$(9x-\arcsin^2 x) \sim 9x.$$

Third, $\tan 3x \sim 3x$ as $x \to 0$.
Finally,

$$\lim_{x\to 0} \frac{\ln(1+9x-\arcsin^2 x)}{\tan 3x} = \lim_{x\to 0} \frac{9x}{3x} = 3.$$

4. Evaluate $\lim\limits_{x\to\infty} x\cdot(\ln(x+2)-\ln x)$.

Solution: In view of the logarithm properties,

$$\ln(x+2)-\ln x=\ln\frac{x+2}{x}=\ln(1+\frac{2}{x}).$$

Since $2/x$ is an infinitesimal function as $x\to\infty$, then $\ln(1+\frac{2}{x})\sim\frac{2}{x}$.

Therefore,

$$\lim_{x\to\infty} x\cdot(\ln(x+2)-\ln x)=\lim_{x\to\infty}\frac{\ln(1+\frac{2}{x})}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{2}{x}}{\frac{1}{x}}=2.$$

5. Find $\lim\limits_{x\to\infty}(x\cdot\ln\sqrt{\frac{x+1}{x-1}})$.

Solution: In a similar way,

$$\lim_{x\to\infty} x\cdot\ln\sqrt{\frac{x+1}{x-1}}=\frac{1}{2}\lim_{x\to\infty} x\cdot\ln\frac{x+1}{x-1}=\frac{1}{2}\lim_{x\to\infty} x\cdot\ln\frac{1+1/x}{1-1/x}$$

$$=\frac{1}{2}\lim_{x\to\infty}\frac{\ln(1+1/x)-\ln(1-1/x)}{1/x}=\frac{1}{2}\lim_{x\to\infty}\frac{1/x-(-1/x)}{1/x}=\frac{1}{2}\lim_{x\to\infty}\frac{2/x}{1/x}=1.$$

6. Evaluate $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$.

Solution: By Rule (8) (see Theorem 2, p. 74), $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}=e^A$,

where $A=\lim\limits_{x\to 0}\ln(\cos x)^{\frac{1}{x^2}}=\lim\limits_{x\to 0}\frac{\ln(\cos x)}{x^2}$.

Using the fundamental trigonometric identity

$$\cos^2 x=1-\sin^2 x$$

and in view of the properties of logarithms, we obtain

$$A=\lim_{x\to 0}\frac{\ln(\cos x)}{x^2}=\lim_{x\to 0}\frac{\ln\sqrt{1-\sin^2 x}}{x^2}=\lim_{x\to 0}\frac{\ln(1-\sin^2 x)}{2x^2}.$$

However, $\ln(1-\sin^2 x)\sim-\sin^2 x\sim -x^2$, and so $A=\lim\limits_{x\to 0}\frac{-x^2}{2x^2}=-\frac{1}{2}$.

Therefore, $\lim\limits_{x\to 0}(\cos x)^{1/x^2}=e^{-1/2}=1/\sqrt{e}$.

2.3.4. Theorem 4

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

The statement can be also expressed in the form

$$e^x-1\sim x \quad \text{as} \quad x\to 0.$$

Proof: By the substitution $t=e^x-1$, we obtain

$$e^x=1+t \quad \Rightarrow \quad x=\ln(1+t).$$

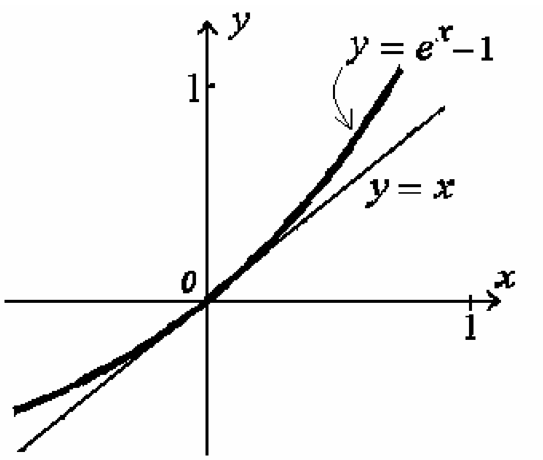
Therefore,

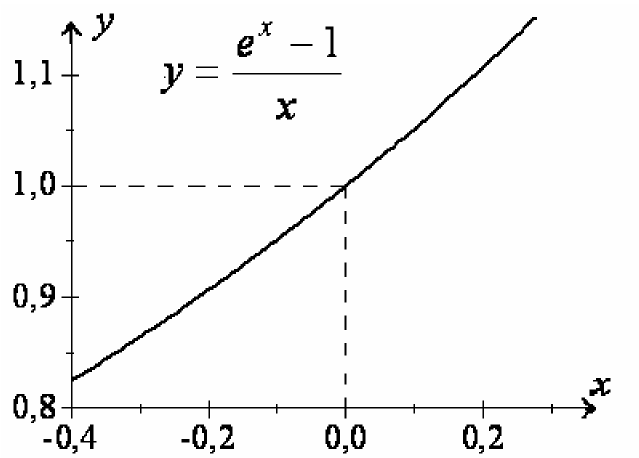
$$\frac{e^x-1}{x}=\frac{t}{\ln(1+t)}.$$

If $x\to 0$ then $t=e^x-1\to e^0-1=0$, and so

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{t\to 0}\frac{t}{\ln(1+t)}=1.$$

Graphic Illustrations:





Note: If $\alpha(x)$ is an infinitesimal function as $x \to a$, then

$$e^{\alpha(x)} - 1 \sim \alpha(x) \quad \text{ as } x \to a,$$

$$\lim_{x \to a} \frac{e^{\alpha(x)} - 1}{\alpha(x)} = 1.$$

For instance, $e^{5x} - 1 \sim 5x$ as $x \to 0$.

