ФЕДЕРАЛЬНОЕ АГЕНТСТВО ПО ОБРАЗОВАНИЮ
Государственное образовательное учреждение высшего профессионального образования
«**ТОМСКИЙ ПОЛИТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ**»

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# HIGHER MATHEMATICS, PART 2

Workbook

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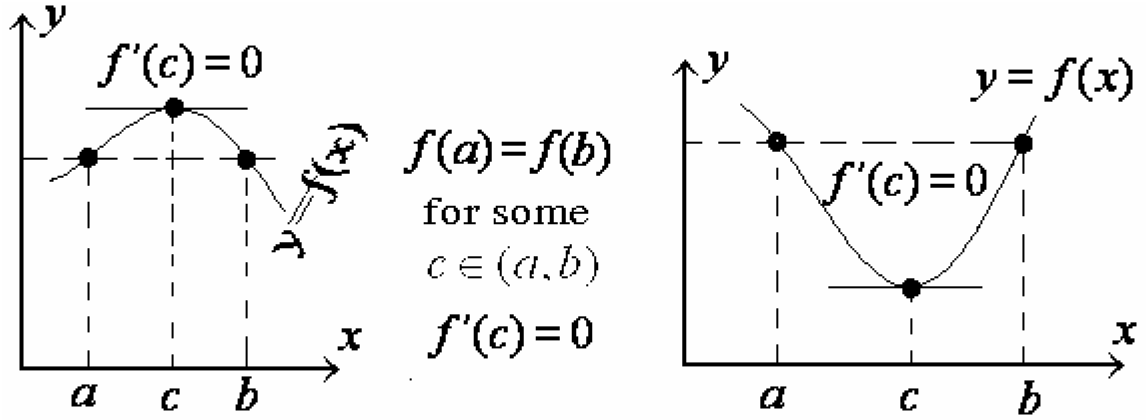
# Fundamental Theorems of Differential Calculus

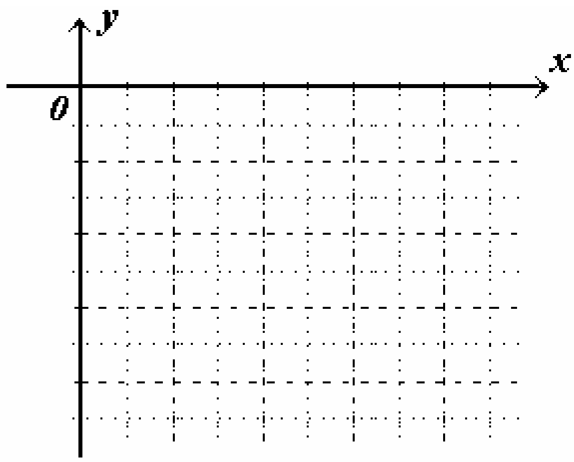
## The Rolle Theorem




**Problem 1**: Consider the function $f(x) = x^2 - x$.

- Prove that it obeys the Rolle Theorem on the interval [0, 1].
- Find the corresponding value of *c*.
- Make a sketch of the graph of *f*, indicating the function values at the endpoints of the interval and *f*(*c*).

**Solution**:

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**Problem 2**: Check whether $f(x) = \sqrt[3]{(x-1)^2}$ satisfies the conditions of the Rolle Theorem on the interval [0, 2].

**Solution**:

**Conclusion**:

**Problem 3**: Consider the function $f(x) = \tan x$ on the interval [0, $\pi$] with equal values at the endpoints:

$$f(0) = f(\pi) = 0.$$

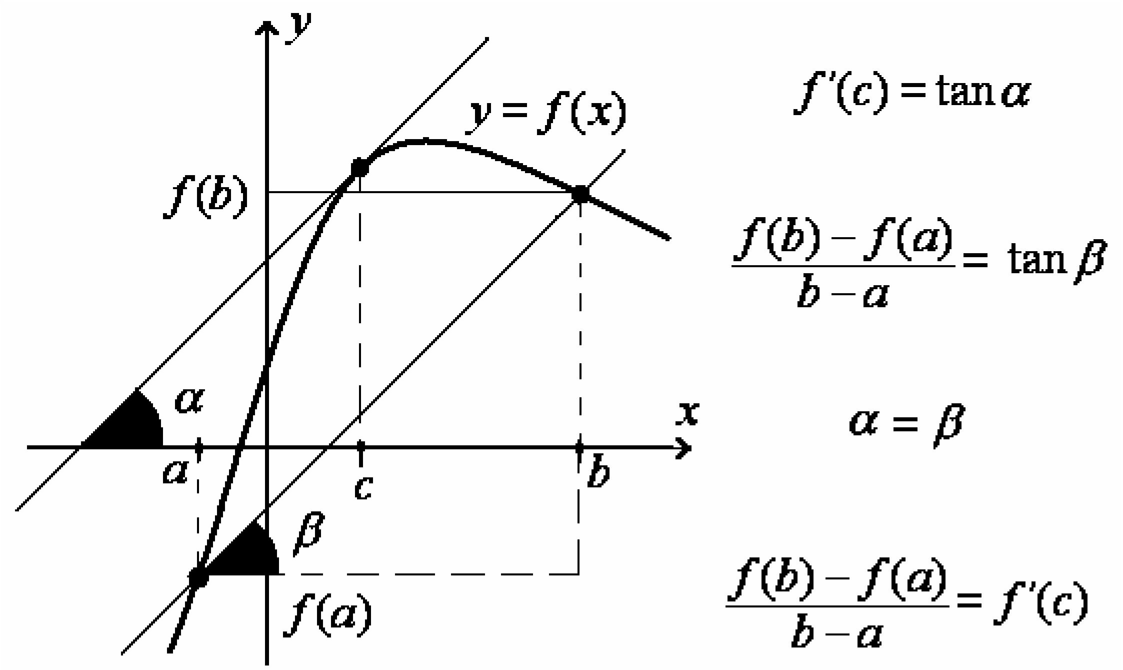
Is the statement below true?

> "In view of the Rolle Theorem there exists a point $c$ such that $0 < c < \pi$ and $f'(c) = 0$".

If it is false, give the correct statement.

**Solution**:

## The Mean Value Theorem




**Problem 4**: Check whether the function $f(x) = x^3 - x$ satisfies on the interval [−2,1] the conditions of the Mean Value Theorem. Give reasons for your answer. In the case if it satisfies, find the corresponding point on the interval.

**Solution**:

**Problem 5**: Prove that the functions

$$f(x) = x^2 \quad \text{and} \quad g(x) = x^3 - 1$$

satisfy on the interval [1, 2] the conditions of the Cauchy Theorem, and hence, there exists a point $c \in (1,2)$ such that

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}. \qquad (*)$$

Find this point.

**Solution**:

**Problem 6**: Prove that the Mean Value Theorem is a special case of the Cauchy Theorem.

**Solution**:

**Hints**:

Problem 1.

1) Check the validity of each of the following statements:
   - the function is defined and continuous on the closed interval [0, 1];
   - the function is differentiable at each point of the open interval (0, 1);
   - $f(0) = f(1)$.
2) Find the solution of the equation $f'(c) = 0$.

Problem 2-3. Recall the Rolle Theorem. (See [1, Chapter 1, page 4]).

Problem 4. See the Mean Value Theorem, [1, Chapter 1, pages 4-6].

Problem 5.

1) Recall the Cauchy Theorem. (See [1, Chapter 1, page 6]).
2) Find the derivatives of the given functions.
3) Solve equation (*) in respect to $c$.

Problem 6. Consider a special case when $g(x) = x$.

# The L'Hopital Rule

If $\dfrac{f(x)}{g(x)} \Rightarrow \dfrac{0}{0}$ or $\dfrac{f(x)}{g(x)} \Rightarrow \dfrac{\infty}{\infty}$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

**Example 1**: Evaluate $\lim\limits_{x\to+\infty} \dfrac{\ln x}{x}$.

**Solution**: Since $\ln x \to \infty$ and $x \to \infty$ as $x \to +\infty$, so $\dfrac{\ln x}{x}$ is the indeterminate form $\dfrac{\infty}{\infty}$.

Therefore, by the L'Hopital Rule,

$$\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{(\ln x)'}{x'} = \lim_{x\to+\infty} \frac{1/x}{1} = 0.$$

**Solution in a short form**:

$$\lim_{x\to+\infty} \frac{\ln x}{x} = (\frac{\infty}{\infty}) = \lim_{x\to+\infty} \frac{(\ln x)'}{x'} = \lim_{x\to+\infty} \frac{1/x}{1} = 0.$$

**Problems 7-18**: Evaluate each of the following limits by applying the L'Hopital Rule.

7. $\lim\limits_{x\to 0} \dfrac{\ln 5x}{\cot 3x} =$

8. $\lim\limits_{x\to 0} \dfrac{x\cos x - \sin x}{x^3} =$

9. $\lim\limits_{x\to 1} \dfrac{x^4 - 2x^3 + 3x^2 + 5x - 7}{x^3 + 4x - 5} =$

10. $\lim\limits_{x\to \pi} \dfrac{\tan x}{\tan 4x} =$

11. $\lim_{x \to +\infty} \frac{e^{3x}}{x^7} =$

12. $\lim_{x \to -\infty} \frac{e^{3x}}{x^7} =$

13. $\lim_{x \to +\infty} \frac{(\ln x)^3}{x} =$

14. $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} =$

15. $\lim_{x \to 1} (x-1) \tan \frac{\pi x}{2} =$

16. $\lim_{x \to 1} \ln x \ln(x-1) =$

17. $\lim_{x \to 0} (1 - \cos x) \cot x =$

18. $\lim_{x \to 2} (\frac{1}{x-2} - \frac{3}{x^2 + 2x - 8}) =$

**Hints**: See the L'Hopital Rule [1, Chapter 1, pages 6-9].

Problems 7 - 14. Check whether the expression under the sign of limit is the indeterminacy of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If it is, use the L'Hopital Rule.
To solve some problems one needs to apply the above procedure repeatedly.

Problems 15 - 18. Each of these forms can be reduced to one of the forms: either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

**Example 2**: Evaluate $\lim_{x\to 0} x^{\tan x}$.

**Solution**: This limit contains an indeterminacy of the form $0^0$. Let $y = x^{\tan x}$. Then $\ln y = \tan x \ln x$, and so we have

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \tan x \ln x = \lim_{x\to 0} \frac{\ln x}{\cot x},$$

which is the indeterminacy of the form $\frac{\infty}{\infty}$.

By the L'Hopital rule

$$\lim_{x\to 0} \frac{\ln x}{\cot x} = \lim_{x\to 0} \frac{1/x}{-1/\sin^2 x}$$

$$= -\lim_{x\to 0} \frac{\sin^2 x}{x} = -\lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \sin x = -1 \cdot 0 = 0.$$

Therefore, $$\lim_{x\to 0} x^{\tan x} = e^{\lim_{x\to 0} \ln y} = e^0 = 1.$$

**Problems 19 - 24**: Transform each of the following indeterminate forms to one of the forms, $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and evaluate the limits by the L'Hopital Rule.

