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«**ТОМСКИЙ ПОЛИТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ**»

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HIGHER MATHEMATICS, PART 2

TextBook

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The textbook consists of 7 chapters, each of which concentrates on a topic of Differential and Integral Calculus. The important concepts of Calculus are explained and illustrated by figures and examples. The textbook can be helpful for students who want to understand and be able to use standard differentiation and integration techniques, make numerical approximations, analyze the behavior of a function, operate with complex numbers, solve ordinary differential equations, and so on.
The textbook is prepared at the Department of Higher Mathematics. It covers the topics to be studied in the second semester and it is intended for students who have been studied the basic Mathematics.

Reviewed by Victor V. Konev, Professor of Tomsk State University, D.Sc.

Preface

This textbook is intended mainly for students who have already studied the basic Mathematics and need to study and practice using the methods of Differential and Integral Calculus. All the important concepts of Calculus are explained and there are exercises of each point to concentrate on those methods, which students need to use but which often cause difficulty. The mathematical language used is as simple as possible.

The textbook covers the topics to be studied in the second semester.

1. The Fundamental Theorems of Differential Calculus.
2. Investigation of Functions.
3. Indefinite Integrals.
4. Definite Integrals. Geometric Applications of Definite Integrals.
5. Improper Integrals. Convergence and Divergence of Improper Integrals
6. Functions of Several Variables.
7. Complex Numbers.
8. Ordinary Differential Equations of the First Order.

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Chapter 1

FUNDAMENTAL THEOREMS OF DIFFERENTIAL CALCULUS

1.1. The Rolle Theorem

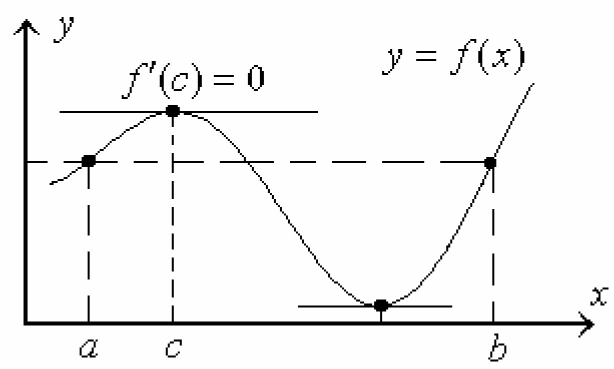


Fig. 1

Theorem: Let a function $f(x)$ be defined and continuous on a closed interval $[a,b]$ and be differentiable at each point of the open interval (a,b).

If $f(a) = f(b)$, then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.

The idea of a **Proof** is evident from Fig.1. By assumption, the function $f(x)$ is continuous on the closed interval $[a,b]$ and $f(a) = f(b)$, so $f(x)$ attains either its maximum or minimum at some point $x = c$ of the open interval (a,b). The tangent line of the function $y = f(x)$ at this point is a horizontal line. Hence, its slope is equal to zero, that is, the derivative $f'(c) = 0$.

Note that the Rolle Theorem does not claim where c can be found on (a,b); it claims only that there exists at least one point c such that $f'(c) = 0$.

As for the curve in Fig. 1, there are two points satisfying the equation $f'(x) = 0$.

1.2. The Mean Value Theorem

Theorem: Let a function $f(x)$ be defined and continuous on a closed interval $[a,b]$ and be differentiable on the open interval (a,b).

Then there exists a point $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (1)$$

Proof: Consider the auxiliary function

$$\Phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a),$$

which satisfies the conditions of the Rolle Theorem.

Indeed, $\Phi(x)$ is the sum of the functions defined and continuous on $[a,b]$ and differentiable on (a,b). Moreover, $\Phi(a)=\Phi(b)=0$.
Therefore, by the Rolle theorem, there exists some point $c\in(a,b)$ such that

$$\Phi'(c)=f'(c)-\frac{f(b)-f(a)}{b-a}=0.$$

Hence, the theorem.
Note that the Mean Value Theorem does not claim where c can be found on (a,b).

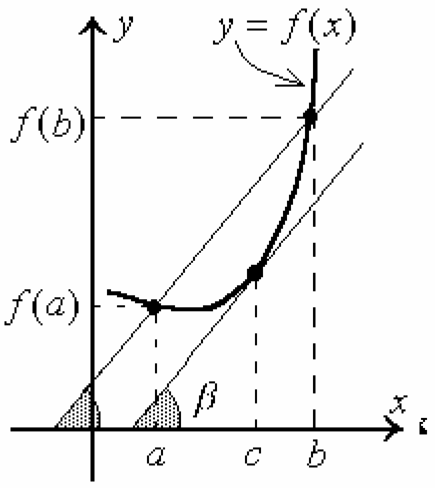


Fig. 2

Geometric Interpretation

- The difference quotient $\frac{f(b)-f(a)}{b-a}$ equals the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$.
- The derivative $f'(c)$ equals the slope of the tangent passing through the point $(c, f(c))$.

Hence, the theorem asserts that the secant line through $(a, f(a))$ and $(b, f(b))$ is parallel to the tangent at some point $(c, f(c))$, where $a<c<b$. (See Fig. 2.)

Corollary 1: The Rolle Theorem is a special case of the Mean Value Theorem:
If $f(a)=f(b)$, then $f'(c)=0$ for some $c\in(a,b)$.

Corollary 2: If $f'(x)=0$ for all points of some interval (a,b), then $f(x)$ is a constant on (a,b).
Proof: Let x and x_0 be any points on (a,b).
Then by the theorem,

$$f(x)-f(x_0)=f'(c)(x-x_0),$$

where c is some point between x_0 and x.
But $f'(c)=0$ and hence, $f(x)=f(x_0)$ for any $x\in(a,b)$.

Corollary 3: If functions $f(x)$ and $g(x)$ are such that $f'(x)=g'(x)$ for all $x\in(a,b)$, then $f(x)=g(x)+C$, where C is a constant.

Proof: Noting that

$$(f(x)-g(x))' = f'(x)-g'(x)=0,$$

by Corollary 2, we obtain $f(x)-g(x)=C$.

Example 1: One can easily check that the functions $\arctan x$ and $\arcsin\dfrac{x}{\sqrt{1+x^2}}$ have the same derivative $1/(1+x^2)$.

Therefore, by Corollary 3, $\arcsin(x/\sqrt{1+x^2}) = \arctan x + C$.
Setting $x=0$, we find the value of the constant: $0=0+C$.
Thus,

$$\arcsin\frac{x}{\sqrt{1+x^2}} = \arctan x.$$

Example 2: By the same argument, equations

$$\left(\arctan\frac{x}{\sqrt{1-x^2}}\right)' = \frac{1}{\sqrt{1-x^2}} = (\arcsin x)'$$

yield

$$\arctan\frac{x}{\sqrt{1-x^2}} = \arcsin x, \qquad (C=0).$$

1.3. The Cauchy Theorem

Let functions $f(x)$ and $g(x)$ be defined and continuous on a closed interval $[a,b]$ and be differentiable on the open interval (a,b), and $g'(x)\neq 0$ for $a<x<b$.
Then there exists a point $c\in(a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}. \qquad (2)$$

Proof: Note that $g(b)\neq g(a)$. Otherwise, by the Rolle Theorem, $g'(x)=0$ for some $x=c$, that contradicts to the assumption $g'(x)\neq 0$.
Consider the function

$$\Phi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a)).$$

It satisfies the conditions of the Rolle Theorem:

- $\Phi(x)$ is defined and continuous on $[a,b]$;
- $\Phi(x)$ has the derivative on (a,b):

$$\Phi'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x);$$

- $\Phi(a) = \Phi(b) = 0$.

Hence, according to the Rolle Theorem, there exists a point $c \in (a,b)$ such that $\Phi'(c) = 0$, which implies the desired result:

$$f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0.$$

1.4. The L'Hopital Rule

The definition of the derivative of a function is based on the concept of the limit of the ratio of infinitesimal quantities. The rules of differentiation and the derivatives of the basic functions are derived by making use of limits.
On the other hand, one of the most powerful tools for finding limits of functions is connected with application of derivatives. The corresponding algorithm is named the L'Hopital Rule.

The L'Hopital Rule for an indeterminate form $\frac{0}{0}$.

Let functions $f(x)$ and $g(x)$ be defined and differentiable on (a,b), and $g'(x) \neq 0$ for all $a < x < b$.
Assume that
1) $\lim\limits_{x\to a} f(x) = 0 = \lim\limits_{x\to a} g(x)$, and
2) there exists $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ (finite or not).

Then there exists $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ and the equality is true:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \qquad (3)$$

Proof: In view of assumption 1) we put

$$f(a) = 0 \quad \text{and} \quad g(a) = 0.$$

Then by the Cauchy Theorem, we have

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)},$$

where $a < c < x$.
Taking the limit in this equality and noting that $c \to a$ as $x \to a$, we come to (3).

Note: If the fraction $\dfrac{f'(x)}{g'(x)}$ is again an indeterminate form $\dfrac{0}{0}$, then the L'Hopital Rule can be applied repeatedly.

Example 1: The expression $\dfrac{\ln(1+5x)}{x}$ is the indeterminate form $\dfrac{0}{0}$ as $x \to 0$.

By the L'Hopital Rule,

$$\lim_{x\to 0}\frac{\ln(1+5x)}{x} = \lim_{x\to 0}\frac{(\ln(1+5x))'}{x'} = \lim_{x\to 0}\frac{5/(1+5x)}{1} = 5.$$

Example 2: If $x \to 0$, then $\dfrac{x - \sin x}{x^3}$ represents the indeterminate form $\dfrac{0}{0}$.

In order to find the limit we need to apply the L'Hopital Rule several times:

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{(x-\sin x)'}{(x^3)'} = \lim_{x\to 0}\frac{1-\cos x}{3x^2} = \frac{1}{3}\lim_{x\to 0}\frac{(1-\cos x)'}{(x^2)'}$$

$$= \frac{1}{3}\lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{6}\lim_{x\to 0}\frac{(\sin x)'}{x'} = \frac{1}{6}\lim_{x\to 0}\frac{\cos x}{1} = \frac{1}{6}.$$

The L'Hopital Rule for an indeterminate form $\dfrac{\infty}{\infty}$.

Let functions $f(x)$ and $g(x)$ be defined and differentiable on (a,b), and let $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$.

Assume that there exists $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ (finite or not).

Then there exists also $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ and

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Proof: An indeterminate form $\frac{\infty}{\infty}$ can be easily reduced to the form $\frac{0}{0}$ by means of a simple transformation: $\frac{f(x)}{g(x)} = \frac{1/g(x)}{1/f(x)} \Rightarrow (\frac{0}{0})$.

Let $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = A$. Then

$$A = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{1/g(x)}{1/f(x)} = \lim_{x\to a} \frac{(1/g(x))'}{(1/f(x))'}.$$

By the rule of differentiation of a quotient, we have

$$(\frac{1}{g(x)})' = -\frac{g'(x)}{g^2(x)} \qquad \text{and} \qquad (\frac{1}{f(x)})' = -\frac{f'(x)}{f^2(x)}.$$

Then making use of the properties of limits we obtain

$$A = \lim_{x\to a} \frac{(1/g(x))'}{(1/f(x))'} = \lim_{x\to a} \frac{f^2(x)}{g^2(x)} \frac{g'(x)}{f'(x)}$$

$$= \left(\lim_{x\to a} \frac{f(x)}{g(x)} \right)^2 \lim_{x\to a} \frac{g'(x)}{f'(x)} = A^2 \lim_{x\to a} \frac{g'(x)}{f'(x)}.$$

Therefore,

$$1 = A \lim_{x\to a} \frac{g'(x)}{f'(x)}, \quad \text{and hence,} \quad A = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Example 3: In order to find $\lim\limits_{x\to 1+0} \frac{\ln(x-1)}{\tan(\pi x/2)}$ one has to expand the indeterminate form $\frac{\infty}{\infty}$.

Applying the L'Hopital Rule we obtain:

$$\lim_{x\to 1+0} \frac{\ln(x-1)}{\tan(\pi x/2)} = \frac{2}{\pi} \lim_{x\to 1+0} \frac{1/(x-1)}{1/\cos^2(\pi x/2)} = \frac{2}{\pi} \lim_{x\to 1+0} \frac{\cos^2(\pi x/2)}{x-1}.$$

This is the indeterminate form $\frac{0}{0}$, and we can apply the L'Hopital Rule once more:

$$\frac{2}{\pi} \lim_{x\to 1+0} \frac{\cos^2(\pi x/2)}{x-1} = \lim_{x\to 1+0} \frac{2\cos(\pi x/2)\sin(\pi x/2)}{1} = 0.$$

Summary: The L'Hopital Rule yields the same formula for both indeterminate forms, $\frac{0}{0}$ and $\frac{\infty}{\infty}$. This rule can also be applied for limits at infinity:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)} \tag{4}$$

as well as for unilateral limits:

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}\frac{f'(x)}{g'(x)} \quad \text{and} \quad \lim_{x\to-\infty}\frac{f(x)}{g(x)}=\lim_{x\to-\infty}\frac{f'(x)}{g'(x)}.$$

In each case the conditions of the theorem has to be changed properly.
For instance, the condition $\lim_{x\to a} f(x)=0=\lim_{x\to a} g(x)$ is replaced by $\lim_{x\to\infty} f(x)=0=\lim_{x\to\infty} g(x)$, *etc.*

1.4.1. The Other Indeterminate Forms

Consider the following forms of an indeterminacy:

1) $0\cdot\infty$, 2) $\infty-\infty$, 3) 1^{∞}, 0^0, ∞^0.

All these forms can be reduced to one of the above-considered forms: $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

1) Let $f(x)\to 0$ and $g(x)\to\infty$ as $x\to a$.
   Then an indeterminate form $f\cdot g=0\cdot\infty$ can be easily transformed to the form $\frac{0}{0}$ as $f\cdot g=\frac{f}{1/g}$, or to the form $\frac{\infty}{\infty}$ as $f\cdot g=\frac{g}{1/f}$.
2) If $f(x)\to\infty$ and $g(x)\to\infty$ as $x\to a$, then

$$f-g=\frac{1}{\frac{1}{f}}-\frac{1}{\frac{1}{g}}=\frac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{f\,g}}.$$

Thus, the indeterminate form $\infty-\infty$ has been reduced to the form $0/0$.

3) In view of the formula $f^{g}=e^{g\ln f}$ each of the forms, 1^{∞}, 0^0 and ∞^0, can be transformed to the product $g(x)\ln f(x)$, which is the indeterminate form $0\cdot\infty$ considered above. (See Case 1.)

Indeed,

- Let $f^g = 1^\infty$, that is, $f(x) \to 1$ and $g(x) \to \infty$. Then $g(x)\ln f(x) = \infty \cdot 0$.
- Let $f^g = 0^0$, that is, $f(x) \to 0$ and $g(x) \to 0$. Then $g(x)\ln f(x) = 0 \cdot \infty$.
- Let $f^g = \infty^0$, that is, $f(x) \to \infty$ and $g(x) \to 0$. Then $g(x)\ln f(x) = 0 \cdot \infty$.

Thus, if $\lim\limits_{x\to a} g(x)\ln f(x) = B$, then $\lim\limits_{x\to a} (f(x))^{g(x)} = e^B$.

Example 4: $\lim\limits_{x\to 0} x\ln x$ contains the indeterminate form $0 \cdot \infty$. However, it can be easily transformed to the form $\dfrac{\infty}{\infty}$, as $x\ln x = \dfrac{\ln x}{x^{-1}}$.

By making use of the L'Hopital Rule we obtain

$$\lim_{x\to 0} \frac{\ln x}{x^{-1}} = \lim_{x\to 1} \frac{(\ln x)'}{(x^{-1})'} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = -\lim_{x\to 0} x = 0.$$

Example 5: $\lim\limits_{x\to 0} (1+2\sin x)^{\frac{3}{4x}}$ contains the indeterminate form 1^∞.

We can use the identity $\lim f = e^{\lim \ln f}$.

Taking the logarithm of the expression under the sign of the limit we obtain

$$\ln(1+2\sin x)^{\frac{3}{4x}} = 3\frac{\ln(1+2\sin x)}{4x}.$$

By the L'Hopital Rule we evaluate the limit of this expression:

$$\lim_{x\to 0} 3\frac{\ln(1+2\sin x)}{4x} = \frac{3}{4}\lim_{x\to 0} \frac{(\ln(1+2\sin x))'}{x'}$$

$$= \frac{3}{4}\lim_{x\to 0} \frac{\dfrac{2\cos x}{1+2\sin x}}{1} = \frac{3}{2}.$$

Finally, we obtain:

$$\lim_{x\to 0} (1+2\sin x)^{\frac{3}{4x}} = e^{\frac{3}{2}} = \sqrt{e^3}.$$

1.5. The Taylor Formula

1.5.1. The Taylor Formula for Polynomials

Theorem 1: For any x_0 a polynomial $P(x)$ of degree n can be represented as

$$P(x) = P(x_0) + \frac{P'(x_0)}{1!}(x - x_0) + ... + \frac{P^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$= \sum_{k=0}^{n} \frac{P^{(k)}(x_0)}{k!}(x - x_0)^k, \qquad (5)$$

where $P'(x_0)$, $P''(x_0)$, … are the derivatives of $P(x)$ at the point x_0.

Note: Formula (5) is called the **Taylor Formula for polynomials**.

Proof: Any polynomial of degree n can be written as follows:

$$P(x) = \sum_{k=0}^{n} a_k (x - x_0)^k . \qquad (6)$$

Therefore, we have to prove that $a_k = \dfrac{P^{(k)}(x_0)}{k!}$ for $0 \le k \le n$.

First, the equality $a_0 = P(x_0)$ follows from (6) when $x = x_0$.

Then let us find the *k*th derivative of the polynomial $P(x)$ at the point $x = x_0$.

One can easily see that sum (6) contains just one term, whose *k*th derivative at the point $x = x_0$ is not equal to zero: $\left(a_k (x - x_0)^k\right)^{(k)} = a_k\, k!$

The *k*th derivative of other terms of this sum either equals zero for any x or contains the factor $(x - x_0)$, which vanishes as ever $x = x_0$.

Thus, $P^{(k)}(x_0) = a_k\, k!$ and hence, the theorem.

Example: Represent the polynomial $P(x)$ in powers of x, if

$$P(x) = 1 + 8(x - 2) + 6(x - 2)^2 + (x - 2)^3.$$

Solution: The Taylor Formula with $x_0 = 0$ gives the answer in the general form:

$$P(x) = P(0) + P'(0)\, x + \frac{P''(0)}{2} x^2 + \frac{P'''(0)}{6} x^3 .$$

It remains to find $P(0)$ and $P^{(k)}(0)$:

- $P(x) = 1 + 8(x - 2) + 6(x - 2)^2 + (x - 2)^3 \quad \Rightarrow \quad P(0) = 1.$

- $P'(x) = 8 + 12(x-2) + 3(x-2)^2$ $\Rightarrow$ $P'(0) = -4$.
- $P''(x) = 12 + 6(x-2)$ $\Rightarrow$ $P''(0) = 0$.
- $P'''(x) = 6$ $\Rightarrow$ $P'''(0) = 6$.

Thus, $P(x) = 1 - 4x + x^3$.

1.5.2. The Taylor Formula with the Remainder

Theorem 2: Let a function $f(x)$ be n times differentiable at a point x_0. Then $f(x)$ can be represented by the Taylor Formula

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x), \qquad (7)$$

where $f^{(0)}(x_0) = f(x_0)$ by definition, and $R_n(x)$ is a function such that

$$R_n(x_0) = R_n'(x_0) = R_n''(x_0) = ... = R_n^{(n)}(x_0) = 0. \qquad (8)$$

Note that $R_n(x)$ is called the **remainder**.

Proof: The remainder $R_n(x)$ is the difference between $f(x)$ and the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

By the argument used in the proof of Theorem 1, we obtain

$$P_n(x_0) = f(x_0) \quad \text{and} \quad P_n^{(k)}(x_0) = \frac{f^{(k)}(x_0)}{k!}k! = f^{(k)}(x_0).$$

Therefore,

$$R_n^{(k)}(x_0) = f^{(k)}(x_0) - P_n^{(k)}(x_0) = f^{(k)}(x_0) - f^{(k)}(x_0) = 0$$

for $0 \le k \le n$, which implies formulas (8).

In a special case when $x_0 = 0$, the Taylor formula is named the Maclaurin formula:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x) \qquad (9)$$

Now let us come back to formula (7) and rewrite it in terms of differentials. We need to recall the relevant definitions.

The difference $x - x_0 = \Delta x$ can be considered as an increment of the argument; then $f(x) - f(x_0) = \Delta f(x)$ is the corresponding increment of the function.

By definition $dx = \Delta x$, that is, the differential of the argument equals the increment. The *k*th differential of the argument is defined as $dx^k \equiv (dx)^k$.

The differential of $f(x)$ at the point $x = x_0$ is $df(x_0) = f'(x_0)dx$, and the kth differential of $f(x)$ at this point is $d^k f(x_0) = f^{(k)}(x_0)dx^k$.

The Taylor Formula has the simplest form in terms of differentials:

$$\Delta f(x) = df(x_0) + \frac{d^2 f(x_0)}{2!} + \frac{d^3 f(x_0)}{3!} + \ldots + \frac{d^n f(x_0)}{n!} + R_n(x). \qquad (10)$$

The Taylor Formula has diverse applications. Most often it is used tor approximation of transcendental functions by polynomials. In this case the polynomial $P_n(x)$ is an approximation to $f(x)$, whereas $R_n(x)$ is an error of the approximation. Such conclusion has the following background.

If $f(x)$ is a continuous function on some interval, then so is the remainder $R_n(x)$. In view of the fact that $R_n(x_0) = R_n'(x_0) = 0$, the remainder is small enough in some vicinity of the point x_0. Moreover, the remainder $R_n(x)$ is an infinitesimal whose order of smallness is greater than n as $x \to x_0$, that is,

$$\lim_{x \to x_0} \frac{R_n(x)}{(x - x_0)^n} = 0.$$

This statement can be easily proved by applying the L'Hopital rule n times and taking into account equalities (8):

$$\lim_{x \to x_0} \frac{R_n(x)}{(x - x_0)^n} = \lim_{x \to x_0} \frac{R_n'(x)}{n(x - x_0)^{n-1}} = \ldots = \lim_{x \to x_0} \frac{R_n^{(n)}(x)}{n!} = 0.$$

Whenever we deal with approximations, we need to control the errors.

One of the ways is based on the **Lagrange form of the remainder**:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \qquad (11)$$

where c is some point between x and x_0.

If $|x - x_0| < 1$ and $f^{(n+1)}(x) \le M$, then

$$\left|R_n(x)\right| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}\right| \le \left|M\,\frac{(x-x_0)^{n+1}}{(n+1)!}\right| \;\to\; 0$$

very quickly as $n \to \infty$.

Therefore, the more n, the better approximation to $f(x)$ by the polynomial $P_n(x)$.

The Taylor Formula with the Lagrange form of the remainder can be written as follows:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \qquad (12)$$

In a special case when $n = 0$, this formula implies the Mean Value Theorem over the interval $[x, x_0]$:

$$f(x) = f(x_0) + f'(c)(x-x_0).$$

1.5.3. Applications of the Taylor Formula

All the formulas below follow from the Maclaurin Formula. All we need to find the expansion for a specific function $f(x)$ is the general form of the nth derivative of $f(x)$. The remainders in all the cases are written in the Lagrange form.

1) Let $f(x) = e^x$. Then

$$f^{(n)}(x) = e^x \text{ and } \quad f^{(n)}(0) = 1 \quad \text{for } n \ge 0.$$

Therefore,

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \text{Error.}} \qquad (13)$$

$$\text{Error} = R_n(x) = \frac{e^c}{(n+1)!}x^{n+1},$$

where c is a point between zero and x.

If $x < 0$, then $e^c < 1$ and

$$|\,\text{Error}\,| < \frac{1}{(n+1)!}|\,x\,|^{n+1}.$$

2) If $f(x) = \sin x$, then

$$f^{(k)}(x) = \sin(x + \frac{k\pi}{2}) \quad \text{and} \quad f^{(k)}(0) = \sin\frac{k\pi}{2}.$$

If $k = 2n-1$, then

$$f^{(2n-1)}(0) = (-1)^{n-1},$$

while if $k = 2n$, then

$$f^{(2n)}(0) = 0.$$

Therefore,

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \text{Error.}} \tag{14}$$

$$|\,\text{Error}\,| < \frac{|x|^{2n}}{(2n)!} \quad \text{for} \quad x < 0,$$

$$|\,\text{Error}\,| < \frac{|x|^{2n+1}}{(2n+1)!} \quad \text{for} \quad x > 0.$$

3) If $f(x) = \cos x$, then

$$f^{(k)}(x) = \cos(x + \frac{k\pi}{2}) \quad \text{and} \quad f^{(k)}(0) = \cos\frac{k\pi}{2},$$

that is, $f^{(2n+1)}(0) = 0$ and $f^{(2n)}(0) = (-1)^n$.

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \text{Error.}} \tag{15}$$

$$|\,\text{Error}\,| < \frac{|x|^{2n+2}}{(2n+2)!} \quad \text{for any } x.$$

4) Let $f(x) = \arctan x$. Then

$$\boxed{\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \text{Error.}} \tag{16}$$

If $0 < x < 1$, then

$$|\,\text{Error}\,| < \frac{1}{2n+1}|x|^{2n+1}.$$

5) Let $f(x)=\ln(1+x)$, where $x>-1$. Then $f(0)=1$,

$$f^{(n)}(x)=\frac{(-1)^{n-1}}{(1+x)^n}(n-1)!,\ \text{and}\ f^{(n)}(0)=(-1)^{n-1}(n-1)!$$

$$\boxed{\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...+(-1)^{n-1}\frac{x^n}{n}+\text{Error.}} \qquad (17)$$

$$\text{Error}=R_n(x)=(-1)^n\frac{x^{n+1}}{n+1}(1+c)^{-n-1},$$

where c is a point between zero and x.

$$\text{If}\quad 0<x\le 1,\quad \text{then}\quad |\,\text{Error}\,|<\frac{1}{n+1}x^{n+1}.$$

6) Let $f(x)=(1+x)^m$, where m is any rational number. Then

$$f^{(n)}(x)=m(m-1)...(m-n+1)(1+x)^{m-n},$$

$$f^{(n)}(0)=m(m-1)...(m-n+1).$$

Therefore,

$$\boxed{(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+...+\frac{m(m-1)...(m-n+1)}{n!}x^n+\text{Error}} \qquad (18)$$

$$\text{If}\quad m=n,\quad \text{then}\quad \text{Error}=0.$$

Example 1: Calculate approximately $\sqrt[3]{e}$.

Solution: Formula (13) for $n=1,2,3$ and $x=1/3$ yields successively:

1) $e^x\approx 1+x \quad\Rightarrow\quad \sqrt[3]{e}=e^{1/3}\approx 1+\frac{1}{3}=\frac{4}{3}\approx 1.333;$

2) $e^x\approx 1+x+\frac{x^2}{2} \quad\Rightarrow\quad \sqrt[3]{e}\approx 1+\frac{1}{3}+\frac{1}{2!}\frac{1}{3^2}=\frac{25}{18}\approx 1.389;$

3) $e^x\approx 1+x+\frac{x^2}{2}+\frac{x^3}{6} \quad\Rightarrow$

$$\sqrt[3]{e}\approx 1+\frac{1}{3}+\frac{1}{2!}\frac{1}{3^2}+\frac{1}{3!}\frac{1}{3^3}=\frac{113}{81}\approx 1.395.$$

Compare the answers with the exact result $\sqrt[3]{e}=1.3956...$

Example 2: Calculate approximately $\sin 18°$.

Solution: First, it is necessary to convert degrees to radians: $18° = \pi/10$. Then formula (14) for $n = 1, 2$ and $x = \pi/10$ yields successively:

1) $\sin x \approx x \quad \Rightarrow \quad \sin\frac{\pi}{10} \approx \frac{\pi}{10} \approx 0.314$;

2) $\sin x = x - \frac{x^3}{3!} \quad \Rightarrow \quad \sin\frac{\pi}{10} \approx \frac{\pi}{10} - \frac{1}{6}\left(\frac{\pi}{10}\right)^3 \approx 0.3089$.

The exact result is $0.3090...$

Example 3: Calculate approximately $\sqrt[3]{10}$.

Solution: It is necessary to transform the problem for applying formula (18):

$$\sqrt[3]{10} = \sqrt[3]{8+2} = \sqrt[3]{8(1+1/4)} = 2\sqrt[3]{1+1/4}.$$

Now formula (18) for $n = 2$ and $x = 1/4$ yields

$$\sqrt[3]{1+1/4} = (1+1/4)^{1/3} \approx 1 + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{2}\frac{1}{3}(\frac{1}{3}-1)\frac{1}{4^2} = \frac{155}{144}.$$

Therefore, $\sqrt[3]{10} = 2\sqrt[3]{1+1/4} \approx 155/72 \approx 2.1528$.

The exact result is $2.1544...$

Example 4: Suppose we need to calculate $\sqrt{e}$, using an approximating polynomial.

In order to estimate an error bound, we can use the Lagrange form of the remainder. Since e^x is the increasing function and $0 < c < 0.5$, so $e^c < 2$.