Examples:

1. Evaluate $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$.

Solution:

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = 2 \lim_{x \to 0} \frac{e^{2x} - 1}{2x} = 2.$$

2. Evaluate $\lim_{x \to 0} \frac{e^{\sin x} - 1}{4x}$.

Solution: Since $(e^{\sin x} - 1) \sim \sin x$ as $x \to 0$, then

$$\lim_{x \to 0} \frac{e^{\sin x} - 1}{4x} = \frac{1}{4} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{4}.$$

3. Evaluate $\lim_{x \to 0} \frac{e^{\sqrt{x}} - 1 + x}{\tan x}$.

Solution: First, $(e^{\sqrt{x}} - 1) \sim \sqrt{x}$ and $\tan x \sim x$ as $x \to 0$.
Second, x is an infinitesimal function of a higher order of smallness with respect to $\sqrt{x}$, and hence

$$\sqrt{x} + x \sim \sqrt{x} \quad \text{ as } x \to 0.$$

Finally,

$$\lim_{x \to 0} \frac{e^{\sqrt{x}} - 1 + x}{\tan x} = \lim_{x \to 0} \frac{\sqrt{x}}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x}} = \infty.$$

4. Evaluate $\lim_{x \to 1} \frac{e^x - e}{x^2 - 1}$.

Solution: Setting $t = x - 1$ and noting that $t \to 0$ as $x \to 1$ we obtain

$$\lim_{x \to 1} \frac{e^x - e}{x^2 - 1} = \lim_{t \to 0} \frac{e^{t+1} - e}{t(t+2)} = e \lim_{t \to 0} \frac{e^t - 1}{t(t+2)}$$

$$= e \lim_{t \to 0} \frac{e^t - 1}{t} \cdot \lim_{t \to 0} \frac{1}{t+2} = e \cdot 1 \cdot \frac{1}{2} = \frac{e}{2}.$$

5. Evaluate $\lim\limits_{x\to 0}\dfrac{e^{x^2}-\cos 2x}{x^2}$.

Solution: Since $\cos 2x \sim 1-\dfrac{(2x)^2}{2}=1-2x^2$ and $e^{x^2}-1\sim x^2$ as $x\to 0$, we get

$$\lim_{x\to 0}\frac{e^{x^2}-\cos 2x}{x^2}=\lim_{x\to 0}\frac{e^{x^2}-1+2x^2}{x^2}=\lim_{x\to 0}\frac{x^2+2x^2}{x^2}=3.$$

6. Evaluate $\lim\limits_{x\to 0}\dfrac{e^{-4x}-\cos 2x+4x}{x\ln(1+x)+\tan x^2}$.

Solution: By making use of the following relations of equivalence

$$\cos 2x \sim 1-2x^2,$$

$$e^{-4x}-1\sim -4x,$$

$$\ln(1+x)\sim x,$$

$$\tan x^2 \sim x^2$$

as $x\to 0$ we obtain

$$\lim_{x\to 0}\frac{e^{-4x}-\cos 2x+4x}{x\ln(1+x)+\tan x^2}=\lim_{x\to 0}\frac{-4x+2x^2+4x}{x\cdot x+x^2}=\lim_{x\to 0}\frac{2x^2}{2x^2}=1.$$

7. Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 3x+\ln(1-2x)+4\arctan 5x}{7x-2\arcsin 2x+e^{6x}-1}$.

Solution: Likewise, in view of the relations of equivalence:

$$\sin 3x\sim 3x,$$

$$\ln(1-2x)\sim -2x,$$

$$\arctan 5x\sim 5x,$$

$$\arcsin 2x\sim 2x,$$

$$e^{6x}-1\sim 6x,$$

it is not difficult to evaluate the given indeterminate form:

$$\lim_{x\to 0}\frac{\sin 3x+\ln(1-2x)+4\arctan 5x}{7x-2\arcsin 2x+e^{6x}-1}$$

$$=\lim_{x\to 0}\frac{3x+(-2x)+4\cdot 5x}{7x-2\cdot 2x+6x}=\lim_{x\to 0}\frac{21x}{9x}=\frac{7}{3}.$$

2.3.5. Theorem 5

For any number n

$$\lim_{x\to 0}\frac{(1+x)^n-1}{x}=n\,.$$

This statement can be also written in the form

$$(1+x)^n-1\sim nx \quad \text{as} \quad x\to 0\,.$$

Proof: Let $t=(1+x)^n-1$. Then

$$(1+x)^n=1+t \quad \Rightarrow \quad n\ln(1+x)=\ln(1+t)\,.$$

Therefore,

$$\frac{(1+x)^n-1}{x}=\frac{t}{x}=\frac{t}{x}\cdot\frac{\ln(1+x)}{\ln(1+x)}$$

$$=n\frac{t}{n\ln(1+x)}\cdot\frac{\ln(1+x)}{x}=n\frac{t}{\ln(1+t)}\cdot\frac{\ln(1+x)}{x}.$$

Note that $t=(1+x)^n-1\to 0$ as $x\to 0$.

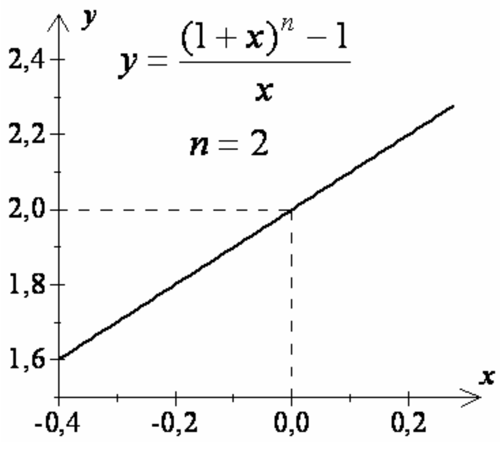
By Theorem 3,

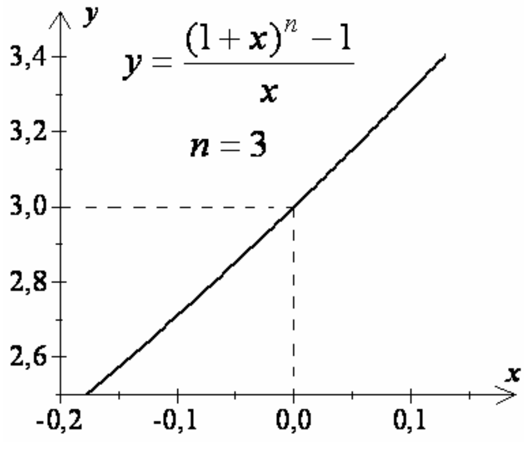
$$\frac{\ln(1+x)}{x}\to 1 \quad \text{as} \quad x\to 0 \quad \text{and} \quad \frac{1}{\ln(1+t)}\to 1 \quad \text{as } t\to 0\,.$$