**Solution:**

19. $\lim_{x\to +\infty} x^{\frac{1}{x}}$

20. $\lim_{x \to 0} (\cot x)^{\frac{1}{\ln x}}$

21. $\lim_{x \to 0} (1 - \sin^2 3x)^{\frac{5}{x^2}}$

22. $\lim_{x \to 1} x^{\frac{1}{1-x}}$

23. $\lim_{x \to 0} (\tan x)^{\sin x}$

24. $\lim_{x \to 0} (\sin x)^{x}$

---

**Hints**:
See the Other Indeterminate Forms in [1, Chapter 1, pages 9-10].

<u>Problems 19 - 24:</u> In order to find the limit of $f(x)$ as $x \to a$ you can first try to find $\lim_{x \to a} \ln f(x)$ and then use the following formular:

$$\lim_{x \to a} f(x) = e^{\lim_{x \to a} \ln f(x)}.$$

---

## The Taylor Formula

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+R_n(x)$$

### The Taylor Formula for Polynomials

$$P(x)=P(x_0)+\frac{P'(x_0)}{1!}(x-x_0)+\ldots+\frac{P^{(n)}(x_0)}{n!}(x-x_0)^n.$$

### The Maclaurin Formula

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k+R_n(x).$$

**Example 3**: Represent the polynomial $P(x)$ in powers of $(x-1)$, if

$$P(x)=-2+4(x+2)-3(x+2)^2+(x+2)^3.$$

**Solution**: The Taylor Formula with $x_0=1$ gives the answer in the general form:

$$P(x)=P(1)+P'(1)(x-1)+\frac{P''(1)}{2}(x-1)^2+\frac{P'''(1)}{6}(x-1)^3.$$

It remains to find $P(1)$ and $P^{(k)}(1)$.

$P(x)=-2+4(x+2)-3(x+2)^2+(x+2)^3 \quad\Rightarrow\quad P(1)=10$

$P'(x)=4-6(x+2)+3(x+2)^2 \quad\Rightarrow\quad P'(1)=13.$

$P''(x)=-6+6(x+2) \quad\Rightarrow\quad P''(1)=12.$

$P'''(x)=6 \quad\Rightarrow\quad P'''(0)=6.$

Thus, $P(x)=10+13(x-1)+6(x-1)^2+(x-1)^3.$

**Problems 25 - 27:** Represent the polynomial

$$P(x)=2(x-3)+5(x-3)^2-(x-3)^3$$

as the one in powers of $(x-x_0)$, if

25) $x_0=2$;

26) $x_0=0$;

27) $x_0=-1$.

**Solution**:

25. Let $x_0 = 2$.

26. Let $x_0 = 0$.

27. Let $x_0 = -1$.

---

**Hint**: See page 11 in [1] and the above example.

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**Problems 28 - 30**: Find several initial non-zero terms of the Taylor expansion in some vicinity of the point $x = x_0$ for each of the following functions:

28) $f(x) = \sin x$, $x_0 = \pi/6$, $n = 3$;

29) $f(x) = e^x$, $x = -2$, $n = 4$;

30) $f(x) = \sqrt{x}$, $x = 4$, $n = 3$.

**Solution**:

28.

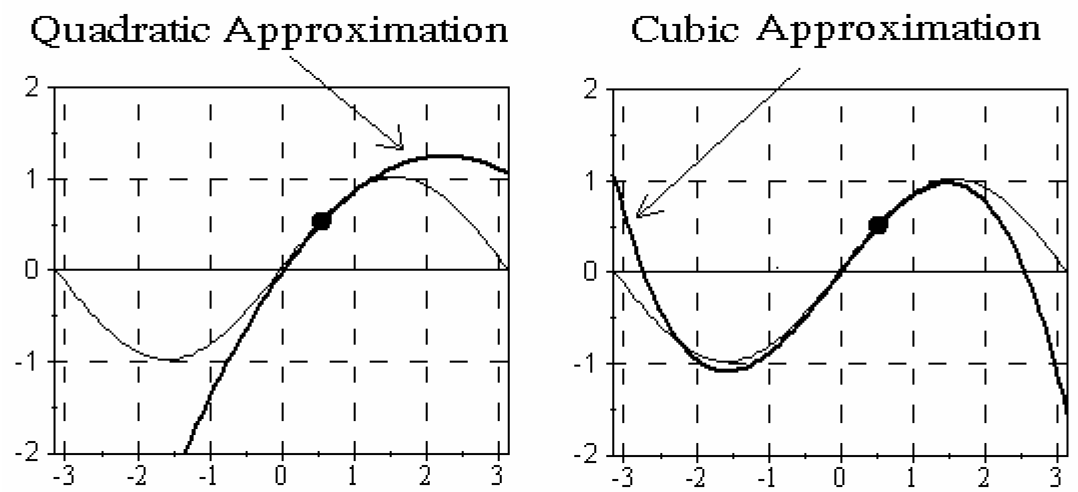
29.

30.

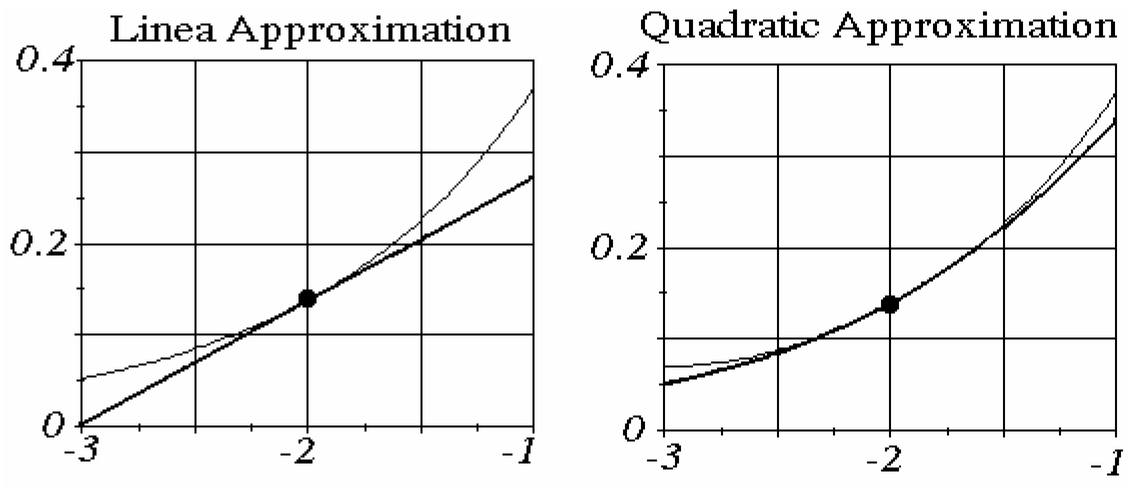
**Hint:** See ( [1], Chapter 1, pages 12-15).

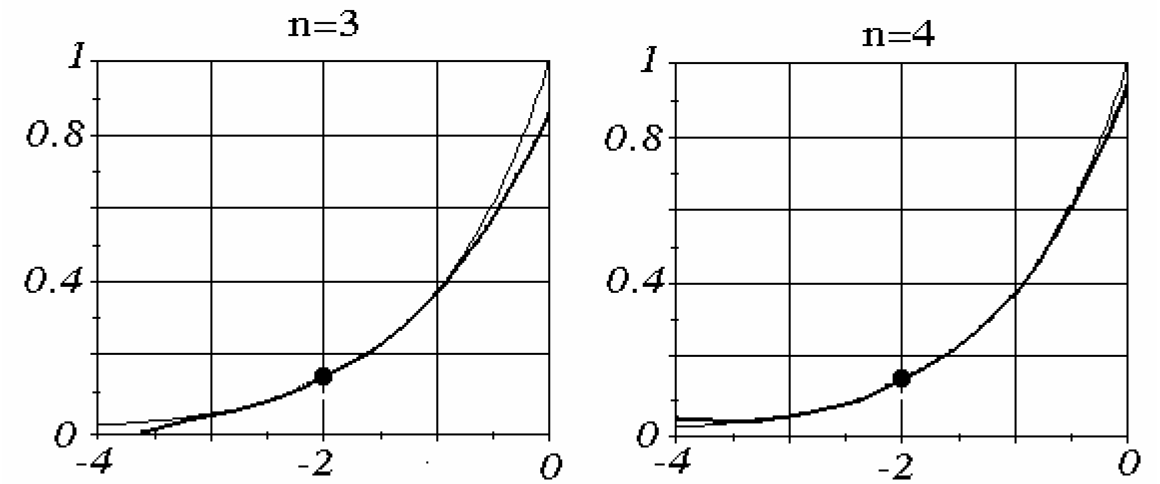
**Illustrations:**

Problem 28. The figures below illustrate graphically the approximate representations of the sine function by the Taylor Polynomial in a vicinity of $x_0 = \pi/6$. The curve $y = \sin x$ is also shown.




Problem 29. At the figures below you can see the approximate representations of the function $f(x) = e^x$ by the Taylor Polynomial in a vicinity of $x_0 = -2$. There are also the graphs of the function $y = e^x$ in these figures.






**Problem 31 - 35**: Find the Maclaurin Expansion for each of the following functions:

31) $f(x) = e^{-4x}$;

32) $f(x) = \sin x^2$;

33) $f(x) = \ln(1-3x)$;

34) $f(x) = \cos^2 x$;

35) $f(x) = x^2 \tan^{-1}(5x)$;

**Solution**:

31.

32.

33.

34.

35.

**Hints**:
Problem 31. Use expansion (13) and substitute (- 4 *x*) for *x*. (See [1, Chapter 1, page 14]).

Problem 32: Use expansion (14) and substitute $x^2$ for *x*. (See [1, Chapter 1, page 14]).

Problem 33: Solve the problem in a similar way as above.

Problem 34: Use the identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ and formula (15). (See [1, Chapter 1, page 14]).

Problem 35: First find the expansion for $\tan^{-1}(5x)$. Then multiply the result by $x^2$.

**Note**: Simplification of the formulas is not required.

**Problem 36:** Compute $1/\sqrt{e}$ approximately with the accuracy 0.02 by applying the appropriate Taylor Polynomial without using a calculator. Compare the answer with the exact result.

**Solution**:

**Problem 37**: Calculate approximately $\cos 9^\circ$ and compare your results (for different values of $n$) with the exact one.

**Solution**:

**Problem 38:** Calculate approximately $\ln 1.12$ by using the Taylor Polynomial of the third degree; estimate the error bound by making use of the Lagrange form for the remainder.

**Solution**:

**Problems 39 - 41**: Prove the following approximating formulas:

39) $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2$;

40) $\sqrt[3]{1+x} = 1 + \frac{1}{3}x - \frac{1}{9}x^2$;

41) $\ln(1+x) = x - \frac{1}{2}x^2$,

where $|x| < 1$. Estimate the errors of these approximations.

**Solution**:

39.

40.

41.

**Hint**: Use formulas (17) and (18). (See [1, Chapter 1]).

## Graphs of Functions

In order to investigate a function $f(x)$ you can follow the scheme below:

I. Find the domain of the function.

II. Determine whether the function has any symmetry.

III. Determine critical points by solving the equation $f'(x)=0$ and finding the points in which the derivative $f'(x)$ does not exist.
Each critical point must be checked whether it is an extreme point or not.