Therefore,

$$R_n(\frac{1}{2}) = \frac{e^c}{(n+1)!}\cdot\frac{1}{2^{n+1}} < \frac{1}{(n+1)!\,2^n},$$

which yields

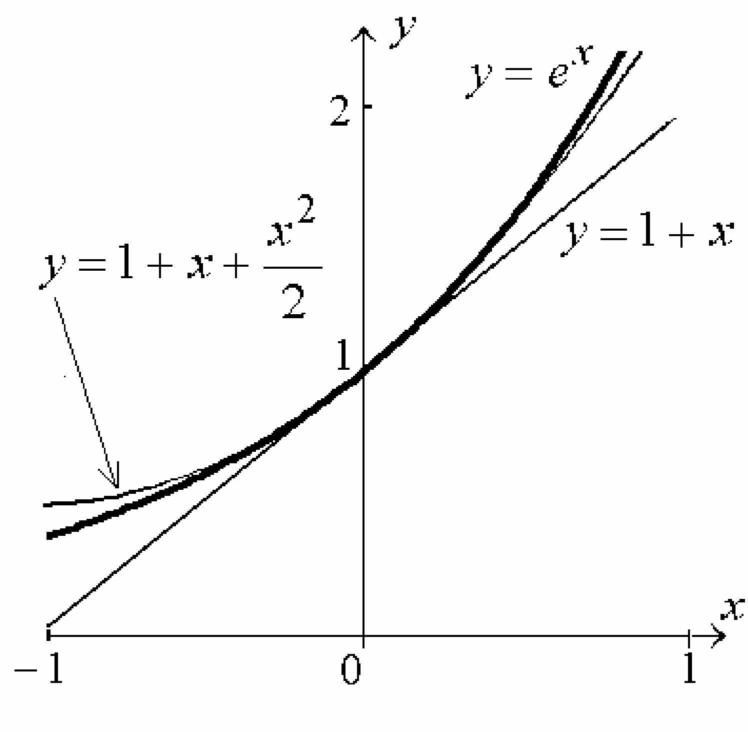
$$R_1(0.5) < 0.25, \quad R_2(0.5) < 1/24 \approx 0.04, \quad R_3(0.5) < 1/192 \approx 0.005,$$

etc.

Thus, the approximating polynomial of the third degree yields a value of $\sqrt{e}$ with an error bound of at most 0.005.

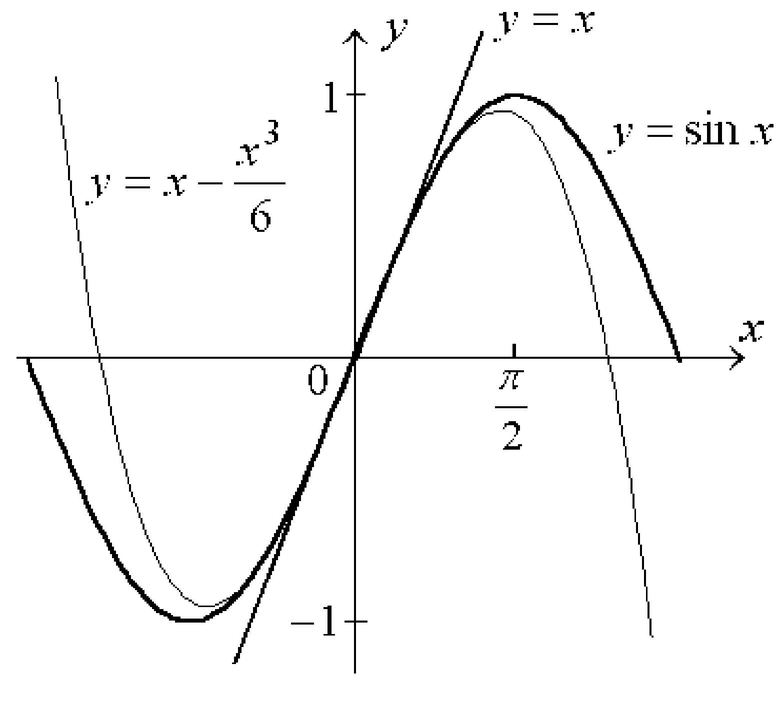
Summary: The table below contains a list of approximating formulas for some functions in a vicinity of the point $x=0$. The formulas are illustrated by drawings.

Functions	First Approximation	Close Approximation
e^x	$1+x$	$1+x+\frac{x^2}{2}$
$\sin x$	x	$x-\frac{x^3}{6}$
$\cos x$	$1-\frac{x^2}{2}$	$1-\frac{x^2}{2}+\frac{x^4}{24}$
$\tan x$	x	$x+\frac{x^3}{3}$
$\ln(1+x)$	x	$x-\frac{x^2}{2}$
$\arctan x$	x	$x-\frac{x^3}{3}$
$\frac{1}{1+x}$	$1-x$	$1-x+x^2$
$\sqrt{1+x}$	$1+\frac{x}{2}$	$1+\frac{x}{2}-\frac{x^2}{8}$



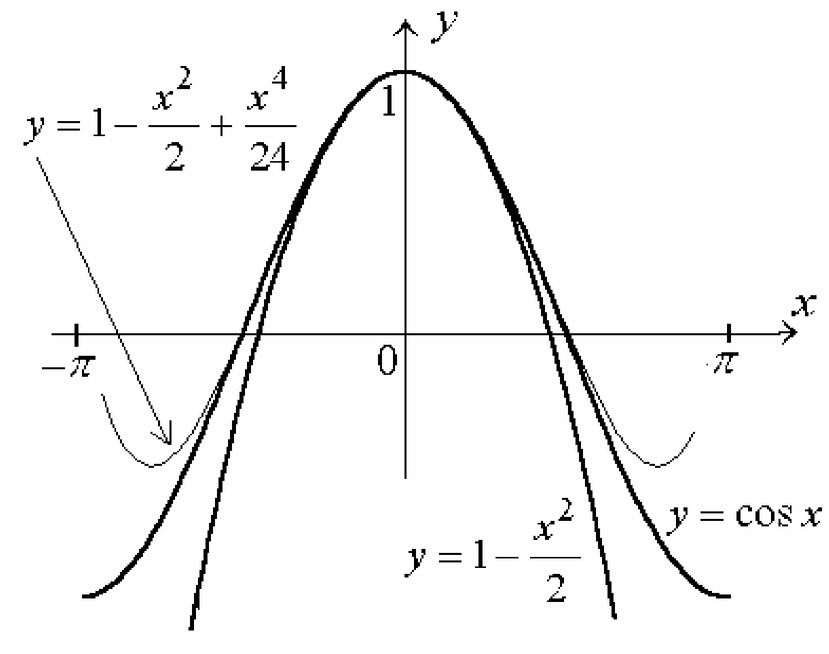
y
$y = e^x$
2
$y = 1 + x + \frac{x^2}{2}$
$y = 1 + x$
1
x
−1
0
1

Fig. 3



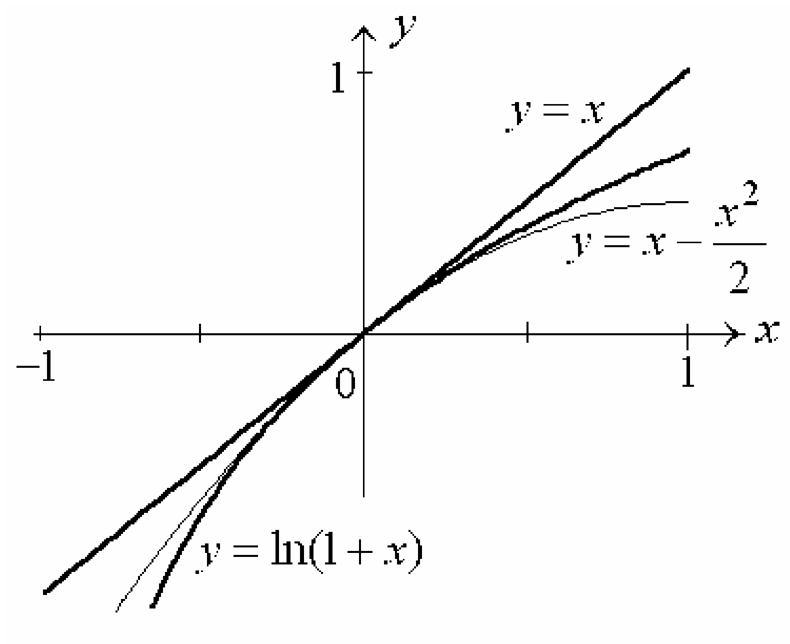
y
$y = x$
1
$y = \sin x$
$y = x - \frac{x^3}{6}$
x
0
$\frac{\pi}{2}$
−1

Fig. 4



y
$y = 1 - \frac{x^2}{2} + \frac{x^4}{24}$
1
x
$-\pi$
0
π
$y = \cos x$
$y = 1 - \frac{x^2}{2}$

Fig. 5



y
1
$y = x$
$y = x - \frac{x^2}{2}$
x
-1
0
1
$y = \ln(1 + x)$

Fig. 6

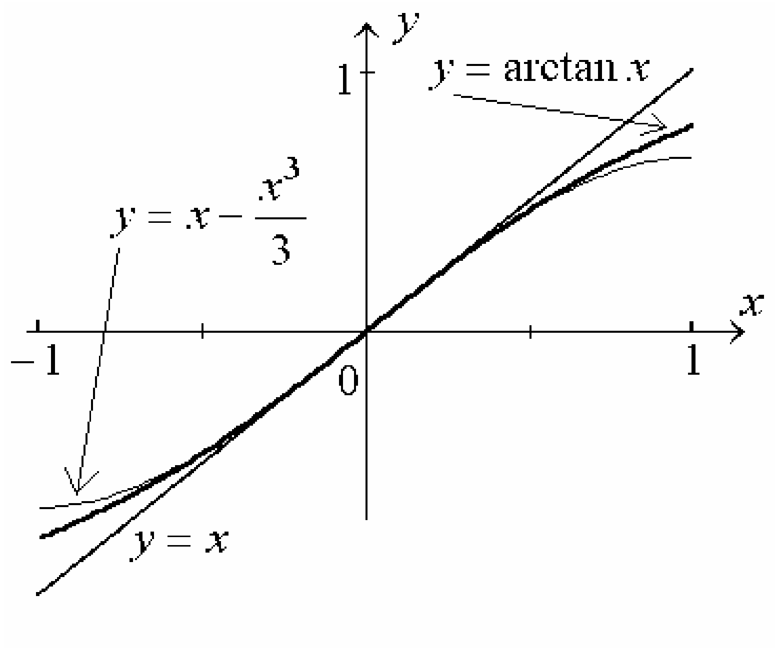


Fig. 7

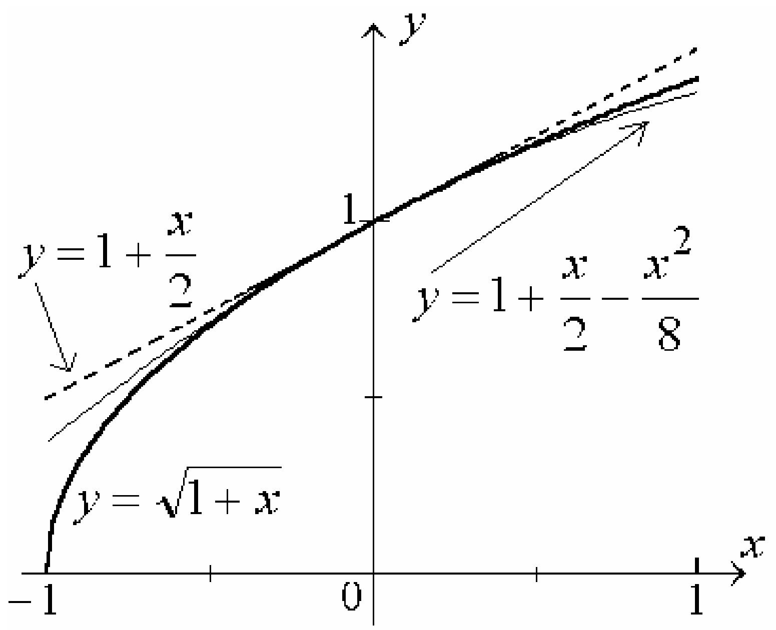


Fig. 8

1.6. Graphs of Functions

1.6.1. Symmetry of Functions

Even-Odd Symmetry: Assume that for every x in the domain of a function $f(x)$, $(-x)$ also enters into the domain. A function is called an **even** function if $f(-x) = f(x)$ for all x in its domain.

The graph of an even function is symmetric with respect to y-axis.

A function is called an **odd** function if $f(-x) = -f(x)$ for all x in its domain.

The graph of an odd function is symmetric with respect to the origin.
Examples of even and odd functions are shown in Fig. 9.

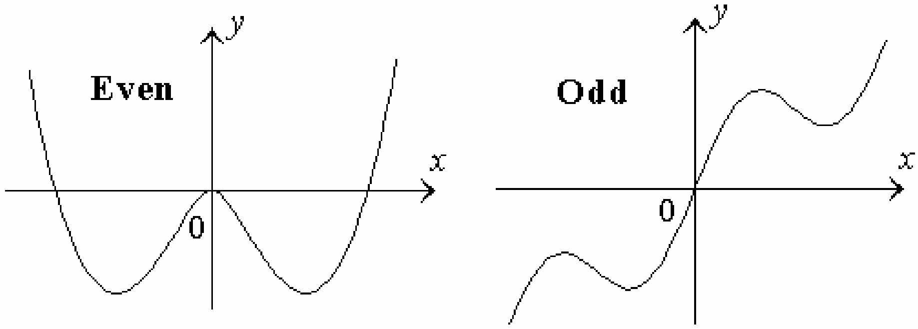


Fig. 9

One should keep in mind the following properties of even or odd functions:

- The sum of even functions is an even function and so is the product of any number of even functions.
- The sum of odd functions is an odd function.
- The product of two odd functions is an even function.
- The product of an even function and an odd function is an odd one.

Most functions are neither even nor odd.

Periodic Symmetry: A function is said to be **periodic** if there exists a positive number T such that $f(x+T) = f(x)$ for all x in its domain. The smallest positive number T is the **period** of the function.
All trigonometric functions are periodic functions and so are their combinations. An example of a periodic function is given in Fig. 10.

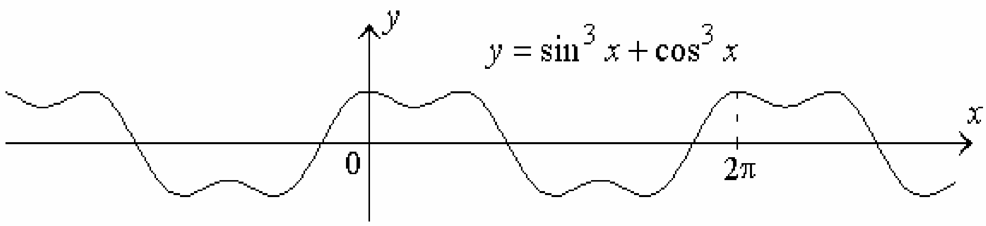


Fig. 10

1.6.2. Increasing and Decreasing Functions

Intervals of increasing and decreasing of a function can be easily found by the sign of its derivative.

Theorem: Let a function $f(x)$ be defined and differentiable on some interval (a,b).

If $f'(x) \geq 0$ at all points of the interval, then $f(x)$ is a monotone increasing function on (a,b).

If $f'(x) \leq 0$ at all points of the interval, then $f(x)$ is a monotone decreasing function on (a,b).

Proof: Let $x_1 \in (a,b)$, $x_2 \in (a,b)$ and $x_1 < x_2$. Then by the Mean Value Theorem

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1),$$

where $x_1 < c < x_2$.

If $f'(c) \geq 0$, then $f(x_2) \geq f(x_1)$ and hence, $f(x)$ is a monotone increasing function on (x_1, x_2).

Likewise, if $f'(c) \leq 0$ then $f(x)$ is a monotone decreasing function on (x_1, x_2). But x_1 and x_2 are arbitrary points on (a,b).

Hence, the theorem.

Example 1: Find the intervals of monotonicity of the function

$$f(x) = 3x^4 + 16x^3 - 6x^2 - 48x + 1.$$

Solution: First, we find the derivative of $f(x)$:

$$f'(x) = 12x^3 + 48x^2 - 12x - 48 = 12(x^3 + 4x^2 - x - 4).$$

Then we solve the equation $f'(x) = 0$ by factoring:

$$x^3 + 4x^2 - x - 4 = (x+4)(x+1)(x-1) = 0.$$

The derivative is positive for $-4 < x < -1$ and for $x > 1$.

The derivative is negative for $x < -4$ and for $-1 < x < 1$.

Therefore, the given function is monotonically increasing on each of the intervals $(-4, -1)$ and $(1, +\infty)$, and monotonically decreasing on $(-\infty, -4)$ and $(-1, 1)$.

1.6.3. Maxima and Minima of Functions

It is said that a function $f(x)$ has a **local** or **relative maximum** at a certain point in its domain, if the value of the function at this point is greater than or equal to the values at all other points in some vicinity of the point.
A function $f(x)$ has a **local** or **relative minimum** at a point c, if $f(c) \le f(x)$ for all x in some vicinity of the point c.
The **global maximum** (**minimum**) is the highest (lowest) value, which a function attains on the given domain.
An **extreme** point is the point where the function attains either its maximum or minimum.

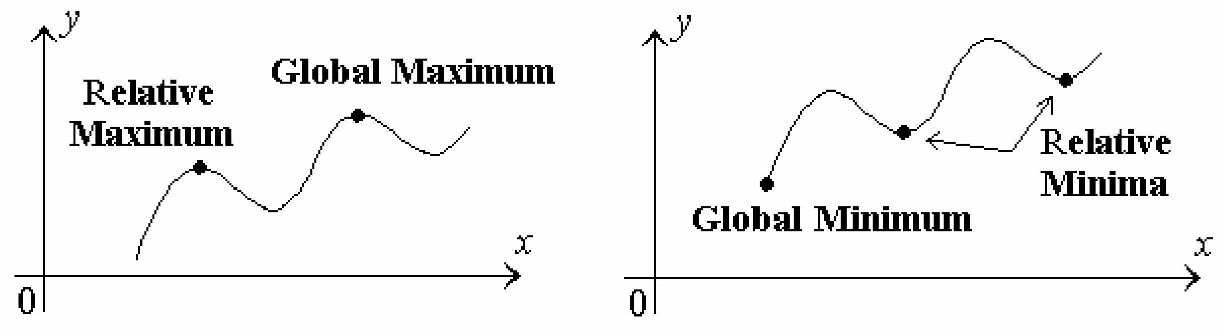


Fig. 11

Usually the domain of a function can be divided into a finite number of intervals of monotonicity of the function. The derivative of the function has the same sign for all inner points of these intervals. But the derivative either equals zero or does not exist at the partition points. (See the figure below.)

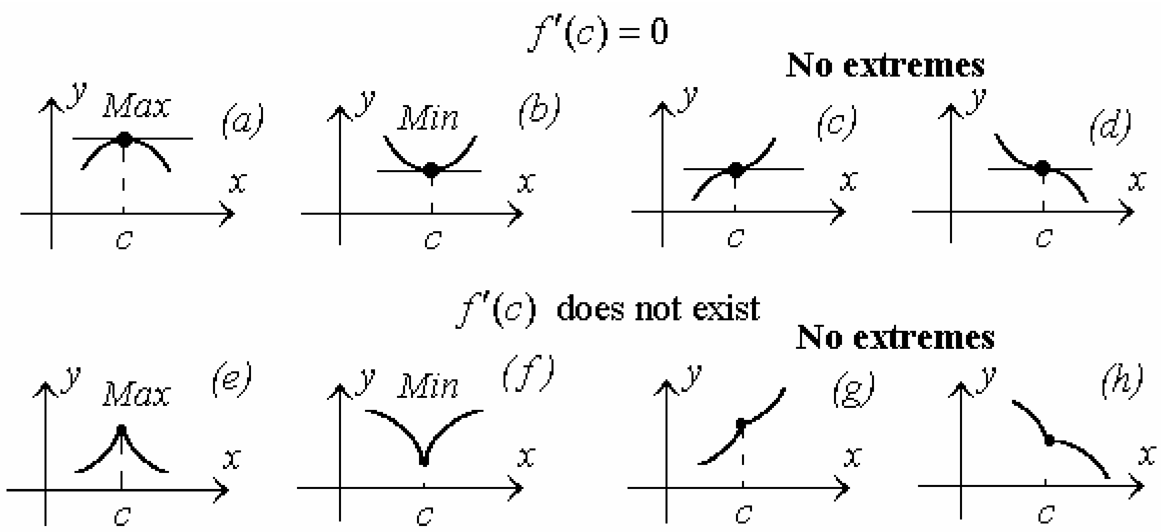


Fig. 12

A point c is called a **critical** point for a function $f(x)$, if either $f'(c)=0$ or $f'(c)$ does not exist.

Therefore, in order to find extreme values of a function $f(x)$ we have to determine critical points by solving the equation $f'(x)=0$. At these points the tangents are horizontal. We need also to determine points, where the derivative of $f(x)$ does not exist.

Then we have to check whether a critical point is an extreme point, following the rules:

- At a point of maximum the derivative changes its sign from positive to negative.
- At a point of minimum the derivative changes its sign from negative to positive.
- If the derivative holds its sign, passing through a point, then the point is not an extreme one.

There is another method to solve this problem, which is based on the investigation of a behavior of a differentiable function by applying the Taylor Formula:

$$\Delta f(x) = df(c) + \frac{d^2 f(c)}{2!} + \frac{d^3 f(c)}{3!} + \ldots + \frac{d^n f(c)}{n!} + R_n(x).$$

The first differential $df(c)$ gives the first approximation for the increment of the function. However, at a critical point $df(c) = f'(c)dx = 0$, and hence, we have to take into account the second differential. Then for small enough dx we have

$$\Delta f(x) = \frac{1}{2} d^2 f(c).$$

Since $d^2 f(c) = f''(c)\,d^2 x$ and $d^2 x > 0$, so $\Delta f(x)$ has the same sign as $f''(c)$.

If $f''(c) > 0$, then $\Delta f(x) > 0$ regardless of the sign of Δx. Therefore, the point c is a point of a relative minimum.

If $f''(c) < 0$, then $\Delta f(x) < 0$. Therefore, the function has a relative maximum at the point c.

If $f''(c) = 0$, then the second derivative test does not give any answer, and it is necessary to take into consideration the next term of the Taylor Formula.

1.6.4. Curvature of Curve

A curve $y = f(x)$ is said to be **concave** (or concave downwards) on some interval (a,b), if $f''(x) > 0$ at all points of the interval.

If $f''(x) < 0$ on (a,b), then the curve is **convex** (or concave upwards).

A point of **inflection** is the point of changing of curvature from convex to concave or vice versa. This point separates the concave and convex arcs of a curve.

At a point of inflection the second derivative either equals zero or does not exist.

In order to determine the points of inflection, we have to find the solution of the equation $f''(x) = 0$. We have also to determine the points, where the second derivative of $f(x)$ does not exist.

Then we need to check whether the obtained point is a point of inflection.

As above, one can use the following rules:

- At a point of inflection the second derivative changes its sign.
- If it changes the sign from plus to minus, then the curvature is changing from concave to convex.
- If it changes the sign from minus to plus, then the curvature is changing from convex to concave.
- If the second derivative holds its sign, then the point is not a point of inflection.

Some examples of concave curves are shown in Fig. 12 (*b*), (*e*), (*g*).

Fragments of convex curves are represented by Fig. 12 (*a*), (*f*), (*h*).

Points of inflection are shown in Fig. 12 (*c*), (*d*).

1.6.5. Asymptotes

An **asymptote** is a straight line approached by a given curve as one of the variables in the equation of the curve approaches infinity.

Asymptotes can be vertical, horizontal or inclined.

If $f(x) \to \infty$ as $x \to a$, then there exists the vertical asymptote, which is described by the equation $x = a$. In this case they say about the asymptotic behavior of the curve as $x \to a$.

If $f(x) \to b$ as $x \to \infty$, then there exists the horizontal asymptote, whose equation is $y = b$.

The general equation of an inclined asymptote is the following:

$$y = kx + b. \tag{19}$$

Assume that a curve $y = f(x)$ asymptotically approaches line (19) as $x \to \infty$, that is, $f(x) \approx kx + b$ as $x \to \infty$.

Therefore, $k \approx \frac{f(x)}{x} - \frac{b}{x}$, and we obtain by the limit process

$$k = \lim_{x \to \infty} \frac{f(x)}{x}. \tag{20}$$

Likewise, $b \approx f(x) - kx$ implies

$$b = \lim_{x \to \infty} (f(x) - kx) \tag{21}$$

Thus, if there exist finite limits (20) and (21), then the curve $y = f(x)$ has the inclined asymptote $y = kx + b$.

Do not forget that the short form $x \to \infty$ describes two cases: $x \to +\infty$ and $x \to -\infty$.

For instance, if $f(x) = e^x$, then there exists the asymptote $y = 0$ as $x \to -\infty$ but there is not any asymptote as $x \to +\infty$.

Example: Find the asymptotes for the function $f(x) = \frac{3x^2}{x-7}$.

Solution: The function $f(x) \to \infty$ as $x \to 7$.

Therefore, there is the vertical asymptote $x = 7$.

One can easily get that

$$k = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{3x^2}{(x-7)x} = 3,$$

$$b = \lim_{x \to \infty} (f(x) - kx) = \lim_{x \to \infty} (\frac{3x^2}{x-7} - 3x)$$

$$= \lim_{x \to \infty} \frac{3x^2 - 3x^2 + 21x}{x-7} = 21.$$

Therefore, there is the inclined asymptote $y = 3x + 21$.

Chapter 2

FUNCTIONS OF SEVERAL VARIABLES

2.1. Introduction

The basic concepts of the theory of functions of several variables are the same or can be formulated like that of a single variable. Many definitions of a function of one variable can be easily generalized to functions of two or more than two variables.
However, some complications arise in the computation and interpretation of results.
Let us begin from the simplest concepts.

Distance Between Points
Any point P in the $xy-$plane can be described by the ordered pair (x,y) of real numbers. The distance between two points $P(x,y)$ and $P_0(x_0,y_0)$ is $\rho(P,P_0)=\sqrt{(x-x_0)^2+(y-y_0)^2}$.

In order to describe a point in three-dimensional space, it is necessary to operate with a triplet (x,y,z) of numbers, so that the distance between points $P(x,y,z)$ and $P_0(x_0,y_0,z_0)$ is

$$\rho(P,P_0)=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\ .$$

In a similar way a point in multidimensional space can be represented by n numbers $x_1,x_2,...,x_n$. The generalized formula for the distance between points $P(x_1,x_2,...,x_n)$ and $P(a_1,a_2,...,a_n)$ looks like above:

$$\rho(P,P_0)=\sqrt{(x_1-a_1)^2+(x_2-a_2)^2+...+(x_n-a_n)^2}\ . \qquad (1)$$

Definition of Functions
Let $P(x_1,x_2,...,x_n)$ be a point of some set D.
If each point of D is associated with one value of a variable u, then it is said that a function u of variables $x_1,x_2,...,x_n$ is defined on the set D.
Recall that a function of one variable is denoted as $y=f(x)$. A function of several variables is denoted just in the same manner using the function notation by the equality

$$u=f(x_1,x_2,...,x_n)$$

or in a short form as $u=f(P)$.

The set D is called the **domain** of definition, and the set of all values of u is called the **range** of a function.
In particular, a function of two independent variables is usually denoted as $z = f(x, y)$. The equation $z = f(x, y)$ can be interpreted graphically as a surface in three-dimensional space.
The domain of definition of a function of two variables is some set of points in $xy-$plane.
Example: The domain D of the function

$$z = \sqrt{x^2 + y^2 - 1} + \sqrt{2 - x^2 - y^2}$$

is the ring domain $D = (x| \, (x^2 + y^2 \ge 1) \cap (x^2 + y^2 \le 2))$, that means any values of x and y such that $1 \le x^2 + y^2 \le 2$.

Some examples of domains are shown in Fig. 1.

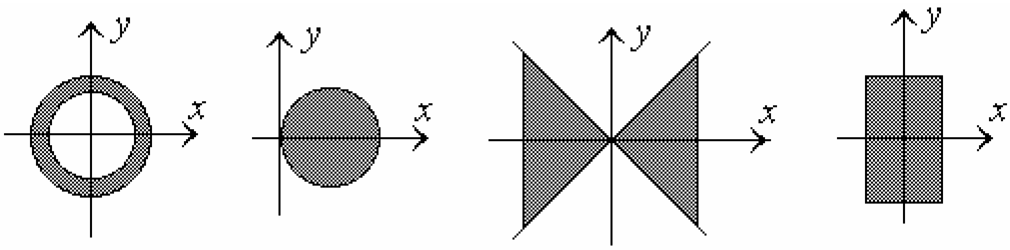


Fig. 1

2.2. Limits of Functions of Several Variables

The mathematical statement

$$\lim_{x \to a} f(x) = A$$

for a function of the single variable means that the difference between $f(x)$ and A vanishes as the distance between points x and a on the number line is getting smaller and smaller.
The definition as well as the properties of limits of a function of one variable can be easily generalized to functions of more than one variable.
Moreover, the limit of a function of several independent variables can be defined just in the same way as in case of a function of one variable.

Let $f(P)$ be a function of several variables, which is defined in some vicinity of a point P_0.
The limit of $f(P)$ as P tends to P_0 is equal to A if and only if for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that $f(P)$ obeys the inequality

$$|f(P) - A| < \varepsilon,$$

whenever the distance $\rho(P,P_0)$ between points P and P_0 obeys the inequality

$$|\rho(P,P_0)|<\delta.$$

This statement is denoted as

$$\lim_{P\to P_0} f(P)=A. \qquad (2)$$

In a particular case of a function of two variables one uses the natural notation

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y)=A \qquad (3)$$

If limit (3) exists, then

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y)=\lim_{x\to a}\lim_{y\to b} f(x,y)=\lim_{y\to b}\lim_{x\to a} f(x,y). \qquad (4)$$

All properties of limits hold for functions of several variables:

- If there exists $\lim_{P\to P_0} f(P)$ and c is any number, then

$$\lim_{P\to P_0} cf(P)=c\lim_{P\to P_0} f(P). \qquad (5)$$

- If there exist both limits, $\lim_{P\to P_0} f(P)$ and $\lim_{P\to P_0} g(P)$, then there exist the limits of the sum, product and quotient of functions such that
  - The limit of the sum of functions is the sum of the limits of the functions:

$$\lim_{P\to P_0}(f(P)\pm g(P))=\lim_{P\to P_0} f(P)\pm\lim_{P\to P_0} g(P). \qquad (6)$$

  - The limit of the product of functions is the product of the limits of the functions:

$$\lim_{P\to P_0} f(P)g(P)=\lim_{P\to P_0} f(P)\lim_{P\to P_0} g(P). \qquad (7)$$

  - The limit of the quotient of functions is the quotient of the limits of the functions, provided $\lim_{P\to P_0} g(P)\neq 0$:

$$\lim_{P\to P_0}\frac{f(P)}{g(P)}=\frac{\lim_{P\to P_0} f(P)}{\lim_{P\to P_0} g(P)}. \qquad (8)$$

Example: Find the limit of the function $f(x,y)=\frac{\sin xy}{x(1+y)}$ as $(x,y)\to(0,3)$.