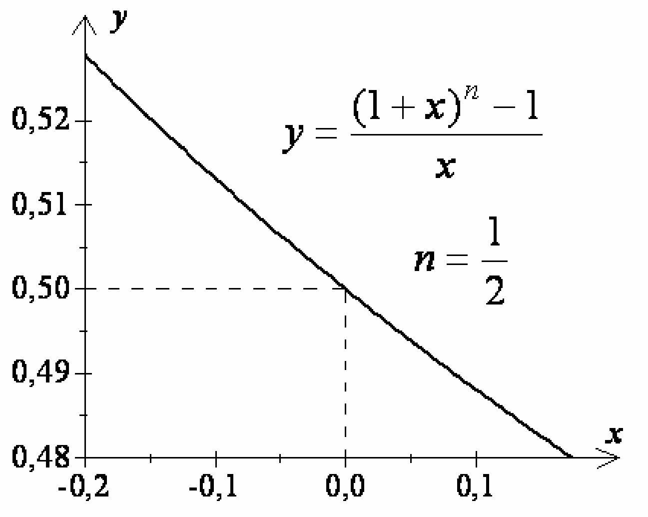
Hence,

$$\frac{(1+x)^n-1}{x}=n\frac{t}{\ln(1+t)}\cdot\frac{\ln(1+x)}{x}\to n \quad \text{as } x\to 0\,.$$

Graphic Illustrations:







Note: If $\alpha(x)$ is an infinitesimal function as $x \to a$, then

$$(1+\alpha(x))^n - 1 \sim n \cdot \alpha(x) \quad \text{(as } x \to a\text{)},$$

$$\lim_{x \to a} \frac{(1+\alpha(x))^n - 1}{\alpha(x)} = n.$$

For instance,

$$\sqrt[4]{1+\cos x} - 1 \sim \frac{1}{4}\cos x \quad \text{as} \quad x \to \frac{\pi}{2},$$

$$\lim_{x \to 0} \frac{(1+\sin^2 3x)^5 - 1}{\sin^2 3x} = 5.$$

Examples:

1. Evaluate $\lim\limits_{x \to 0} \frac{\sqrt{1+3x} - 1}{x}$.

Solution: By Theorem 5 (case $n = \frac{1}{2}$),

$$\lim_{x \to 0} \frac{\sqrt{1+3x} - 1}{x} = 3 \lim_{x \to 0} \frac{\sqrt{1+3x} - 1}{3x} = 3 \cdot \frac{1}{2} = \frac{3}{2}.$$

2. Evaluate $\lim\limits_{x \to 0} \frac{\sqrt[5]{1-2x} - 1}{\sin x}$.

Solution: By Theorem 5 (case $n = 1/5$) and in view of the relation of equivalence $\sin x \sim x$ (as $x \to 0$), we obtain

$$\lim_{x \to 0} \frac{\sqrt[5]{1-2x} - 1}{\sin x} = -2 \lim_{x \to 0} \frac{\sqrt[5]{1-2x} - 1}{-2x} = -2 \cdot \frac{1}{5} = -\frac{2}{5}.$$

3. Evaluate $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+4x}-1}{\sqrt{1-7x}-1}$.

Solution: By Theorem 5 (cases $n=1/3$ and $n=1/2$),

$$\frac{\sqrt[3]{1+4x}-1}{4x}\to\frac{1}{3}\quad\text{and}\quad\frac{-7x}{\sqrt{1-7x}-1}\to 2\quad\text{as}\quad x\to 0.$$

Therefore,

$$\frac{\sqrt[3]{1+4x}-1}{\sqrt{1-7x}-1}=-\frac{4}{7}\cdot\frac{\sqrt[3]{1+4x}-1}{4x}\cdot\frac{-7x}{\sqrt{1-7x}-1}\to-\frac{4}{7}\cdot\frac{1}{3}\cdot 2=-\frac{8}{21}.$$

4. Evaluate $\lim\limits_{x\to 0}\dfrac{(1+6x)^{2002}-1}{(1-3x)^{2002}-1}$.

Solution:

$$\lim_{x\to 0}\frac{(1+6x)^{2002}-1}{(1-3x)^{2002}-1}=-\frac{6}{3}\lim_{x\to 0}\frac{(1+6x)^{2002}-1}{6x}\cdot\frac{-3x}{(1-3x)^{2002}-1}$$

$$=-2\lim_{x\to 0}\frac{(1+6x)^{2002}-1}{6x}\cdot\lim_{x\to 0}\frac{-3x}{(1-3x)^{2002}-1}$$

$$=-2\cdot 2002\cdot\frac{1}{2002}=-2.$$

5. Evaluate $\lim\limits_{x\to 0}\dfrac{(1+2x)^{100}-1}{\sqrt[5]{1+20x}-1}$.

Solution: Since $x\to 0$,

$$(1+2x)^{100}-1\sim 2x\cdot 100=200x\quad\text{and}$$

$$\sqrt[5]{1+20x}-1\sim 20x\cdot\frac{1}{5}=4x.$$

Therefore,

$$\lim_{x\to 0}\frac{(1+2x)^{100}-1}{\sqrt[5]{1+20x}-1}=\lim_{x\to 0}\frac{200x}{4x}=50.$$

6. Evaluate $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+3x}-x-\cos 4x}{x^2}$.

Solution: Apply the relations of equivalence to evaluate the limit:

$$\sqrt[3]{1+3x}\sim 1+\frac{3x}{3}=1+x\quad(\text{as}\quad x\to 0),$$

$$\cos 4x\sim 1-(4x)^2/2=1-8x^2\quad(\text{as}\quad x\to 0),$$

$$\lim_{x\to 0}\frac{\sqrt[3]{1+3x}-x-\cos 4x}{x^2}=\lim_{x\to 0}\frac{x-x-8x^2}{x^2}=-8\lim_{x\to 0}\frac{x^2}{x^2}=-8.$$