IV. Find the intervals of monotonicity of the function.

V. Determine the points of inflection, that is, find the solution of the equation $f''(x)=0$. It is necessary also to find the points, where the second derivative $f''(x)$ does not exist. Each of these points must be checked whether it is a point of inflection or not.

VI. Find the intervals where the curve $y=f(x)$ is concave, and where it is convex.

VII. Find the asymptotes for the function.

**Problem 42**: Find out which of the following functions

$$|x|,\quad x^2,\quad x^3,\quad x^4,\quad x^5,\quad \sqrt[3]{x},\quad (x+1),$$

$$\sin x,\qquad \cos x,\qquad \sin^2 x,\qquad \sin^{-1} x,\qquad \tan^7 x,\qquad x\cos x,$$

$$\ln x,\ |\ln x|,\qquad \ln|x|,\qquad \ln(x^2+1),\ \ln(x+1),\quad e^{-x^2},\qquad e^x,\quad e^{-x}$$

are even, odd, or neither even nor odd.
Give reasons for each answer.
**Solution**: (See [3, pages 38-43]).
Even functions:

Odd functions:

Neither even nor odd functions:

Reasoning:

**Problems 43 - 46:** For each of the problems below find out whether the function is increasing or decreasing in some vicinity of the indicated points.

| | | | | |
|---|---|---|---|---|
| 43) | $f(x) = x^2$, | $x_0 = 5$, | $x_2 = -5$, | $x_3 = 0$. |
| 44) | $f(x) = x^3$, | $x_0 = 5$, | $x_2 = -5$, | $x_3 = 0$. |
| 45) | $f(x) = \sin x$, | $x_0 = \pi/2$, | $x_0 = \pi/2$, | $x_3 = 0$. |
| 46) | $f(x) = x \ln x$, | $x_0 = 1/e$, | $x_0 = e$, | $x_3 = 1$. |

**Solution**:

43. $f(x) = x^2$.

a) $x_0 = 5$,

b) $x_2 = -5$,

c) $x_3 = 0$,

44. $f(x) = x^3$.

a) $x_0 = 5$,

b) $x_2 = -5$,

c) $x_3 = 0$,

45. $f(x) = \sin x$.

a) $x_0 = \pi/2$,

b) $x_0 = \pi/2$,

c) $x_3 = 0$,

46. $f(x) = x \ln x$.

a) $x_0 = 1/e$,

b) $x_0 = e$,

c) $x_3 = 1$,

**Problems 47 - 52:** Divide the domain of the function into a finite number of intervals for which the function is strictly monotone. Indicate the intervals where the function is increasing and those where it is decreasing.

47) $f(x) = x^2(x-3)$.

48) $f(x) = (x-3)\sqrt{x}$.

49) $f(x) = x\ln x$.

50) $f(x) = e^{x^2-4x}$.

51) $f(x) = \dfrac{e^x}{x}$.

52) $f(x) = \dfrac{x}{x+4}$.

**Solution:**

47. $(x^2(x-3))' =$

48. $((x-3)\sqrt{x})' =$

49. $(x\ln x)' =$

50. $(e^{x^2-4x})' =$

51. $(\dfrac{e^x}{x})' =$

52. $\left(\frac{x}{x+4}\right)' =$

---

**Problems 53 - 59**: Find all local extrema of $f(x)$ and determine which of them are local minimums and which are local maximums.

53) $f(x) = x^2 - 6x + 7$.

54) $f(x) = \dfrac{x^2 - 2x + 2}{x-1}$.

55) $f(x) = x - \ln(1+x)$.

56) $f(x) = x\ln^2 x$.

57) $f(x) = x^2 e^{-x}$.

58) $f(x) = xe^x$.

59) $f(x) = x - \tan^{-1} x$.

**Solution**:

53) $(x^2 - 6x + 7)' =$

Conclusion:

54) $\left(\dfrac{x^2 - 2x + 2}{x-1}\right)' =$

Conclusion:

55) $(x - \ln(1+x))' =$

Conclusion:

56) $(x\ln^2 x)' =$

Conclusion:

57) $(x^2 e^{-x})' =$

Conclusion:

58) $(xe^x)' =$

Conclusion:

59) $(x - \tan^{-1} x)' =$

Conclusion:

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**Problems 60 – 63:** For each o f the following problems find the global maximum and the global minimum.

60) $f(x) = \sqrt{x(10-x)}$

61) $f(x) = 2x^3 + 3x^2 - 12x + 1, \quad -1 \le x \le 5.$

62) $f(x) = 2x^3 + 3x^2 - 12x + 1, \quad -10 \le x \le 12$

63) $f(x) = \frac{1}{5}x^5 - \frac{10}{3}x^3 + 9x.$

**Solution**:

60. $(\sqrt{x(10-x)})' =$

$\begin{cases} x_{\min} = \\ y_{\min} = \end{cases}$ $\begin{cases} x_{\max} = \\ y_{\max} = \end{cases}$

61. $(2x^3+3x^2-12x+1)'=$

$\begin{cases} x_{\min} = \\ y_{\min} = \end{cases}$ $\begin{cases} x_{\max} = \\ y_{\max} = \end{cases}$

62. $(2x^3+3x^2-12x+1)'=$

$\begin{cases} x_{\min} = \\ y_{\min} = \end{cases}$ $\begin{cases} x_{\max} = \\ y_{\max} = \end{cases}$

63. $(\frac{1}{5}x^5-\frac{10}{3}x^3+9x)'=$

$\begin{cases} x_{\min} = \\ y_{\min} = \end{cases}$ $\begin{cases} x_{\max} = \\ y_{\max} = \end{cases}$

---

**True or False Quiz**:: Give the conclusion about each of the following statements.

64) $x > 2\ln x$ for any $x > 0$.

65) $x > 3\ln x$ for any $x > 0$.

66) What is the maximum real number $n$ such that $x > n\ln x$ for any $x > 0$?

67) $x+\frac{1}{x} \geq 2$ for any $x > 0$.

68) $e^x > 1+x$ for any $x \neq 0$.

69) $x-\frac{x^3}{6} < \sin x < x$ for any $x > 0$.

**Solution:**

64.

Conclusion:

65.

Conclusion:

66. Hint: What is the absolute minimum of $f(x) = x - n\ln x$?

Conclusion:

67. Hint: Let $f(x) = x + \frac{1}{x} - 2$. Then determine the absolute minimum of $f(x)$ for $x > 0$.

Conclusion:

68.

Conclusion:

69. Hint: in this case you need to consider two function,

$$f_1(x) = x - \frac{x^3}{6} - \sin x \quad \text{and} \quad f_2(x) = \sin x - x.$$

Conclusion:

**Problems 70 - 72**: Find all values of $x$ which are the point of inflection. Find the values of $x$ where $f(x)$ increases the most rapidly; where $f(x)$ is decreasing most rapidly.

70) $f(x) = x^3 - 6x^2 + 12x + 4$,

71) $f(x) = x^2 \ln x$,

72) $f(x) = \tan^{-1} x - x$.

**Solution**:

70. $f''(x) = (x^3 - 6x^2 + 12x + 4)'' =$

71. $f''(x) = (x^2 \ln x)'' =$

72. $f''(x) = (\tan^{-1} x - x)'' =$

---

**Problems 73 - 80:** Find all asymptotes for the following functions**.** Sketch the graphs of the functions indicating the asymptotes. No need to determine extremes, concavity and so on.

73) $f(x) = \dfrac{x^2}{x^2 - 9}$,

74) $f(x) = \dfrac{x^2 + 4}{\sqrt{x^2 - 4}}$,

75) $f(x) = \dfrac{1}{x^2 + 9}$,

76) $f(x) = \dfrac{x}{x^2 + 2x - 3}$,

77) $f(x) = e^{2x}$,

78) $f(x) = \dfrac{1}{e^{x} - 1}$,

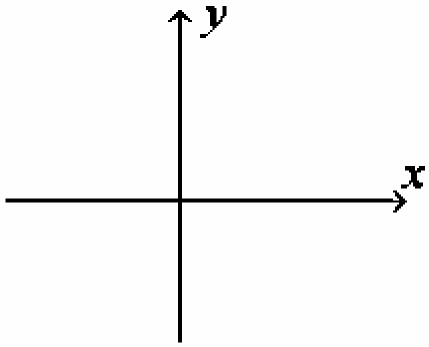
79) $f(x) = e^{-x^2} + 4$,

80) $f(x) = \dfrac{\sin x}{x}$.

**Solution**:

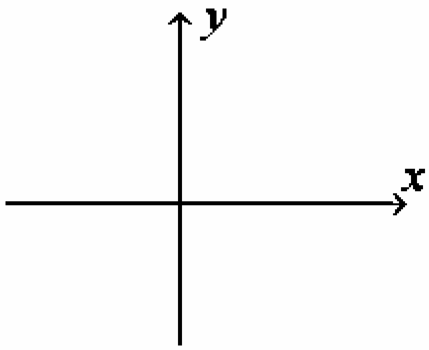
73. $k =$

$b =$



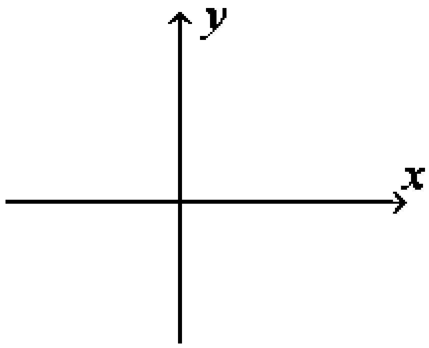

74. $k =$

$b =$

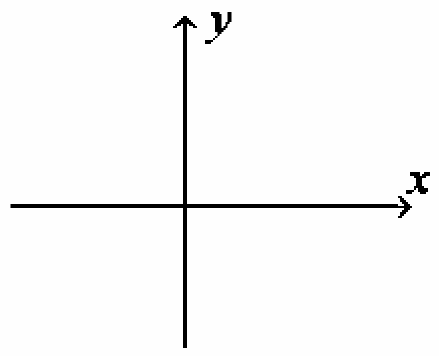



75. $k =$

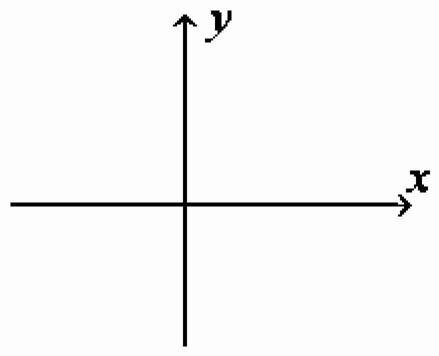
$b =$



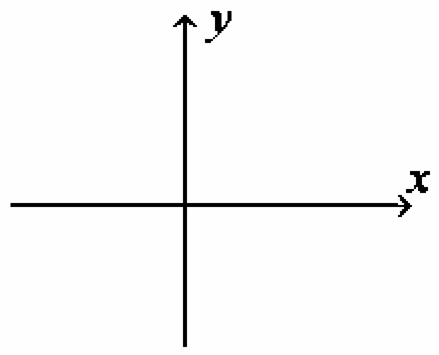
76. $k =$
$b =$



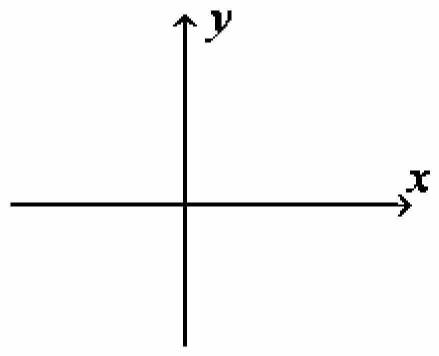

77. $k =$
$b =$



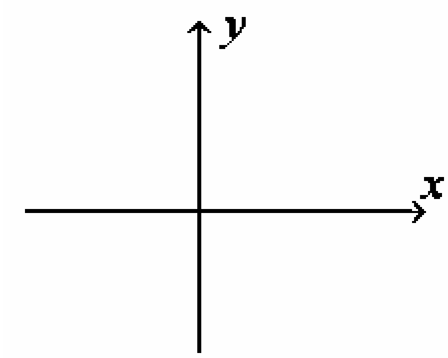

78. $k =$
$b =$




79. $k =$
$b =$



80. $k =$
$b =$




For Problems 81 - 85 sketch the graphs of the functions. State as much information about the function as you can.