Solution:

1) In view of (4) we have to hold fixed one of the variables in order to take the limit with respect to the second variable.

Let us hold fixed y as x approaches zero:

$$\lim_{x\to 0}\frac{\sin xy}{x(1+y)}=\frac{1}{(1+y)}\lim_{x\to 0}\frac{\sin xy}{x}=\frac{y}{(1+y)}.$$

2) Now let $y\to 3$:

$$\lim_{y\to 3}\frac{y}{(1+y)}=\frac{3}{4}.$$

It does not matter whether we hold fixed x or y. By interchanging of the order of a passage to the limit we obtain the same result as above:

$$\lim_{y\to 3}\frac{\sin xy}{x(1+y)}=\frac{\sin 3x}{4x} \qquad \Rightarrow$$

$$\lim_{x\to 0}\lim_{y\to 3}\frac{\sin xy}{x(1+y)}=\lim_{x\to 0}\frac{\sin 3x}{4x}=\frac{3}{4}.$$

Thus, the given function tends to $3/4$ as (x,y) approaches $(0,3)$.

Naturally, there are such functions, which have no limits at some points. For instance, consider the limit of the function $f(x)=\frac{x^2-y^2}{x^2+y^2}$ as $(x,y)\to(0,0)$.

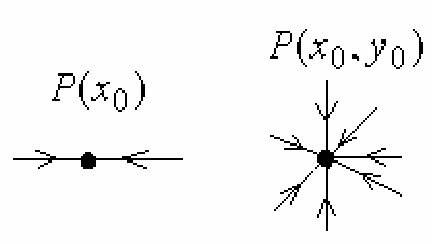
Note that

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2-y^2}{x^2+y^2}=\lim_{x\to 0}1=1,$$

while

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x^2-y^2}{x^2+y^2}=\lim_{y\to 0}(-1)=-1.$$

The results differ from each other. Hence, the given function has no a limit at the point $(0,0)$.



It is appropriate to mention here that the limit of a function of one variable exists if and only if the left-hand and right-hand limits equal with each other.
So, as for the example above, there is nothing new; the only difference is that there is an infinite number of directions of passages to the limit point but not just two.

2.3. Continuity of Functions of Several Variables

The concept of continuity of functions of one variable does not require any modification with reference to functions of several variables.
A function $f(P)$ is called continuous at a point P_0 if there exists the finite limit of $f(P)$ that equals the value of the function at the point P_0: $\lim\limits_{P\to P_0} f(P) = f(P_0)$.
A function $f(P)$ is said to be continuous on some set D, if it is continuous at each point of D. Otherwise, if $f(P)$ is not continuous, e.g., at a point P_1, it is said that the function $f(P)$ is discontinuous at the point P_1 or that $f(P)$ has a discontinuity at the point P_1.
The points of discontinuity can form lines or surfaces.

Examples:

- The function $z = \tan xy$ is not defined on the lines $xy = (2k+1)\pi/2$, where k is any integer. The lines of discontinuity are the set of hyperbolas.
- The function $u = \dfrac{x - z^2}{2x + y - 3x}$ is not defined in the plane $2x + y - 3z = 0$, which is the plane of discontinuity.

Continuous functions have the same properties, no matter how many of variables.

- The sum of a finite number of continuous functions is a continuous function as well as the product of a finite number of continuous functions is a continuous function.
- The quotient of two continuous functions is a continuous function wherever the denominator is non-zero.
- All elementary functions are continuous in their domains.

2.4. Partial Derivatives

The derivative of a function of one variable is defined as the limit of the quotient of the increment $\Delta f = f(x+\Delta x) - f(x)$ of the function to an increment Δx of the argument as $\Delta x \to 0$:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

The partial derivatives of a function of several variables are defined in a similar way.

For convenience sake consider a function of two independent variables.

The partial derivative of $u = f(x, y)$ with respect to x is defined as

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}. \qquad (9)$$

The definition of the partial derivative of $z = f(x, y)$ with respect to y looks like above:

$$\frac{\partial f(x,y)}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}. \qquad (10)$$

In a short form partial derivatives are denoted by the symbols f'_x, f'_y or u'_x, u'_y.

Partial derivatives have the same properties as ordinary derivative as well as all rules of differentiation hold.

Note that when one takes the partial derivative, e.g., with respect to x, it is necessary to hold the other variables as constants.

Example: Find the partial derivatives of $f(x, y)$ with respect to x and y, if $f(x,y) = x^2 y^3 + 5\sin x - e^{\sqrt{y}}$.

Solution:

$$f'_x = 2xy^3 + 5\cos x,$$

$$f'_x = 3x^2 y^2 - e^{\sqrt{y}} \frac{1}{2\sqrt{y}}.$$

Partial derivatives of higher orders are defined in a similar way as ordinary higher derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

They are also denoted as f''_{x^2}, f''_{y^2}, f''_{xy}, f''_{yx} correspondingly.

Partial derivatives like f''_{xy}, f''_{yx} are called mixed partial derivatives.

There exists the theorem according to that mixed partial derivatives do not depend on the order of differentiation provided that the partial derivatives are continuous functions. We are going to consider only functions, which obey such conditions.
Therefore,

$$u''_{xy} = u''_{yx}\ , \qquad u'''_{x^2 y} = u'''_{xyx} = u'''_{yx^2}\ , \qquad etc.$$

2.5. Total Differentials

Let $u = f(x, y)$ be a function of two independent variables.

Increments of the argument are called the differentials of the independent variables:

$$dx = \Delta x \qquad \text{and} \qquad dy = \Delta y .$$

The total differential of a function $u = f(x, y)$ is defined as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy . \tag{11}$$

This definition of the differential can be easily generalized for a function u of n independent variables:

$$du = \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} dx_k = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ldots + \frac{\partial u}{\partial x_n} dx_n . \tag{12}$$

The properties of differentials of functions of several variables differ nothing from that of one variable:

$$d(u \pm v) = du \pm du ,$$
$$d(u \cdot v) = udv + vdu ,$$
$$d\left(\frac{u}{v}\right) = \frac{udv - vdu}{v^2} .$$

Theorem: Let the functions $A(x, y)$ and $B(x, y)$ have continuous partial derivatives to the second order inclusive. Then the expression of the form

$$A(x, y)dx + B(x, y)dy$$

is the total differential of some function $u = f(x, y)$ if and only if

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} . \tag{13}$$

Proof: Let us prove the necessity of condition (13).
Assume that

$$du = A(x, y)dx + B(x, y)dy.$$

Then from definition (11) it follows that

$$A(x, y) = \frac{\partial u}{\partial x} \quad \text{and} \quad B(x, y) = \frac{\partial u}{\partial y}.$$

Therefore,

$$\frac{\partial A(x, y)}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} \quad \text{and} \quad \frac{\partial B(x, y)}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}.$$

However, the mixed partial derivatives u''_{xy} and u''_{yx} equal each other because they are continuous functions.

Hence, $A'_y = B'_x$.

2.6. Differentials of Higher Orders

The nth differentials of arguments are the nth power of the first differentials:

$$dx^2 = (dx)^2, \quad dy^2 = (dy)^2, \ldots \quad , \quad dx^n = (dx)^n, \quad dy^n = (dy)^n.$$

The second differential of a function is the differential of the first differential; the third differential is the differential of the second differential, and so on:

$$d^2u = d(du), \quad d^3u = d(d^2u), \ldots, \quad d^nu = d(d^{n-1}u).$$

If $u = f(x, y)$, then

$$d^2u = d(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy) = d(\frac{\partial u}{\partial x}dx) + d(\frac{\partial u}{\partial y}dy)$$

$$= (\frac{\partial^2 u}{\partial x^2}dx^2 + \frac{\partial^2 u}{\partial y \partial x}dydx) + (\frac{\partial^2 u}{\partial x \partial y}dxdy + \frac{\partial^2 u}{\partial y^2}dy^2) \qquad (14)$$

$$= \frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x \partial y}dxdy + \frac{\partial^2 u}{\partial y^2}dy^2$$

due to equality of the mixed partial derivatives.

The nth differential of a function can be simply obtained by the following formal rule:

$$d^nu = (dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial x})^n u. \qquad (15)$$

In such a way the sum has to be raised to nth power. Then the parentheses have to be removed, putting the symbol u from the right of each of the symbols like $\frac{\partial}{\partial x}$. Finally, we have to interpret the exponents as the orders of derivatives and differentials.

Example: Find the third differential of a function of two variables.

Solution: The first step:

$$d^3u = (dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y})^3 u.$$

The second step:

$$d^3u = (dx^3\frac{\partial^3}{\partial x^3} + 3dx^2\,dy\frac{\partial^3}{\partial x^2\partial y} + 3dx\,dy^2\frac{\partial^3}{\partial x\partial y^2} + dy^3\frac{\partial^3}{\partial y^3})u.$$

The final step:

$$d^3u = \frac{\partial^3 u}{\partial x^3}dx^3 + 3\frac{\partial^3 u}{\partial x^2\partial y}dx^2\,dy + 3\frac{\partial^3 u}{\partial x\partial y^2}dx\,dy^2 + \frac{\partial^3 u}{\partial y^3}dy^3.$$

2.7. Derivatives of Composite Functions

Let $u = f(x_1, x_2, ..., x_n)$ be a composite function of the variables $x_1, x_2, ..., x_n$, where $x_1 = x_1(t)$, $x_2 = x_2(t)$, …, $x_n = x_n(t)$ all are functions of the variable t. Then the complete derivative is

$$\frac{du}{dt} = \sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\frac{dx_k}{dt}. \qquad (16)$$

If the function u is also an explicit function of t, that is, $u = f(x_1, x_2, ..., x_n, t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\frac{dx_k}{dt}. \qquad (17)$$

In particular, let $u = f(x, y, t)$ with $x = x(t)$ and $y = y(t)$. Then

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}. \qquad (18)$$

Example: Find $\frac{du}{dt}$, if $u = e^{5x}y^3$ with $x = \sin t$ and $y = t^2$.

Solution:

$$\frac{du}{dt} = 5e^{5x}\cos t \cdot y^3 + e^{5x}3y^2 2t = 5e^{5\sin t}\cos t \cdot t^6 + 6e^{5\sin t}t^5.$$

2.8. Derivatives of Implicit Functions

- Let a function $y = y(x)$ be defined by an implicit function:

$$F(x, y) = 0. \quad (19)$$

Then the total differential of F is $dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$.

Therefore, the derivative of y with respect to x can be expressed through the partial derivatives as follows:

$$\frac{dy}{dx} = -\frac{F'_x}{F'_y}. \quad (20)$$

- Let a function $z = z(x, y)$ of two variables be defined by an implicit function:

$$F(x, y, z) = 0.$$

As above, the total differential of F is equal to zero:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0. \quad (21)$$

In order to find, for instance, the partial derivative of z with respect to x, we divide both sides of this equality by dx and hold the variable y as a constant. In this case the ratio $\frac{dz}{dx}$ has to be considered as the partial derivative $\frac{\partial z}{\partial x}$, and hence

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} \quad (22a)$$

The other partial derivatives can be found in a similar way:

$$\frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}, \ \frac{\partial y}{\partial x} = -\frac{F'_x}{F'_y}, \ \frac{\partial y}{\partial z} = -\frac{F'_z}{F'_y}, \quad etc. \quad (22b)$$

Example: Find the partial derivatives of z with respect to x and y if

$$xy^2z^3 + \sqrt{z}\ln x - y/z = 0.$$

Solution: First, let us find the partial derivatives of the function

$$F(x, y, z) = xy^2z^3 + \sqrt{z}\ln x - y/z:$$

$$F'_x = y^2z^3 + \sqrt{z}/x, \qquad F'_y = 2xyz^3 - 1/z,$$

$$F'_z = 3xy^2z^2 + \ln x/(2\sqrt{z}) + y/z^2.$$

Then, we use formulas (22):

$$\frac{\partial z}{\partial x} = -\frac{y^2z^3 + \sqrt{z}/x}{3xy^2z^2 + \ln x/(2\sqrt{z}) + y/z^2},$$

$$\frac{\partial z}{\partial y} = -\frac{2xyz^3 - 1/z}{3xy^2z^2 + \ln x/(2\sqrt{z}) + y/z^2}.$$

2.9. Geometric Interpretation of Partial Derivatives

Consider a function of two variables.
The equation of a surface in three-dimension space can be written as

$$z = f(x, y). \qquad (23)$$

This equation can also be represented in the implicit form as follows:

$$F(x, y, z) = 0. \qquad (24)$$

Assume that there exist the partial derivatives of z at some point $P_0(x_0, y_0, z_0)$ on the surface, that is, the surface is smooth enough in the vicinity of the point P_0.
There is an infinite number of lines that are tangents to the surface at this point. These lines form a plane called the **tangent plane** to the surface at the given point.
An equation of the tangent plane can be written in the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \qquad (25)$$

Here A, B and C are components of a normal vector to the surface at the point P_0.
In order to determine this vector we consider anther way to get the equation of the tangent plane.
Let $P(x, y, z)$ be any point on the surface. If the point $P(x, y, z)$ approaches $P_0(x_0, y_0, z_0)$, that is,

$$\Delta x = x - x_0 \to 0,\ \Delta y = y - y_0 \to 0, \qquad \Delta z = z - z_0 \to 0,$$

then the vector $\Delta\mathbf{r} = \{\Delta x, \Delta y, \Delta z\}$ tends to the vector $d\mathbf{r} = \{dx, dy, dz\}$, which is coplanar to the tangent plane.
By equation (24), the differential of F at the point $P_0(x_0, y_0, z_0)$ is equal to zero. Hence, in view of formula (21)

$$\frac{\partial F(x_0, y_0, z_0)}{\partial x} dx + \frac{\partial F(x_0, y_0, z_0)}{\partial y} dy + \frac{\partial F(x_0, y_0, z_0)}{\partial z} dz = 0. \qquad (26)$$

This equation states the orthogonality condition of the vectors $d\mathbf{r} = \{dx, dy, dz\}$ and $\mathbf{N} = \{F'_x(P_0), F'_y(P_0), F'_z(P_0)\}$, where the vector $d\mathbf{r}$ lies in the tangent plane.

Therefore, the partial derivatives of F at the point P_0 are the components of a normal vector to the tangent plane and so to the considered surface at this point:

$$A = \frac{\partial F(x_0, y_0, z_0)}{\partial x}, \qquad B = \frac{\partial F(x_0, y_0, z_0)}{\partial y}, \qquad C = \frac{\partial F(x_0, y_0, z_0)}{\partial z}.$$

Then formula (25) yields the equation of the tangent plane to surface (24) at the given point:

$$F'_x(P_0)(x - x_0) + F'_y(P_0)(y - y_0) + F'_z(P_0)(z - z_0) = 0. \qquad (27)$$

Now we can also write the equations of the straight line passing through the point P_0 and being perpendicular to the surface $F(x, y, z) = 0$:

$$\frac{x - x_0}{F'_x(x_0, y_0, z_0)} = \frac{y - y_0}{F'_y(x_0, y_0, z_0)} = \frac{z - z_0}{F'_z(x_0, y_0, z_0)}. \qquad (28)$$

If the surface is defined by equation (23) in the explicit form, then

$$F(x, y, z) = z - f(x, y)$$

and hence,

$$F'_x = -f'_x, \quad F'_y = -f'_y \quad \text{and} \quad F'_z = 1. \qquad (29)$$

Example: Find the equation of the tangent plane to the surface of the paraboloid of revolution $z = x^2 + y^2$ at the point $(-4, 3, 25)$.

Solution: Using formulas (29) we find the partial derivatives of

$$F(x, y, z) = z - x^2 - y^2$$

at the given point: $F'_x(P_0) = 8$, $F'_y(P_0) = -6$ and $F'_z(P_0) = 1$.

In view of (27) the equation of the tangent plane is

$$8(x + 4) - 6(y - 3) + (z - 25) = 0$$

or equivalently

$$8x - 6y + z + 25 = 0.$$

2.10. Maxima and Minima of Functions of Two Variables

The definitions of the maximum and minimum of a function of several variables are just the same as in case of function of one variable.

For instance, a function $f(P)$ has a **relative maximum** at a point P_0, if $f(P) \le f(P_0)$ for all points P in some vicinity of the point P_0.

An **extreme** point is the point where the function attains either maximum or minimum.

The problem of determining the maximum and minimum of some differentiable function can be solved by using of the Taylor Formula.

The main idea is quite clear: if the difference $\Delta f(P_0)$ holds its sign in some vicinity of P_0, then P_0 is an extreme point. Otherwise, the function $f(P)$ has neither maximum nor minimum at this point.

Let $z = f(x, y)$ be a given function. We begin with the first approximation: $\Delta f(P_0) \approx df(P_0) = f'_x(P_0)dx + f'_y(P_0)dy$.

Even if one of these partial derivatives is not equal zero, then the sign of $\Delta f(P_0)$ depends on the signs of the increments dx and dy.

Hence, all the partial derivatives of $f(P)$ either equal zero or do not exist at the extreme point.

To find critical points we need to solve the following equations simultaneously:

$$f'_x(x, y) = 0 \qquad \text{and} \qquad f'_y(x, y) = 0. \tag{30}$$

Note that the tangent planes at such critical points are parallel to the $xy-$plane.

Then we have to take into account the next term in the Taylor formula.

Using the form of the second differential one can prove the following rule.

Rule: Let the partial derivatives of the second order $f''_{x^2}(P_0)$, $f''_{xy}(P_0)$, $f''_{yx}(P_0)$ and $f''_{y^2}(P_0)$ be the elements of the determinant:

$$D = \begin{vmatrix} f''_{x^2}(P_0) & f''_{xy}(P_0) \\ f''_{yx}(P_0) & f''_{y^2}(P_0) \end{vmatrix} \tag{31}$$

where P_0 is a critical point.

- If $D > 0$ and $f''_{x^2}(P_0) > 0$, then P_0 is a point of a relative minimum.
- If $D > 0$ and $f''_{x^2}(P_0) < 0$, then P_0 is a point of a relative minimum.
- If $D < 0$, then function $f(x, y)$ has a saddle point at P_0.
- If $D = 0$, then the rule does not give any answer.

Chapter 3

INDEFINITE INTEGRALS

3.1. Primitives

Integrals, together with **Derivatives**, are the basic objects of **Calculus**. Indefinite integrals are defined through **Primitives** (or **Antiderivatives**).

The function $F(x)$ is called a **primitive** (or **antiderivative**) of a function $f(x)$ if

$$F'(x) = f(x) \qquad (1)$$

for all x in the domain of $f(x)$.

In other words a **primitive** of $f(x)$ is a function whose derivative equals the given function $f(x)$.

Example 1: The function $F(x)$ is a primitive of $F'(x)$.

Example 2: The function $\ln(1+x^2)$ is a primitive of $\dfrac{2x}{1+x^2}$ since

$$(\ln(1+x^2))' = \frac{1}{1+x^2} \cdot (1+x^2)' = \frac{2x}{1+x^2} \qquad \text{for all } x \in R.$$

Primitives have the following important **property**:

📖 Let $F_1(x)$ and $F_2(x)$ be primitives of f, that is,

$$F_1'(x) = F_2'(x) = f(x)$$

for all x in the domain of $f(x)$.

Then there exists a constant C such that

$$F_1(x) = F_2(x) + C.$$

Indeed, $F_1'(x) = F_2'(x)$ by definition. Therefore, the derivative of the difference between functions $F_1(x)$ and $F_2(x)$ is equal to zero for all x on the given interval:

$$(F_1 - F_2)' = F_1' - F_2' = 0.$$

Hence, the difference $F_1 - F_2$ equals a constant by the corollary to the Mean Value Theorem. (See Chapter 1, page 5.)

In general, if a function has one primitive, then it has an infinite number of primitives.

However, if we know one primitive $F(x)$ of the function $f(x)$, then we know all primitives of f. The set of all primitives of f can be represented as $F(x) + C$, where C is an arbitrary constant.

Example 3: Both functions, $F_1 = (x+1)^2$ and $F_2 = x^2 + 2x - 4$, are primitives of $f(x) = 2(x+1)$ for all $x \in R$.

One can easily check that the difference between the primitives is a constant:

$$F_1 - F_2 = (x+1)^2 - (x^2 + 2x - 4)$$
$$= (x^2 + 2x + 1) - (x^2 + 2x - 4) = 5.$$

3.2. The Definition and Properties of Indefinite Integrals

The set of all primitives $F(x)$ of $f(x)$ is called the **indefinite integral** of the function $f(x)$.

The **indefinite integral** of $f(x)$ is denoted by the symbol $\int f(x)dx$, which is read as "The integral of $f(x)$ with respect to x".

$$\int f(x)dx = F(x) + C$$
if and only if $F'(x) = f(x)$.

- The function $f(x)$ under the integral sign is called the **integrand**.
- The x is the **integration variable**.
- The symbol dx is the differential of x.
- An arbitrary constant C is said to be a **constant of integration**.

All indefinite integrals have the following **properties**:

- Differentiation is the inverse operation to indefinite integration:

$$(\int f(x)dx)' = f(x), \tag{1a}$$

$$d\int f(x)dx = f(x)dx. \tag{1b}$$

  - These formulas follow from the definition of indefinite integrals and can be easily memorized using the following rule:
    Symbols d and $\int$ cancel each other if they follow one after another.

- Integration of the derivative of $f(x)$ yields the function $f(x)$:

$$\int f'(x)dx = \int df(x) = f(x) + C. \tag{2}$$

  - This property is evident since the function $f(x)$ is a primitive of $f'(x)$.

Note that integration is the inverse operation to differentiation. However do not forget to add a constant of integration when integration is the last operation!

The following two general formulas allow us to transform a given integral into another integral or integrals.

📖 A constant factor can be taken outside the integral sign.

$$\int cf(x)dx = c\int f(x)dx. \qquad (3)$$

📖 The integral of an algebraic sum of functions equals the algebraic sum of the integrals of each of the functions.

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx. \qquad (4)$$

☝ Both these properties are based on the properties of derivatives. Indeed,

$(\int cf(x)dx)' = cf(x)$ and $(c\int f(x)dx)' = c(\int f(x)dx)' = cf(x)$.

Therefore, both sides in equality (3) represent primitives of the same function.

Property (4) can be obtained in a similar way since the derivative of a sum of functions equals the sum of derivatives of each of the functions.

📖 Let $\int f(x)dx = F(x) + \mathrm{C}$.

Then

$$\int f(u)du = F(u) + \mathrm{C} \qquad (5)$$

for any differentiable function $u = u(x)$.

☝ This property is based on the invariance of the form of the first differential, according to which the differential formula $dF(x) = F'(x)dx$ holds for any composite function $F(u(x))$:

$$dF(u) = F'(u)du.$$

Advice: Try to memorize and understand all these rules.

Let us consider some elementary examples to illustrate the definition and properties of indefinite integrals before going on.

Examples:

- $\frac{d}{dx}\int \sin^3 3x dx = \sin^3 3x.$ | property (1) |
- $\int \frac{dx}{\cos^2 x} = \int (\tan x)' dx = \tan x + C.$ | property (2) |

- $\int 6x^2 dx = 6\int x^2 dx = 6\dfrac{x^3}{3} + C = 2x^3 + C$. | property (3) |

- $\int (2x-3)dx = 2\int xdx - 3\int dx$ | properties (3) and (4) |

$$= 2\frac{x^2}{2} - 3x + C = x^2 - 3x + C.$$

- $\int \dfrac{\ln^4 x}{x} dx = \int (\ln x)^4 d(\ln x) = \dfrac{\ln^5 x}{5} + C$. | property (5) |

3.3. A Table of Common Integrals

Let us recall the derivatives of elementary functions. For instance, the power rule states that

$$(x^k)' = kx^{k-1}.$$

This formula can be transformed as follows.
First, we substitute $(n+1)$ for k:

$$(x^{n+1})' = (n+1)x^n.$$

Then we divide both sides of the equality by $(n+1)$) (provided that $n \neq -1$) and read the formula from right to left:

$$x^n = (\frac{x^{n+1}}{n+1})'.$$

Therefore, the function $\dfrac{x^{n+1}}{n+1}$ is a primitive of x^n, so the power rule for integration is the following:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

The derivatives of all elementary functions can be treated likewise. Then the table of derivatives can be easily transformed into the table of common integrals.
Thus, we have a list of common indefinite integrals.

Table 1

Derivatives	Integrals
$x^n = \left(\frac{x^{n+1}}{(n+1)}\right)'$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$ $\int dx = x + C$
$\frac{1}{x} = (\ln x)'$	$\int \frac{dx}{x} = \ln\lvert x\rvert + C$ $(x \neq 0)$
$a^x = \left(\frac{a^x}{\ln a}\right)'$ $e^x = (e^x)'$	$\int a^x dx = \frac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$ $\int e^x dx = e^x + C$
$\sin x = (-\cos x)'$	$\int \sin x dx = -\cos x + C$
$\cos x = (\sin x)'$	$\int \cos x dx = \sin x + C$
$\frac{1}{\cos^2 x} = (\tan x)'$	$\int \frac{dx}{\cos^2 x} = \tan x + C \qquad (x \neq \frac{\pi}{2} + \pi n)$
$\frac{1}{\sin^2 x} = (-\cot x)'$	$\int \frac{dx}{\sin^2 x} = -\cot x + C \qquad (x \neq \pi n)$
$\frac{1}{\sqrt{1-x^2}} = \begin{cases}(\arcsin x)' \\ (-\arccos x)'\end{cases}$	$\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases}\arcsin x + C \\ -\arccos x + C\end{cases} \qquad (\lvert x\rvert \leq 1)$
$\frac{1}{1+x^2} = \begin{cases}(\arctan x)' \\ (-\cot^{-1} x)'\end{cases}$	$\int \frac{dx}{1+x^2} = \begin{cases}\arctan x + C \\ -\cot^{-1} x + C\end{cases}$

Comment on # 2:

If $x > 0$, then $\ln\lvert x\rvert = \ln x$ and $(\ln\lvert x\rvert)' = \frac{1}{x}$.

If $x < 0$, then $\ln\lvert x\rvert = \ln(-x)$ and $(\ln\lvert x\rvert)' = (\ln(-x))' = \frac{1}{-x}(-1) = \frac{1}{x}$.

Therefore, the function $\frac{1}{x}$ is a primitive of $\ln\lvert x\rvert$ in both cases.

Corollary 1: Each of the functions, $\arcsin x$ and $(-\arccos x)$, is a primitive of $\dfrac{1}{\sqrt{1-x^2}}$. Therefore, the difference between them should be equal to a constant:

$$\arcsin x-(-\arccos x)=\arcsin x+\arccos x=C\ .$$

Setting $x=0$ we find the constant C:

$$C=\arcsin 0+\arccos 0=0+\pi/2=\pi/2\,.$$

Thus,

$$\arcsin x+\arccos x=\pi/2\ .$$

Corollary 2: In a similar way one can get one more formula of elementary mathematics:

$$\arctan x+\cot^{-1}x=\pi/2\,.$$

The result for any particular integral can often be written in many different forms.

In order to solve successfully integration problems, it is necessary to know:

- the properties of integrals;
- the table of common integrals;
- the techniques used for manipulation with integrals.

The best way to acquire enough knowledge of the integral formulas is to use them as many times as possible. Knowledge of the formulas develops the ability to recognize them.

3.4. Techniques of Integration

Evaluating integrals is much more difficult than evaluating derivatives. As for derivatives, there is a systematic procedure based on the chain rule that effectively allows any derivative to be worked out. However, there is not any similar procedure for integrals.

One of the main problems is that it is difficult to know what kinds of functions will be needed to evaluate a particular integral. When we work out a derivative, we always end up with functions that are of the same kind or simpler than the ones we started with. But when we work out integrals, we often end up needing to use functions that are much more complicated than the ones we started with.

Whenever the specific integration formulas do not apply, we have to transform the problem into another problem or problems. One can try to manipulate the integrand algebraically, separate the integrand, if possible, put any constant factors outside of the sign of the integral by making use of the properties of integrals, and so on.

The basic techniques of integration are algebraic manipulation, substitutions, integration by parts, and the method of partial fractions.

3.4.1. Integration by Substitution

The technique of substitutions helps to reduce many integrals to common indefinite integrals, which are given in Table 1.
For convenience sake all substitutions may be subdivided into two classes:

- $u = g(x)$,
- $x = u(t)$.

In both cases we change the variable of integration - in one way or another.
As a rule, the substitution $u = g(x)$ is used when a given integral has the following structure:

$$\int f(g(x))g'(x)dx.$$

Then the substitution $u = g(x)$ implies $du = g'(x)dx$, so that we obtain

$$\int f\big(g(x)\big)g'(x)dx = \int f(u)du. \qquad (6)$$

Therefore, the initial integration problem is transformed into another integration problem. However if we can not integrate the function $f(u)$, then another method of integration may be required.

On the other hand, the substitution $x = u(t)$ gives another way of transformation of a given integral.

Now let $\int f(x)dx$ be a given integral.

Then the substitution $x = u(t)$ implies $dx = u'(t)dt$, and we obtain

$$\int f(x)dx = \int f\big(u(t)\big)\,u'(t)\,dt. \qquad (7)$$

As above, we expect that the new integral is easier evaluated. Otherwise, another substitution or integration method may be needed.
As a matter of fact, formulas (6) and (7) give the reverse transformation to each other. They are called the **substitution formulas**.

The technique of substitution is quite general and can be used in a wide variety of problems.
In particular, one can generalize the table of common integrals applying the technique of substitutions. Consider, for instance, the power rule:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1).$$

Let $u(t)$ be any differentiable function. If we use the substitution $x = u$, then the power rule can be formulated as follows:

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1).$$

This formula is exactly the same as the original power rule. The only difference is the interpretation of the symbol u as a function of a variable t and so $du = u'dt$. Therefore, we obtain the following generalized power rule:

$$\int u^n(t)u'(t)dt = \frac{u^{n+1}(t)}{n+1} + C \qquad (n \neq -1).$$

One can interpret each of the common integrals in a similar way by considering the variable of integration as a function.