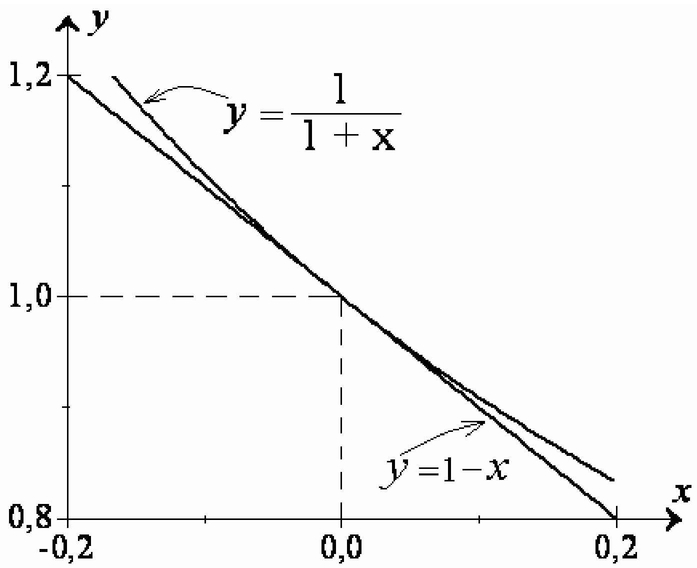
2.3.5.1. Calculus of Approximations

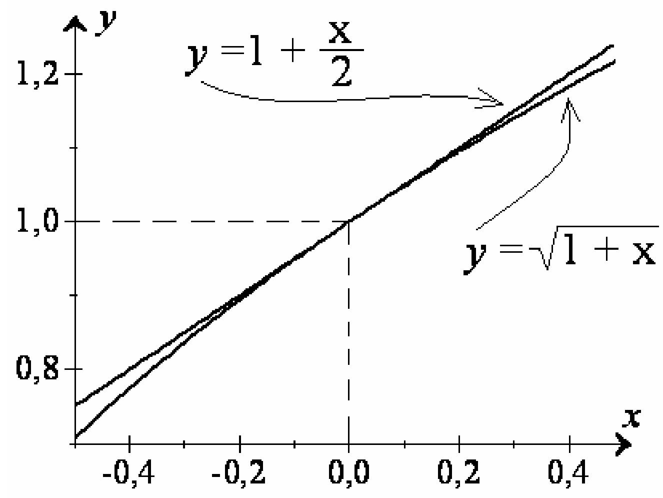
Theorem 5 for $n = -1, \frac{1}{2}, \frac{1}{3}$ yields the following approximation formulas:

$$\frac{1}{1+x} \approx 1 - x,$$

$$\sqrt{1+x} \approx 1 + \frac{x}{2}.$$

The graphic illustrations below give a pictorial presentation about a range of accuracy of the approximations. We see that each of the above formulas can be considered as the first approximation which is valid only in an immediate neighborhood of the zero point. Note that the straight lines $y = 1 - x$ and $y = 1 + \frac{x}{2}$ are the tangents to the curves $y = \frac{1}{1+x}$ and $y = \sqrt{1+x}$, correspondingly, both at the point $x = 0$.





Other Approximations in an immediate neighborhood of the point $x = 0$:

$$\frac{1}{1-x} \approx 1 + x,$$

$$\sqrt[3]{1+x} \approx 1 + \frac{x}{3},$$

$$\sqrt[n]{1+x} \approx 1 + \frac{x}{n}.$$

Numerical Examples:

Approximation formulas for $f(x)$	Approximate values of $f(x)$	Exact values of $f(x)$
$\frac{1}{1+x} \approx 1 - x$	$\frac{1}{1.1} = \frac{1}{1+0.1} \approx 1 - 0.1 = 0.9$	0.90909…
$\frac{1}{1-x} \approx 1 + x$	$\frac{1}{0.95} = \frac{1}{1-0.05} \approx 1 + 0.05 = 1.05$	1.09545…
$\sqrt{1+x} \approx 1 + \frac{x}{2}$	$\sqrt{1.2} = \sqrt{1+0.2} \approx 1 + \frac{0.2}{2} = 1.1$	0.9397…
	$\sqrt{0.9} = \sqrt{1-0.1} \approx 1 - \frac{0.1}{2} = 0.95$	0.94868…
$\sqrt[3]{1+x} \approx 1 + \frac{x}{3}$	$\sqrt[3]{1.15} = \sqrt[3]{1+0.15} \approx 1 + \frac{0.15}{3} = 1.05$	1.04769…

In order to calculate an approximate value of $\sqrt[3]{120}$, it is necessary to represent the given number in the form

$$\sqrt[3]{125-5} = \sqrt[3]{125(1-\frac{5}{125})} = 5\sqrt[3]{1-0.04},$$

and then to apply the corresponding approximation formula.

$$\sqrt[3]{120} = 5\sqrt[3]{1-0.04} \approx 5(1-\frac{0.04}{3}) = 5 - \frac{0.2}{3} = 4.9333...$$

The exact value of $\sqrt[3]{120}$ is 4.93242…

2.4. Summary: Infinitesimal Analysis

Infinitesimal functions are notions of fundamental importance, because many concepts in the theory of limits can be expressed in terms of infinitesimals. In particular, an infinite large function is the reciprocal quantity of an infinitesimal one. The limit of a function can also be defined by making use of the concept of infinitesimal quantities.

Due to the concept of equivalent infinitesimal functions, evaluation of indeterminate forms involving, for example, logarithmic or trigonometric functions, can be reduced to operations with simple power functions.

The main purpose of the limit process is evaluation of indeterminate forms. The underlying cause of arising difficulties consists in a variety of indeterminate forms:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0\cdot\infty, \quad \infty-\infty, \quad 1^{\infty}, \quad \infty^{0}, \quad \text{and} \quad 0^{0}.$$

The indeterminate form $\frac{0}{0}$ represents the ratio of two infinitesimal quantities. The others can be algebraically transformed to the indeterminate forms $\frac{0}{0}$.

Really, if f and g are infinite large quantity then

$$\frac{f}{g}=\frac{1/g}{1/f}=[\frac{0}{0}].$$

In a similar way,

$$0\cdot\infty=0\cdot\frac{1}{0}=[\frac{0}{0}] \qquad \text{and} \qquad \infty-\infty=f-g=\frac{1}{\frac{1}{f}}-\frac{1}{\frac{1}{g}}=\frac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{f}\cdot\frac{1}{g}}=[\frac{0}{0}].$$

By making use of the formula $f^{g}=e^{g\ln f}$, expressions 1^{∞}, ∞^{0}, and 0^{0} are reduced to the indeterminate form $0\cdot\infty$ considered above, and hence, each of them can be presented in the form of $\frac{0}{0}$:

$$\text{if } f(x)\to 1 \text{ and } g(x)\to\infty, \text{ then } g\ln f=\infty\cdot 0=\frac{0}{1/\infty}=[\frac{0}{0}];$$

$$\text{if } f(x)\to\infty \text{ and } g(x)\to 0, \text{ then } g\ln f=0\cdot\infty=\frac{0}{1/\infty}=[\frac{0}{0}];$$

$$\text{if } f(x)\to 0 \text{ and } g(x)\to 0, \text{ then } g\ln f=0\cdot\infty=\frac{0}{1/\infty}=[\frac{0}{0}].$$