**Hints**:

- Domain.
- Symmetry.
- Where is the function increasing, decreasing, concave, convex?
- Where are the extreme points, and the points of inflection?
- Are there asymptotes? How does the function approach them?
- Do not decide on a scale for axes until you have come up with a picture of the function. You need not use the same scales for both the $x-$ and $y-$axes.
- Try to find the $x$ and $y$ intercepts.
- It is convenient to represent the intermediate results by two tables.

The examples of these tables are given below.

**<u>Table 1</u>**. Consider a function $y = f(x)$, which is determined on an interval $[a, \infty)$. Let $x_1$ and $x_2$ be critical points.

| $x$ | $a$ | $[a, x_1)$ | $x_1$ | $(x_1, x_2)$ | $x_2$ | $(x_2, +\infty)$ |
|---|---|---|---|---|---|---|
| $y$ | 9 | ↘ | 2 | ↗ | $\pm\infty$ | ↗ |
| $y'$ | - | - | 0 | + | does not exist | + |
| Conclusion | | $f(x)$ is decreasing | min | $f(x)$ is increasing | discontinuity point | $f(x)$ is increasing |

**<u>Table 2</u>**. Let $x_3$ and $x_4$ be points of inflection.

| $x$ | $a$ | $[a, x_3)$ | $x_3$ | $(x_3, x_4)$ | $x_4$ | $(x_4, +\infty)$ |
|---|---|---|---|---|---|---|
| $y''$ | - | - | 0 | + | does not exist | - |
| Conclusion | | curve is convex | point of inflection | Curve is concave | point of inflection | curve is convex |

**Problems 81 – 85**:

81) $f(x) = x^2 + \dfrac{2}{x}$,

82) $f(x) = \dfrac{x}{x^2 + 4}$,

83) $f(x) = (2 + x^2)e^{-x^2}$,

84) $f(x) = x \ln x^2$,

85) $f(x) = \dfrac{x}{\ln x}$.

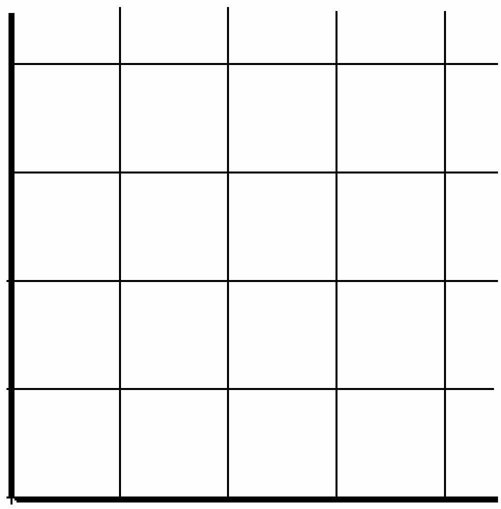
**Solution**:

81. $f(x) = x^2 + \dfrac{2}{x}$.

Domain:

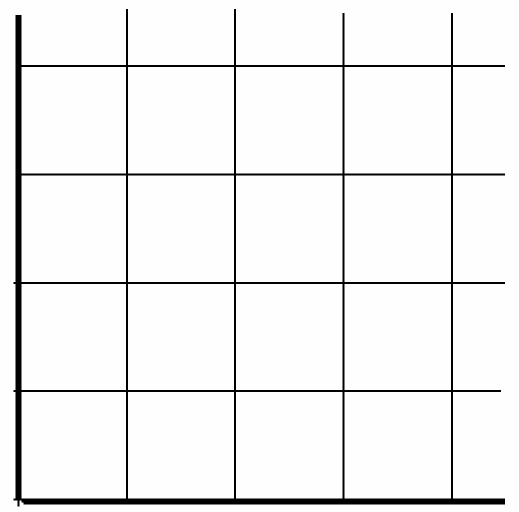
Symmetry:

Critical points:



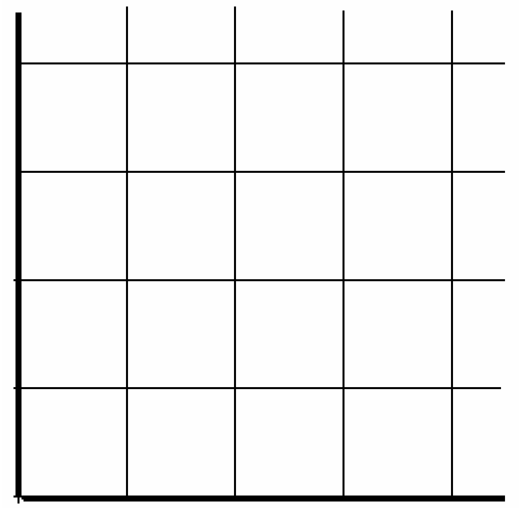
82. $f(x) = \dfrac{x}{x^2 + 4}$

Domain:
Symmetry:
Critical points:



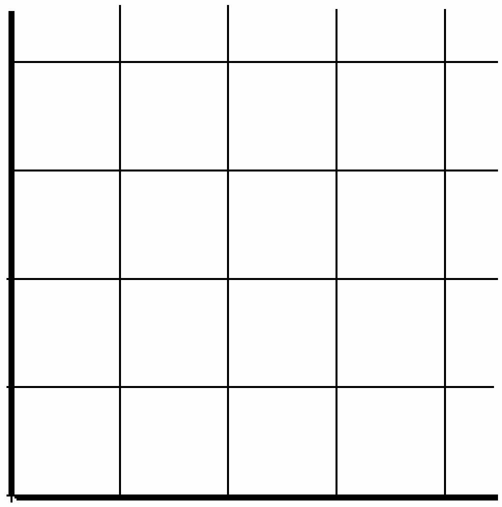
83. $f(x) = (2 + x^2)e^{-x^2}$

Domain:
Symmetry:
Critical points:



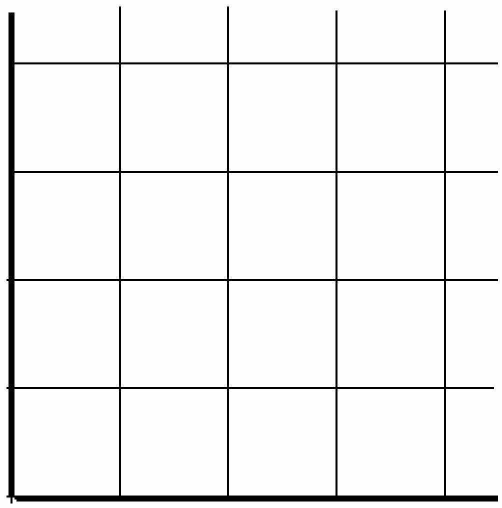
84. $f(x) = x \ln x^2$

Domain:
Symmetry:
Critical points:



85. $f(x) = \dfrac{x}{\ln x}$

Domain:
Symmetry:
Critical points:



## Functions of Several Variables

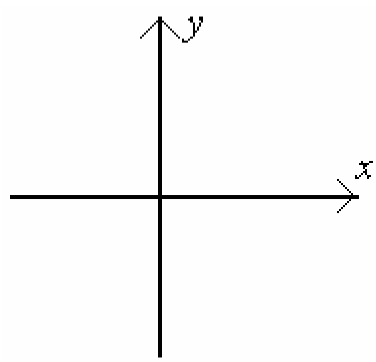
**Problems 1 – 8:** Find the domain of definition of a function of two variables:

1) $z=\sqrt{9-x^2-y^2}$ ,
2) $z=\sqrt{x\sin y}$ ,
3) $z=\sqrt{(9-x^2-y^2)(x^2+y^2-4)}$ ,
4) $z=\dfrac{1}{x^2+y^2}$ ,
5) $z=\dfrac{1}{x^2-y^2}$ ,
6) $z=\ln(x-y)$ ,
7) $z=\ln(x-y^2)$ ,
8) $z=\sqrt{4-x^2}+\sqrt{4-y^2}$ .

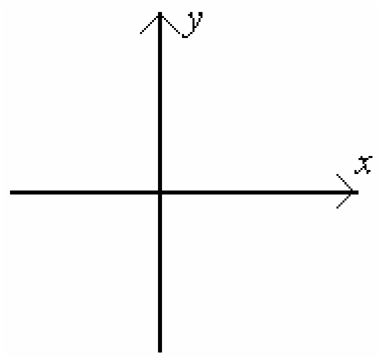
Show each of the domains in $xy-$plane.

**Solution**:

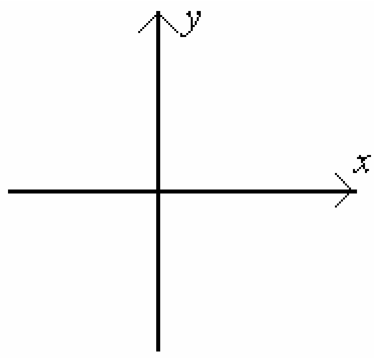
1. $z=\sqrt{9-x^2-y^2}$ .



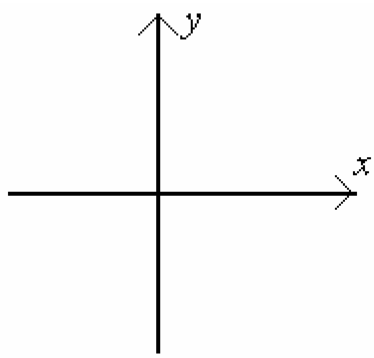

2. $z=\sqrt{x\sin y}$ .



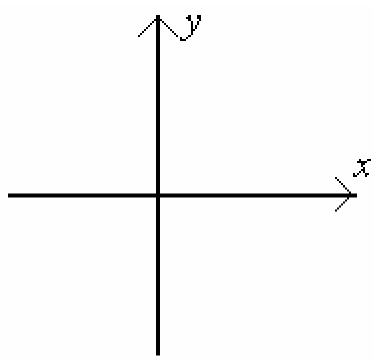
3. $z=\sqrt{(9-x^2-y^2)(x^2+y^2-4)}$.