3.4.1.1. Examples of Integrating by Substitution

Example 1: Each of the following integrals

1) $\int \frac{(\arctan x)^4}{1+x^2} dx$,

2) $\int \frac{dx}{(\arctan x)(1+x^2)}$,

3) $\int e^{(\arctan x)} \frac{dx}{1+x^2}$

can be written as

$$\int f(\arctan x)\frac{dx}{1+x^2} = \int f(\arctan x)(\arctan x)'dx$$
$$= \int f(\arctan x)d(\arctan x).$$

Therefore, the substitution $u = \arctan x$ is fairly suitable for each of them:

1) $\int \frac{(\arctan x)^4}{1+x^2} dx = \int (\arctan x)^4 d(\arctan x) = \int u^4 du = \frac{u^5}{5} + C$.

Once the solution has been found in terms of u, one needs to replace u in it by the corresponding function of x. So the final solution is the following:

$$\int \frac{(\arctan x)^4}{1+x^2} dx = \frac{(\arctan x)^5}{5} + C.$$

2) $$\int \frac{dx}{(\arctan x)(1+x^2)} = \int \frac{d(\arctan x)}{\arctan x}$$
$$= \int \frac{du}{u} = \ln|u| + C = \ln|\arctan x| + C.$$

3) $$\int e^{(\arctan x)} \frac{dx}{1+x^2} = \int e^{(\arctan x)} d(\arctan x)$$
$$= \int e^u du = e^u + C = e^{\arctan x} + C.$$

One can easily check these solutions by differentiating. Let us check, e.g., the last integral:

$$(e^{\arctan x})' = e^{\arctan x}(\arctan x)' = \frac{e^{\arctan x}}{1+x^2}.$$ That is O.K.

Example 2: Both integrals, $\int \frac{\sin(\ln x)}{x} dx$ and $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$, are easily evaluated by using of the substitution $u = \ln x$. They are just common integrals:

$$\int \frac{\sin(\ln x)}{x} dx = \int \sin u du = -\cos u + C = -\cos(\ln u) + C,$$
$$\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C = \arcsin(\ln x) + C.$$

Example 3: Each of the integrals below is reduced to the table integral

$$\int \frac{dx}{\cos^2 u} = \tan u + C,$$

using the appropriate substitution:

- $\int \frac{dx}{\cos^2(3x-4)} = \frac{1}{3}\tan(3x-4) + C$ ($u = 3x-4$, $du = 3dx$).
- $\int \frac{dx}{\sqrt{x}\cos^2(\sqrt{x})} = 2\tan(\sqrt{x}) + C$ ($u = \sqrt{x}$, $du = \frac{dx}{2\sqrt{x}}$).
- $\int \frac{x^4 dx}{\cos^2(x^5)} = \frac{1}{5}\tan(x^5) + C$ ($u = x^5$, $du = 5x^4 dx$).
- $\int \frac{dx}{x\cos^2(\ln x)} = \tan(\ln x) + C$ ($u = \ln x$, $du = \frac{dx}{x}$).

- $\int \frac{e^x dx}{\cos^2(e^x)} = \tan e^x + C \qquad\qquad (u = e^x,\ du = e^x dx).$

The formal substitution into the integral really is not necessary.

3.4.1.2. Some Important Integrals

Problem 1: Evaluate the following integral: $\int \frac{dx}{a^2 + x^2}$.

Solution: Let us make the substitution $x = at$. Then

$$\begin{aligned}\int \frac{dx}{a^2 + x^2} &= \int \frac{adt}{a^2 + a^2 t^2} = \frac{a}{a^2} \int \frac{dt}{1 + t^2} \\ &= \frac{1}{a} \arctan t + C = \frac{1}{a} \arctan \frac{x}{a} + C.\end{aligned} \tag{8}$$

Problem 2: Find the integral $\int \frac{dx}{\sqrt{a^2 - x^2}}$.

Solution: By making use of the same substitution $x = at$ we get:

$$\begin{aligned}\int \frac{dx}{\sqrt{a^2 - x^2}} &= \int \frac{adt}{\sqrt{a^2 - a^2 t^2}} = \int \frac{adx}{a\sqrt{1 - t^2}} \\ &= \int \frac{dt}{\sqrt{1 - t^2}} = \arcsin t + C = \arcsin \frac{x}{a} + C.\end{aligned} \tag{9}$$

Problem 3: Prove the following formula:

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2}) + C. \tag{10}$$

Proof: The formula can be verified by differentiation. We have only to check whether the derivative of the function $\ln(x + \sqrt{x^2 \pm a^2})$ equals the integrand.

$$\begin{aligned}(\ln(x + \sqrt{x^2 \pm a^2}))' &= \frac{1}{x + \sqrt{x^2 \pm a^2}} (1 + \frac{1}{2\sqrt{x^2 \pm a^2}} 2x) \\ &= \frac{1}{x + \sqrt{x^2 \pm a^2}} \frac{\sqrt{x^2 \pm a^2} + x}{\sqrt{x^2 \pm a^2}} = \frac{1}{\sqrt{x^2 \pm a^2}}.\end{aligned}$$

That is true and hence the formula.

3.4.2. Integration by Parts

The formula for integration by parts states that

$$\int udv = uv - \int vdu \qquad (11)$$

for any differentiable functions $u(x)$ and $v(x)$.

This formula allows us to transform one problem of integration into another.

If one of the two integrals, $\int udv$ or $\int vdu$, is easily evaluated, it can be used to find the other one. This is the main idea of the method of integration by parts.

Formula (11) can be derived in the following way:

$$d(uv) = udv + vdu \quad \Rightarrow \quad udv = d(uv) - vdu \quad \Rightarrow$$

$$\int udv = \int d(uv) - \int vdu \quad \Rightarrow \quad \int udv = uv - \int vdu.$$

In practice, the procedure of integrating by parts consists of the following steps:

- First, we introduce intermediary functions $u(x)$ and $v'(x)$ to represent the function $f(x)$ as the product of the factors $u(x)$ and $v'(x)$, so that $f(x)dx = u(x)v'(x)dx = u(x)dv$ and

  $$\int f(x)dx = \int udv.$$

  For example, one can set $u(x) = f(x)$, which implies $v'(x) = 1$.
- Next we need to differentiate $u(x)$ and integrate $v'(x)$ to get the differential $du = u'(x)dx$ and the function $v(x) = \int v'(x)dx$ respectively. Note that a constant of integration can be taken zero at this step ($C = 0$).
- Then we use formula (11) and try to evaluate the integral $\int vdu$.

<u>FAQ (Frequently Asked Questions)</u>: Why do we prefer to deal with the integral $\int vdu$ instead of the initial one?

<u>Answer</u>: It depends on the choice of $u(x)$ whether the integral $\int vdu$ is easier to evaluate in comparison with the initial one. We assume that there exists the right choice.

The main problem one faces when dealing with the method of integration by parts is the choice of the intermediary functions. There is no general rule to follow it. It is a matter of experience and nothing more. At first in order to understand better this technique, it is necessary to make any choice and perform the calculations. If the new integral is simpler than the given one, then the choice is a good one; otherwise, go back and make another choice.

In such a way one can easily appreciate whether the choice of $u(x)$ is the best one. It is possible that you need to evaluate a few integrals before you will start to feel the right choice.

One can apply the following **criteria** to make the right choice.

A: The integral of v' should be easy for evaluation.

B: The derivative of $u(x)$ should be a simple function. Moreover, it is desirable that $u'(x)$ would be more simple function than $u(x)$.

The following examples illustrate the most common cases in which we need to use the technique of integration by parts.

Example 1: Evaluate the integral $\int x^2 \ln x dx$.

Solution: Consider some variants of representation of the above integrand as the product udv.

1) $u = \ln x$, $v' = x^2$ $\Rightarrow$ $du = \frac{dx}{x}$, $v = \int x^2 dx$;

2) $u = x$, $v' = x \ln x$ $\Rightarrow$ $du = dx$, $v = \int x \ln x dx$;

3) $u = x^2$, $v' = \ln x$ $\Rightarrow$ $du = 2x dx$, $v = \int \ln x dx$;

4) $u = x \ln x$, $v' = x$ $\Rightarrow$ $du = d(x \ln x)$, $v = \int x dx$;

5) $u = x^2 \ln x$, $v' = 1$ $\Rightarrow$ $du = d(x^2 \ln x)$, $v = \int dx$.

Let us discuss these choices in detail this time.
Both hypotheses, 2) and 3), do not satisfy criterion A, because it is not clear how to integrate $\ln x$, while hypotheses 4) and 5) contradict to criterion B.
Similar reasons suggest that the first way only is appropriate. Indeed,

- The power function x^2 is easily integrated and its primitive is

$$v = \int x^2 dx = \frac{x^3}{3} \qquad (C = 0).$$

- The derivative of the transcendental function $\ln x$ is the rational function:

$$(\ln x)' = x^{-1}.$$

Therefore, in view of formula (11) we finally get

$$\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^3 \frac{dx}{x} = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C.$$

Example 2: Evaluate $\int \arctan x dx$.

Solution: Let $u = \arctan x$ and $v' = 1$. Then $du = \dfrac{dx}{1+x^2}$ and $v = x$.

We integrate by parts:

$$\int \arctan x dx = x \arctan x - \int \frac{x}{1+x^2} dx .$$

To evaluate the new integral we use the substitution $z = 1 + x^2$, which implies $dz = d(1+x^2) = 2xdx$, and so $xdx = \dfrac{1}{2} dz$.

Therefore, $\displaystyle \int \frac{x}{1+x^2} dx = \frac{1}{2} \int \frac{dz}{z} = \frac{1}{2} \ln |z| = \frac{1}{2} \ln(1+x^2)$.

Hence, the final solution is the following:

$$\int \arctan x dx = x \arctan x - \int \frac{x}{1+x^2} dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C .$$

In a similar way one can integrate the product of a polynomial $P(x)$ and any inverse trigonometric function, as well as the product of a polynomial $P(x)$ and the logarithmic function.

> Each of the following function
> $P(x) \arcsin x$,
> $P(x) \arccos x$,
> $P(x) \arctan x$,
> $P(x) \cot^{-1} x$ and
> $P(x) \ln x$
> can be integrated by parts.
> The inverse trigonometric function (or $\ln x$) should be chosen as $u(x)$ and $v'(x) = P(x)$.

It is not always so easy. Sometimes one has to integrate by parts more than once to obtain the result.

Example 3: Evaluate $\int x \ln^2 x dx$.

Solution: Let $u = \ln^2 x$ and $dv = xdx$. Then $du = \dfrac{2 \ln x dx}{x}$ and $v = \dfrac{x^2}{2}$.

The formula of integration by parts gives

$$\int x\ln^2 x dx = \frac{x^2}{2}\ln^2 x - \int x\ln x dx.$$

Now we integrate by parts a second time, setting $u = \ln x$ and $dv = xdx$.

After integration and differentiation, we get $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$.

Therefore,

$$\int x\ln x dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int xdx = \frac{x^2}{2}\ln x - \frac{x^2}{4}.$$

The final result is the following:

$$\int x\ln^2 x dx = \frac{x^2}{2}\ln^2 x - \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right) + C$$

$$= \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C = \frac{x^2}{4}(2\ln^2 x - 2\ln x + 1) + C.$$

Example 4: Evaluate $\int x^2 e^x dx$.

Solution: We have to make the right choice between differentiation and integration of x^2. Note that every differentiation of a polynomial decreases its degree, and hence, the polynomial vanishes after a few steps, while integration of a polynomial increases its degree.

Therefore, the right choice is the following:

$u = x^2$ and $dv = e^x dx$ $\Rightarrow$ $du = 2xdx$ and $v = e^x$.

The formula of integration by parts yields:

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx. \quad (12)$$

We need to integrate by parts once more.

Let $u = x$ and $dv = e^x dx$ which imply $du = dx$ and $v = e^x$.

Therefore,

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x.$$

From here and equality (12) we obtain

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C.$$

The examples above illustrate that the single integration by parts can not be enough to obtain the answer, and so some extra work may be needed, e.g., another integration by parts or using some other techniques.

The last example can be generalized:

Each of the following integrals

$$\int P(x)e^{ax}dx,$$

$$\int P(x)\sin ax dx \text{ and}$$

$$\int P(x)\cos ax dx$$

can be evaluated using the integration by parts.

In order to get the solution, it is necessary to use integration by parts n times if the degree of the polynomial equals n.

The summary table below includes some suggested substitutions and formulas.

Table 2

Integrals	Substitutions	Basic Formulas
$\int P(x)\begin{pmatrix}\arcsin x\\ \arccos x\\ \arctan x\\ \cot^{-1} x\\ \ln x\end{pmatrix}dx$	$u=\begin{cases}\arcsin x\\ \arccos x\\ \arctan x\\ \cot^{-1} x\\ \ln x\end{cases}$ $dv = P(x)dx$	$du=\begin{cases}\frac{dx}{\sqrt{1-x^2}}\\ -\frac{dx}{\sqrt{1-x^2}}\\ \frac{dx}{1+x^2}\\ -\frac{dx}{1+x^2}\\ \frac{dx}{x}\end{cases}$ $v=\int P(x)dx$
$\int P(x)\begin{pmatrix}e^{ax}\\ \sin ax\\ \cos ax\end{pmatrix}dx$	$u = P(x)$ $dv=\begin{pmatrix}e^{ax}\\ \sin ax\\ \cos ax\end{pmatrix}dx$	$du = P'(x)dx$ $v=\frac{1}{a}\begin{cases}e^{ax}\\ -\cos ax\\ \sin ax\end{cases}$

By making use of integration by parts we sometimes come to an equation for the integral but not an explicit formula. However, by solving this equation we obtain the desired result. Let us consider a typical problem of such a kind.

Problem 4: Find the integral $I = \int e^{ax}\cos(bx)dx$.

Solution: Let $u = e^{ax}$ and $dv = \cos bx dx$, so that $du = ae^{ax}dx$ and $v = \sin bx/b$. The formula of integration by parts gives

$$\int e^{ax}\cos bx dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx dx.$$

The new integral is similar to the initial one. Let us integrate by parts the second time. Note that we have to use again e^{ax} as u. Otherwise, we would come back to the original integral and nothing more.

Thus, now let $u = e^{ax}$ and $dv = \sin bx dx$. Then $du = ae^{ax}dx$ and $v = -\cos bx/b$.

In this case we have

$$\int e^{ax}\sin bx dx = -\frac{1}{b}e^{ax}\cos bx - (-\frac{a}{b}\int e^{ax}\cos bx dx).$$

Combining both formulas yields

$$\int e^{ax}\cos bx dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx dx).$$

This equality can be considered as a linear equation with respect to the given integral I:
$$I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I.$$

By combining of similar terms and making use of simple algebraic manipulations, we get

$$(b^2 + a^2)I = e^{ax}(b\sin bx + a\cos bx) \qquad \Rightarrow$$

$$I = \frac{a\cos bx + b\sin bx}{a^2 + b^2}e^{ax}.$$

Hence, the final solution is

$$\int e^{ax}\cos bx dx = \frac{a\cos bx + b\sin bx}{a^2 + b^2}e^{ax} + C. \qquad (13)$$

In a similar way one can obtain another formula of this kind:

$$\int e^{ax}\sin bx dx = \frac{a\sin bx - b\cos bx}{a^2 + b^2}e^{ax} + C. \qquad (14)$$

3.5. Integration of Rational Functions

3.5.1. Main Definitions

Let us start from the definition chain:

Rational Functions → Proper Fractions → Partial Fractions

A **rational function** is a function that can be expressed as the ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}.$$

A rational function $\frac{P(x)}{Q(x)}$ is said to be a **proper fraction** if the degree of the polynomial $P(x)$ is less than that of $Q(x)$.

For example, the following functions

$$\frac{x^3}{2x+7}, \qquad \frac{3x-2}{5x^3+x-1}, \qquad \frac{1}{(x+5)^4}.$$

are the rational functions. Furthermore, the last two functions are the proper fractions.

Fractions of the following form

$$1)\quad \frac{1}{(x-a)^n} \qquad (n \geq 1), \tag{15}$$

$$2)\quad \frac{Ax+B}{(x^2+px+q)^n} \qquad (n \geq 1) \tag{16}$$

are called the **partial fractions**, where the quadratic polynomial x^2+px+q is assumed to be irreducible, that is, the discriminant $D = p^2 - 4q$ is negative.

The problem of integration of rational functions can be subdivided into several separate problems such as:

1) Integration of partial fractions.
2) Decomposition of a proper fraction into a sum of partial fractions.
3) Reduction of any rational function to a proper fraction.

Consider the procedure of integration of a ration function $f(x) = \frac{P(x)}{Q(x)}$.

It comprises the following steps.

- Assume that $f(x)$ is a proper fraction. Otherwise it is necessary first to perform the **polynomial long division** in order to represent the function

$f(x)$ as a sum of some polynomial and the remainder term (which is a proper fraction). Any polynomial is easily integrated, so in both cases we can deal only with proper fractions.
Therefore, the problem of integration of rational functions can always be reduced to the one of integration of proper fractions, keeping in mind that any rational function either is a proper fraction or can be expressed through a proper fraction.

- In order to decompose the given function (or the remainder term) into the sum of partial fractions, the denominator $Q(x)$ has to be factored into irreducible polynomials, that is, linear and irreducible quadratic polynomials. The corresponding method is called Decomposition of Rational Functions into a Sum of Partial Fractions (in short form: **Partial Fraction Decomposition**).
- To integrate each of the obtained partial fractions.

3.5.2. Integration of Partial Fractions

We attach importance to the partial fractions because any proper fraction can be decomposed into a sum of partial fractions.
Partial fractions of the first type (expression (15)) are easily integrated in view of common integrals:

$$\int \frac{dx}{x-a} = \ln|x-a| + C, \tag{16}$$

$$\int \frac{dx}{(x-a)^n} = \frac{1}{(-n+1)(x-a)^{n-1}} + C \qquad (n \neq 1). \tag{17}$$

In order to integrate partial fractions of the second type (expression (16)), one has to complete the square for the polynomial $x^2 + px + q$, e.g., making use of the substitution $t = x + p/2$, that is, $x = t - p/2$.
Hence,

$$x^2 + px + q = (t - \frac{p}{2})^2 + p(t - \frac{p}{2}) + q$$

$$= t^2 - pt + \frac{p^2}{4} + pt - \frac{p^2}{2} + q = t^2 + (q - \frac{p^2}{4}) = t^2 + a^2,$$

where the positive constant $q - \frac{p^2}{4}$ is denoted as a^2.

Thereto, $dx = dt$ and $Ax + B = At + (B - A\frac{p}{2}) = At + B_1$, where $B_1 = B - p/2$.

Then we apply the properties of integrals to obtain

$$\int \frac{Ax+B}{(x^2+px+q)^n}dx = \int \frac{At+B_1}{(t^2+a^2)^n}dt$$
$$= A\int \frac{tdt}{(t^2+a^2)^n} + B_1\int \frac{dt}{(t^2+a^2)^n}. \tag{18}$$

The first integral on the right-hand side is easily evaluated:

$$\int \frac{tdt}{(t^2+a^2)^n} = \frac{1}{2}\int \frac{d(t^2)}{(t^2+a^2)^n}$$
$$= \frac{1}{2}\int \frac{d(t^2+a^2)}{(t^2+a^2)^n} = \begin{cases} \frac{1}{2}\ln(t^2+a^2)+C, & \text{if} \quad n=1; \\ \frac{1}{2(-n+1)(t^2+a^2)^{n-1}}+C, & \text{if} \quad n>1. \end{cases}$$

Now let us apply the technique of integration by parts to find integrals

$$I_n = \int \frac{dt}{(t^2+a^2)^n} \qquad (n \ge 1). \tag{19}$$

Let $u = \dfrac{1}{(t^2+a^2)^n}$ and $dv = dt$.

Then $du = \dfrac{-2ntdt}{(t^2+a^2)^{n+1}}$ and $v = t$.

Therefore,

$$\int \frac{dt}{(t^2+a^2)^n} = \frac{t}{(t^2+a^2)^n} - (-2n)\int \frac{t^2}{(t^2+a^2)^{n+1}}dt$$
$$= \frac{t}{(t^2+a^2)^n} + 2n\int \frac{(t^2+a^2)-a^2}{(t^2+a^2)^{n+1}}dt$$
$$= \frac{t}{(t^2+a^2)^n} + 2n\int \frac{dt}{(t^2+a^2)^n} - 2na^2\int \frac{dt}{(t^2+a^2)^{n+1}}.$$

Then we combine the similar terms and express the integral I_{n+1} through the integral I_n:

$$\int \frac{dt}{(t^2+a^2)^{n+1}} = \frac{1}{2na^2}\left((2n-1)\int \frac{dt}{(t^2+a^2)^n} + \frac{t}{(t^2+a^2)^n}\right). \tag{20}$$

This recurrence formula allows us:

- to find integral I_2 if integral I_1 is known (setting $n = 1$),
- to find integral I_3 if integral I_2 is known (setting $n = 2$), and so on.

Note that integral I_1 can be found by using of simple methods. (See equality (8).)

$$I_1 = \int \frac{dt}{t^2 + a^2} = \frac{1}{a}\arctan\frac{t}{a} + C.$$

Then setting $n = 1$ we have by recurrence formula (20)

$$I_2 = \int \frac{dt}{(t^2 + a^2)^2} = \frac{1}{2a^2}\left(\frac{1}{a}\arctan\frac{t}{a} + \frac{t}{t^2 + a^2}\right) + C, \quad etc.$$

Thus, the problem of integration of partial fractions is completely solved.

3.5.3. Partial Fraction Decomposition

3.5.3.1. The Main Idea of the Method

In simple cases the decomposition of proper fractions into a sum of partial fractions can be easily obtained by means of algebraic manipulations.

Here are typical **examples**:

- $$\frac{1}{(x-a)(x-b)} = \frac{b-a}{(b-a)(x-a)(x-b)} = \frac{(x-a)-(x-b)}{(b-a)(x-a)(x-b)} = \frac{1}{b-a}\left(\frac{1}{x-b} - \frac{1}{x-a}\right).$$

- $$\frac{1}{(x^2-49)} = \frac{1}{(x-7)(x+7)} = \frac{1}{14}\left(\frac{1}{x-7} - \frac{1}{x+7}\right).$$

- $$\frac{1}{x(x^2+4)} = \frac{1}{4}\frac{4}{x(x^2+4)} = \frac{1}{4}\frac{(x^2+4)-x^2}{x(x^2+4)} = \frac{1}{4}\left(\frac{(x^2+4)}{x(x^2+4)} - \frac{x^2}{x(x^2+4)}\right) = \frac{1}{4}\left(\frac{1}{x} - \frac{x}{x^2+4}\right).$$

In more complicated cases one has to use the Method of Partial Fractions Decomposition.

The main idea of this method can be illustrated by the following simple example.

Example 1:

1) The sum of partial fractions, $\frac{2}{x-1}$ and $\frac{5}{x+4}$, can be combined into a more complicated fraction:

$$\frac{2}{x-1}+\frac{5}{x+4}=\frac{2(x+4)+5(x-1)}{(x-1)(x+4)}=\frac{7x+3}{(x-1)(x+4)}.$$

When we read this formula from left to right, we say about reduction of fractions to the common denominator.

We can also read the same formula from right to left:

$$\frac{7x+3}{(x-1)(x+4)}=\frac{2}{x-1}+\frac{5}{x+4}.$$

In this case we say about decomposition of the compound fraction into the sum of partial fractions.

2) Let us assume that we need to decompose the fraction $\frac{7x+3}{(x-1)(x+4)}$ into partial fractions. It looks in a general form as follows:

$$\frac{7x+3}{(x-1)(x+4)}=\frac{A}{x-1}+\frac{B}{x+4},$$

where A and B are undetermined constants.

If we multiply across by $(x-1)(x+4)$, then we get

$$7x+3=A(x+4)+B(x-1).$$

This equality is the equation for constants A and B but at the same time it is the identity with respect to x. So one can substitute any value for x to find the constants.

Setting $x=1$ we get the equality $10=5A$ which implies $\underline{A=2}$.

Setting $x=-4$, we obtain $(-25)=-5B \Rightarrow \underline{B=5}$.

Therefore, $$\frac{7x+3}{(x-1)(x+4)}=\frac{2}{x-1}+\frac{5}{x+4}.$$

as it was desired.

The Method of Partial Fractions Decomposition proceeds in the opposite direction in comparison with the reduction to a common denominator, that is, it transforms a compound fraction into a sum of partial fractions.

Partial Fractions Decomposition is the reverse procedure to reduction to the common denominator.

3.5.3.2. Partial Fraction Decomposition: The Main Rules

There are a few rules to decompose any proper fraction into a sum of partial fractions.

Rule 1: Let $\dfrac{P(x)}{Q(x)}$ be a proper fraction and let $Q(x) = (x-a)\,Q_1(x)$.

Then there exists a unique proper fraction $\dfrac{P_1(x)}{Q_1(x)}$ and a unique constant A, such that the given proper fraction can be represented in the form

$$\frac{P(x)}{Q(x)} = \frac{A}{x-a} + \frac{P_1(x)}{Q_1(x)}.$$

Note that the degree of a polynomial $Q_1(x)$ is less than the degree of the given polynomial $Q(x)$: $\text{degree}(Q_1) = \text{degree}(Q) - 1$.

Then one can apply this rule to the proper fraction $P_1(x)/Q_1(x)$, if the denominator $Q_1(x)$ includes a linear factor, that is, $Q_1(x) = (x-b)\,Q_2(x)$. Therefore,

$$\frac{P(x)}{(x-a)(x-b)Q_2(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{P_2(x)}{Q_2(x)}.$$

Each such transformation decreases the degree of the denominator of the proper fraction.

Corollary: If the denominator $Q(x)$ of the proper fraction $\dfrac{P(x)}{Q(x)}$ consists of n different linear factors, that is, $Q(x) = (x-a_1)(x-a_2)...(x-a_n)$, then

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a_1)(x-a_2)...(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + ... + \frac{A_n}{x-a_n}.$$

One can say that each linear factor $(x-a_k)$ in the denominator of the proper fraction yields the partial fraction $\dfrac{A_k}{x-a_k}$, where A_k is a constant.

The structure of decomposition of any proper fraction depends only on the factors, which the denominator consists of. For instance, both fractions below have the same structure of decomposition into partial fractions:

$$\frac{1}{x(x-3)(x+2)} = \frac{A_1}{x} + \frac{A_2}{x-3} + \frac{A_3}{x+2}, \qquad (21)$$

$$\frac{5x-1}{x(x-3)(x+2)} = \frac{A_1}{x} + \frac{A_2}{x-3} + \frac{A_3}{x+2}. \qquad (22)$$

The numerator determines numerical values of the constants A_1, A_2, A_3.
Let us find, e.g., numerical values of the constants in decomposition (21).
First, we multiply both sides by $x(x-3)(x+2)$:

$$1 = A_1(x-3)(x+2) + A_2 x(x+2) + A_3 x(x-3).$$

One can see that all fractions have disappeared.
Then we take for x such values that make some of the terms vanish:

$$x = 0 \quad \Rightarrow \quad 1 = A_1(-3)2 = -6A_1 \quad \Rightarrow \quad A_1 = -1/6,$$
$$x = 3 \quad \Rightarrow \quad 1 = 15A_2 \quad \Rightarrow \quad A_2 = 1/15,$$
$$x = -2 \quad \Rightarrow \quad 1 = 10A_3 \quad \Rightarrow \quad A_3 = 1/10.$$

Finally, it remains to put the constants back into the original partial fractions:

$$\frac{1}{x(x-3)(x+2)} = -\frac{1}{6x} + \frac{1}{15(x-3)} + \frac{1}{10(x+2)}.$$

If the denominator of a proper fraction includes nth power of the factor $(x-a)$, then one can use the following rule of decomposition into a sum of partial fractions:

Rule 2: Let $\dfrac{P(x)}{Q(x)}$ be a proper fraction and let $Q(x) = (x-a)^n \, Q_1(x)$.

Then there exists a unique proper fraction $P_1(x)/Q_1(x)$ and unique constants $A_1, A_2, ..., A_n$ such that the given rational function can be represented in the form

$$\frac{P(x)}{(x-a)^n Q_1(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + ... + \frac{A_n}{(x-a)^n} + \frac{P_1(x)}{Q_1(x)}.$$

Then one can apply the above rules to the proper fraction $\dfrac{P_1(x)}{Q_1(x)}$, if the denominator $Q_1(x)$ includes a linear factor (repeated or not).

Example 2: The decomposition of any proper fraction $\dfrac{P(x)}{Q(x)}$ with denominator $Q(x) = (x-a)(x-b)^3$ has the following form:

$$\frac{P(x)}{(x-a)(x-b)^3} = \frac{A}{x-a} + \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \frac{B_3}{(x-b)^3}.$$

Example 3: Decompose the fraction $\dfrac{1}{(x+1)(x-4)^2}$ into partial fractions.

Solution: By applying the above rules we have:

$$\frac{1}{(x+1)(x-4)^2}=\frac{A_1}{x+1}+\frac{A_2}{x-4}+\frac{A_3}{(x-4)^2}.$$

Then we multiply both sides by $(x+1)(x-4)^2$:

$$1=A_1(x-4)^2+A_2(x+1)(x-4)+A_3(x+1).$$

To solve this equation with respect to A_1, A_2 and A_3, we take for x a few values:

$$x=-1 \;\Rightarrow\; 1=25A_1 \qquad\Rightarrow\qquad \underline{A_1=1/25};$$

$$x=4 \;\Rightarrow\; 1=5A_3 \qquad\Rightarrow\qquad \underline{A_3=1/5};$$

$$x=0 \;\Rightarrow\; 1=16A_1-4A_2+A_3=16/25-4A_2+1/5 \;\Rightarrow\; \underline{A_2=-1/25}.$$

Thus,

$$\frac{1}{(x+1)(x-4)^2}=\frac{1}{25}\left(\frac{1}{x+1}-\frac{1}{x-4}+\frac{5}{(x-4)^2}\right).$$

Consider now the case when the denominator of a proper fraction includes the irreducible factor (x^2+px+q).

Rule 3: Let $\dfrac{P(x)}{Q(x)}$ be a proper fraction and let

$$Q(x)=(x^2+px+q)\,Q_1(x).$$

Then there exists a unique proper fraction $\dfrac{P_1(x)}{Q_1(x)}$ and unique constants A and B such that the given rational function can be represented in the form

$$\frac{P(x)}{(x^2+px+q)Q_1(x)}=\frac{Ax+b}{x^2+px+q}+\frac{P_1(x)}{Q_1(x)}.$$

Note that $\text{degree}(Q_1)=\text{degree}(Q)-2$.