2.5. Table of Often Used Equivalent Infinitesimal Functions

Relations of Equivalency as $x \to 0$
$\sin x \sim x$
$1 - \cos x \sim \frac{x^2}{2}$
$\tan x \sim x$
$\arcsin x \sim x$
$\arctan x \sim x$
$\ln(1+x) \sim x$
$e^x - 1 \sim x$
$(1+x)^n - 1 \sim nx$
$1 - \frac{1}{1+x} \sim x$
$\frac{1}{1-x} - 1 \sim x$
$\sqrt{1+x} - 1 \sim \frac{x}{2}$
$\sqrt[3]{1+x} - 1 \sim \frac{x}{3}$
$\sqrt[n]{1+x} - 1 \sim \frac{x}{n}$

2.5.1. Review Exercises

In Problems 1 through 30 evaluate indeterminate forms:

1.	$\lim_{x\to 3}\frac{\sin(x^2-9)}{3-x}$	2.	$\lim_{x\to -2}\frac{\sin(x+2)}{8+x^3}$
3.	$\lim_{x\to 1}\frac{\tan x\pi}{x-1}$	4.	$\lim_{x\to 1}\frac{\tan x\pi}{\sqrt{x}-1}$
5.	$\lim_{x\to 1}\frac{\tan(x-1)}{\sqrt{x}-1}$	6.	$\lim_{x\to\infty} x^2\tan\frac{1}{x^2}$
7.	$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{\tan 7x}$	8.	$\lim_{x\to 1}\frac{1+\cos\pi x}{\sin 3\pi x}$
9.	$\lim_{x\to 0}\frac{1-\cos 8x}{x\sin 3x}$	10.	$\lim_{x\to 0}\frac{1-\sqrt{\cos 4x}}{1-\cos\sqrt{2x}}$
11.	$\lim_{x\to 1}\frac{\arcsin(1-x^2)}{1-\sqrt{x}}$	12.	$\lim_{x\to 2}\frac{\arctan(x^2-3x+2)}{x^2-x-2}$
13.	$\lim_{x\to\infty}(\frac{x^2+3x-1}{x+1}-\frac{x^2-3x+1}{2x+1})$	14.	$\lim_{x\to\infty}(\frac{x^2+3x-1}{x+1}-\frac{x^2-3x+1}{x-5})$
15.	$\lim_{x\to\infty}\left(\frac{x-3}{x}\right)^{5x+2}$	16.	$\lim_{x\to\infty}\left(\frac{x-2}{x+7}\right)^{3x}$
17.	$\lim_{x\to 3}(5x-14)^{\frac{2x}{3-x}}$	18.	$\lim_{x\to 0}(\cos 4x)^{\frac{1}{\sin^2 3x}}$
19.	$\lim_{x\to 0}\frac{\ln(1-4x^3)}{x^2\sin 5x}$	20.	$\lim_{x\to 0}\frac{\ln(1+\arcsin\sqrt{x})}{\tan 3\sqrt{x}}$
21.	$\lim_{x\to\infty} x\cdot(\ln(2x-5)-\ln 2x)$	22.	$\lim_{x\to\infty}(x\cdot\ln\sqrt[3]{\frac{x-4}{x+3}})$
23.	$\lim_{x\to 2}\frac{e^{x-2}-1}{x^2-4}$	24.	$\lim_{x\to 0}\frac{e^{\tan 4x}-1}{\sin 5x}$
25.	$\lim_{x\to 0}\frac{e^{3x}-\cos 4x+5x}{\ln(1-7x)+\tan^3 x}$	26.	$\lim_{x\to 0}\frac{e^{-x}-2\arctan 3x-1}{5\arcsin 4x+2\sqrt{x}\sin\sqrt{x}}$
27.	$\lim_{x\to 0}\frac{\sqrt{1-4x}-1}{\tan 2x}$	28.	$\lim_{x\to 0}\frac{\sqrt[4]{1-6x^2}-1}{x\ln(1-3x)}$
29.	$\lim_{x\to 0}\frac{\sqrt[5]{1-10x}-1}{\sqrt{1+4x}-1}$	30.	$\lim_{x\to 0}\frac{(1-4x)^{1000}-1}{(1+8x)^{1000}-1}$

The limit of the form

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

is named the instantaneous rate of change of $f(x)$ at the point x.

In Problems 31 through 42 find the instantaneous rate of change of the given functions.

N	$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$
31.	$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$
32.	$\lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$
33.	$\lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x}$
34.	$\lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$
35.	$\lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x}$
36.	$\lim_{\Delta x \to 0} \frac{\tan(x+\Delta x) - \tan x}{\Delta x}$
37.	$\lim_{\Delta x \to 0} \frac{\cot(x+\Delta x) - \cot x}{\Delta x}$
38.	$\lim_{\Delta x \to 0} \frac{\arcsin(x+\Delta x) - \arcsin x}{\Delta x}$
39.	$\lim_{\Delta x \to 0} \frac{\arccos(x+\Delta x) - \arccos x}{\Delta x}$
40.	$\lim_{\Delta x \to 0} \frac{\tan^{-1}(x+\Delta x) - \tan^{-1} x}{\Delta x}$
41.	$\lim_{\Delta x \to 0} \frac{\cot^{-1}(x+\Delta x) - \cot^{-1} x}{\Delta x}$

2.6. Continuity of Functions

2.6.1. Basic Definitions

A function $f(x)$ is called **continuous** at a point *a*, if there exists a finite limit of $f(x)$ as $x \to a$, which is equal to the value of the function at the point *a*,

$$\lim_{x\to a} f(x) = f(a).$$

A function $f(x)$ is said to be continuous on some set *D*, if $f(x)$ is continuous at each point of *D*. Otherwise, if $f(x)$ is not continuous, e.g., at a point *b*, they say that the function $f(x)$ is **discontinuous** at the point *b*, or that $f(x)$ has a **discontinuity** at the point *b*.

Points of discontinuity are classified by the difference between one-sided limits,

$$|\lim_{x\to a-0} f(x) - \lim_{x\to a+0} f(x)|.$$

This difference is called the **jump** of the function at the point *a*.
If the jump is a finite number, then $f(x)$ has an **ordinary discontinuity** at $x = a$. The point *a* is said to be a point of the function discontinuity of the **first kind**.