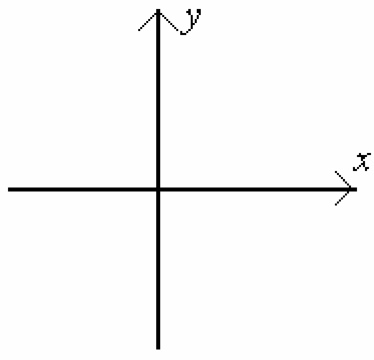

4. $z=\dfrac{1}{x^2+y^2}$.



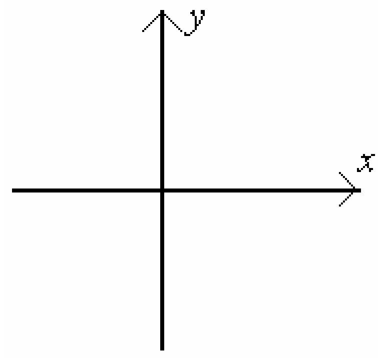

5. $z=\dfrac{1}{x^2-y^2}$.



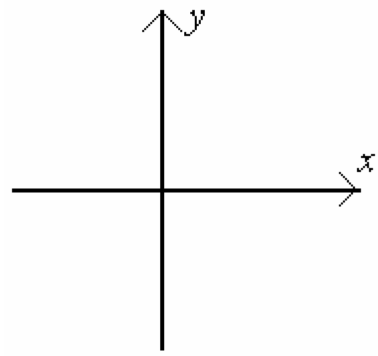

6. $z=\ln(x-y)$.



7. $z = \ln(x - y^2)$.




8. $z = \sqrt{4 - x^2} + \sqrt{4 - y^2}$.

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**Hint**: See Chapter 2, pages 23-24 in [1].

---

**Problems 9 – 15**: Find the partial derivatives of the following functions:

9) $z = \dfrac{x - y}{x + y}$ ,

10) $z = \sqrt{x^2 - y^3}$,

11) $z = \arctan \dfrac{y}{x}$,

12) $z = x^y$,

13) $z = \ln(x + \sqrt{x^2 + y^2})$,

14) $u = z^{x\sqrt{y}}$ ,

15) $u = (x \sin y)^z$.

**Solution**:

9. $z'_x =$

$z'_y =$

10. $z'_x =$
$z'_y =$

11. $z'_x =$
$z'_y =$

12. $z'_x =$
$z'_y =$

13. $z'_x =$
$z'_y =$

14. $z'_x =$
$z'_y =$

15. $z'_x =$
$z'_y =$

**Problems 16 - 20**: Find $\frac{du}{dt}$, if

16) $u = \frac{x}{y}$ with $x = e^{-5t}$ and $y = \ln t$,

17) $u = \ln\sin\frac{x}{\sqrt{y}}$ with $x = 3t^2$ and $y = \sqrt{t^2+1}$,

18) $u = xyz$ with $x = t^4 - t^2$, $y = \ln(t+1)$, and $z = \tan^2 t$,

19) $u = x^y + \ln^2 t$ with $x = \sin t$ and $y = \cos t$,

20) $u = \frac{z}{\sqrt{x^2+y^2}}$ with $x = 5\cos t$, $y = 5\sin t$, and $z = \pi$.

**Solution:**

16. $\frac{du}{dt} =$

17. $\frac{du}{dt} =$

18. $\dfrac{du}{dt} =$

19. $\dfrac{du}{dt} =$

20. $\dfrac{du}{dt} =$

---

**Hint**: See Chapter 2, page 30 in [1].

---

**Problem 21**: Find the derivative of the function $y$ with respect to $x$, if it satisfies the equation

$$x^3 e^{5y} - \sin(y^2 x) = 0 .$$

**Solution**:

---

**Hint**:

First find the partial derivatives of $F(x, y) = x^3 e^{5y} - \sin(y^2 x)$ with respect to $x$ and $y$.

Then apply the rule $\dfrac{dy}{dx} = -\dfrac{F'_x(x, y)}{F'_y(x, y)}$.

See ([1], Chapter 2, page 30, formula (20)).

---

**Problems 22 - 23**: Find the equation of the tangent plane to a given surface $F(x, y, z) = 0$ at a point $P_0(x_0, y_0, z_0)$.

Find the equations of the straight line passing through the point $P_0$ perpendicular to the surface.

22) $\dfrac{x^2}{9} + \dfrac{y^2}{16} - \dfrac{z^2}{8} = 0,$ $\quad P_0(3, 4, 4)$;

23) $x^2 + y^2 + z^2 - 12z = 0,$ $\quad P_0(3\sqrt{3}, 3, 6)$.

**Solution**:

22.

23.

---

**Hint**: Evaluate the partial derivatives of $F$ with respect to $x, y$ and $z$ at the point $P_0$. Then use formulas (27) and (28) in [1]. (See details in Chapter 2, pages 31-33, Geometric Interpretation of Partial Derivatives).

---

**Problems 24 –27**: Investigate for extreme points the following functions of two variables:

24) $z = (x-1)^2 + 2y^2$,

25) $z = (x-1)^2 - 2y^2$,

26) $z = x^4 + y^4 - 2x^2 + 4xy - 2y^2$,

27) $z = 1 - (x^2 + y^2)^{2/3}$.

**Solution**:

24.

25.

26.

27.

---

**Hint**: See Maximum and Minimum of Functions of Two Variables in [1], Chapter 2, page 33.

---

## Indefinite Integrals

## I. A table of common integrals

See [1], Chapter 3, pages 34-38.

**Problems 1 – 12:** Evaluate the following indefinite integrals. Check your answers by differentiating your expressions for the integrals.

1) $\int (4x^3 - \frac{2}{x})dx$

2) $\int (2x+1)^{2000} dx$

3) $\int \frac{dx}{\sqrt[3]{7x-2}}$

4) $\int 5^{3x+2} dx$

5) $\int \frac{dx}{3-4x}$

6) $\int \frac{dx}{5+9x^2}$

7) $\int \frac{dx}{\sqrt{4-3x^2}}$

8) $\int \frac{dx}{\cos^2(5x+1)}$

9) $\int (5^{2x} + \cos 3x - \frac{8}{\sin^2 4x})dx$

10) $\int \frac{(x-1)^2}{\sqrt{x}} dx$

11) $\int \sin(5x-3)dx$

12) $\int (\cos 2x - 3)dx$

**Solution**:

1. $\int (4x^3 - \frac{2}{x})dx = 4\int x^3 dx - 2\int \frac{dx}{x}$

$$= 4\frac{x^4}{4} - 2\ln|x| + C = x^4 - 2\ln|x| + C.$$

Check-up: $(x^4 - 2\ln|x|)' = (x^4)' - (2\ln|x|)' = 4x^3 - \frac{2}{x}$. That is true.

2. $\int (2x+1)^{2000} dx =$

Check-up:

3. $\int \frac{dx}{\sqrt[3]{7x-2}} =$

Check-up:

4. $\int 5^{3x+2} dx =$

Check-up:

5. $\int \frac{dx}{3-4x} =$

Check-up:

6. $\int \frac{dx}{5+9x^2} =$

Check-up:

7. $\int \frac{dx}{\sqrt{4-3x^2}} =$

Check-up:

8. $\int \frac{dx}{\cos^2(5x+1)} =$

Check-up:

9. $\int (5^{2x} + \cos 3x - \frac{8}{\sin^2 4x})dx =$

Check-up:

10. $\int \frac{(x-1)^2}{\sqrt{x}} dx =$

Check-up:

11. $\int \sin(5x-3)dx$

Check-up:

12. $\int (\cos 2x - 3)dx$

Check-up:

## II. Integration by Substitution

See [1], Chapter 3, pages 38-42.

**Problems 13 – 24:** Evaluate the following indefinite integrals by using of the appropriate substitution. Check your answers by differentiating the expressions for the corresponding integrals.

13) $\int \frac{\sqrt{\arctan x}}{1+x^2}dx$

14) $\int \frac{dx}{(1+4x^2)\arctan 2x}$

15) $\int 5^{\arcsin x}\frac{dx}{\sqrt{1-x^2}}$

16) $\int \frac{\cos(\ln x)}{x}dx$

17) $\int \frac{dx}{x\sqrt{1-5\ln x}}$

18) $\int \frac{xdx}{\sin^2(x^2+1)}$

19) $\int \frac{\tan(\sqrt{x})dx}{\sqrt{x}}$

20) $\int \frac{dx}{x\ln^4 x}$

21) $\int \sin(2x)e^{\cos^2 x}dx$

22) $\int \frac{e^x dx}{25+e^{2x}}$

23) $\int \frac{e^x dx}{\sqrt{4-e^{2x}}}$

24) $\int (6x^5-1)\cos(x-x^6)dx$

**Solution**:

13. Substitution $u=\arctan x$, $du=\frac{dx}{1+x^2}$,

$$\int \frac{\sqrt{\arctan x}}{1+x^2}dx=\int \sqrt{u}du=\int u^{1/2}du=\frac{2}{3}u^{\frac{3}{2}}+C=\frac{2}{3}\arctan^{\frac{3}{2}}x+C.$$

Check-up: $(\frac{2}{3}\arctan^{\frac{3}{2}}x)'=\frac{2}{3}\cdot\frac{3}{2}\arctan^{\frac{1}{2}}x\frac{1}{1+x^2}=\frac{\sqrt{\arctan x}}{1+x^2}$. That is true.

14. Substitution $u=$ $du=$

$$\int \frac{dx}{(1+4x^2)\arctan 2x}=$$

Check-up:

15. Substitution $u =$ $du =$

$$\int 5^{\arcsin x} \frac{dx}{\sqrt{1-x^2}} =$$

Check-up:

16. Substitution $u =$ $du =$

$$\int \frac{\cos(\ln x)}{x} dx =$$

Check-up:

17. Substitution $u =$ $du =$

$$\int \frac{dx}{x\sqrt{1-5\ln x}} =$$

Check-up:

18. Substitution $u =$ $du =$

$$\int \frac{x dx}{\sin^2(x^2+1)} =$$

19. Substitution $u =$ $du =$

$$\int \frac{\tan(\sqrt{x}) dx}{\sqrt{x}} =$$

Check-up:

20. Substitution $u =$ $du =$

$$\int \frac{dx}{x \ln^4 x} =$$

21. Substitution $u =$ $du =$

$$\int \sin(2x) e^{\cos^2 x} dx =$$

Check-up:

22. Substitution $u =$ $du =$

$$\int \frac{e^x dx}{25+e^{2x}} =$$

Check-up:

23. Substitution $u =$ $du =$

$$\int \frac{e^x dx}{\sqrt{4-e^{2x}}} =$$

Check-up:

24. Substitution $u =$ $du =$

$$\int (6x^5 - 1)\cos(x - x^6)dx =$$

Check-up:

---

## III. Integration by Parts

See [1], Chapter 3, pages 42-47.