Then one can apply the above rules to the proper fraction $\dfrac{P_1(x)}{Q_1(x)}$, if its denominator includes either linear or irreducible factors.

Example 4: The proper fraction $\frac{1}{(x-3)(x^2-x+2)}$ is decomposed into partial fractions as follows:

$$\frac{1}{(x-3)(x^2-x+2)} = \frac{A_1}{(x-3)} + \frac{A_2x+B_2}{(x^2-x+2)}.$$

As above we get the equality

$$1 = A_1(x^2-x+2) + (A_2x+B_2)(x-3)$$

and solve it with respect to A_1, A_2 and B_2:

$$x=3 \quad \Rightarrow \quad 1 = 8A_1 \quad \Rightarrow \quad A_1 = 1/8;$$

$$x=0 \quad \Rightarrow \quad 1 = 2A_1 - 3B_2 \quad \Rightarrow \quad \frac{3}{4} = -3B_2 \quad \Rightarrow B_2 = -1/4;$$

$$x=1 \;\Rightarrow\; 1 = 2A_1 + (A_2+B_2)(-2) \;\Rightarrow\; 1 = \frac{1}{4} - 2A_2 + \frac{1}{2} \;\Rightarrow\; A_2 = -\frac{1}{8}.$$

Thus, $$\frac{1}{(x-3)(x^2-x+2)} = \frac{1}{8}\left(\frac{1}{(x-3)} - \frac{x+2}{(x^2-x+2)}\right).$$

At last, we need only to consider the case when the denominator of a proper fraction includes n times repeated irreducible factor (x^2+px+q).

Rule 4: Let $\frac{P(x)}{Q(x)}$ be a proper fraction and let

$$Q(x) = (x^2+px+q)^n\, Q_1(x).$$

Then there exists a unique proper fraction $\frac{P_1(x)}{Q_1(x)}$ and unique constants $A_1, A_2, ..., A_n$ and $B_1, B_2, ..., B_n$ such that the given rational function can be represented in the form

$$\frac{P(x)}{(x^2+px+q)^n Q_1(x)} = \frac{A_1x+B_1}{x^2+px+q} + \frac{A_2x+B_2}{(x^2+px+q)^2} + \dots$$

$$+ \frac{A_nx+B_n}{(x^2+px+q)^n} + \frac{P_1(x)}{Q_1(x)}.$$

Here $\text{degree}(Q_1) = \text{degree}(Q) - 2n$.

Example 5: The partial decomposition technique gives

$$\frac{1}{(x-3)(x^2-x+2)^2} = \frac{A_1x+B_1}{x^2-x+2} + \frac{A_2x+B_2}{(x^2-x+2)^2} + \frac{A_3}{x-3}.$$

3.5.3.3. Factoring

One of the steps of decomposition of a proper fraction into a sum of partial fractions consists of factoring of the denominator $Q(x)$.

It is appropriate to mention here t**he fundamental theorem of algebra:**

Every polynomial can be factored into linear factors (polynomials of degree 1) and irreducible polynomials of degree 2.

Some **Examples** of **Factoring:**

- The polynomial $x^3 - 5x^2 - x - 15$ can be factored into a linear factor and an irreducible factor of degree 2:
$$x^3 - 5x^2 - x - 15 = (x-3)(x^2 - 2x + 5).$$
- The polynomial $x^2 + 6x + 9$ has a twice repeated linear factor (of degree 1):
$$x^2 + 6x + 9 = (x+3)^2.$$
- The polynomial $x^4 + 2x^2 + 1$ has a twice repeated irreducible factor of degree 2:
$$x^4 + 2x^2 + 1 = (x^2 + 1)^2.$$
- Both factors of the polynomial $x^4 + 1$ are irreducible ones of degree 2:
$$x^4 + 1 = (x^4 + 2x^2 + 1) - 2x^2 = (x^2 + 1)^2 - 2x^2$$
$$= (x^2 + 1 - \sqrt{2}x)(x^2 + 1 + \sqrt{2}x).$$

FAQ: How can we know whether a quadratic polynomial is irreducible or it can be factored further into two linear factors?
Answer: A reducible quadratic polynomial has two zeros or one repeated zero; an irreducible quadratic polynomial has no zeros. So if the quadratic formula results in a negative expression under the radical (the discriminant), the associated polynomial is irreducible.

- The quadratic polynomial $x^2 - 5x + 4$ has two zeros: $x_1 = 1$ and $x_2 = 4$. Therefore, it can be factored into two linear factors as follows:
$$x^2 - 5x + 4 = (x-1)(x-4).$$
- The quadratic polynomial $(x^2 - 4x + 4)$ has one repeated zero: $x_1 = x_2 = 2$. Therefore, $x^2 - 4x + 4 = (x-2)^2$.
- Using the quadratic formula for the polynomial $(x^2 - 2x + 4)$ yields:
$$x_{1,2} = 1 \pm \sqrt{1-4} = 1 \pm \sqrt{-3}.$$
Since the discriminant is negative, the polynomial is irreducible.

3.5.4. Polynomial Long Division

Let $\frac{P(x)}{Q(x)}$ be a rational function, and let the degree of the polynomial $P(x)$ be greater than or equal to the degree of the polynomial $Q(x)$. Then there exist the uniquely determined polynomials $S(x)$ and $R(x)$, such that the rational function $\frac{P(x)}{Q(x)}$ can be represented in the form

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $\frac{R(x)}{Q(x)}$ is a proper fraction.

The polynomial $S(x)$ is called the quotient; the term $Q(x)$ is the divisor and the expression $R(x)$ is called the remainder. In the special case when the remainder equals zero, it is said that $Q(x)$ divides evenly into $P(x)$.

Let us consider the division algorithm in detail for particular examples.

Example 6: Perform polynomial long division if

$$f(x) = \frac{5x^3 - x^2 + 4x + 7}{x^2 + 3x - 1}.$$

First, we write the expression in a form of long division:

$$5x^3 - x^2 + 4x + 7 \quad \Big| \; \underline{x^2 + 3x - 1}$$

Next we divide the leading term $5x^3$ in the numerator of the given polynomial by the leading term x^2 of the divisor, and write the answer $5x$ under the line:

$$5x^3 - x^2 + 4x + 7 \quad \Big| \; \frac{x^2 + 3x - 1}{5x}$$

Now we multiply the term $5x$ to the divisor $x^2 + 3x - 1$, and write the answer

$$5x(x^2 + 3x - 1) = 5x^3 + 15x^2 - 5x$$

under the numerator polynomial, lining up the terms of equal degree:

$$\begin{array}{l|l} 5x^3 \quad - x^2 + 4x + 7 & \underline{x^2 + 3x - 1} \\ 5x^3 + 15x^2 - 5x & 5x \end{array}$$

Then subtract the last line from the line above it:

$$\begin{array}{l} - \begin{array}{r} 5x^3 \quad -x^2+4x+7 \\ \underline{5x^3+15x^2-5x \quad\quad} \\ -16x^2+9x+7 \end{array} \end{array} \qquad \begin{array}{|l} x^2+3x-1 \\ \hline 5x \end{array}$$

Now we have to repeat the procedure: to divide the leading term $(-16x^2)$ of the polynomial in the last line by the leading term x^2 of the divisor to obtain (-16), and add this term to the $5x$ under the line on the right-hand side:

$$\begin{array}{l} - \begin{array}{r} 5x^3 \quad -x^2+4x+7 \\ \underline{5x^3+15x^2-5x \quad\quad} \\ -16x^2+9x+7 \end{array} \end{array} \qquad \begin{array}{|l} x^2+3x-1 \\ \hline 5x-16 \end{array}$$

Then multiply the term (-16) by the divisor x^2+3x-1, and write the answer

$$-16(x^2+3x-1) = -16x^2-48x+16$$

under the last line polynomial, lining up terms of equal degree:

$$\begin{array}{l} - \begin{array}{r} 5x^3 \quad -x^2+4x+7 \\ \underline{5x^3+15x^2-5x \quad\quad} \\ -16x^2 \quad +9x+7 \\ -16x^2-48x+16 \end{array} \end{array} \qquad \begin{array}{|l} x^2+3x-1 \\ \hline 5x-16 \end{array}$$

Subtract the last line from the line above it:

$$\begin{array}{l} - \begin{array}{r} 5x^3 \quad -x^2+4x+7 \\ \underline{5x^3+15x^2-5x \quad\quad} \\ - \begin{array}{r} -16x^2 \quad +9x+7 \\ \underline{-16x^2-48x+16} \\ 57x-9 \end{array} \end{array} \end{array} \qquad \begin{array}{|l} x^2+3x-1 \\ \hline 5x-16 \end{array}$$

At the next step we would divide the term $57x$ by the leading term x^2 of the divisor, not yielding a polynomial expression.
Therefore, the division procedure is terminated. The remainder is in the last line: $57x-9$, and the quotient is $5x-16$. One can see that the remainder $(57x-9)$ has degree 1, which is less than the degree of the divisor.

Thus, we finally get:

$$\frac{5x^3 - x^2 + 4x + 7}{x^2 + 3x - 1} = (5x - 16) + \frac{57x - 9}{x^2 + 3x - 1}.$$

The easiest way to check the answer algebraically is to multiply both sides by the divisor:

$$5x^3 - x^2 + 4x + 7 = (5x - 16)(x^2 + 3x - 1) + (57x - 9).$$

Then we multiply out and simplify the right side:

$$\begin{aligned} 5x^3 - x^2 + 4x + 7 &= (5x - 16)(x^2 + 3x - 1) + (57x - 9) \\ &= 5x^3 + 15x^2 - 5x - 16x^2 - 48x + 16 + 57x - 9 \\ &= 5x^3 - x^2 + 4x + 7. \end{aligned}$$

Thus, we have the identity and so the answer is correct.

Example 7: Perform polynomial long division if $f(x) = \dfrac{x^3 - 4x^2 - x - 6}{x^2 - x + 2}$.

In a similar way as above we get:

$$\begin{array}{rl|l} & x^3 - 4x^2 - x - 6 & x^2 - x + 2 \\ \cline{3-3} - & x^3 - x^2 + 2x & x - 3 \\ \hline & -3x^2 + 3x - 6 & \\ - & -3x^2 + 3x - 6 & \\ \hline & 0 & \end{array}$$

In this case, the remainder equals zero, so $(x - 3)$ divides evenly into $(x^2 - x + 2)$.

Therefore, $$\frac{x^3 - 4x^2 - x - 6}{x^2 - x + 2} = x - 3.$$

Multiplying both sides by the divisor yields:

$$x^3 - 4x^2 - x - 6 = (x^2 - x + 2)(x - 3).$$

By polynomial long division, the polynomial $x^3 - 4x^2 - x - 6$ is factored, that is, it is written as the product of polynomials with lower degrees.

Summary example: Evaluate the integral $\displaystyle\int \frac{x^4 + 1}{x^3 - 9x} dx$ using the technique of integrating rational functions.

Solution: Since the degree of the numerator is greater than that of the denominator, we have to perform the polynomial long division to get

$$\frac{x^4+1}{x^3-9x} = x + \frac{9x^2+1}{x^3-9x}.$$

Next we factor the denominator:

$$x^3 - 9x = x(x^2-9) = x(x-3)(x+3).$$

Then we use the method of partial fractions to split the fraction $\frac{9x^2+1}{x^3-9x}$ into easily integrable ones:

$$\frac{9x^2+1}{x^3-9x} = \frac{9x^2+1}{x(x-3)(x+3)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+3}.$$

Now we simplify this equality to get

$$9x^2+1 = A(x-3)(x+3) + Bx(x+3) + Cx(x-3).$$

To solve this equation with respect to the constants we take for x a few values:

$$x = 0 \qquad \Rightarrow \qquad 1 = -9A \qquad \Rightarrow \qquad A = -1/9;$$
$$x = 3 \qquad \Rightarrow \qquad 82 = 18B \qquad \Rightarrow \qquad B = 41/9,$$
$$x = -3 \qquad \Rightarrow \qquad 82 = 18c \qquad \Rightarrow \qquad C = 41/9.$$

Therefore,

$$\frac{9x^2+1}{x^3-9x} = \frac{1}{9}\left(-\frac{1}{x} + \frac{41}{x-3} + \frac{41}{x+3}\right),$$

which implies

$$\frac{x^4+1}{x^3-9x} = x + \frac{9x^2+1}{x^3-9x} = x + \frac{1}{9}\left(-\frac{1}{x} + \frac{41}{x-3} + \frac{41}{x+3}\right).$$

Finally, we get:

$$\int \frac{x^4+1}{x^3-9x}dx = \int \left(x - \frac{1}{9x} + \frac{41}{9(x-3)} + \frac{41}{9(x+3)}\right)dx$$
$$= \int x\,dx - \frac{1}{9}\int \frac{dx}{x} + \frac{41}{9}\left(\int \frac{dx}{x-3} + \int \frac{dx}{x+3}\right)$$
$$= \frac{x^2}{2} - \frac{1}{9}\ln|x| + \frac{41}{9}(\ln|x-3| + \ln|x+3|) + C$$
$$= \frac{x^2}{2} - \frac{1}{9}\ln|x| + \frac{41}{9}\ln|x^2-9| + C.$$

3.6. Integration of Trigonometric Functions

3.6.1. Integrals of the Form $\int \sin^m x \cos^n x\, dx$

We consider here two cases: either both exponents, m and n, are even numbers or at least one of them is odd.

Case 1: Let m and n be even numbers, that is, $m = 2k$ and $n = 2l$.

Then the powers of sines and cosines can be reduced step by step, using the following trigonometric identities:

$$2\sin^2 x = 1 - \cos 2x, \tag{23a}$$

$$2\cos^2 x = 1 + \cos 2x, \tag{23b}$$

$$2\sin x \cos x = \sin 2x. \tag{24}$$

Indeed,

$$\int \sin^{2k} x \cos^{2l} x dx = \int (\sin^2 x)^k (\cos^2 x)^l dx$$

$$= \frac{1}{4}\int (1 - \cos 2x)^k (1 + \cos 2x)^l dx.$$

By removing parentheses, we obtain the sum of simpler integrals, some of which have to be further simplified in a similar way as above.

Case 2: Let n be an odd number: $n = 2k + 1$.
Then for any number m we get:

$$\int \sin^m x \cos^{2k+1} x dx = \int \sin^m x \cos^{2k} x \cos x dx$$

$$= \int \sin^m x (1 - \sin^2 x)^k \cos x dx.$$

This form suggests the substitution $t = \sin x$, which implies $dt = \cos x dx$, and so

$$\int \sin^m x \cos^{2k+1} x dx = \int t^m (1 - t^2)^k dt.$$

If m is an odd number, then by making use of the substitution $t = \cos x$, we obtain

$$\int \sin^{2k+1} x \cos^n x dx = -\int (1 - t^2)^k t^n dt.$$

Thus, the problem of integration is reduced to a simple procedure of term-by-term integration of a linear combination of power functions.

Examples:

- $$\int \cos^2 x dx = \frac{1}{2}\int (1+\cos 2x)dx = \frac{1}{2}(x+\frac{1}{2}\sin 2x)+C.$$
- $$\int \sin^2 3x \cos^2 3x dx = \frac{1}{4}\int \sin^2 6x dx \qquad |\text{ by formula (24) }|$$
$$= \frac{1}{8}\int (1-\cos 12x)dx = \frac{1}{8}(x-\frac{1}{12}\sin 12x)+C.$$
- $$\int \cos^5 x dx = \int (1-\sin^2 x)^2 \cos x dx \qquad |\text{ by substitution } t=\sin x \;|$$
$$= \int (1-2t^2+t^4)dt = t-\frac{2}{3}t^3+\frac{1}{5}t^5+C$$
$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$
- $$\int \frac{\sin^5 x}{\cos x}dx = \int \frac{(1-\cos^2 x)^2}{\cos x}\sin x dx \qquad |\text{ by substitution } t=\cos x \;|$$
$$= -\int \frac{(1-t^2)^2}{t}dt = \int (\frac{1}{t}-2t+t^3)dt$$
$$= \ln|t| - t^2 + \frac{t^4}{4} + C$$
$$= \ln|\cos x| - \cos^2 x + \frac{\cos^4 x}{4} + C.$$

3.6.2. Integration of Powers of Trigonometric Functions

3.6.2.1. Integrals of the Form $\int \frac{\mathbf{dx}}{\mathbf{\sin^n x}}$ and $\int \frac{\mathbf{dx}}{\mathbf{\cos^n x}}$

The power n is assumed to be a natural number. So there are two possible cases.

Case 1: Let n be an odd number, that is, $n = 2k-1$.

In this case, both problems of integration, $\int \frac{dx}{\sin^n x}$ and $\int \frac{dx}{\cos^n x}$, can be solved by using of the substitutions $\cos x = t$ or $\sin x = t$, correspondingly:

$$\int \frac{dx}{\sin^{2k-1} x} = \int \frac{\sin x dx}{\sin^{2k} x} = \int \frac{\sin x dx}{(1-\cos^2 x)^k} = -\int \frac{d(\cos x)}{(1-\cos^2 x)^k} = -\int \frac{dt}{(1-t^2)^k},$$

$$\int \frac{dx}{\cos^{2k-1} x} = \int \frac{\cos x dx}{(1-\sin^2 x)^k} = \int \frac{dt}{(1-t^2)^k}.$$

Hence, the given integrals are transformed to integrals of proper fractions.

Case 2: Let n be an even number, that is, $n = 2k$.

The integral $\int \frac{dx}{\sin^n x}$ can be transformed by using of the trigonometric identities:

$$\frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = 1 + \cot^2 x \qquad \Rightarrow$$

$$\frac{1}{\sin^{2k} x} = (\frac{1}{\sin^2 x})^{k-1} \frac{1}{\sin^2 x} = (1 + \cot^2 x)^{k-1} \frac{1}{\sin^2 x}$$

$$\Rightarrow$$

$$\int \frac{dx}{\sin^{2k} x} = \int (1 + \cot^2 x)^{k-1} \frac{dx}{\sin^2 x} = -\int (1 + t^2)^{k-1} dt,$$

where $t = \cot x$.

As above, the integral $\int \frac{dx}{\cos^n x}$ can be evaluated by the substitution $t = \tan x$:

$$\int \frac{dx}{\cos^{2k} x} = \int (1 + \tan^2 x)^{k-1} \frac{dx}{\cos^2 x} = \int (1 + t^2)^{k-1} dt.$$

Thus, we have the integral of a polynomial.

Examples:

- $$\int \frac{dx}{\cos x} = \int \frac{\cos x dx}{\cos^2 x} = \int \frac{d(\sin x)}{1 - \sin^2 x} = \frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x} + C.$$

- $$\int \frac{dx}{\cos^4 x} = \int (1 + \tan^2 x) \frac{dx}{\cos^2 x}$$

$$= \int (1 + t^2) dt = t + \frac{t^3}{3} + C = \tan x + \frac{\tan^3 x}{3} + C.$$

3.6.2.2. Integrals of the Form $\int \tan^n x\,dx$ and $\int \cot^n x\,dx$

As usual, the power n is assumed to be natural unless otherwise is stipulated.
Note that the given integrals are easily evaluated for $n=1$ and $n=2$. For instance,

$$\int \tan^2 x dx = \int \frac{1-\cos^2 x}{\cos^2 x} dx = \int \frac{dx}{\cos^2 x} - \int dx$$
$$= \tan x - \int dx = \tan x - x + C.$$

Hence, the problem of integration consists in lowering of the power n of tangents and cotangents, that can be easily carried out by using of trigonometric identities:

$$\int \tan^n x dx = \int \tan^{n-2} x\left(\frac{1}{\cos^2 x}-1\right)dx = \int \tan^{n-2} x \frac{dx}{\cos^2 x} - \int \tan^{n-2} x dx.$$

Taking into account that

$$\int \tan^{n-2} x \frac{dx}{\cos^2 x} = \int \tan^{n-2} x d(\tan x) = \frac{\tan^{n-1} x}{n-1} + C,$$

we obtain the following reduction formula:

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx. \qquad (25)$$

Therefore, the problem of integration of $\tan^n x$ is reduced to that of integration of $\tan^{n-2} x$. In this way one can lower any natural power n to 1 or zero.
Similarly, one can also get the reduction formula for the cotangent function:

$$\cot^n x = \cot^{n-2} x \cot^2 x = \cot^{n-2} x\left(\frac{1}{\sin^2 x}-1\right) \qquad \Rightarrow$$

$$\int \cot^n x dx = \int \cot^{n-2} x \frac{dx}{\sin^2 x} - \int \cot^{n-2} x dx \qquad \Rightarrow$$

$$\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx. \qquad (26)$$

In particular, $\int \cot^3 x dx = -\frac{\cot^2 x}{2} - \int \cot x dx = -\frac{\cot^2 x}{2} - \ln|\sin x| + C$.

3.6.3. Integration of Products of Sines and Cosines

Each of the following integrals

$$\int \sin ax \cos bx dx, \qquad \int \sin ax \sin bx dx \quad \text{and} \quad \int \cos ax \cos bx dx$$

can be easily evaluated with the help of trigonometric identities:

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta)), \tag{27}$$

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)), \tag{28}$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta)), \tag{29}$$

Examples:

- $\int \sin 2x \cos x dx = \frac{1}{2}\int (\sin 3x + \sin x) dx = -\frac{1}{6}\cos 3x - \frac{1}{2}\cos x + C$.
- $\int \sin 5x \sin 3x dx = \frac{1}{2}\int (\cos 2x - \cos 8x) dx = \frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C$.

Sometimes it is necessary to apply identities (27) – (29) more than once to obtain the final result.

Example: In order to evaluate $\int \sin 2x \cos 3x \cos 4x dx$ it is necessary to transform the product of trigonometric function into their linear combination.
By identity (29) we have

$$\cos 3x \cos 4x = \frac{1}{2}(\cos x + \cos 7x).$$

Then we use identity (27):

$$\begin{aligned}\sin 2x \cos 3x \cos 4x &= \frac{1}{2}\sin 2x(\cos x + \cos 7x)\\ &= \frac{1}{2}(\sin 2x \cos x + \sin 2x \cos 7x)\\ &= \frac{1}{4}(\sin x + \sin 3x + \sin(-5x) + \sin 9x).\end{aligned}$$

Hence,

$$\int \sin 2x \cos 3x \cos 4x dx = -\frac{1}{4}(\cos x + \frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x + \frac{1}{9}\cos 9x) + C.$$

3.6.4. Rational Expressions of Trigonometric Functions

3.6.4.1. General Substitution $\mathbf{t = \tan\frac{x}{2}}$

Let $P(x, y)$ and $Q(x, y)$ be polynomials with respect to variables x and y.

The quotient $R(x, y) = \dfrac{P(x, y)}{Q(x, y)}$ of two polynomials is a rational expression of x and y.

Likewise, the quotient

$$R(\sin x, \cos x) = \frac{P(\sin x, \cos x)}{Q(\sin x, \cos x)}$$

is called a rational expression of sine and cosine.

Note that all the other trigonometric functions are rational functions of sine and cosine.

Example 1: Such expressions as

$$\frac{2 - 3\sin x}{7 - 4\cos^2 x + 2\sin x}, \qquad \frac{1}{1 + \sqrt{3}\cos^5 x}, \qquad \frac{\cos x}{2 + 5\cos^3 x \sin x}$$

are rational ones of sine and cosine, but the expression $\dfrac{1}{1 + \sqrt{\cos x}}$ is not that.

Theorem: Let $R(\sin x, \cos x)$ be a rational expression of sine and cosine. Then there exists a rational function $f(t)$ such that

$$\int R(\sin x, \cos x)dx = \int f(t)dt\,.$$

Note: Any integral of a rational function can be evaluated. Therefore, the theorem states that any integral of a rational expression $R(\sin x, \cos x)$ can be transformed into the integral of a rational function and hence, can also be evaluated.

Proof: Let $t = \tan\dfrac{x}{2}$. Then $\sin x$ and $\cos x$ can be expressed through rational functions with respect to t by using of double-angle formulas:

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2})} = \frac{2\tan\frac{x}{2}}{1 + \tan^2(\frac{x}{2})} = \frac{2t}{1 + t^2}, \tag{30}$$

$$\cos x = \frac{\cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})}{\cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2})} = \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} = \frac{1-t^2}{1+t^2}. \qquad (31)$$

Moreover, from $x = 2\arctan t$ it follows that $dx = \frac{2dt}{1+t^2}$.

Therefore,

$$\int R(\sin x, \cos x)dx = \int R(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})\frac{2dt}{1+t^2} = \int f(t)dt,$$

where $f(t) = R(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})\frac{2}{1+t^2}$ is some rational function.

This completes the proof.

Example 2: By applying the substitution $t = \tan\frac{x}{2}$ we get

$$\int \frac{dx}{\sin x} = \int \frac{1}{\frac{2t}{1+t^2}} \frac{2dt}{(1+t^2)} = \int \frac{dt}{t} = \ln|t| + C = \ln|\tan\frac{x}{2}| + C.$$

Note that

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin(x+\frac{\pi}{2})} = \int \frac{d(x+\frac{\pi}{2})}{\sin(x+\frac{\pi}{2})} = \ln|\tan(\frac{x}{2}+\frac{\pi}{4})| + C.$$

Example 3: Find $\int \frac{dx}{2+\cos x - \sin x}$

Solution: Let $t = \tan\frac{x}{2}$. Using simple algebraic manipulations we obtain:

$$\int \frac{dx}{2+\cos x - \sin x} = \int \frac{1}{2+\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= 2\int \frac{dt}{2(1+t^2)+1-t^2-2t} = 2\int \frac{dt}{t^2-2t+3} = 2\int \frac{dt}{(t+1)(t-3)}.$$

The technique of partial fraction decomposition gives

$$2\int\frac{dt}{(t+1)(t-3)}=\frac{1}{2}\left(\int\frac{dt}{t-3}-\int\frac{dt}{t+1}\right)$$

$$=\frac{1}{2}(\ln|t-3|-\ln|t+1|)+C=\frac{1}{2}\ln\left|\frac{t-3}{t+1}\right|+C.$$

It remains to substitute $\tan\frac{x}{2}$ for t:

$$\int\frac{dx}{2+\cos x-\sin x}=\frac{1}{2}\ln\left|\frac{\tan x/2-3}{\tan x/2+1}\right|+C.$$

3.6.4.2. Other Substitutions

General substitution $t=\tan\frac{x}{2}$ enables us to evaluate integrals of the form $\int R(\sin x,\cos x)dx$ but very often in a complicated way.

However, there are a few specific cases when a rational expression $R(\sin x,\cos x)$ has even-odd symmetry. In these cases, integrals $\int R(\sin x,\cos x)dx$ can be transformed into integrals of rational functions by another trigonometric substitutions, which turn out often to be more preferable for integration of rational functions.

Let us consider these cases.

Case 1: If

$$R(-\sin x,\cos x)=-R(\sin x,\cos x),$$

then one can apply the substitution $t=\cos x$.

Case 2: If

$$R(\sin x,-\cos x)=-R(\sin x,\cos x),$$

then the suitable substitution is $t=\sin x$.

Case 3: If

$$R(-\sin x,-\cos x)=R(\sin x,\cos x),$$

then both substitutions, $t=\tan x$ and $t=\cot x$, are suitable.

As an example, let us give reasoning for Case 1.

Proof: The expression $R(\sin x,\cos x)$ is an odd rational function with respect to $\sin x$. Hence, we have

$$R(\sin x,\cos x)=\sin x\frac{R(\sin x,\cos x)}{\sin x}=\sin x\cdot R_1(\sin x,\cos x).$$

Here $R_1(\sin x, \cos x)$ is some even rational function with respect to $\sin x$ containing only even powers of sine.
Hence,

$$R_1(\sin x, \cos x) = R_2(\sin^2 x, \cos x) = R_2(1-\cos^2 x, \cos x).$$

However, the last rational expression is some rational function f with respect to $\cos x$:

$$R_2(1-\cos^2 x, \cos x) = f(\cos x).$$

Therefore, by making use of the substitution $\cos x = t$, we obtain

$$\int R(\sin x, \cos x)dx = \int f(\cos x)\sin x dx = \int f(t)dt$$

Hence, the desired result.
Other cases can be treated similarly.

Example 1: Find $\int \frac{\sin^3 x}{4-\cos^2 x}dx$.

Solution: This is Case 1, that is $R(-\sin x, \cos x) = -R(\sin x, \cos x)$.
Indeed,

$$\frac{(-\sin x)^3}{4-\cos^2 x} = -\frac{\sin^3 x}{4-\cos^2 x}.$$

Then for the substitution $t = \cos x$ we have $dt = -\sin x dx$ and

$$\sin^3 x dx = \sin^2 x \sin x dx = (1-\cos^2 x)\sin x dx = -(1-t^2)dt.$$

Therefore,

$$\int \frac{\sin^3 x}{4-\cos^2 x}dx = -\int \frac{1-t^2}{4-t^2}dt = \int (-1+\frac{3}{4-t^2})dt = -t + 3\int \frac{dt}{4-t^2}$$

$$= -t + \frac{3}{4}\ln\left|\frac{t+2}{t-2}\right| + C = -\cos x + \frac{3}{4}\ln\left|\frac{\cos x+2}{\cos x-2}\right| + C.$$

Example 2: Find $\int \frac{\sin x \cos x}{3\sin x + 1}dx$.

Solution: Here we have Case 2 due to the identity

$$\frac{\sin x(-\cos x)}{3\sin x+1} = -\frac{\sin x \cos x}{3\sin x + 1}.$$

So we make the substitution

$$t = \sin x,$$

which gives $\cos x dx = dt$.

Therefore,

$$\int \frac{\sin x \cos x}{3\sin x + 1} dx = \int \frac{t}{3t+1} dt = \frac{1}{3}\int (1 - \frac{1}{3t+1}) dt$$
$$= \frac{1}{3}(t - \frac{1}{3}\ln|3t+1|) + C$$
$$= \frac{1}{3}(\sin x - \frac{1}{3}\ln|3\sin x + 1|) + C.$$

Example 3: Find $\int \frac{dx}{2\sin x \cos x - 4\sin^2 x + 5}$.