If the one-sided limits $\lim\limits_{x\to a-0} f(x)$ and $\lim\limits_{x\to a+0} f(x)$ are finite and equal to each other but not equal to the value of the function at the point *a*, then *a* is called a point of **removable discontinuity**. To remove discontinuity at a point of removable discontinuity it is necessary to redefine the function at that point or to extend the domain of $f(x)$ to include that point by the supplementary condition:

$$f(a) \equiv \lim_{x\to a-0} f(x) = \lim_{x\to a+0} f(x).$$

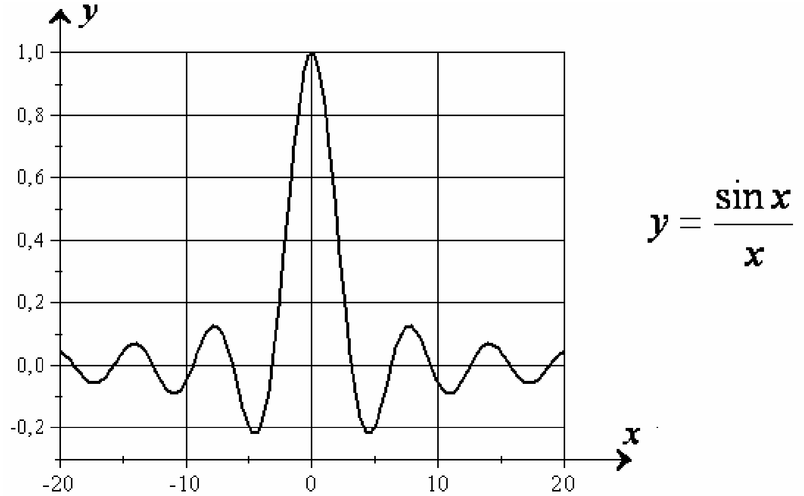
If the jump $|\lim\limits_{x\to a-0} f(x) - \lim\limits_{x\to a+0} f(x)|$ takes an infinite value, or at least one of the one-sided limits does not exists, then the point *a* is a point of **non-removable discontinuity**, or a point of discontinuity of the **second kind**.

Examples

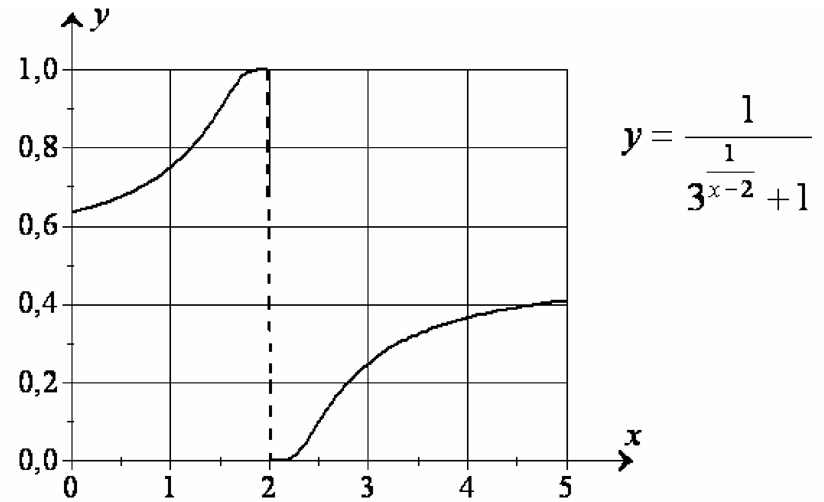
1. In the figure below, there is shown the graph of the function $f(x) = \dfrac{\sin x}{x}$, which is not defined at the point *x* = 0, and so it has an discontinuity at that point. However, there exists the limit of $f(x)$ as $x \to 0$ (see Theorem 1, p. 66): $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

Therefore, the discontinuity at $x = 0$ can be removed by including $x=0$ in the domain of the function and redefining $f(0)=1$,

$$f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{if } x\neq 0\\ 1, & \text{if } x=0\end{cases}$$



2. Consider the function $f(x)=\dfrac{1}{3^{\frac{1}{x-2}}+1}$ defined at all points except for $x=2$. It means that $x=2$ is a point of discontinuity of the function.



Find the one-sided limits at this point.

If $x\to 2-0$ then $x-2<0$ and $\dfrac{1}{x-2}\to-\infty$, which implies

$$f(x)=\frac{1}{3^{1/(x-2)}+1}\ \to\ \frac{1}{3^{-\infty}+1}=\frac{1}{1}=1.$$

If $x \to 2+0$ then $x-2>0$ and $\dfrac{1}{x-2} \to +\infty$, which implies

$$f(x)=\frac{1}{3^{1/(x-2)}+1} \to \frac{1}{3^{+\infty}+1}=\frac{1}{\infty}=0.$$

Thus, the left-sided limit is not equal to the right-sided limit at the point $x=2$; however, the jump has a finite value (number 1).
Hence, $f(x)$ has an ordinary discontinuity at the point $x=2$.

3. Let $f(x)$ be defined as $f(x)=\begin{cases} x, & \text{if } x<0 \\ x^2, & \text{if } 0 \le x < 1 \\ x-1, & \text{if } x \ge 1 \end{cases}$

Since x, x^2, and $x-1$ are continuous functions at all points, discontinuities of the given function $f(x)$ could arise at the linking points, $x=0$ and $x=1$, only.
Find one-sided limits at the point $x = 0$:

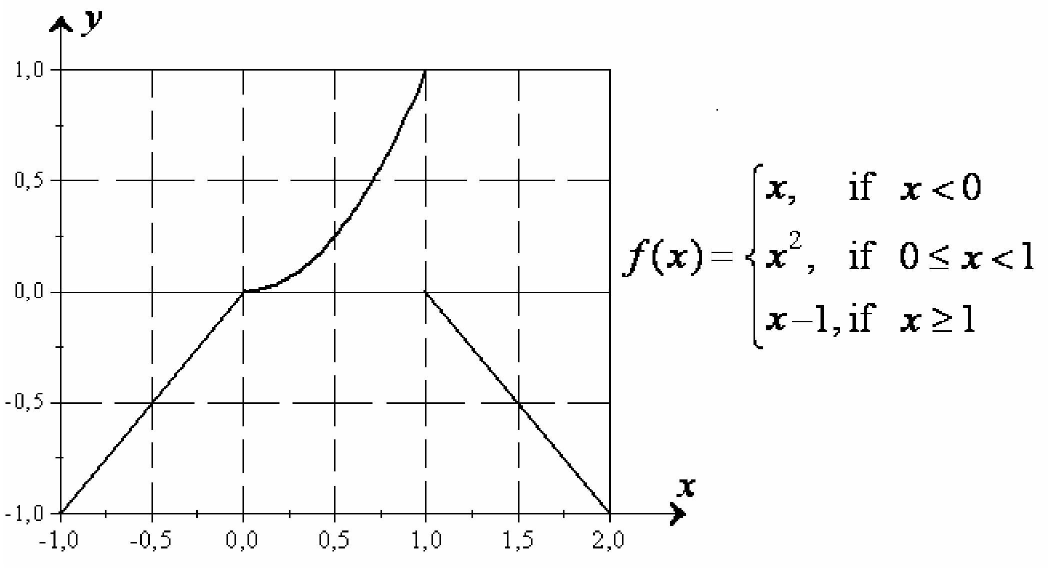
$$\lim_{x\to -0} f(x) = \lim_{x\to -0} x = 0,$$

$$\lim_{x\to +0} f(x) = \lim_{x\to -0} x^2 = 0.$$

By the above definition, $f(0)=0$, so that

$$\lim_{x\to -0} f(x) = \lim_{x\to +0} f(x) = f(0) = 0,$$

which means that $f(x)$ is a continuous function at the point $x=0$.