**Problems 25 – 36:** Evaluate the following indefinite integrals by applying the integration by parts.

25) $\int x \ln x dx$ 26) $\int (x^3 - 5x)\ln x dx$

27) $\int x \ln^2 x dx$ 28) $\int x \tan^{-1} x dx$

29) $\int x e^x dx$ 30) $\int x^2 e^x dx$

31) $\int x \cos 3x dx$ 32) $\int x^2 \sin x dx$

33) $\int \arccos x dx$ 34) $\int x \arcsin x dx$

35) $\int e^{-3x} \cos 2x dx$ 36) $\int e^{4x} \sin 3x dx$

**Solution**:

25. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$$\int x \ln x dx =$$

26. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$\int (x^3 - 5x)\ln x dx =$

27. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$\int x \ln^2 x dx =$

28. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$\int x \tan^{-1} x dx =$

29. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$\int x e^x dx =$

30. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$\int x^2 e^x dx =$

31. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$\int x \cos 3x dx =$

32. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$\int x^2 \sin x dx =$

33. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$\int \arccos x dx =$

34. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$\int x \arcsin x dx =$

35. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$\int e^{-3x} \cos 2x dx =$

36. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$\int e^{4x} \sin 3x dx =$

## IV. Integration of Rational Functions

See [1], Chapter 3, pages 48-60.

**Problems 37 – 42: D**ecompose each of the following rational functions into a sum of partial fractions.

37) $\dfrac{1}{x^2 - x - 6}$

38) $\dfrac{1}{x(x^2 - x - 6)}$

39) $\dfrac{5x - 1}{x(x^2 - x - 6)}$

40) $\dfrac{x^2}{x^2 - x - 6}$

41) $\dfrac{1}{x^2(x - 3)}$

42) $\dfrac{1}{(x - 3)(x^2 + 4)}$

**Solution**:

37. $\dfrac{1}{x^2 - x - 6} = \dfrac{1}{(x + 2)(x - 3)} = \dfrac{A}{x + 2} + \dfrac{B}{x - 3} \quad \Rightarrow$

$$A(x - 3) + B(x + 2) = 1,$$

Let $x = -2$. Then $-5A = 1$, $A = -1/5$.

Let $x = 3$. Then $-B = 1$, $B = -1$.

Therefore, $$\frac{1}{x^2 - x - 6} = \frac{1}{5}(\frac{1}{x - 3} - \frac{1}{x + 2}).$$

38. $\dfrac{1}{x(x^2 - x - 6)} =$

39. $\dfrac{5x - 1}{x(x^2 - x - 6)} =$

40. $\dfrac{x^2}{x^2 - x - 6} =$

41. $\dfrac{1}{x^2(x-3)} =$

42. $\dfrac{1}{(x-3)(x^2+4)} =$

---

**Hint**:

- The degree of the numerator must be lower than that of the denominator; otherwise perform the polynomial long division.
- The denominator must be factored, so that every factor is either a linear factor or an irreducible quadratic polynomial with real coefficients. (See [3, pp. 18-20]).
- Then break up the rational fraction into the sum of the partial fractions.

---

**Problems 43 – 48:** Integrate the following rational functions.

43) $\dfrac{1}{4x^2-9}$ 44) $\dfrac{1}{x^2-x-6}$

45) $\dfrac{5x-1}{x(x^2-x-6)}$ 46) $\dfrac{x^2}{x^2-x-6}$

47) $\dfrac{1}{(x-3)(x^2+4)}$ 48) $\dfrac{x}{(x-3)^2(x^2+4)}$

Solution:

43. $$\int \frac{dx}{4x^2-9} = \int \frac{dx}{(2x-3)(2x+3)}$$
$$= \frac{1}{6}\int \left(\frac{1}{2x-3} - \frac{1}{2x+3}\right)dx = \frac{1}{6}\int \frac{dx}{2x-3} - \frac{1}{6}\int \frac{dx}{2x+3}$$
$$= \frac{1}{12}(\ln|2x-3| - \ln|2x+3|) + C = \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C$$

44. $\displaystyle\int \frac{dx}{x^2-x-6} =$

45. $\int \frac{5x-1}{x(x^2-x-6)}dx =$

46. $\int \frac{x^2}{x^2-x-6}dx =$

47. $\int \frac{dx}{(x-3)(x^2+4)} =$

48. $\int \frac{x}{(x-3)^2(x^2+4)}dx =$

---

## V. Integration of Trigonometric Functions

See [1], Chapter 3, pages 60-64.

**Problems 49 – 56:** Integrate the following trigonometric functions.

49) $\int \sin^2 3x dx$

50) $\int \cos^3 2x dx$

51) $\int \frac{\cos^3 5x}{\sin 5x}dx$

52) $\int \frac{dx}{\sin 7x}$

53) $\int \frac{dx}{\sin^4 3x}$

54) $\int \tan^3 4x dx$

55) $\int \cos x \cos 4x dx$

56) $\int \cos 5x \sin 2x dx$

**Solution**:

49. $$\int \sin^2 3x dx = \frac{1}{2}\int (1-\cos 6x)dx$$
$$= \frac{1}{2}(\int dx - \int \cos 6x dx) = \frac{1}{2}(x - \frac{1}{6}\sin 6x) + C.$$

50. $\int \cos^3 2x dx =$

51. $\int \frac{\cos^3 5x}{\sin 5x} dx =$

52. $\int \frac{dx}{\sin 7x} =$

53. $\int \frac{dx}{\sin^4 3x} =$

54. $\int \tan^3 4x dx =$

55. $\int \cos x \cos 4x dx =$

56. $\int \cos 5x \sin 2x dx =$

---

**Problems 57 – 62:** Integrate the following rational expressions of trigonometric functions.

57) $\int \frac{\cos^3 x}{25+\sin^2 x} dx$

58) $\int \frac{\sin 4x \cos 4x}{2-\cos 4x} dx$

59) $\int \frac{dx}{6\sin x \cos x + 8\cos^2 x + 1}$

60) $\int \frac{dx}{\cos x + \sin x}$

61) $\int \frac{dx}{3+\cos x + 2\sin x}$

62) $\int \frac{1+\tan x}{1-\tan x} dx$

**Solution**:

57. $\int \frac{\cos^3 x}{25+\sin^2 x} dx =$

58. $\int \frac{\sin 4x \cos 4x}{2-\cos 4x} dx =$

59. $\int \frac{dx}{6\sin x \cos x + 8\cos^2 x + 1}$

60. $\int \frac{\cos x}{2+\cos x - \sin x} dx$

61. $\int \frac{dx}{3+\cos x + 2\sin x}$

62. $\int \frac{1+\tan x}{1-\tan x} dx$

---

Hints:
<u>Problems 57-59</u>. See [1, Chapter 3, pages 67-69, Other Substitutions].

<u>Problems 60-62</u>. Use the general substitution $t = \tan \frac{x}{2}$.

## VI. Integrals Involving Radicals

**Problems 63 – 72:** Evaluate

63) $\int \frac{dx}{\sqrt{x+1}-4}$

64) $\int \frac{\sqrt{x}-1}{\sqrt[3]{x}+1}dx$

65) $\int \frac{dx}{\sqrt{2x-x^2-2}}$

66) $\int \frac{dx}{x\sqrt{1-x^3}}$

67) $\int \frac{dx}{x^4\sqrt{x^2-1}}$

68) $\int \frac{dx}{\sqrt{x^2+8x-9}}$

69) $\int \sqrt{4-x^2}\,dx$

70) $\int \sqrt{x-4x^2}\,dx$

71) $\int \frac{x+1}{\sqrt{(x^2+1)^3}}dx$

72) $\int \sqrt[4]{1+x^3}\,dx$

Solution:

63. $$\int \frac{dx}{\sqrt{x+1}-4} = \left| \begin{matrix} x+1=t^2 \\ dx=2tdt \end{matrix} \right| = \int \frac{2tdt}{t-4} = 2\int \frac{(t-4+4)dt}{t-4}$$
$$= 2\int dt + 8\int \frac{dt}{t-4} = 2t + 8\ln|t-4| + C$$
$$= 2\sqrt{x+1} + 8\ln|\sqrt{x+1}-4| + C$$

64. $\int \frac{\sqrt{x}-1}{\sqrt[3]{x}+1}dx =$

65. $\int \frac{dx}{\sqrt{2x-x^2-2}} =$

66. $\int \frac{dx}{\sqrt{x^2+8x-9}} =$

67. $\int \frac{dx}{x^4\sqrt{x^2-1}} =$

68. $\int \frac{dx}{x\sqrt{1-x^3}} =$

69. $\int \sqrt{4-x^2}\,dx =$

70. $\int \sqrt{x-4x^2}\,dx =$

71. $\int \frac{x+1}{\sqrt{(x^2+1)^3}}\,dx =$

72. $\int \sqrt[4]{1+x^3}\,dx =$

---

Hints:

Problem 64. First use the substitution $x = t^6$. Then the previous methods enable us to solve the integral.

Problems 65-66. In order to solve integrals of these type complete the square of the quadratic expression, then apply the appropriate substitutions.
Problems 67-72.  See Chapter 3, pages 70-76 in [1].

---

# Definite Integrals

## The Fundamental Theorems of Calculus

$$\frac{d}{dx}\int_a^x f(t)dt = f(x),$$

$$\int_a^b f(t)dt = F(x)\Big|_a^b = F(b) - F(a), \text{ where } F(x) = \int f(x)dx.$$

**Problems 1 – 8:** Find the derivatives of the following functions

1) $F(x) = \int_e^{x^5} \frac{t}{\ln t} dt$;

2) $F(x) = \int_1^{x^2} \sqrt{1-t^3} dt$;

3) $F(x) = \int_0^{\sqrt{x}} e^{-t^3} dt$;

4) $F(x) = \int_x^1 \sqrt{2+t^4} dt$;

5) $F(x) = \int_{1/x}^1 \ln t dt$;

6) $F(x) = \int_{x^2}^x e^{-t^2} dt$;

7) $F(x) = \int_{1/x}^{1/x^2} \cos t^2 dt$;

8) $F(x) = \int_{1/x}^{\sqrt{x}} \cos t^2 dt$.

**Solution**:

1. $$F'(x) = \frac{d}{dx}\int_e^{x^5} \frac{t}{\ln t} dt = \frac{x^5}{\ln x^5} 5x^4 = \frac{5x^9}{5\ln x} = \frac{x^9}{\ln x}.$$

2. $$F'(x) = \frac{d}{dx}\int_1^{x^2} \sqrt{1-t^3} dt =$$

3. $$F'(x) = \frac{d}{dx}\int_0^{\sqrt{x}} e^{-t^3} dt =$$

4. $F'(x) = \frac{d}{dx}\int_{x}^{1}\sqrt{2+t^4}\,dt =$

5. $F'(x) = \frac{d}{dx}\int_{1/x}^{1}\ln t\,dt =$

6. $F'(x) = \frac{d}{dx}\int_{x^2}^{x}e^{-t^2}\,dt =$

7. $F'(x) = \frac{d}{dx}\int_{1/x}^{1/x^2}\cos t^2\,dt =$

8. $F'(x) = \frac{d}{dx}\int_{1/x}^{\sqrt{x}}\cos t^2\,dt =$

---

**Hints**:
To solve Problems 4–8 use the following properties of definite integrals:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx,$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Then apply the first Fundamental Theorem of Calculus. (See Chapter 4, pages $77-81$ in [1]).
Recall also the differentiation rule of composite functions. (See [2, p. 87]).