Solution: This is Case 3 since:

$$\frac{1}{2(-\sin x)(-\cos x) - 4(-\sin x)^2 + 5} = \frac{1}{2\sin x \cos x - 4\sin^2 x + 5},$$

that is, $R(-\sin x, -\cos x) = R(\sin x, \cos x)$.

First, we transform the integrand:

$$\frac{1}{2\sin x \cos x - 4\sin^2 x + 5} = \frac{1}{2\sin x \cos x - 4\sin^2 x + 5(\sin^2 x + \cos^2 x)}$$
$$= \frac{1}{\sin^2 x + 2\sin x \cos x + 5\cos^2 x} = \frac{1}{\cos^2 x(\tan^2 x + 2\tan x + 5)}.$$

Therefore, the rational expression of sine and cosine has been transformed into the rational function of t by the above formula. So we have reduced the initial problem to integration of the rational function:

$$\int \frac{dx}{2\sin x \cos x - 4\sin^2 x + 5} = \int \frac{dt}{t^2 + 2t + 5}$$
$$= \int \frac{d(t+1)}{(t+1)^2 + 4} = \frac{1}{2}\arctan\frac{t+1}{2} + C = \frac{1}{2}\arctan\frac{\tan x + 1}{2} + C$$

For convenience sake, let us summarize the main results. The table gives substitutions and basic formulas for all the cases.

Table 3

Properties	Substitutions	Basic Formulas
$R(-\sin x, \cos x) = -R(\sin x, \cos x)$	$\cos x = t$	$-\sin x dx = dt$ $\sin^2 x = 1 - t^2$
$R(\sin x, -\cos x) = -R(\sin x, \cos x)$	$\sin x = t$	$\cos x dx = dt$ $\cos^2 x = 1 - t^2$
$R(-\sin x, -\cos x) = R(\sin x, \cos x)$	$\tan x = t$	$\frac{dx}{\cos^2 x} = dt$ $\sin^2 x = \frac{t^2}{1+t^2}$ $\cos^2 x = \frac{1}{1+t^2}$
	$\cot x = t$	$\frac{dx}{\sin^2 x} = -dt$ $\sin^2 x = \frac{1}{1+t^2}$ $\cos^2 x = \frac{t^2}{1+t^2}$
Any rational expression $R(\sin x, \cos x)$	$\tan \frac{x}{2} = t$	$dx = \frac{2dt}{1+t^2}$ $\sin x = \frac{2t}{1+t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$

3.7. Integrals Involving Rational Exponents

1. Integrals with rational exponents $x^{\frac{1}{n}}$ can be transformed to integrals of rational functions by making use the substitution $x = u^n$, which implies $\sqrt[n]{x} = u$ and $dx = nu^{n-1}du$.

   Example 1: Let $\int \frac{dx}{\sqrt{x}+3}$ be a given integral.

   The substitution $x = u^2$ yields $\sqrt{x} = u$ and $dx = 2udu$, so that
$$\int \frac{dx}{\sqrt{x}+3} = 2\int \frac{udu}{u+3} = 2\int \frac{(u+3-3)du}{u+3}$$
$$= 2\int du - 6\int \frac{du}{u+3} = 2u - 6\ln|u+3| + C$$
$$= 2\sqrt{x} - 6\ln|\sqrt{x}+3| + C.$$

2. Integrals with a few rational exponents can be evaluated by the substitution $x = u^n$, where n is the least common multiple of the denominators of the exponents.

   Example 2: Consider the integral $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

   The substitution $x = u^6$ allows us to get rid of both square and cube radical signs without getting new fractional exponents. Then $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$ and $dx = 6u^5du$, so that
$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = 6\int \frac{u^5du}{u^3+u^2} = 6\int \frac{u^3du}{u+1}.$$
   This integral of the rational function can be easily evaluated by employing a polynomial long division:
$$\int \frac{u^3du}{u+1} = \int (u+1)^2du - 3\int (u+1)du + 3\int du - \int \frac{du}{u+1}$$
$$= \frac{(u+1)^3}{3} - \frac{3(u+1)^2}{2} + 3u - \ln|u+1| + C$$
$$= \frac{(\sqrt[6]{x}+1)^3}{3} - \frac{3(\sqrt[6]{x}+1)^2}{2} + 3\sqrt[6]{x} - \ln|\sqrt[6]{x}+1| + C.$$

Therefore,

$$\int \frac{dx}{\sqrt[3]{u}+\sqrt{x}} = \frac{(\sqrt[6]{x}+1)^3}{3} - \frac{3(\sqrt[6]{x}+1)^2}{2} + 3\sqrt[6]{x} - \ln|\sqrt[6]{x}+1| + C.$$

3. Integrals involving expressions of the form $\sqrt[n]{\dfrac{ax+b}{cx+d}}$ can be evaluated by the substitution $\dfrac{ax+b}{cx+d} = u^n$, which eliminates the radical sign and yields x as a rational function of u: $x = \dfrac{u^n d - b}{a - u^n c}$.

3.8. Integrals Involving Radicals $\sqrt{a^2 \pm x^2}$ or $\sqrt{x^2 - a^2}$

Consider integrals that involve the following radicals:

$$\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2} \quad \text{or} \quad \sqrt{x^2 - a^2}.$$

In order to eliminate the radical sign, one needs to use appropriate substitutions, e.g., trigonometric substitutions.

Problem 1: Eliminate the radical sing for $\sqrt{a^2 - x^2}$.

Solution: The trigonometric identity $1 - \sin^2 x = \cos^2 x$ suggests the substitution $x = a\sin u$. Indeed,

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 u}$$

$$= \sqrt{a^2(1 - \sin^2 u)} = \sqrt{a^2 \cos^2 u} = a\cos u.$$

Note: The same idea works for the cosine-substitution: $x = a\cos u$. In this case $\sqrt{a^2 - x^2} = a\sin u$.

Problem 2: Eliminate the radical sign for $\sqrt{a^2 + x^2}$.

Solution: The trigonometric identity

$$1 + \tan^2 u = \frac{1}{\cos^2 u},$$

hints at the substitution $x = a\tan u$. Then

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2 u} = \sqrt{a^2(1 + \tan^2 u)} = \sqrt{\frac{a^2}{\cos^2 u}} = \frac{a}{\cos u}.$$

Note: One can also use the substitution $x=\cot u$, which gives $\sqrt{a^2+x^2}=\dfrac{a}{\sin u}$.

Problem 3: Eliminate the radical sign for $\sqrt{x^2-a^2}$.
Solution: Since the difference

$$\frac{1}{\sin^2 u}-1=\frac{1-\sin^2 u}{\sin^2 u}=\frac{\cos^2 u}{\sin^2 u}=\cot^2 u$$

is the perfect square, the substitution $x=\dfrac{a}{\sin u}$ is suitable for eliminating of the radical:

$$\sqrt{x^2-a^2}=\sqrt{\frac{a^2}{\sin^2 u}-a^2}=\sqrt{a^2(\frac{1}{\sin^2 u}-1)}=\sqrt{a^2\cot^2 u}=a\cot u.$$

Note: The identity $\dfrac{1}{\cos^2 u}-1=\tan u$ suggests the substitution $x=\dfrac{a}{\cos u}$, which is also suitable for eliminating of the radical $\sqrt{x^2-a^2}$.
In this case $\sqrt{x^2-a^2}=a\tan u$.

The following examples illustrate applications of the above trigonometric substitutions for elimination of radical signs.

Example 1: Find $\displaystyle\int\frac{\sqrt{3-x^2}}{x^2}dx$.
Solution: Let $x=\sqrt{3}\sin u$. Then

$$\int\frac{\sqrt{3-x^2}}{x^2}dx=\int\frac{\sqrt{3}\cos u}{3\sin^2 u}\sqrt{3}\cos u du$$

$$=\int\cot^2 u du=\int(\frac{1}{\sin^2 u}-1)du=-\cot u-u+C$$

The solution is found in terms of u, and we have to express it in terms of x:

$$x=\sqrt{3}\sin u \qquad\Rightarrow\qquad u=\arcsin\frac{x}{\sqrt{3}},$$

$$\cot u = \frac{\cos u}{\sin u} = \frac{\sqrt{1-\sin^2 u}}{\sin u} = \frac{\sqrt{1-\sin^2 \arcsin\frac{x}{\sqrt{3}}}}{\sin \arcsin\frac{x}{\sqrt{3}}} = \frac{\sqrt{1-\frac{x^2}{3}}}{\frac{x}{\sqrt{3}}} = \frac{\sqrt{3-x^2}}{x}.$$

Therefore, the final solution is

$$\int \frac{\sqrt{3-x^2}}{x^2}dx = -\frac{\sqrt{3-x^2}}{x} - \arcsin\frac{x}{\sqrt{3}} + C.$$

Example 2: Find $\int \frac{\sqrt{9+x^2}}{x^4}dx$.

Solution: Let $x = 3\tan u$. Then $dx = \frac{3du}{\cos^2 u}$ and

$$\sqrt{9+x^2} = \sqrt{9+9\tan^2 u} = \sqrt{9(1+\tan^2 u)} = \sqrt{\frac{9}{\cos^2 u}} = \frac{3}{\cos u}.$$

Therefore,

$$\int \frac{\sqrt{9+x^2}}{x^4}dx = \int \frac{9}{81\tan^4 u \cos^3 u}du$$

$$= \frac{1}{9}\int \frac{\cos u du}{\sin^4 u} = \frac{1}{9}\int \frac{dt}{t^4} \qquad | \text{substitution} \quad t = \sin u \,|$$

$$= -\frac{1}{27t^3} + C = -\frac{1}{27\sin^3 u} + C.$$

It remains to express the answer in terms of x:

$$x = 3\tan u \qquad \Rightarrow \qquad u = \arctan\frac{x}{3},$$

$$\sin u = \frac{\sin u}{\sqrt{\cos^2 u + \sin^2 u}} = \frac{\tan u}{\sqrt{1+\tan^2 u}}$$

$$= \frac{\tan(\arctan\frac{x}{3})}{\sqrt{1+\tan^2(\arctan\frac{x}{3})}} = \frac{\frac{x}{3}}{\sqrt{1+(\frac{x}{3})^2}} = \frac{x}{\sqrt{9+x^2}}.$$

Finally we have

$$\int \frac{\sqrt{9+x^2}}{x^4}dx = -\frac{(9+x^2)\sqrt{9+x^2}}{27x^3} + C.$$

Example 3: Find $\int \frac{dx}{x^2\sqrt{x^2-5}}$

Solution: Let $x = \frac{\sqrt{5}}{\sin u}$. Then $dx = -\frac{\sqrt{5}\cos u}{\sin^2 u}du$,

$$\sqrt{x^2-5} = \sqrt{\frac{5}{\sin^2 u} - 5} = \sqrt{\frac{5(1-\sin^2 u)}{\sin^2 u}} = \sqrt{5}\cot u$$

Therefore,

$$\int \frac{dx}{x^2\sqrt{x^2-5}} = -\int \frac{\sqrt{5}\cos u \sin^2 u du}{5\sqrt{5}\cot u \sin^2 u} = -\frac{1}{5}\int \sin u du = \frac{\cos u}{5} + C.$$

Now we have to return to the initial variable x:

$$x = \frac{\sqrt{5}}{\sin u} \Rightarrow \qquad \sin u = \frac{\sqrt{5}}{x} \qquad \Rightarrow$$

$$\cos u = \sqrt{1-\sin^2 u} = \sqrt{1-(\frac{\sqrt{5}}{x})^2} = \sqrt{1-\frac{5}{x^2}} = \frac{\sqrt{x^2-5}}{x}.$$

Therefore,

$$\int \frac{dx}{x^2\sqrt{x^2-5}} = \frac{\sqrt{x^2-5}}{5x} + C.$$

Problem 4: Eliminate the radical sign for $\sqrt{\pm x^2 + px + q}$.

Solution: In order to evaluate an integral of expression involving the radical of this type, one has to complete the square of the quadratic trinomial. Then the previous methods can be used to solve the integrals.

Example 4: Consider the radical $\sqrt{x^2 - 6x + 10}$.

Let us transform the quadratic polynomial under the radical sign to get a perfect square:

$$x^2 - 6x + 25 = (x^2 - 6x + 9) + 16 = (x-3)^2 + 4^2.$$

Then we can use the tangent-substitution $x - 3 = 4\tan u$ to solve the problem.

3.9. Integrals of the Form $\int x^{m}(a+bx^{n})^{p}\,dx$

Chebyshev proved the following **theorem**:
Let m, n and p be rational numbers.
Then the following integral

$$\int x^{m}(a+bx^{n})^{p}\,dx$$

is evaluated in terms of elementary functions if and only if there is an integer among the numbers p, $\frac{m+1}{n}$ and $\frac{m+1}{n}+p$.

Proof: Consider three cases.

1) Let the number p be an integer, and let s be the least common multiple of the denominators of the exponents m and n. Then by substitution $x=t^{s}$, the given integral can be transformed to the integral of a rational function. Therefore, it can be evaluated in terms of elementary functions.

2) Let the number $\frac{m+1}{n}$ be an integer. By making the substitution $x^{n}=z$, that is, $x=z^{1/n}$ we get

$$\int x^{m}(a+bx^{n})^{p}\,dx=\int z^{\frac{m}{n}}(a+bz)^{p}\,\frac{1}{n}z^{\frac{1}{n}-1}dz=\frac{1}{n}\int z^{\frac{m+1}{n}-1}(a+bz)^{p}\,dz\,.$$

If s is the denominator of the rational number p, then by substitution $a+bz=t^{s}$ we obtain the integral of a rational function.

3) Let the number $\frac{m+1}{n}+p$ be an integer. The last integral can be written as:

$$\int z^{\frac{m+1}{n}-1}(a+bz)^{p}\,dz=\int z^{\frac{m+1}{n}+p-1}\left(\frac{a+bz}{z}\right)^{p}dz\,.$$

Therefore, by substitution $\frac{a+bz}{z}=t^{s}$ it is transformed to the integral of a rational function (s is the denominator of the rational number p).

Thus, all the cases are investigated.

In conclusion, we give the table of substitutions for all these cases.

Table 4

Integer	Substitutions
p	$x = u^s$ s is the least common multiple of the denominators of the rational exponents m and n
$\frac{m+1}{n}$	$a + bx^n = t^s$ s is the denominator of the rational exponent p
$\frac{m+1}{n} + p$	$\frac{a}{x^n} + b = t^s$ s is the denominator of the rational exponent p

Example: Consider $\int \sqrt[4]{1+x^2}\,dx$.

Here all numbers, $p = 1/4$, $\frac{m+1}{n} = \frac{0+1}{2} = \frac{1}{2}$ and $\frac{m+1}{n} + p = \frac{3}{4}$, are not integer. Hence, the given integral cannot be expressed through a finite number of elementary functions.

3.10. Some Irreducible Integrals

Integrals of rational functions are evaluated straightforward, and the answer is expressed in terms of rational functions, logarithms, and inverse trigonometric functions.

But it is still possible to find even fairly simple looking integrals that just cannot be done in terms of elementary functions such as exponentials, logarithms, trigonometric functions and so on.

Liouville showed that the integrals given below cannot be expressed in terms of a finite number of elementary functions:

$$\int e^{-x^2}dx, \quad \int \frac{e^x}{x}dx, \quad \int \frac{\sin x}{x}dx, \quad \int \frac{\cos x}{x}dx, \quad \int \frac{dx}{\ln x}.$$

Each of the following integrals is also irreducible:

$$\int x^x dx, \quad \int \frac{\arctan x}{x}dx, \quad \int \frac{\ln x}{x+1}dx.$$

3.11. Extended List of Common Indefinite Integrals

The table below gives the list of the most important indefinite integrals.

Table 5

$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \frac{dx}{x-a} = \ln\lvert x-a\rvert + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$	
$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$	
$\int \frac{dx}{\cos^2 x} = \tan x + C$	$\int \frac{dx}{\sin^2 x} = -\cot x + C$
$\int \frac{dx}{\sqrt{a^2-x^2}} = \begin{cases} \arcsin\frac{x}{a} + C \\ -\arccos\frac{x}{a} + C \end{cases}$	$\int \frac{dx}{a^2+x^2} = \begin{cases} \frac{1}{a}\arctan\frac{x}{a} + C \\ -\frac{1}{a}\cot^{-1}\frac{x}{a} + C \end{cases}$
$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2}) + C$	$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a} + C$
$\int \tan x dx = -\ln\lvert\cos x\rvert + C$	$\int \cot x dx = \ln\lvert\sin x\rvert + C$
$\int \frac{dx}{\sin x} = \ln\lvert\tan\frac{x}{2}\rvert + C$	$\int \frac{dx}{\cos x} = \ln\lvert\tan(\frac{x}{2} + \frac{\pi}{4})\rvert + C$
$\int e^{ax}\sin bx dx = \frac{a\sin bx - b\cos bx}{a^2+b^2}e^{ax} + C$	
$\int e^{ax}\cos bx dx = \frac{a\cos bx + b\sin bx}{a^2+b^2}e^{ax} + C$	
$\int \frac{dx}{(x^2+a^2)^{n+1}} = \frac{1}{2a^2}(\int \frac{dx}{(x^2+a^2)^n} + \frac{x}{(x^2+a^2)^n})$	
$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$	

Chapter 4
DEFINITE INTEGRALS
4.1. The Geometric Definition of Definite Integrals

The mathematical concept of definite integrals can be understood better by considering the following problem.

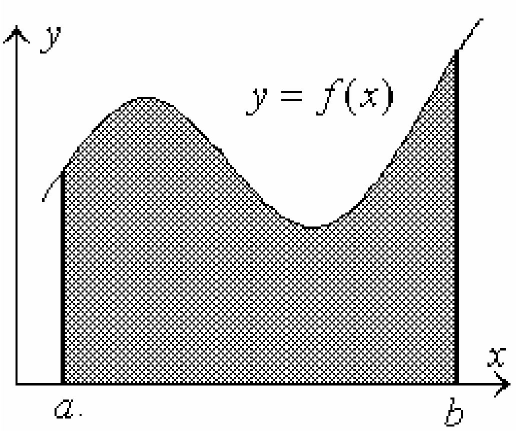


Fig. 1

Problem: Let a function $y = f(x)$ be positive defined on a closed interval $[a,b]$. Find the area of the region under the curve $y = f(x)$ bounded by the x-axis and the lines $x = a$ and $x = b$. (See Fig.1.)

Solution: The main idea is very simple: parts form a whole.

- First, we partition the interval $[a,b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2], \ldots,$ $[x_{n-1}, x_n]$ by arbitrary points $x_1, x_2, ..., x_{n-1}$ of the partition, as it is shown in Fig. 2.

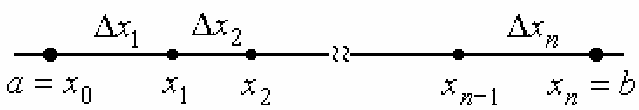


Fig. 2

- Next, we draw vertical lines at the partition points to approximate the region by n rectangles. The area of each rectangle equals the product of its base and height, and it can be easily found.

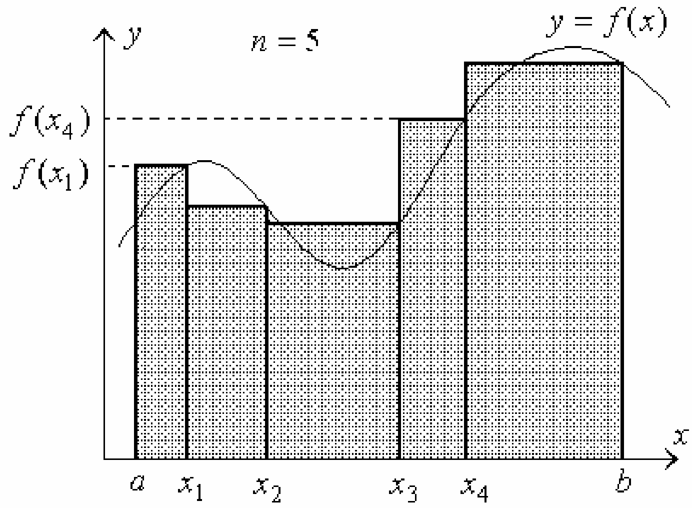


Fig. 3

The base of each rectangle is the difference between one value of x and the previous value of x:

$\Delta x_1 = x_1 - x_0$,

$\Delta x_2 = x_2 - x_1, \ldots,$

$\Delta x_n = x_n - x_{n-1}$.

The heights of the rectangles are equal to $y_k = f(x_k)$, where index k varies from 1 to n.

- Then, we sum up the areas of all rectangles to find approximately the total area A of the region bounded by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

$$A \approx \sum_{k=1}^{n} f(x_k)\Delta x_k \,. \qquad (1)$$

The above sum is known as the Riemann Sum.

- By comparing Fig. 3 and Fig. 4 one can easily see that approximation (1) is getting better when the number of approximating rectangles increases.

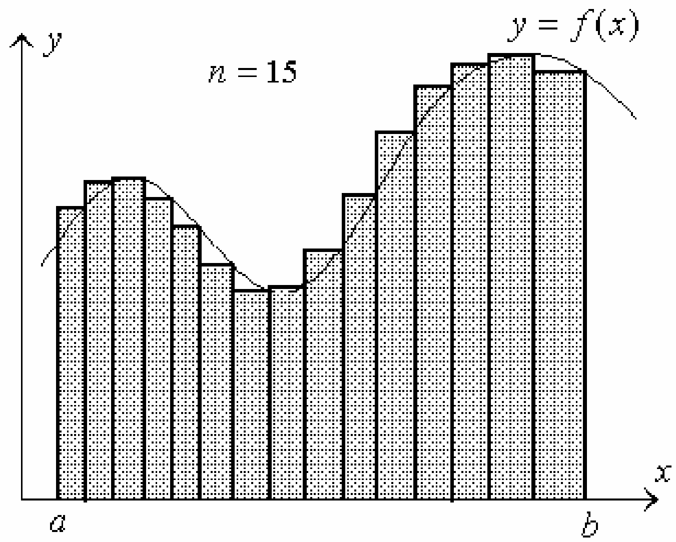


Fig. 4

If the number of the rectangles tends to infinity, so that all the bases of the rectangles tend to zero, then sum (1) gives the area under the curve **exactly**.

Note that the last condition can be written for a short as "maximum $\Delta x \to 0$" because in this case all bases $\Delta x_k \to 0$ $(k = 1, 2, ..., n)$ and the number of the rectangles $n \to \infty$.
Therefore,

$$A = \lim_{\max \Delta x \to o} \sum_{k=1}^{n} f(x_k)\Delta x_k \,. \qquad (2)$$

If this limit exists, no matter how the partition points x_k are chosen, then it is called a **definite integral** of $f(x)$ over the interval $[a, b]$.
A definite integral is denoted as an indefinite integral but with upper and lower limits:

$$\int_a^b f(x)dx = \lim_{\max \Delta x \to o} \sum_{k=1}^{n} f(x_k)\Delta x_k \,. \qquad (3)$$

The numbers a and b are said to be **lower** and **upper limits** correspondingly.

4.2. The Algebraic Definition of Definite Integrals

Let $f(x)$ be a function defined on a closed interval $[a,b]$. Consider a partition of the interval $[a,b]$ taking points $x_1, x_2, ..., x_{n-1}$ such that

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b.$$

The sum of the products $f(x_k)\Delta x_k$ is called the Riemann Sum, where Δx_k denotes the difference between two successive partition points, that is, $\Delta x_k = x_k - x_{k-1}, \quad k \in N$.

Let $n \to \infty$ and all $\Delta x_k \to 0$. If the limit of the Riemann Sums exists and does not depend on a choice of the points x_k of the partition, then it is called a definite integral of the function $f(x)$ over the interval $[a,b]$:

$$\int_a^b f(x)dx = \lim_{\max \Delta x \to o} \sum_{k=1}^{n} f(x_k)\Delta x_k . \qquad (4)$$

The process of computing an integral is called **integration** and the approximate computation of an integral is called **numerical integration**.

4.3. Properties of Definite Integrals

The following properties are based on the definition of definite integrals.

1. The variable of integration is a dummy variable, that is, an integral is independent of the choice of a symbol denoting the variable of integration:
$$\int_a^b f(x)dx = \int_a^b f(t)dt .$$
2. For any constant *c* and any function $f(x)$ we have:
$$\int_a^b cf(x)dx = c\int_a^b f(x)dx .$$
3. The integral of a sum of integrable functions over the interval $[a,b]$ is equal to the sum of the integrals of the addends over $[a,b]$:
$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx .$$
4. By definition $\int_a^a f(x)dx = 0$.
5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

6. $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx, \qquad (a < b).$

7. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

   This formula is quite evident if $c \in [a,b]$ (see Fig. 5), but it holds true when $c \notin [a,b]$ provided that all the above integrals exist.

8. $\int_a^b f(x)dx = f(\bar{x})\,(b-a), \quad (a < \bar{x} < b).$ (See Fig. 6.)

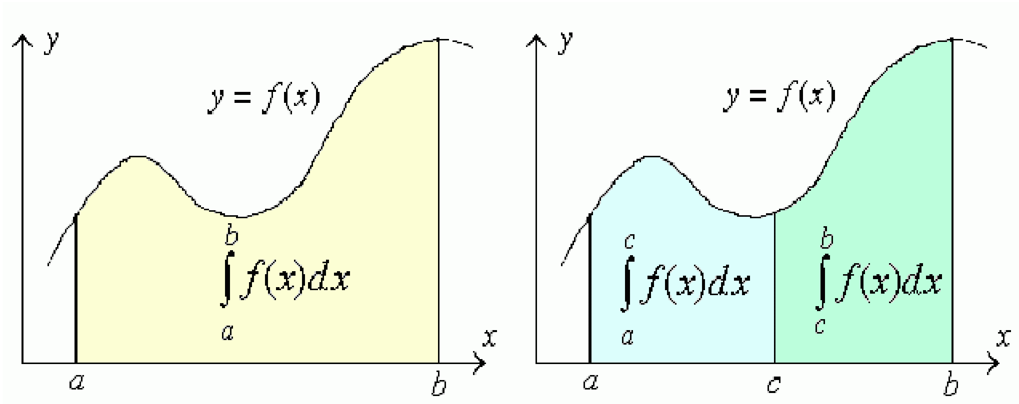


Fig. 5

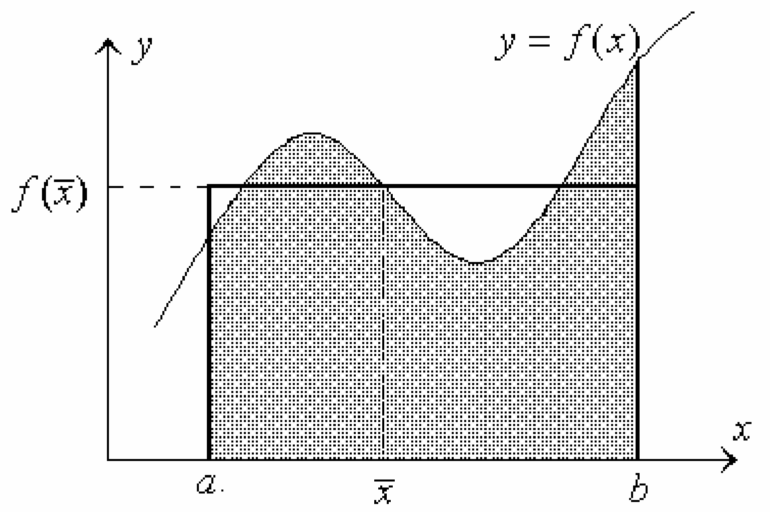


Fig. 6

4.4. The Fundamental Theorems of Calculus

1. If the function $f(x)$ is continuous on (a,b), then the function $\int\limits_a^x f(t)dt$ is a primitive of $f(x)$ for any $x \in (a,b)$:

$$\frac{d}{dx}\int\limits_a^x f(t)dt = f(x). \qquad (5)$$

2. If the function $f(x)$ is continuous on a closed interval $[a,b]$ and $F(x)$ is a primitive of $f(x)$ on the interval $[a,b]$, then

$$\int\limits_a^b f(t)dt = F(x)\Big|_a^b = F(b) - F(a). \qquad (6)$$

Proof: Let us recall the definition of the derivative:

$$\frac{d\varphi(x)}{dx} = \lim_{\Delta x\to 0}\frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x}.$$

Therefore, by Property 7,

$$\frac{d}{dx}\int\limits_a^x f(t)dt = \lim_{\Delta x\to 0}\frac{\int\limits_a^{x+\Delta x} f(t)dt - \int\limits_a^x f(t)dt}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\int\limits_a^x f(t)dt + \int\limits_x^{x+\Delta x} f(t)dt - \int\limits_a^x f(t)dt}{\Delta x} = \lim_{\Delta x\to 0}\frac{\int\limits_x^{x+\Delta x} f(t)dt}{\Delta x}.$$

Applying Property 8 to the interval $[x, x+\Delta x]$ we find that

$$\int\limits_x^{x+\Delta x} f(t)dt = f(\bar{x})\Delta x,$$

where $\bar{x} \in (x, x+\Delta x)$ and $\bar{x} \to x$ as $\Delta x \to 0$.

By combining these results, we get

$$\frac{d}{dx}\int\limits_a^x f(t)dt = \lim_{\Delta x\to 0}\frac{f(\bar{x})\Delta x}{\Delta x} = f(x).$$

Therefore, the function

$$F(x)=\int_a^x f(t)dt+C \qquad (7)$$

is a primitive of $f(x)$.
This is the first fundamental theorem of calculus.
Setting $x=a$, we find the constant C:

$$F(a)=\int_a^a f(t)dt+C \qquad \Rightarrow \qquad C=F(a).$$

Hence,

$$\int_a^x f(t)dt=F(x)-F(a).$$

Setting $x=b$, we get the second fundamental theorem of calculus:

$$\int_a^b f(t)dt=F(b)-F(a).$$

Therefore, both fundamental theorems of calculus are proved.
The Fundamental Theorems of Calculus bind a definite integral of $f(x)$ over the interval $[a,b]$ with an indefinite integral of $f(x)$. All we need only is to evaluate $F(x)$ at b and to subtract $F(x)$ evaluated at a from it.

Examples: Evaluate each of the following integrals:

1) $\int_0^{\pi/12}\cos 2x dx$, 2) $\int_2^5 (3x^2-\frac{7}{x})dx$.