Find one-sided limits at the point $x = 1$:

$$\lim_{x\to 1-0} f(x) = \lim_{x\to 1-0} x^2 = 1,$$

$$\lim_{x\to 1+0} f(x) = \lim_{x\to 1+0}(x-1) = 0 .$$

The limits are not equal to each other but they both have finite values (the jump equals 1). Therefore, the function $f(x)$ has an ordinary discontinuity at the point $x=1$.

4. Consider the function $f(x) = 5^{\frac{1}{x-3}}$, which is continuous at all points except for $x=3$, where the function is not defined.
   Find one-sided limits at the point $x = 3$.
   If $x \to 3-0$ then $\dfrac{1}{x-3} \to -\infty$, and so $f(x) = 5^{\frac{1}{x-3}} \to 5^{-\infty} = 0$.
   If $x \to 3+0$ then $\dfrac{1}{x-3} \to +\infty$, and so $f(x) = 5^{\frac{1}{x-3}} \to 5^{+\infty} = +\infty$.
   Therefore, $x = 3$ is a point of discontinuity of the second kind.

2.6.2. Properties of Continuous Functions

> The sum of a finite number of continuous functions is a continuous function.
>
> The product of a finite number of continuous functions is a continuous function.
>
> The quotient of two continuous functions is a continuous function except for the points where the denominator is equal to zero.

Let us prove, for example, the product property.
If $f(x)$ and $g(x)$ are continuous functions at a point *a*, then

$$\lim_{x\to a} f(x) = f(a) \quad \text{and} \quad \lim_{x\to a} g(x) = g(a) .$$

By the properties of limits of functions,

$$\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) = f(a)\cdot g(a) ,$$

which required to be proved.

> **Theorem**: All elementary functions are continuous in their domains.

To prove this statement it is necessary to show that each elementary function $f(x) \to f(a)$ for any number a in the domain of *f*. Below we give a few examples to demonstrate the validity of the theorem.

Proof:

1. The power function x^n is a continuous function at each point in the domain of x^n. Indeed,

$$x^n = (x^n - a^n) + a^n$$
$$= (x-a)(x^{n-1} + x^{n-2}a + x^{n-2}a^2 + \ldots + a^{n-1}) + a^n \to a^n \quad \text{as} \quad x \to a.$$

2. The exponential function e^x is a continuous function at each point a, since
$$e^x = (e^x - e^a) + e^a = e^a(e^{x-a} - 1) + e^a$$
$$\sim\ e^a(x-a) + e^a \ \to\ e^a \quad \text{as} \quad x \to a.$$

3. The logarithmic function $\ln x$ is a continuous function at each point $x = a > 0$:
$$\ln x = (\ln x - \ln a) + \ln a = \ln\frac{x}{a} + \ln a = \ln(1 + \frac{x-a}{a}) + \ln a$$
$$\sim \frac{x-a}{a} + \ln a \to \ln a \quad \text{as} \quad x \to a.$$

4. The sine function $\sin x$ is a continuous function at each point a, since
$$\sin x = (\sin x - \sin a) + \sin a = 2\sin\frac{x-a}{2}\cos\frac{x+a}{2} + \sin a$$
$$\sim\ 2\cdot\frac{x-a}{2}\cdot\cos a + \sin a \ \to\ \sin a \qquad \text{as} \quad x \to a.$$

5. The cosine function $\cos x$ is a continuous function at each point a, since
$$\cos x = (\cos x - \cos a) + \cos a = -2\sin\frac{x-a}{2}\sin\frac{x+a}{2} + \sin a$$
$$\sim\ -2\cdot\frac{x-a}{2}\cdot\sin a + \cos a \ \to\ \cos a \qquad \text{as} \quad x \to a.$$

6. The tangent function $\tan x$ is a continuous function at each point a in the domain of $\tan x$, since $\tan x = \dfrac{\sin x}{\cos x}$ is the ratio of two continuous functions (provided that $\cos x \neq 0$).

7. Likewise, $\cot x$ is a continuous function at each point a in the domain of $\cot x$ as the ratio of two continuous functions (by the quotient property): $\cot x = \dfrac{\cos x}{\sin x}$.

8. To prove the property of continuity for a trigonometric function, one can apply the corresponding substitutions such as
$$t = \arcsin x, \quad t = \arccos x, \quad \text{and so on.}$$

2.7. Selected Solutions

Problem 2.

$$\lim_{x\to -2}\frac{\sin(x+2)}{8+x^3}=\lim_{x\to -2}\frac{x+2}{8+x^3}=\lim_{x\to -2}\frac{x+2}{(x+2)(x^2-2x+4)}=\frac{1}{12}.$$

Problem 3.

$$\lim_{x\to 1}\frac{\tan x\pi}{x-1}=\lim_{t\to 0}\frac{\tan(t\pi+\pi)}{t}=\lim_{t\to 0}\frac{\tan t\pi}{t}=\lim_{t\to 0}\frac{t\pi}{t}=\pi .$$

Problem 6.

$$\lim_{x\to\infty}x^2\tan\frac{1}{x^2}=\lim_{t\to 0}\frac{\tan t}{t}=1.$$

Problem 8.

$$\lim_{x\to 1}\frac{1+\cos\pi x}{\sin 3\pi x}=\lim_{t\to 0}\frac{1+\cos(t\pi+\pi)}{\sin(3t\pi+3\pi)}=\lim_{t\to 0}\frac{1-\cos t\pi}{-\sin 3t\pi}$$

$$=-\lim_{t\to 0}\frac{\frac{(t\pi)^2}{2}}{3t\pi}=-\frac{\pi}{6}\lim_{t\to 0}\frac{t^2}{t}=0.$$