---

**Problems 9 – 20**: Evaluate the following integrals

9) $\int_{0}^{\pi} \sin\frac{x}{3}\,dx$

10) $\int_{3}^{4} \frac{dx}{2x-1}$

11) $\int_{\pi/16}^{\pi/8} \frac{dx}{\sin^2 4x}$

12) $\int_{4}^{9} (3t^2 - \frac{7}{t} + \sqrt{t})dt$

13) $\int_{0}^{2\sqrt{3}} \frac{dx}{4+x^2}$

14) $\int_{0}^{1/4} \frac{dx}{\sqrt{1-4x^2}}$

15) $\int_{0}^{1} e^{-2x}dx$

16) $\int_{-x}^{x} e^{t}dx$

17) $\int_{4}^{9} \frac{1+\sqrt{z}}{z^2}dz$

18) $\int_{3}^{4} (y-2)^{99}dy$

19) $\int_{0}^{1} \frac{dx}{x^2-4x+5}$

20) $\int_{-3}^{3} \frac{dx}{x^2-4}$

**Solution**:

9. $\int_{0}^{\pi} \sin\frac{x}{3}\,dx = 3\int_{0}^{\pi} \sin\frac{x}{3}\,d\frac{x}{3} = -3\cos\frac{x}{3}\Big|_{0}^{\pi}$

$$= -3(\cos\frac{\pi}{3} - \cos 0) = -3(\frac{1}{2} - 1) = \frac{3}{2}$$

10. $\int_{3}^{4} \frac{dx}{2x-1} =$

11. $\int_{\pi/16}^{\pi/8} \frac{dx}{\sin^2 4x} =$

12. $\int_{4}^{9} (3t^2 - \frac{7}{t} + \sqrt{t})dt =$

13. $\int_{0}^{2\sqrt{3}} \frac{dx}{4+x^2} =$

14. $\int_{0}^{1/4} \frac{dx}{\sqrt{1-4x^2}} =$

15. $\int_{0}^{1} e^{-2x} dx =$

16. $\int_{-x}^{x} e^{t} dx =$

17. $\int_{4}^{9} \frac{1+\sqrt{z}}{z^2} dz =$

18. $\int_{3}^{4} (y-2)^{99} dy =$

19. $\int_{0}^{1} \frac{dx}{x^2-4x+5} =$

20. $\int_{-3}^{3} \frac{dx}{x^2-4} =$

---

Hint: Apply the second Fundamental Theorem of Calculus.

**Integration by Substitution**

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt,$$

where $\varphi(t) = x$, $\varphi(\alpha) = a$ and $\varphi(\beta) = b$.

**Problems 21 – 28**: By using the appropriate substitutions evaluate the following integrals

21) $\int_1^4 \frac{dx}{1+\sqrt{x}}$

22) $\int_0^5 \frac{dx}{2x+\sqrt{3x+1}}$

23) $\int_0^{\ln 6} \sqrt{e^t+3}dt$

24) $\int_0^{\ln 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}dx$

25) $\int_{1/\sqrt{2}}^1 \frac{\sqrt{1-x^2}}{x^2}dx$

26) $\int_1^2 \frac{\sqrt{y^2-1}}{y}dy$

27) $\int_0^{\pi/2} \frac{\cos x dx}{5-\cos^2 x - 2\sin^2 x}$

28) $\int_0^\pi \frac{dx}{3+2\cos x}$

**Solution**:

21. Substitution $x = t^2$, $dx = 2tdt$, $\alpha = \sqrt{1} = 1$, $\beta = \sqrt{4} = 2$.

$$\int_1^4 \frac{dx}{1+\sqrt{x}} = \int_1^2 \frac{2tdt}{1+t} = 2\int_1^2 \frac{(t+1-1)dt}{1+t} = 2(\int_1^2 dt - \int_1^2 \frac{dt}{1+t})$$

$$= 2(t - \ln|t+1|)\Big|_1^2 = 2(2-\ln 3 - 1 + \ln 2) = 2(1-\ln\frac{3}{2})$$

22. Substitution

$$\int_0^5 \frac{dx}{2x+\sqrt{3x+1}} =$$

23. Substitution

$$\int_0^{\ln 6} \sqrt{e^t+3}dt =$$

24. Substitution

$$\int_0^{\ln 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}\,dx =$$

25. Substitution

$$\int_{1/\sqrt{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx =$$

26. Substitution

$$\int_1^2 \frac{\sqrt{y^2-1}}{y}\,dy =$$

27. Substitution

$$\int_0^{\pi/2} \frac{\cos x dx}{5-\cos^2 x-2\sin^2 x} =$$

28. Substitution

$$\int_0^{\pi} \frac{dx}{3+2\cos x} =$$

---

**Hints**:

Problem 22. $3x+1=t^2$

Problem 23. $e^t+3=y^2$

Problem 24. $e^x-1=t^2$

Problem 25. $x=\sin t$

Problem 26. $y^2-1=t^2$

Problem 27. $\sin x=t$

Problem 28. $\tan\frac{x}{2}=t$.

---

**Integration by Parts:**

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

**Problems 29 – 32**: Using appropriate substitutions evaluate the following integrals

29) $\int_1^e \ln x dx$

30) $\int_e^{e^2} x^5 \ln x dx$

31) $\int_0^{\pi/6} x \sin 3x dx$

32) $\int_0^1 x^3 e^{-2x} dx$

33) $\int_1^e (x+3)\ln^2 x dx$

34) $\int_0^{\pi/2} x^2 \cos x dx$

35) $\int_0^1 \arctan x dx$

36) $\int_0^1 \arcsin x dx$

**Solution**:

29. $\begin{cases} u = \ln x \\ dv = dx \end{cases}$ $\begin{cases} du = \dfrac{dx}{x} \\ v = \int dx = x \end{cases}$

$$\int_1^e \ln x dx = x \ln x \Big|_1^e - \int_1^e x \frac{dx}{x}$$

$$= e \ln e - \ln 1 - \int_1^e dx = e \ln e - x\Big|_1^e = e \ln e - e + 1$$

30. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$$\int_e^{e^2} x^5 \ln x dx =$$

31. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$$\int_0^{\pi/6} x \sin 3x dx =$$

32. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$$\int_0^1 x^3 e^{-2x} dx =$$

33. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$$\int_1^e (x+3)\ln^2 x dx =$$

34. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

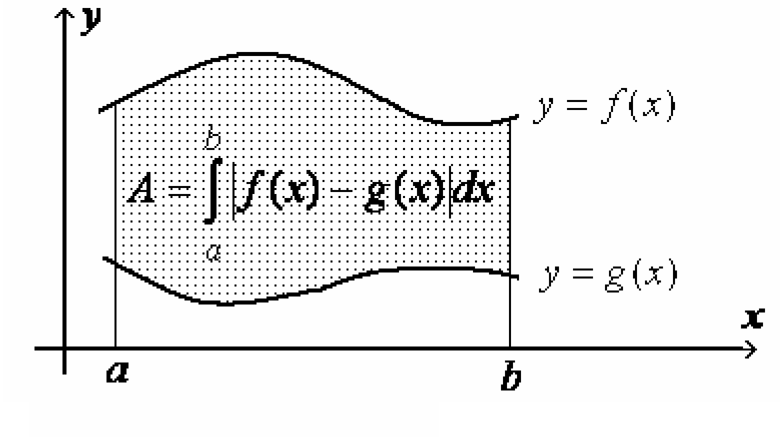
$$\int_0^{\pi/2} x^2 \cos x dx =$$

35. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$$\int_0^1 \arctan x dx =$$

36. $\begin{cases} u = \\ dv = \end{cases}$ $\begin{cases} du = \\ v = \end{cases}$

$$\int_0^1 \arcsin x dx =$$

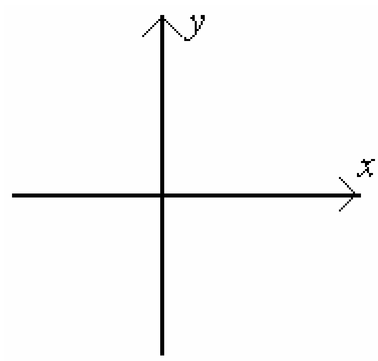



The area $A$ of a region bounded by the graphs of the functions $y = f(x)$ and $y = g(x)$, and by the vertical lines $x = a$ and $x = b$.

---

**Problem 33**: Find the area of the region bounded by the graph of the function $y = 4x - x^2$ and the $x-$axis.

**Solution**:

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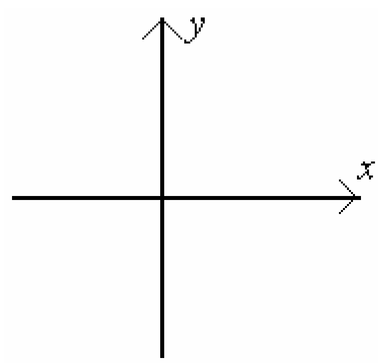
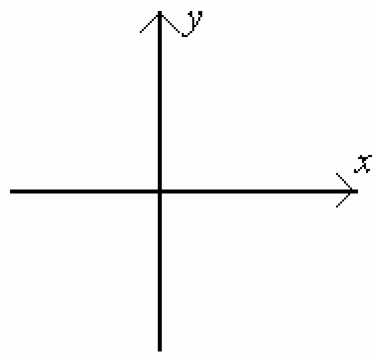
**Problem 34**: Find the area of the region bounded by the graph of the function $y = \ln x$, the $x-$axis and the vertical line $x = e$.

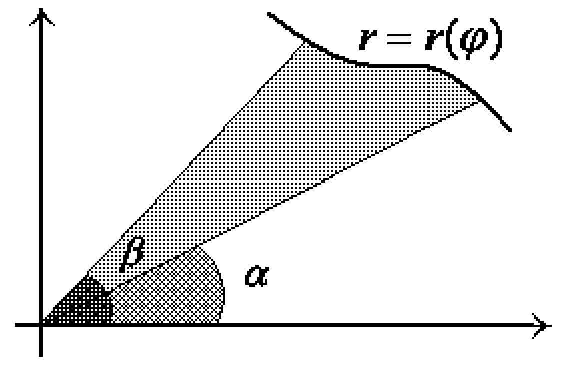
**Solution**:



**Problem 35**: Find the area of the region bounded by the graphs of the functions $y^2 = 4x$ and $x^2 = 4y$.

**Solution**:

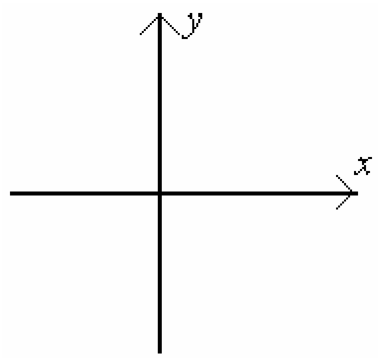



The area $A$ of a region bounded by the graph $r = r(\varphi)$ and the rays $\varphi = \alpha$ and $\varphi = \beta$:

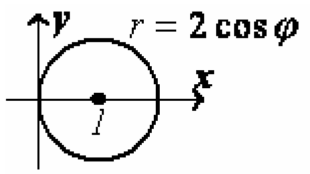
**Problem 36**: Find the area of the region bounded by the graph of the function $r = 2\cos\varphi$.

**Solution**:




**Hint**:

Problem 36.