Solution:

1) $$\int_0^{\pi/12}\cos 2x dx=\frac{1}{2}\sin 2x\Big|_0^{\pi/12}=\frac{1}{2}(\sin\frac{\pi}{6}-\sin 0)=\frac{1}{4}.$$

2) $$\int_2^5 (3x^2-\frac{7}{x})dx=3\int_2^5 x^2dx-7\int_2^5\frac{dx}{x}=(x^3-7\ln x)\Big|_2^5$$

$$=(5^3-7\ln 5)-(2^3-7\ln 2)=117-7\ln\frac{5}{2}.$$

4.5. Techniques of Integration

This section contains a review of the major techniques of integration including substitution method and integration by parts.

4.5.1. Substitution Method

Theorem: Let $f(x)$ be a continuous function on the interval $[a, b]$. Assume that a function $x = \varphi(t)$ has a continuous derivative on the interval $[\alpha, \beta]$.

If $\varphi(\alpha) = a$ and $\varphi(\beta) = b$, then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt \qquad (8)$$

Proof: Let $F(x)$ be a primitive of $f(x)$ on the interval $[a, b]$.

Applying the fundamental theorem of calculus and the properties of primitives we have

$$\int_a^b f(x)dx = F(b) - F(a) = F(\varphi(\beta)) - F(\varphi(\alpha)) = \int_\alpha^\beta dF(\varphi(t))$$

$$= \int_\alpha^\beta F'(\varphi(t))d\varphi(t) = \int_\alpha^\beta f(\varphi(t))d\varphi(t) = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

Formula (8) allows us to change the variables of integration in definite integrals just as in the case of indefinite integrals, but in addition we have to replace the limits of integration.

Note that it is not necessary to return to the initial variable x.

Example 1: Evaluate $\int_1^e \frac{\ln x}{x}dx$.

Solution: Let $t = \ln x$. Then the equalities $\begin{cases} x = 1 \\ x = e \end{cases}$ imply $\begin{cases} t = \ln 1 = 0 \\ t = \ln e = 1. \end{cases}$

Therefore, the interval of integration from 1 to *e* is replaced by the interval $[0, 1]$:

$$\int_1^e \frac{\ln x}{x}dx = \int_0^1 tdt = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}.$$

Example 2: Evaluate $\int_2^3 x^2 e^{x^3} dx$.

Solution: By applying the substitution $t = x^3$, we have $dt = 3x^2 dx$.
Then we find the lower and upper limits of integration:

$$\begin{cases} x = 2 \\ x = 3 \end{cases} \Rightarrow \begin{cases} t = 2^3 = 8 \\ t = 3^3 = 27. \end{cases}$$

Therefore,

$$\int_2^3 x^2 e^{x^3} dx = \frac{1}{3}\int_8^{27} e^t dt = \frac{1}{3} e^t \Big|_8^{27} = \frac{1}{3}(e^{27} - e^8) = \frac{1}{3} e^8 (e^{19} - 1).$$

4.5.2. Integration by Parts

The formula for integration by parts for definite integrals states that

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du \qquad (9)$$

for any differentiable functions $u(x)$ and $v(x)$.

The following example refers to the case when we need to use the method of integration by parts and the substitution technique.

Example: Evaluate $\int_0^1 \arcsin x dx$.

Solution: Let $u = \arcsin x$ and $dv = dx$. Then $du = \dfrac{dx}{\sqrt{1-x^2}}$ and $v = x$.

Therefore,

$$\int_{1/2}^{1} \arcsin x dx = x \arcsin x \Big|_{1/2}^{1} - \int_{1/2}^{1} \frac{x dx}{\sqrt{1-x^2}}.$$

In view of the fact that $\arcsin 1 = \pi/2$ and $\arcsin(1/2) = \pi/6$, we obtain

$$x \arcsin x \Big|_{1/2}^{1} = \arcsin 1 - \frac{1}{2}\arcsin\frac{1}{2} = \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{6}.$$

The integral on the right-hand side can be evaluated by substitution of the variable. One natural substitution is the following. We introduce a new variable t in order to eliminate the radical sign of the integrand.

Let $t^2 = 1 - x^2$.
Then $\sqrt{1-x^2} = t$ and $tdt = -xdx$.
The new limits of integration are as follows:
The lower limit equals $\sqrt{1-(1/2)^2} = \sqrt{3/4} = \sqrt{3}/2$.
The upper limit equals $\sqrt{1-1^2} = 0$.

Thus, we get $\int\limits_{1/2}^{1} \frac{x}{\sqrt{1-x^2}} dx = -\int\limits_{\sqrt{3}/2}^{0} \frac{tdt}{t} = \int\limits_{0}^{\sqrt{3}/2} dt = t\Big|_{0}^{\sqrt{3}/2} = \frac{\sqrt{3}}{2}$.

By combining these results, we finally obtain

$$\int\limits_{1/2}^{1} \arcsin x dx = \frac{5\pi}{6} - \frac{\sqrt{3}}{2}.$$

4.6. Geometric Applications of Definite Integrals

4.6.1. The Area of a Region

One of the problems of such a kind has been considered in section 4.1.

Let us recall the main idea: The given region is represented by an infinite number of rectangles, whose altitudes depend on x-coordinate, and the definite integral of the altitude gives the area of this region.

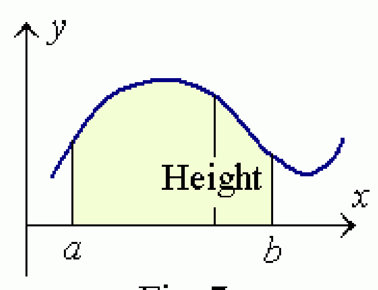


Fig. 7

Problem 1: Given a functions $y = f(x)$ defined over a closed interval $[a,b]$, find the area A of the region bounded by the graph of this function, the x-axis and the vertical lines $x = a$ and $x = b$.
Solution: The altitude of the rectangle with base in the vicinity of the point x is equal to the absolute value of $f(x)$ - it does not matter whether the curve $y = f(x)$ lies above or below the x-axis.
Therefore, we have the following formula for the area of the given region:

$$A = \int\limits_{a}^{b} |f(x)| dx.$$

Note: If the graph lies below the x-axis, then $f(x)<0$ and $\int_a^b f(x)dx = -A$.

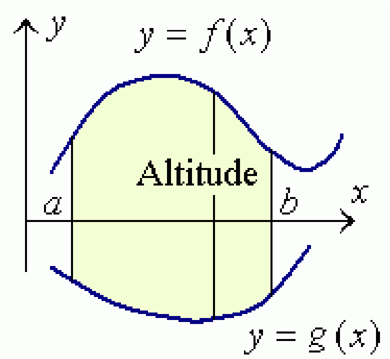


Fig. 8

Problem 2: Given two functions, $y=f(x)$ and $y=g(x)$, defined over a closed interval $[a,b]$, find the area A of the region bounded by their graphs and the vertical lines $x=a$ and $x=b$.

Solution: This region can be represented by an infinite number of rectangles whose altitudes are equal to the absolute value of the difference between $f(x)$ and $g(x)$. Therefore,

$$A=\int_a^b \left| f(x)-g(x) \right| dx. \qquad (10)$$

Problem 3: Let a function be specified in the polar system of coordinates as $r=r(\varphi)$; find the area A of the region bounded by the graph $r=r(\varphi)$ and the rays $\varphi=\alpha$ and $\varphi=\beta$.

Solution: This region can be represented by an infinite number of sectors. (See Fig. 9).

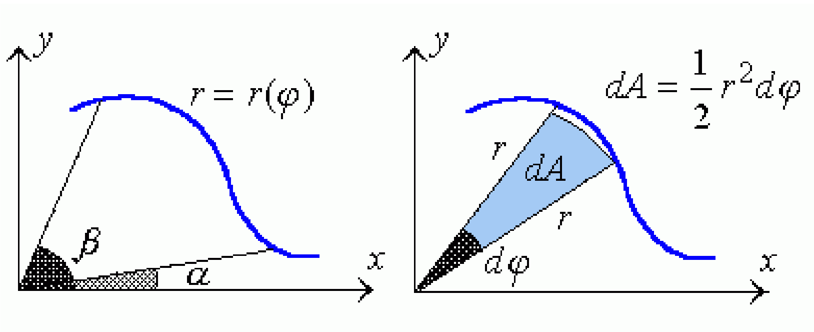


Fig. 9

The area of an arbitrary sector is $dA=\frac{1}{2}r^2 d\varphi$.

Therefore, the area of the whole region is

$$A=\frac{1}{2}\int_\alpha^\beta r^2 d\varphi. \qquad (11)$$

Example: Find the area of the region bounded by the graphs of the functions $y = 3x$ and $y = x^2$.

Solution: First, let us make a sketch of this region.

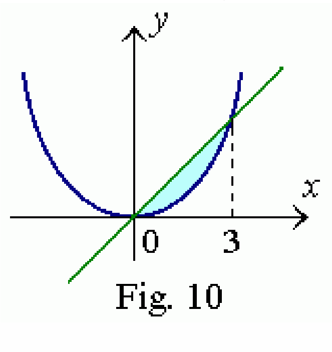


Fig. 10

Then we find the points of intersection, solving the equation $3x = x^2$. This equation has two roots: $x_1 = 0$ and $x_2 = 3$, which give the limits of integration.

Finally, we obtain

$$A = \int_0^3 (3x - x^2)dx = \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^3 = \frac{27}{2} - 9 = \frac{9}{2}.$$

4.6.2. The Arc Length of a Curve

Problem 1: Given a curve $y = f(x)$ in the $xy-$plane, find the arc length of the curve between the given values of x.

Solution: The given arc can be subdivided by partition points into an infinite number of portions of the curve, and each of the portions can be represented by a line segment.

Look at the Fig. 6, where some partition of the arc is shown. There is also an arbitrary portion with the approximating line segment in expanded scale.

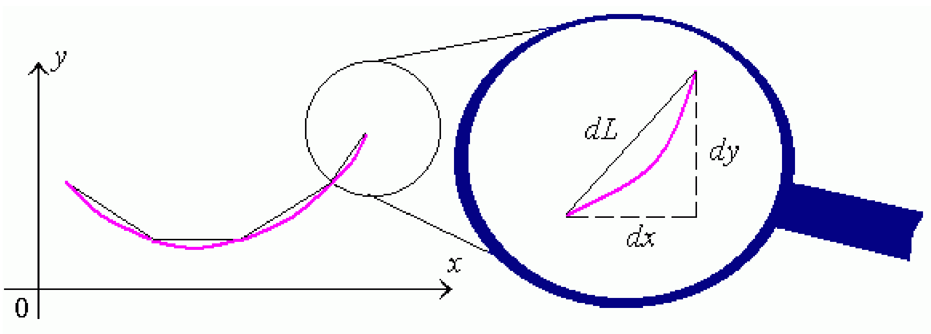


Fig. 11

Let dx and dy be Cartesian coordinates of an arbitrary segment. Then its length dL can be found by the Pythagorean Theorem: $dL = \sqrt{(dx)^2 + (dy)^2}$. Therefore,

$$dL = \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \sqrt{1 + (y')^2}\, dx,$$

where y' is the derivative of the function $y = f(x)$ with respect to x.

The total length of the arc equals the sum of all lengths of the portions, and hence, the definite integral of $\sqrt{1+(y')^2}$ with respect to x.

Therefore, the arc length of the curve between points a and b of the x-axis is given by the following formula:

$$L = \int_a^b \sqrt{1+(y')^2}\, dx \qquad (12)$$

Another solution: The length of an arbitrary portion of the arc can be written by the Pythagorean Theorem as $dL = \sqrt{(dx)^2 + (dy)^2}$ which implies

$$dL = \sqrt{1+(\frac{dy}{dx})^2}\, dx = \sqrt{1+(y')^2}\, dx,$$

where y' is the derivative of the function $y = f(x)$ with respect to x.

Therefore, $L(x)$ is a primitive of $\sqrt{1+(y')^2}$:

$$L(x) = \int \sqrt{1+(y')^2}\, dx + C.$$

Since $L(a) = 0$ and $L(b) = L$, so $L = \int_a^b \sqrt{1+(y')^2}\, dx$.

Problem 2: Let a given curve be defined parametrically in three-dimensional space: $\begin{cases} x = x(t), \\ y = y(t), \\ z = z(t). \end{cases}$

Find the arc length of the curve between the given values of t.

Solution: As above by the Pythagorean Theorem, we have

$$dL = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \quad \Rightarrow \quad dL = \sqrt{\frac{(dx)^2 + (dy)^2 + (dz)^2}{(dt)^2}} dt$$

$$\Rightarrow$$

$$dL = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt \quad \Rightarrow$$

$$dL = \sqrt{(x')^2 + (y')^2 + (z')^2}\, dt,$$

where x', y' and z' are derivatives of the functions $x(t)$, $y(t)$ and $z(t)$ correspondingly with respect to t.

If the end-points of arc are determined by the values t_1 and t_2 of the parameter t, then the arc length of the curve is given as

$$L = \int_{t_1}^{t_2} \sqrt{(x')^2 + (y')^2 + (z')^2}\, dt\,. \qquad (13)$$

This formula gives the general solution of finding the arc length of a curve. In a particular case when the curve lies in the $xy-$plane and the x-coordinate is considered as the parameter t, we have $x = x$, $y = y(x)$ and $z = 0$.
Hence, we return to formula (12).

4.6.3. Volumes of Solids

Problem 1: Given a solid, find the volume of the solid. (See Fig. 12.)
Solution: Assume that the solid is of such a nature that whenever we intersect the solid with a plane perpendicular to the x-axis, the cross-sectional area A is known.
This area A is a function of a point x, which we make the cross-section through. Let $A(x)$ be the cross-sectional area at the point x, and let $A(x) = 0$ for any $x \notin [a,b]$.
By intersecting the solid with planes perpendicular to the x-axis, it can be subdivided into an infinite number of layers. Each of the layers can be represented by a cylinder. The volume of an arbitrary cylinder is $dV = A(x)dx$.

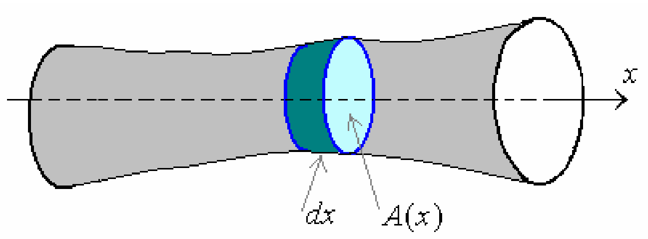


Fig. 12

In a similar way as above we can conclude that the volume of the solid between points a and b is given by the following formula:

$$V = \int_a^b A(x)dx \qquad (14)$$

Note: In order to determine the values of the limits of integration, a and b, one can use the following rules:

The lower limit a of integration is the smallest number such that $A(x)=0$ for all $x<a$.
The upper limit b of integration is the largest number such that $A(x)=0$ for all $x>b$.

Problem 2: Let a curve $y=f(x)$ be defined over a closed interval $[a,b]$; find the volume of the resultant solid of revolution by rotating of the curve about the x-axis.

Solution: If we intersect the solid with a plane perpendicular to the x-axis, **t**hen the cross-section of the solid is a circular disk. The radius of this circular disk is $|f(x)|$. By the formula for the area of a circle, the cross-sectional area of the solid at x equals $A(x)=\pi f^2(x)$, provided that $a<x<b$.
Thus, in view of formula (14), the volume of the solid of revolution is given by

$$V=\int_a^b A(x)dx=\pi\int_a^b f^2(x)dx. \qquad (15)$$

Chapter 5

IMPROPER INTEGRALS

5.1. Basic Definitions

Improper integrals are either integrals with at least one infinite limit of integration or integrals of functions that are unbounded on the interval of integration. For instance, the following integrals are improper:

$$\int_a^{+\infty} f(x)dx, \quad \int_{-\infty}^{b} f(x)dx, \quad \int_{-\infty}^{+\infty} f(x)dx, \quad \int_0^1 \frac{dx}{\sqrt{x}}, \quad \int_1^2 \frac{dx}{2-x}, \quad \int_2^5 \frac{dx}{(x-3)^2}.$$

All improper integrals are defined as limits of the definite integrals. In particular,

$$\int_a^{+\infty} f(x)dx = \lim_{c\to+\infty}\int_a^c f(x)dx, \tag{1}$$

$$\int_{-\infty}^{b} f(x)dx = \lim_{c\to-\infty}\int_c^b f(x)dx. \tag{2}$$

Note that integrals with both infinite limits do not require a special consideration because of

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx. \tag{3}$$

Note also that the integral with the lower infinite limit can be transformed into the integral with the upper infinite limit by substitution $x=-t$:

$$\int_{-\infty}^{b} f(x)dx = \int_{-b}^{+\infty} f(-t)dt. \tag{4}$$

Integrals of unbounded functions are defined in a similar way:
Let $f(x)\to\infty$ as $x\to a$. Then

$$\int_a^b f(x)dx = \lim_{c\to a}\int_c^b f(x)dx. \tag{5}$$

An improper integral is said to be **convergent**, if there exists the limit of the corresponding definite integral. Otherwise, if the limit does not exist or it is infinite, then the improper integral is called **divergent**.

Examples of convergent integrals:

- $$\int_1^\infty \frac{dx}{x^2} = \lim_{c\to\infty}\int_1^c \frac{dx}{x^2} = \lim_{c\to\infty}(-\frac{1}{x})\Big|_1^c = \lim_{c\to\infty}(1-\frac{1}{c}) = 1.$$
- $$\int_0^{+\infty} e^{-5x}dx = \lim_{c\to+\infty}\int_0^c e^{-5x}dx = \lim_{c\to+\infty}(-\frac{1}{5}e^{-5x})\Big|_0^c = \frac{1}{5}\lim_{c\to+\infty}(1-e^{-5c}) = \frac{1}{5}.$$
- $$\int_3^\infty \frac{dx}{x^2-1} = \lim_{c\to\infty}\int_3^c \frac{dx}{x^2-1} = \frac{1}{2}\lim_{c\to\infty}\ln\left|\frac{x-1}{x+1}\right|\Big|_3^c$$
  $$= \frac{1}{2}\lim_{c\to\infty}(\ln\frac{c-1}{c+1} - \ln\frac{2}{4}) = \frac{1}{2}(\ln 1 + \ln 2) = \frac{\ln 2}{2}.$$

Examples of divergent integrals:

- $$\int_1^\infty \frac{dx}{x} = \lim_{c\to\infty}\int_1^c \frac{dx}{x} = \lim_{c\to\infty}\ln|c|\Big|_1^c = \lim_{c\to\infty}\ln|c| = \infty.$$
- $$\int_0^\infty \cos x dx = \lim_{c\to\infty}\int_0^c \cos x dx = \lim_{c\to\infty}\sin x\Big|_0^c = \lim_{c\to\infty}\sin c,$$
  which does not exist.

5.2. Convergence and Divergence of Improper Integrals

If $f(x)$ is a positive defined function on some interval (a,b), then $\int_a^b f(x)dx$ represents the area of the region bounded by the graph of the function $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$.

In order to determine whether or not some integral is convergent we can either evaluate it by the definition or use the **comparison tests**.

The idea of the simplest comparison test is based on the property of integrals: If a curve goes down as a whole, then the area under the curve decreases, and vice versa. This means that:

- If some integral converges, then the integral of a smaller positive function also has to converge.

- If some integral of a positive function diverges, then the integral of a greater function also has to be divergent.

Direct Comparison Test

Let $f(x)$ and $g(x)$ be two functions defined on (a,b) such that

$$0 \le f(x) \le g(x)$$

for any $a < x < b$.

- If $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ also converges.
- If $\int_a^b f(x)dx$ diverges, then $\int_a^b g(x)dx$ also diverges.

Here a and b are finite or infinite numbers, and the function $f(x)$ has the improper behavior at a or b.
According to this test we have to find an integral that is similar to but always less than the original one. If it diverges by any tests, then the original integral also diverges. On the contrary, we can try to find an integral similar to but always greater than the original one. If it converges, then the given integral also converges.

5.2.1. Convergence and Divergence of Integrals with Infinite limits

The integral $\int_a^{+\infty} |f(x)|dx$ equals the area of the region under the curve $y = |f(x)|$ bounded by the x-axis and the vertical line $x = a$. It is evident that this area is infinite, if the function $f(x)$ is not decreasing one.

Divergence Test

If $\lim\limits_{x \to +\infty} f(x) \ne 0$,

then the integral $\int_a^{+\infty} f(x)dx$ diverges.

Note that the implication goes only one way: if $\lim\limits_{x\to+\infty} f(x) = 0$, it does not mean that the integral of $f(x)$ is convergent.

For instance, $\lim\limits_{x\to+\infty} 1/x = 0$, but the integral $\int\limits_1^\infty \frac{dx}{x}$ is divergent. (See the example above.)

One can also compare the rates of decreasing of functions to determine whether some integral converges.

Let $\lim\limits_{x\to+\infty} \left|\frac{f(x)}{g(x)}\right| = \lambda$. There are three possible cases: $0 < \lambda < \infty$, $\lambda = 0$ or $\lambda = \infty$.

- If λ is a finite non-zero number, then the integral

  $\int\limits_a^{+\infty} f(x)dx$ converges if and only if $\int\limits_a^{+\infty} g(x)dx$ converges;

  $\int\limits_a^{+\infty} f(x)dx$ diverges if and only if $\int\limits_a^{+\infty} g(x)dx$ diverges.

  It follows from the fact that the area under the asymptotic part of a curve being multiplied by a finite non-zero number λ holds its finiteness or infinity.

- If $\lambda = 0$, then the convergence of the integral $\int\limits_a^{+\infty} g(x)dx$ implies the convergence of the integral $\int\limits_a^{+\infty} f(x)dx$.

  However, we can say nothing if the integral of $g(x)$ is divergent.

- If $\lambda = \infty$, then the divergence of the integral $\int\limits_a^{+\infty} g(x)dx$ implies the divergence of the integral $\int\limits_a^{+\infty} f(x)dx$.

  However, we can say nothing if the integral of $g(x)$ is convergent.

Limit Comparison Test

Let $f(x)$ and $g(x)$ be two functions defined on $[a,+\infty)$ such that

$$0<\lim_{x\to+\infty}\left|\frac{f(x)}{g(x)}\right|<\infty$$

Then both integrals, $\int_a^{+\infty} f(x)dx$ and $\int_a^{+\infty} g(x)dx$, converge or diverge simultaneously.

To use this test for a given integral we have to find a second integral such that the limit of the ratio of integrands is evaluable.
If the second integral converges (diverges) by any tests, then the original integral also converges (diverges).

Example: Determine whether the integral $\int_1^{+\infty}\frac{\ln x dx}{x^3}$ converges.

Let us compare the given integral with the convergent integral $\int_1^{+\infty}\frac{dx}{x^2}$. (See the example above.)
To apply the limit comparison test we calculate: $\lim_{x\to+\infty}\frac{\ln x/x^3}{1/x^2}=\lim_{x\to+\infty}\frac{\ln x}{x}=0$.
Therefore, the given integral is also convergent.

Note that there are useful integrals of the general form $\int_a^{+\infty}\frac{dx}{x^p}$, which are called p-integrals. They are helpful in comparison tests because of the following theorem:

Theorem

The p-integral $\int_a^{+\infty}\frac{dx}{x^p}$ $\begin{cases}\text{converges, if} & p>1\\ \text{diverges, if} & p\le 1\end{cases}$

Proof: If $p\neq 1$, then

$$\int_a^{+\infty} \frac{dx}{x^p} = \left. \frac{x^{-p+1}}{-p+1} \right|_a^{+\infty} \Rightarrow \begin{cases} \text{convergence, if} & (-p+1) < 0, \\ \text{divergence,} \quad \text{if} & (-p+1) > 0. \end{cases}$$

If $p = 1$, then the integral $\int_a^{+\infty} \frac{dx}{x^p}$ is divergent. (See the example above.)

Example: Determinate whether or not the integral $\int_1^{+\infty} \frac{\sqrt{x}}{2x^4 + 5} dx$ converges.

Solution: We use the comparison test, comparing with the p-integral:

$$\int_1^{+\infty} \frac{\sqrt{x}}{2x^4 + 5} dx < \int_1^{+\infty} \frac{\sqrt{x}}{2x^4} dx = \frac{1}{2} \int_1^{+\infty} \frac{dx}{x^{7/2}}.$$

The p-integral with $p > 1$ converges. Hence, the given integral also converges.

5.2.2. Convergence and Divergence of Integrals of Unbounded Functions

The limit comparison test can be easily adapted for unbounded function. For instance, assume $f(x)$ is unbounded at the point a.

Limit Comparison Test

Let $f(x)$ and $g(x)$ be two functions defined on (a, b) such that

$$0 < \lim_{x \to a} \left| \frac{f(x)}{g(x)} \right| < \infty$$

Then both integrals, $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$, converge or diverge simultaneously.

If the function $f(x)$ is unbounded at $x = b$, then we have to operate with $\lim_{x \to b}$ instead of $\lim_{x \to a}$, changing nothing more. For instance, let $f(x)$ and $g(x)$ be two functions defined on (a, b) such that

$$0 < \lim_{x \to b} \left| \frac{f(x)}{g(x)} \right| < \infty.$$

Then both integrals, $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$, converge or diverge simultaneously.

We need also to modify the p-integrals.

Theorem

The p-integrals $\int_a^b \frac{dx}{(x-a)^p}$ and $\int_a^b \frac{dx}{(b-x)^p}$

$$\begin{cases} \text{converge, if} \quad p<1 \\ \text{diverge,} \quad \text{if} \quad p \ge 1 \end{cases}$$

Proof: If $p \ne 1$, then

$$\int_a^b \frac{dx}{(x-a)^p} = \left.\frac{(x-a)^{-p+1}}{-p+1}\right|_a^b \Rightarrow \begin{cases} \text{convergence, if} \quad (-p+1)>0, \\ \text{divergence,} \quad \text{if} \quad (-p+1)<0. \end{cases}$$

If $p=1$, then $\int_a^b \frac{dx}{x-a} = (\ln|b-a| - \lim_{x\to a} |x-a|)$. Since this limit does not exist so the given integral diverges.
Similar arguments can be used to prove the second part of the theorem.

Example: Consider the integral $\int_2^5 \frac{dx}{x(x^2-4)}$.

The function $\frac{1}{x(x^2-4)}$ is unbounded at $x=2$. Compare the original integral with the divergent p-integral: $\int_2^5 \frac{dx}{x-2}$.

The limit of the ratio of the integrands is a finite number:

$$\lim_{x\to 2} \frac{x-2}{x(x^2-4)} = \lim_{x\to 2} \frac{1}{x(x+2)} = \frac{1}{8}.$$

Thus, we conclude that the given integral diverges by the limit comparison test.

Chapter 6
COMPLEX NUMBERS

A **complex number** is an expression of the form $x+iy$, where x and y are real numbers, and i is the imaginary number such that $i^2=-1$.
Usually a complex number is denoted by a single letter, e.g., $z=x+iy$.
The numbers x and y are called the **real** and **imaginary parts** of z. They are symbolized as $x=\operatorname{Re} z$, $y=\operatorname{Im} z$.
Thus,

$$z=\operatorname{Re} z+i\operatorname{Im} z. \qquad (1)$$

Note that both numbers, $\operatorname{Re} z$ and $\operatorname{Im} z$, are real numbers.
The set of all complex numbers is denoted by the symbol C. Any real number x can be considered as a complex number whose imaginary part equals zero. Therefore, the set of complex numbers includes the set of all real numbers as a subset.
The set of real numbers is a proper subset of the set of complex numbers: $R\subset C$.
If $\operatorname{Re} z=0$, then a number $z=iy$ is said to be **purely imaginary**.

6.1. Algebraic Operations

1. Two complex numbers, $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, are equal to each other if and only if $x_1=x_2$ and $y_1=y_2$:

$$z_1=z_2 \quad \Leftrightarrow \quad \begin{cases} x_1=x_2, \\ y_1=y_2. \end{cases} \qquad (2)$$

2. Complex numbers are added (or subtracted) by adding (or subtracting) the real and imaginary parts correspondingly:

$$(x_1+iy_1)\pm(x_2+iy_2)=(x_1\pm x_2)+i(y_1\pm y_2).$$

   Complex numbers have the same addition properties as real numbers:

$$z_1\pm z_2=z_2\pm z_1,$$
$$(z_1+z_2)+z_3=z_1+(z_2+z_3),$$
$$z+0=z,$$
$$z+(-z)=0.$$

3. In order to multiply complex numbers one has to expand the product and substitute (-1) for i^2:

$$\begin{aligned}(x_1+iy_1)(x_2+iy_2)&=x_1x_2+i(x_1y_2+x_2y_1)+i^2y_1y_2\\ &=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1).\end{aligned}$$

Multiplication properties for complex numbers are the same as for real ones:

$$z_1 z_2 = z_2 z_1,$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3),$$
$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3,$$
$$z \cdot 1 = z,$$
$$z \cdot 0 = 0.$$

4. The number $z^* = x - iy$ is said to be **complex conjugate** of a number $z = x + iy$.

- For any complex number z

$$\operatorname{Re} z = \frac{1}{2}(z + z^*), \qquad \operatorname{Im} z = \frac{1}{2i}(z - z^*). \qquad (3)$$

- For any complex number z the product $z z^*$ is a nonnegative real number:

$$z \cdot z^* = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2.$$

  Therefore, the sum of squares of any real numbers can be factored into linear complex factors: $a^2 + b^2 = (a + ib)(a - ib)$.

- The absolute value of z is denoted by the symbol $|z|$ and defined as

$$|z| = \sqrt{z \cdot z^*} = \sqrt{x^2 + y^2}. \qquad (4)$$

- The complex conjugate is associative and distributive:

$$(z_1 + z_2)^* = z_1^* + z_2^*,$$
$$(z_1 z_2)^* = z_1^* z_2^*.$$

  Indeed,

$$(z_1 + z_2)^* = (x_1 + iy_1 + x_2 + iy_2)^* = ((x_1 + x_2) + i(y_1 + y_2))^*$$
$$= (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = z_1^* + z_2^*,$$

$$(z_1 z_2)^* = ((x_1 + iy_1)(x_2 + iy_2))^*$$
$$= (x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1))^*$$
$$= x_1 x_2 - y_1 y_2 - i(x_1 y_2 + x_2 y_1) = (x_1 - iy_1)(x_2 - iy_2) = z_1^* z_2^*.$$

5. In order to divide any number w by a nonzero complex number z one can multiply w by the complex conjugate number z^* and divide by the real number $|z|^2$:

$$\frac{1}{z} = \frac{z^*}{z \cdot z^*} = \frac{z^*}{|z|^2}.$$

Examples:

- $i^4 = (i^2)^2 = (-1)^2 = 1$.
- $i^5 = i^4 i = i$.
- $(2-3i)(4+i) = 8 + 2i - 12i - 3i^2 = 11 - 10i$.
- $\frac{1}{i} = \frac{i}{i^2} = -i$.
- $\frac{1+2i}{3-4i} = \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} = \frac{3+6i+4i+8i^2}{3^2+4^2} = \frac{-5+10i}{25} = -\frac{1}{5} + \frac{2}{5}i$.