Problem 11.

$$\lim_{x\to 1}\frac{\arcsin(1-x^2)}{1-\sqrt{x}}=\lim_{x\to 1}\frac{1-x^2}{1-\sqrt{x}}=\lim_{x\to 1}\frac{(1-x)(1+x)}{1-\sqrt{x}}$$

$$=\lim_{x\to 1}\frac{(1-\sqrt{x})(1+\sqrt{x})(1+x)}{1-\sqrt{x}}=\lim_{x\to 1}(1+\sqrt{x})(1+x)=4.$$

Problem 12.

$$\lim_{x\to 2}\frac{\arctan(x^2-3x+2)}{x^2-x-2}=\lim_{x\to 2}\frac{x^2-3x+2}{x^2-x-2}$$

$$=\lim_{x\to 2}\frac{(x-2)(x-1)}{(x-2)(x+1)}=\lim_{x\to 2}\frac{x-1}{x+1}=\frac{1}{3}.$$

Problem 14.

$$\lim_{x\to\infty}\left(\frac{x^2+3x-1}{x+1}-\frac{x^2-3x+1}{x-5}\right)=$$

$$=\lim_{x\to\infty}\frac{(x^2+3x-1)(x-5)-(x^2-3x+1)(x+1)}{(x+1)(x-5)}=\lim_{x\to\infty}\frac{-14x+4}{(x+1)(x-5)}=0.$$

Problem 17.

$$\lim_{x\to 3}(5x-14)^{\frac{2x}{3-x}}=\lim_{t\to 0}(1+5t)^{\frac{2t+6}{-t}}=\lim_{t\to 0}(1+5t)^{\frac{1}{5t}(-30)-2}$$

$$=\lim_{t\to 0}(1+5t)^{\frac{1}{5t}(-30)}\lim_{t\to 0}(1+5t)^{-2}=e^{-30}.$$

Problem 18.

$$\lim_{x\to 0}(\cos 4x)^{\frac{1}{\sin^2 3x}}=\lim_{x\to 0}\left(\sqrt{1-\sin^2 4x}\right)^{\frac{1}{\sin^2 3x}}$$

$$=\lim_{x\to 0}\left(1-\sin^2 4x\right)^{\frac{1}{2}\cdot\frac{1}{\sin^2 3x}}=\lim_{x\to 0}\left(1-16x^2\right)^{\frac{1}{18x^2}}=e^{-\frac{16}{18}}=\frac{1}{e^{8/9}}.$$

Problem 19.

$$\lim_{x\to 0}\frac{\ln(1-4x^3)}{x^2\sin 5x}=\lim_{x\to 0}\frac{-4x^3}{x^2\cdot 5x}=-\frac{4}{5}.$$

Problem 21.

$$\lim_{x\to\infty}x\cdot(\ln(2x-5)-\ln 2x)=\lim_{x\to\infty}x\cdot\ln\frac{2x-5}{2x})$$

$$=\lim_{x\to\infty}x\ln(1-\frac{5}{2x})=\lim_{x\to\infty}x(-\frac{5}{2x})=-\frac{5}{2}.$$

Problem 22.

$$\lim_{x\to\infty}(x\cdot\ln\sqrt[3]{\frac{x-4}{x+3}})=\frac{1}{3}\lim_{x\to\infty}(x\cdot\ln\frac{x-4}{x+3})=\frac{1}{3}\lim_{x\to\infty}(x\cdot\ln\frac{1-4/x}{1+3/x})$$

$$=\frac{1}{3}\lim_{t\to 0}(\frac{1}{t}\cdot\ln\frac{1-4t}{1+3t})=\frac{1}{3}\lim_{t\to 0}\frac{1}{t}(\ln(1-4t)-\ln(1+3t))=\frac{1}{3}\lim_{t\to 0}\frac{1}{t}(-7t)=-\frac{7}{3}.$$

Problem 23.

$$\lim_{x\to 2}\frac{e^{x-2}-1}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\frac{1}{4}.$$

Problem 28.

$$\lim_{x\to 0}\frac{\sqrt[4]{1-6x^2}-1}{x\ln(1-3x)}=\lim_{x\to 0}\frac{-\frac{6x^2}{4}}{-3x^2}=\frac{1}{2}.$$

Problem 30.

$$\lim_{x\to 0}\frac{(1-4x)^{1000}-1}{(1+8x)^{1000}-1}=\lim_{x\to 0}\frac{-4000x}{8000x}=-\frac{1}{2}.$$

2.8. Selected Answers

31.	$\lim\limits_{\Delta x\to 0}\dfrac{(x+\Delta x)^n - x^n}{\Delta x} = nx^{n-1}$
32.	$\lim\limits_{\Delta x\to 0}\dfrac{e^{x+\Delta x} - e^x}{\Delta x} = e^x$
33.	$\lim\limits_{\Delta x\to 0}\dfrac{\ln(x+\Delta x) - \ln x}{\Delta x} = \dfrac{1}{x},\quad x>0$
34.	$\lim\limits_{\Delta x\to 0}\dfrac{\sin(x+\Delta x) - \sin x}{\Delta x} = \cos x$
35.	$\lim\limits_{\Delta x\to 0}\dfrac{\cos(x+\Delta x) - \cos x}{\Delta x} = -\sin x$
36.	$\lim\limits_{\Delta x\to 0}\dfrac{\tan(x+\Delta x) - \tan x}{\Delta x} = \dfrac{1}{\cos^2 x}$
37.	$\lim\limits_{\Delta x\to 0}\dfrac{\cot(x+\Delta x) - \cot x}{\Delta x} = -\dfrac{1}{\sin^2 x}$
38.	$\lim\limits_{\Delta x\to 0}\dfrac{\arcsin(x+\Delta x) - \arcsin x}{\Delta x} = \dfrac{1}{\sqrt{1-x^2}}$
39.	$\lim\limits_{\Delta x\to 0}\dfrac{\arccos(x+\Delta x) - \arccos x}{\Delta x} = -\dfrac{1}{\sqrt{1-x^2}}$
40.	$\lim\limits_{\Delta x\to 0}\dfrac{\tan^{-1}(x+\Delta x) - \tan^{-1} x}{\Delta x} = \dfrac{1}{1+x^2}$
41.	$\lim\limits_{\Delta x\to 0}\dfrac{\cot^{-1}(x+\Delta x) - \cot^{-1} x}{\Delta x} = -\dfrac{1}{1+x^2}$

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Limits of Sequences and Functions

Textbook

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