The arc length of the curve $y = f(x)$ in the $xy-$plane between the given values $x = a$ and $x = b$:

$$L = \int_a^b \sqrt{1 + (y')^2}\, dx.$$

---

The arc length of the curve $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ between the given values $t = t_1$ and $t = t_2$:

$$L = \int_{t_1}^{t_2} \sqrt{(x')^2 + (y')^2 + (z')^2}\, dt$$

**Problem 37**: Find the arc length of the curve $y^2 = x^3$ between the points $O(0,0)$ and $P(4,8)$.

**Solution**:

**Problem 38**: Find the arc length of the curve $y = e^x$ between the points $P_1(0,1)$ and $P_2(1,e)$.

**Solution**:

**Problem 39**: Find the arc length of the curve $y = \ln x$ between the given values of $x$: $x = \sqrt{3}$ and $x = \sqrt{8}$.

**Solution**:

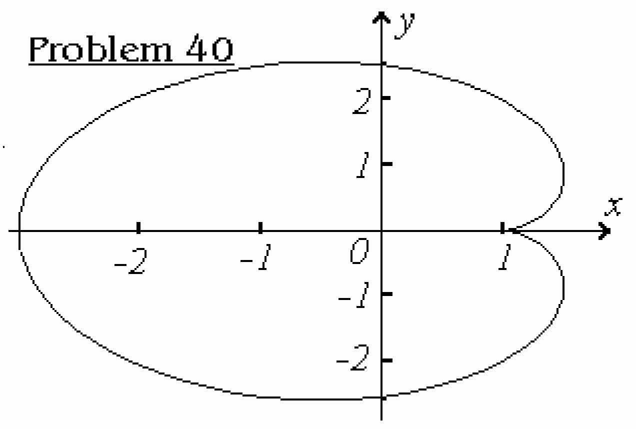
**Problem 40**: Find the arc length of the curve $\begin{cases} x = 2\cos t - \cos 2t \\ y = 2\sin t - \sin 2t \end{cases}$.

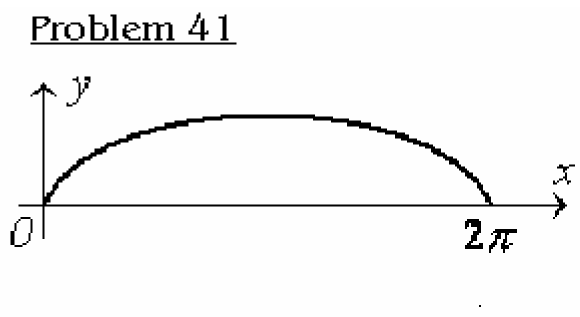
**Solution**:

**Problem 41**: Find the arc length of the curve $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$ between the given values of $t$: $t = 0$ and $t = 2\pi$.

**Solution**:

**Hints**:



# Improper Integrals

**Problems 42 –51**: Determine whether the following integrals converge or not.

42) $\int_1^{\infty} \frac{dx}{\sqrt[3]{x^4+5}}$

43) $\int_1^{\infty} \frac{dx}{x^2+\sqrt[3]{x^4+5}}$

44) $\int_1^{\infty} \frac{dx}{x+\sqrt[3]{x^4+5}}$

45) $\int_0^{\infty} \frac{xdx}{\sqrt{x^5+3}}$

46) $\int_0^{+\infty} xe^{-x}dx$

47) $\int_0^{+\infty} x^{20}e^{-x/10}dx$

48) . $\int_0^{1} \frac{dx}{\sqrt[3]{1-x^4}}$

49) $\int_0^{1} \frac{dx}{\sqrt[3]{1-x^2}}$

50) . $\int_1^{e} \frac{dx}{\ln x}$

51) $\int_1^{+\infty} \sin x dx$

**Solution**:

42. $\int_1^{\infty} \frac{dx}{\sqrt[3]{x^4+5}}$

43. $\int_1^{\infty} \frac{dx}{x^2+\sqrt[3]{x^4+5}}$

44. $\int_1^{\infty} \frac{dx}{x+\sqrt[3]{x^4+5}}$

45. $\int_0^{\infty} \frac{xdx}{\sqrt{x^5+3}}$

46. $\int_0^{+\infty} xe^{-x}dx$

47. $\int_{0}^{+\infty} x^{20} e^{-x/10} dx$

48. $\int_{0}^{1} \frac{dx}{\sqrt[3]{1-x^4}}$

49. $\int_{0}^{1} \frac{dx}{\sqrt[3]{1-x^2}}$

50. $\int_{1}^{e} \frac{dx}{\ln x}$

51. $\int_{1}^{+\infty} \sin x dx$

---

**Problems 52 – 61**: Evaluate each of the following integrals or determine its divergence.

52) $\int_{2}^{\infty} \frac{dx}{x+5}$

53) $\int_{1}^{\infty} \frac{dx}{x^2}$

54) $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$

55) $\int_{-\infty}^{\infty} \frac{dx}{x^2+4x+5}$

56) $\int_{0}^{1/2} \frac{dx}{x \ln x}$

57) $\int_{0}^{1/2} \frac{dx}{x \ln^2 x}$

58) $\int_{-\infty}^{0} e^x dx$

59) $\int_{4}^{+\infty} \frac{dx}{x^2-9}$

**60)** $\int_{-2}^{1} \frac{dx}{x}$

61) $\int_{0}^{1} \frac{dx}{\sqrt{x}}$

**62)** $\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$

63) $\int_{0}^{3} \frac{dx}{(x-1)^2}$

**Solution**:

52. $\int_2^\infty \frac{dx}{x+5}$ is divergent since $\int_2^\infty \frac{dx}{x}$ diverges and $\lim_{x\to+\infty}\left|\frac{x}{x+5}\right| < \infty$.

53. $\int_1^\infty \frac{dx}{x^2}$

54. $\int_{-\infty}^\infty \frac{dx}{x^2+1}$

55. $\int_{-\infty}^\infty \frac{dx}{x^2+4x+5}$

56. $\int_0^{1/2} \frac{dx}{x\ln x}$

57. $\int_0^{1/2} \frac{dx}{x\ln^2 x}$

58. $\int_{-\infty}^0 e^x dx$

59. $\int_4^{+\infty} \frac{dx}{x^2-9}$

60. $\int_{-2}^1 \frac{dx}{x}$

61. $\int_0^1 \frac{dx}{\sqrt{x}}$

62. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

63. $\int_0^3 \frac{dx}{(x-1)^2}$

## First-Order Ordinary Differential Equations

### **Directly Integrable Equations**:

$$y' = f(x).$$

The general solution: $y(x) = \int f(x)dx + C$.

**Problems 1**: Find the general solution of the equation

$$y'(x) = x^2.$$

**Solution**:

---

**Problems 2**: Find the general solution of the equation

$$y'(x) = xe^{x^2}.$$

**Solution**:

---

**Problems 3**: Find the particular solution of the equation

$$y'(x) = x^2.$$

which satisfies the initial condition

$$y(1) = 2.$$

**Solution**:

---

**Problems 4**: Find the particular solution of the equation

$$y'(x) = xe^{x^2}.$$

which satisfies the initial condition

$$y(0) = 3.$$

**Solution**:

## **Separable Equations**:

$$y' = f(x)g(y)$$

The general integral: $\int \frac{dy}{g(y)} = \int f(x)dx + C$.

**Problems 5**: Find the general solution of the equation

$$y' = e^{4x+y}.$$

**Solution**:

---

**Problems 6**: Find the general solution of the equation

$$y' = \frac{x}{y}.$$

**Solution**:

---

**Problems 7**: Find the general solution of the equation

$$xy' - y = y^3.$$

**Solution**:

---

**Problems 8**: Find the general solution of the equation

$$xyy' = 1 - x^2.$$

**Solution**:

---

**Problems 9**: Find the general solution of the equation

$$y' \tan x = y.$$

**Solution**:

**Problems 10**: Find the particular solution of the equation

$$y^2 y' + x^2 = 1$$

which satisfies the initial condition

$$y(1) = 2.$$

**Solution**:

---

**Problems 11**: Find the particular solution of the equation

$$xy' = 2y$$

which satisfies the initial condition

$$y(2) = 4.$$

**Solution**:

---

**Problems 12**: Find the particular solution of the equation

$$(1 + e^x) yy' = e^x$$

which satisfies the initial condition

$$y(0) = 1.$$

**Solution**:

---

**Problems 13**: Find the particular solution of the equation

$$y' \sin x = y \ln y$$

which satisfies the initial condition

$$y(\frac{\pi}{2}) = 1.$$

**Solution**:

## **Homogeneous Equations**:

$$y' = f\left(\frac{y}{x}\right).$$

The substitution $u = y/x$.

**Problems 14**: Find the general solution of the equation

$$xyy' = x^2 + y^2.$$

**Solution**:

---

**Problems 15**: Find the general solution of the equation

$$xy' = -x - y$$

**Solution**:

---

**Problems 16**: Find the particular solution of the equation

$$(x - 2\sqrt{xy})y' = y$$

which satisfies the initial condition

$$y(1) = 1.$$

**Solution**:

---

**Problems 17**: Find the particular solution of the equation

$$y' = e^{y/x} + \frac{y}{x}$$

which satisfies the initial condition

$$y\left(\frac{1}{e}\right) = 0.$$

**Solution**:

## **Linear Equations**:

$$y' + P(x)y = Q(x).$$

$$y = u(x)v(x)$$

**Problems 18**: Find the general solution of the equation

$$y' = \frac{y}{x} + x.$$

**Solution**:

---

**Problems 19**: Find the general solution of the equation

$$y' = x^3 - \frac{2y}{x}.$$

**Solution**:

---

**Problems 20**: Find the particular solution of the equation

$$y' = \frac{y}{1-x^2} + 1 + x$$

which satisfies the initial condition

$$y(0) = 0$$

**Solution**:

---

**Problems 21**: Find the particular solution of the equation

$$y' - y\tan x = \frac{1}{\cos x}$$

which satisfies the initial condition

$$y(0) = 0$$

**Solution**:

## The Bernoulli Equations

$$y'(x) + P(x)y = Q(x)y^n$$

$$y = u(x)v(x)$$

**Problems 22**: Find the general solution of the equation

$$y' = -\frac{y}{x} - xy^2.$$

**Solution**:

---

**Problems 23**: Find the general solution of the equation

$$2xyy' = y^2 - x.$$

**Solution**:

---

**Problems 24**: Find the particular solution of the equation

$$(x^3y - 2x)y' = 2y$$

which satisfies the initial condition

$$y(\frac{1}{2}) = 1.$$

**Solution**:

## Exact Differential Equations

$$P(x, y)dx + Q(x, y)dy = 0,$$

where $P'_y = Q'_x$.

**Problems 25**: Find the general solution of the equation

$$(x + y)dx + (x + 2y)dy = 0.$$

**Solution**:

---

**Problems 26**: Find the general solution of the equation

$$(3x^2 + 6xy^2)dx + (6x^2 y + 4y^3)dy = 0$$

**Solution**:

---

**Problems 27**: Find the general solution of the equation

$$(x^2 + y^2 + 2x)dx + 2xydy = 0$$

**Solution**:

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Higher Mathematics, Part 2

Workbook

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