6.2. The Complex Plane

Properly speaking, a complex number $z = x + iy$ is the ordered pair (x, y) of real numbers. The pair (x, y) can be considered as the Cartesian coordinates of a point in the $xy-$ plane.

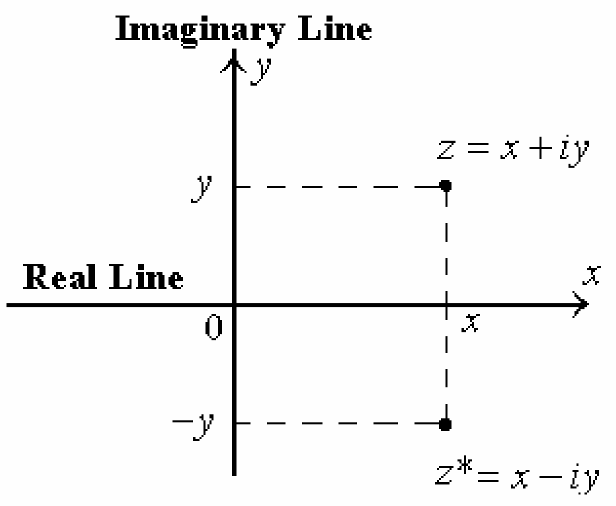


Fig. 1

Therefore, any complex number can be graphically represented by an unique point of the coordinate plane of the two-dimensional Cartesian coordinate system. There is one-to-one correspondence between the set of complex numbers and points in the $xy-$ plane: every point in this complex plane corresponds to a unique complex number, and vice versa.

All real numbers are represented by the points of the x-axis, while all purely imaginary numbers are represented by the points of the y-axis.

These axes are known as the **Real** and **Imaginary Lines** correspondingly.

Complex numbers are added and subtracted in the same way as vectors:

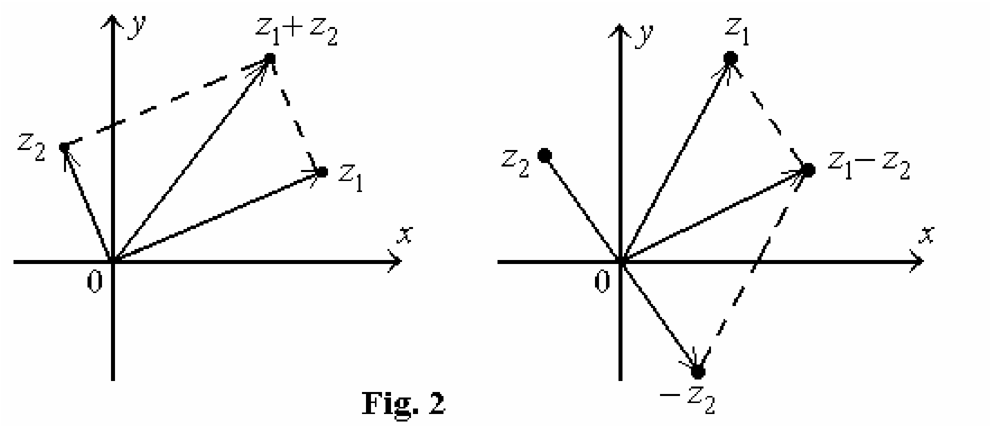


Fig. 2

From the geometrical point of view the absolute value $|z|$ is the distance from the point z to zero-point in the complex plane.

The absolute value $|z_1 - z_2|$ is the distance between points z_1 and z_2 in the complex plane.

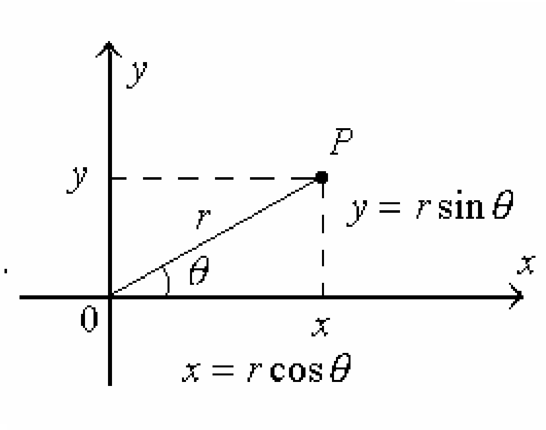


Fig. 3

The point P can also be described by the polar coordinates r and θ, where r is the distance from the origin O to the point P, and θ is the angle that the ray OP makes with the positive direction of x-axis. There exist simple relationships between Cartesian and polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$r = \sqrt{x^2 + y^2}, \qquad \tan\theta = \frac{y}{x},$$

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}.$$

Therefore, a complex number $z = x + iy$ can be written in the polar form as

$$z = r(\cos\theta + i\sin\theta). \qquad (5)$$

In this case, $r = |z|$ is said to be the modulus and $\theta = \arg(z)$ is known as the argument (or phase) of the complex number z.

Note: One has to apply the following rules in order to find $\arg(z)$.

1) If $x > 0$, then $\theta = \arctan(y/x)$,
2) If $x < 0$ while $y > 0$, then $\theta = \pi - \arctan|y/x|$,
3) If $x < 0$ and $y < 0$, then $\theta = -\pi + \arctan(y/x)$.

All these cases are illustrated by simple examples in Fig. 4.

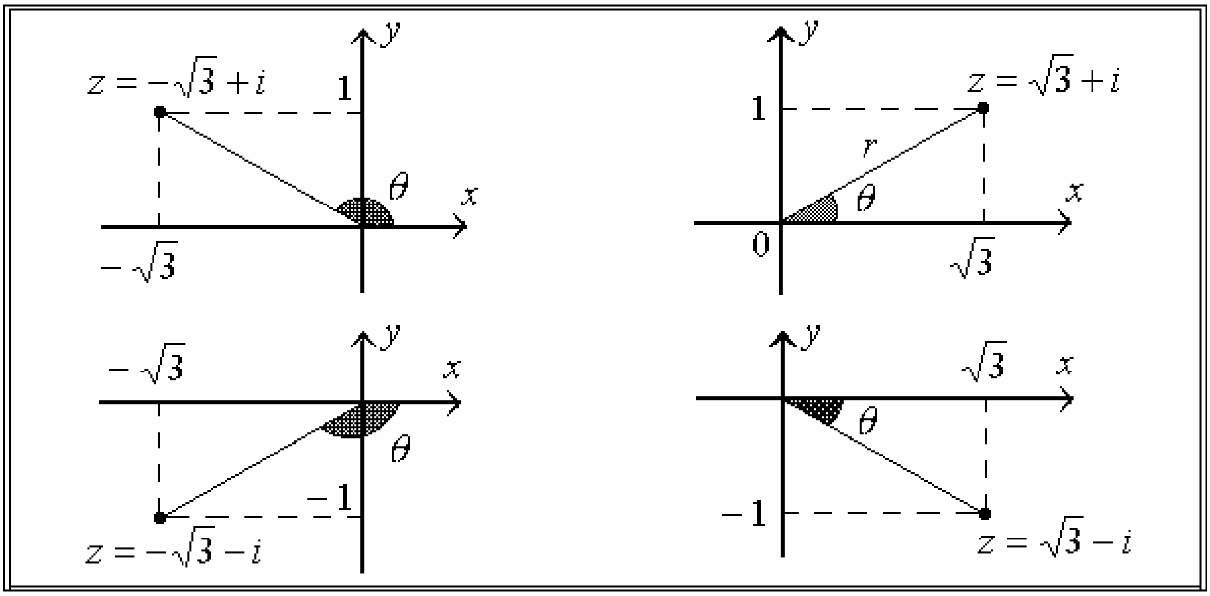


Fig. 4

6.3. The Euler Formula

The Euler Formula states:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi\,. \qquad (6)$$

Due to the Euler Formula the definition of the exponential e^x of a real number x can be generalized to the exponential e^z of a complex number $z = x + iy$:

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)\,. \qquad (7)$$

In a special case, where z is a real number (that is $y = 0$), this formula gives the desired result:

$$e^{x+i0} = e^x(\cos 0 + i\sin 0) = e^x\,.$$

One can easily prove that the exponential of complex numbers have the same properties as the exponential of real numbers. For instance,

$$e^{z_1+z_2} = e^{z_1}e^{z_2}\,.$$

The Euler Formula gives the following representation of a complex number in the polar form:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}\,. \qquad (8)$$

6.3.1. Applications for the Euler Formula

6.3.1.1. Trigonometric Applications

All trigonometric identities can be easily derived by making use of the Euler Formula. The following examples illustrate typical techniques.

1. The identities

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \operatorname{Re}(e^{i\theta}), \qquad (9)$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \operatorname{Im}(e^{i\theta}) \qquad (10)$$

follow from the Euler Formula by adding and subtracting of the equalities

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad (11a)$$

$$e^{-i\theta} = \cos\theta - i\sin\theta. \qquad (11b)$$

2. Let us square both sides of equality (11a):

$$e^{2i\theta} = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + 2i\sin\theta\cos\theta.$$

Since $e^{2i\theta} = \cos 2\theta + i\sin 2\theta$ so by property (2), we get double-angle identities:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \qquad (12a)$$

$$\sin 2\theta = 2\sin\theta\cos\theta. \qquad (12b)$$

3. Consider the product

$$e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

The expression on the left-hand side can be transformed as:

$$e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta). \qquad (13a)$$

On the other hand

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \\ = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha). \qquad (13b)$$

Comparing (13a) with (13b) we conclude that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \qquad (14a)$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha. \qquad (14b)$$

6.3.1.2. Algebraic Applications

1. Let $z = z_1 z_2$. Since $z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$, so

$$|z| = |z_1| \cdot |z_2|, \quad \arg(z) = \arg(z_1) + \arg(z_2). \tag{15}$$

2. Likewise a complex number $z_1 = r_1 e^{i\theta_1}$ is divided by $z_2 = r_2 e^{i\theta_2}$:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2). \tag{16}$$

3. The integer power n of a complex number z can be written in closed form as follows:

$$z^n = |z|^n (\cos n\theta + i \sin n\theta) = |z|^n e^{in\theta}. \tag{17}$$

This formula is known as the DeMoivre Identity.

Examples:

1) Derive the fundamental trigonometric identity:

$$\sin^2\theta + \cos^2\theta = 1.$$

Solution:

Let $z = e^{i\theta}$, then $z \cdot z^* = e^{i\theta} \cdot e^{-i\theta} = 1$.
However, from the Euler Formula it follows that

$$e^{i\theta} \cdot e^{-i\theta} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta.$$

Hence, the desired result.

2) Transform to the polar form the number $z_1 = 3 + i\sqrt{3}$.

Solution:

$$|z_1| = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}, \quad \arg(z_1) = \arctan\frac{\sqrt{3}}{3} = \frac{\pi}{6}.$$

Therefore,

$$z_1 = 2\sqrt{3}e^{i\pi/3}.$$

3) Transform to the polar form the number $z_2 = 2 - 2i$.

Solution:

$$|z_2| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}, \quad \arg(z_2) = \arctan(-1) = -\frac{\pi}{4}.$$

Therefore,

$$z_2 = 2\sqrt{2}\, e^{-i\pi/4}.$$

4) Find the product and quotient of the numbers $z_1 = 3 + i\sqrt{3}$ and $z_2 = 2 - 2i$.

Solution:

- $(3+i\sqrt{3})(2-2i) = 2\sqrt{3}e^{i\frac{\pi}{3}} 2\sqrt{2}e^{-i\frac{\pi}{4}} = 4\sqrt{6}\,e^{i(\frac{\pi}{3}-\frac{\pi}{4})} = 4\sqrt{6}\,e^{i\frac{\pi}{12}}$.
- $\dfrac{(3+i\sqrt{3})}{(2-2i)} = \sqrt{3/2}\,e^{i(\frac{\pi}{3}+\frac{\pi}{4})} = \sqrt{3/2}\,e^{i\frac{7\pi}{12}}$.

6.4. Complex Roots

A complex number can be written in different ways. For instance, for any integer m the angle $\theta + 2\pi m$ corresponds to the same point in the complex plane as the angle θ, so the most common form of $\arg z$ is the following:

$$\arg z = \theta + 2\pi m .$$

Hence, the Euler Formula can be rewritten in the following equivalent form:

$$z = re^{i(\theta + 2\pi m)} .$$

Let us recall that a number t is said to be the nth root of a number z if $t^n = z$. Therefore, in view of the DeMoivre Identity the nth roots of a number z are determined by the following expression:

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta+2\pi m}{n}} = r^{\frac{1}{n}}(\cos\frac{\theta+2\pi m}{n} + i\sin\frac{\theta+2\pi m}{n}) . \qquad (18)$$

There exist n different roots exactly:

$m = 0 \quad \Rightarrow \quad t_1 = \sqrt[n]{r}e^{i\frac{\theta}{n}} = \sqrt[n]{r}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$.

$m = 1 \quad \Rightarrow \quad t_2 = \sqrt[n]{r}e^{i\frac{\theta+2\pi}{n}} = \sqrt[n]{r}(\cos\frac{\theta+2\pi}{n} + i\sin\frac{\theta+2\pi}{n})$.

$m = 2 \quad \Rightarrow \quad t_3 = \sqrt[n]{r}e^{i\frac{\theta+4\pi}{n}} = \sqrt[n]{r}(\cos\frac{\theta+4\pi}{n} + i\sin\frac{\theta+4\pi}{n})$.

…

$m = n-1 \quad \Rightarrow$

$$t_n = \sqrt[n]{r}e^{i\frac{\theta+2\pi(n-1)}{n}} = \sqrt[n]{r}(\cos\frac{\theta+2\pi(n-1)}{n} + i\sin\frac{\theta+2\pi(n-1)}{n}) .$$

The next value of integer m gives the root t_{n+1} that coincides with t_1:

$$t_{n+1} = \sqrt[n]{r}e^{i\frac{\theta+2\pi n}{n}} = \sqrt[n]{r}(\cos\frac{\theta+2\pi n}{n} + i\sin\frac{\theta+2\pi n}{n})$$

$$= \sqrt[n]{r}(\cos(\frac{\theta}{n}+2\pi) + i\sin(\frac{\theta}{n}+2\pi)) = \sqrt[n]{r}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}) = t_1.$$

Note: All the roots have the same modulus $\sqrt[n]{r}$, that is, they lie on the circle with the radius $\sqrt[n]{r}$.

Example: Find the square roots of the number $z = 1 + i\sqrt{3}$.

First, we calculate the modulus r and argument θ of z:

$$r = |1 + i\sqrt{3}| = \sqrt{1+3} = 2,$$

$$\theta = \arctan\sqrt{3} = \frac{\pi}{3}.$$

Then, we need to take the square root of the magnitude r and divide the phase θ by 2: $\sqrt{r} = \sqrt{2}$, $\theta/2 = \pi/6$.

Finally, we use formula (18) with $n = 2$ and $m = 0, 1$:

$$t_1 = \sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{2}(\frac{\sqrt{3}}{2} + \frac{i}{2}) = \frac{\sqrt{6}}{2} + \frac{i\sqrt{2}}{2},$$

$$t_2 = \sqrt{2}(\cos(\frac{\pi}{6}+\pi) + i\sin(\frac{\pi}{6}+\pi)) = \sqrt{2}(-\frac{\sqrt{3}}{2} - \frac{i}{2}) = -\frac{\sqrt{6}}{2} - \frac{i\sqrt{2}}{2}.$$

One can easily check that both numbers, $t_{1,2} = \pm\frac{\sqrt{2}}{2}(\sqrt{3}+i)$, are roots of the given number z: $(\pm\frac{\sqrt{2}}{2}(\sqrt{3}+i))^2 = \frac{2}{4}(3 + 2i\sqrt{3} - 1) = (1 + i\sqrt{3})$.

The figures below illustrate graphically the properties of nth roots. There are shown the square and cube roots of the complex number i in Fig. 5.

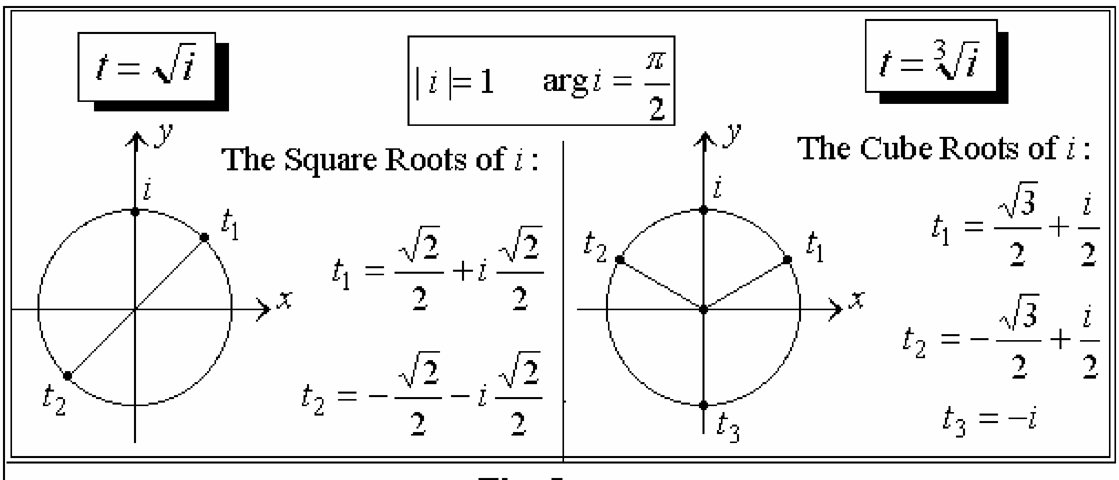


Fig. 5

Verification: We can easily check these results by raising to the second and third powers correspondingly. For instance:

- $(\pm\frac{\sqrt{2}}{2}(1+i))^2 = \frac{1}{2}(1+2i-1) = i$.
- $(\frac{1}{2}(\sqrt{3}+i))^3 = \frac{1}{8}(3\sqrt{3}+9i+3\sqrt{3}i^2+i^3) = \frac{1}{8}(3\sqrt{3}+9i-3\sqrt{3}-i) = i$.

All the cube roots of the number (-1) and the fourth roots of the number 1 are shown in Fig. 6.

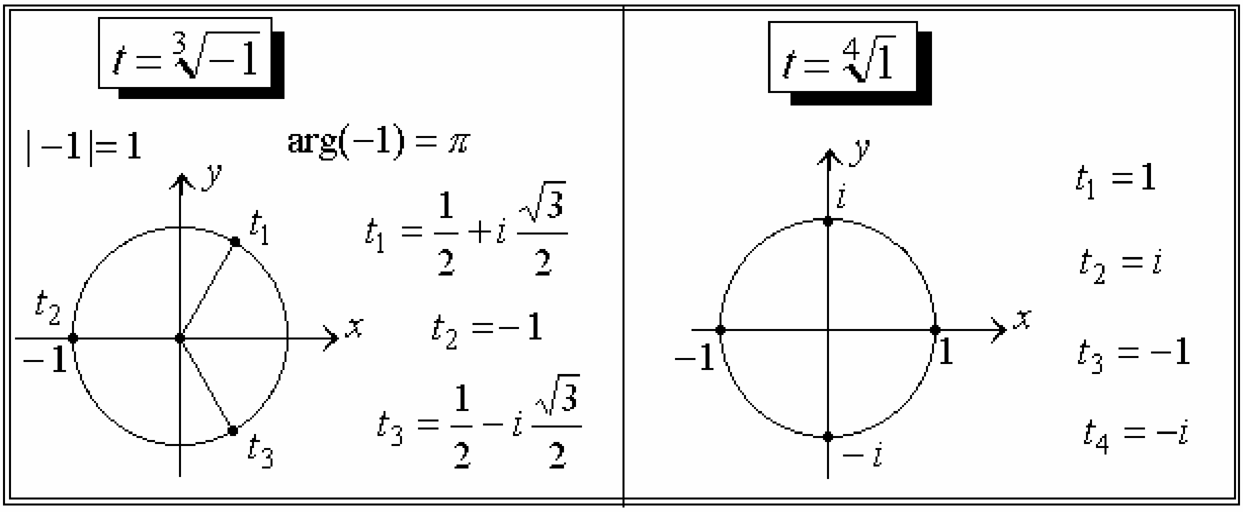


Fig. 6

There is a simple way to plot all n roots of a number z when one of the roots is known. All we need is to divide the circle with radius $|z|^{1/n}$ into n equal parts starting from the point on the circle that corresponds to a root of z. For instance, one of the twelve roots of 1 equals 1 and $2\pi/12 = \pi/6$. The other 11 roots are $\cos\frac{\pi \cdot m}{6} + i\sin\frac{\pi \cdot m}{6}$, ($m$=1, 2, …, 11).

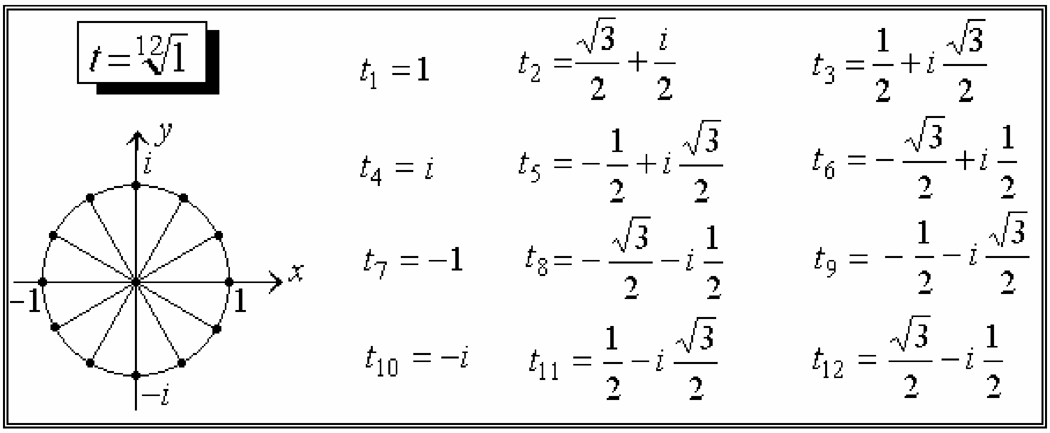


Fig. 7

Chapter 7

FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

7.1. Introduction

Let x be the independent variable, and let y be the dependent variable.

A **differential equation** is an equation, which involves the derivative of a function $y(x)$. The equation may also contain the function itself as well as the independent variable.

The general form of a differential equation of the first order is

$$F(x, y, y') = 0. \quad (1)$$

The solution procedure consists in finding the unknown function $y(x)$, which obeys equation (1) on a given interval.

The general solution of equation (1) is a function $y = \varphi(x, C)$, which is the solution of (1) for any values of a parameter C. By setting $C = const$ we obtain a particular solution of equation (1).

Sometimes the solution can be found in the implicit form only. If the equation

$$\Phi(x, y, C) = 0, \quad (2)$$

determines the general solution of (1), then it is called the general integral of the differential equation.

If there given an initial condition $y(x_0) = y_0$ in addition to equation (1), then it is necessary to find the particular solution, which obeys the initial condition.

Here we consider only such classes of first-order differential equations, which can be solved analytically.

7.2. Directly Integrable Equations

A directly integrable differential equation has the following form:

$$y' = f(x), \quad (3)$$

where $f(x)$ is a given function.

From this equation follows that the function $y(x)$ is a primitive of $f(x)$ and hence

$$y(x) = \int f(x)dx + C. \quad (4)$$

A constant C can be determined from the initial condition, if the one is given.

Example: Find the solution of the equation

$$y'(x) = x + \cos x$$

with the initial condition $y(0) = 1$.

Solution: In view of (4) the general solution is

$$y(x) = \int (x + \cos x)dx + C = \frac{x^2}{2} + \sin x + C .$$

Taking into account the initial condition, we find: $1 = 0 + C$, that is, $C = 1$. Therefore, the function $y(x) = x^2/2 + \sin x + 1$ being the solution of the given equation, satisfies the initial condition.

7.3. Separable Equations

A separable differential equation is an equation of the form

$$y' = f(x)g(y), \tag{5}$$

that is, $y'(x)$ equals the product of given functions, $f(x)$ and $g(y)$, each of which is a function of one variable only.

We can not integrate equation (5) directly because the right-hand side contains an unknown function $y(x)$ together with the variable x.

To separate the variables we rewrite the equation in the form:

$$\frac{dy}{g(y)} = f(x)dx \tag{5a}$$

and then integrate both sides:

$$\int \frac{dy}{g(y)} = \int f(x)dx + C . \tag{6}$$

Thus, the general integral of equation (5) is found.

A differential equations of the form

$$y' = f(ax + by + c) \tag{7}$$

can be reduced to a separable equation by introducing of a new dependent variable $u(x)$ instead of y:

$$u = ax + by + c . \tag{8}$$

Next we have to derive the equation for the variable $u(x)$. By differentiating (8), we obtain $u' = a + by'$, which implies the equation

$$u' = a + b\, f(u)$$

being the separable equation.

Then we obtain $\quad \dfrac{du}{b\, f(u) + a} = dx \quad \Rightarrow \quad \displaystyle\int \frac{du}{b\, f(u) + a} = x + C .$

Example 1: Solve the equation

$$y' = e^{2x-3y}.$$

Solution: The variables can be easily separated:

$$e^{3y}dy = e^{2x}dx.$$

By integrating, we obtain a general integral of the given equation:

$$\frac{1}{3}e^{3y} = \frac{1}{2}e^{2x} + C.$$

By means of simple formula manipulations we can also write the general solution in the explicit form:

$$y = \frac{1}{3}\ln(\frac{3}{2}e^{2x} + C),$$

where the constant $3C_1$ is denoted by C.

Example 2: Find the solution of the equation

$$y' = \cos(x+y), \qquad (9)$$

which obeys the initial condition $y(0) = \pi/2$.

Solution: Let us introduce a new variable:

$$u = x + y.$$

Then from (9) we obtain the separable equation for $u(x)$

$$u' = 1 + \cos u.$$

By separating the variables and integrating, we have:

$$\int \frac{du}{1+\cos u} = x + C.$$

Using the formula $1+\cos u = 2\cos^2 u/2$ we obtain the algebraic equation

$$\tan(u/2) = x + C,$$

which implies

$$u = 2\arctan(x+C).$$

Since $y = u - x$, the general solution of the given equation is the following one: $y = 2\arctan(x+C) - x$.

The initial condition yields: $\pi/2 = 2\arctan C$, so that $C = 1$.

Finally we obtain:

$$y = 2\arctan(x+1) - x.$$

7.4. Homogeneous Equations

If some differential equation can be represented in the following form:

$$y' = f(\frac{y}{x}), \tag{10}$$

then it is called a homogeneous equation.

One of the main methods of solving differential equations is based on introducing a new dependent variable $u(x)$ instead of y. There is no general rule to make the right choice of u because it depends on the form of the equation. That is why it is necessary to consider different classes of equations separately. One of typical techniques of such a kind is illustrated below by solving an homogeneous equation.

The right-hand side of equation (10) suggests the substitution $u = y/x$. Then we have to derive the equation for the new dependent variable u.

To find the derivative of $y = ux$, we use the rule of differentiation of the product:

$$y' = u'x + u .$$

From (10) we obtain the equation

$$u'x + u = f(u),$$

which being rewritten in the form

$$u' = \frac{1}{x}(f(u) - u) \tag{11}$$

is a separable equation. Then the problem of integration is solved just in the same way as above. (See equation (5).)

Example: Solve the equation

$$y' = \frac{y}{x - \sqrt{xy}} . \tag{12}$$

Solution: Since

$$\frac{y}{x - \sqrt{xy}} = \frac{y/x}{1 - \sqrt{y/x}} = f(\frac{y}{x}),$$

the given equation is the homogeneous equation.

To solve this problem, we introduce the variable $u = y/x$ instead of y and derive a differential equation for $u(x)$.

First, $y = ux$, so $y' = u'x + u$. Therefore, by (12),

$$u'x + u = \frac{u}{1 - \sqrt{u}} \qquad \Rightarrow \qquad u'x = \frac{\sqrt{u^3}}{1 - \sqrt{u}} \qquad \Rightarrow$$

$$\frac{1-\sqrt{u}}{\sqrt{u^3}}du = \frac{dx}{x} \quad \Rightarrow \quad \int (u^{-3/2} - \frac{1}{u})du + C = \int \frac{dx}{x} \quad \Rightarrow$$

$$-2/\sqrt{u} - \ln|u| = \ln|x| - C .$$

Replacing u by y/x we obtain the general integral of equation (12):

$$\ln|y| + 2\sqrt{x/y} = C . \tag{13}$$

7.5. Linear Equations

A linear differential equation is an equation, which can be represented as

$$y' + P(x)y = Q(x), \tag{14}$$

where $P(x)$ and $Q(x)$ are given functions.

To solve the equation, we introduce a new dependent variable $u(x)$ instead of y by the equality

$$y = u(x)\,v(x), \tag{15}$$

keeping in mind to determine a function $v(x)$ later.

To derive the differential equation for $u(x)$ we find the derivative $y' = u'v + uv'$ and substitute it into original equation (14):

$$u'v + v'u + P(x)\; u\,v = Q(x) .$$

Next we group the terms and take out the common factor:

$$u'v + u(v' + P(x)\,v) = Q(x) . \tag{16}$$

Now we are ready to determine the function $v(x)$. Let $v(x)$ be a function such that

$$v' + P(x)\,v = 0 . \tag{17}$$

By separating the variables, we obtain the solution of equation (17):

$$\int \frac{dv}{v} = -\int P(x)dx \qquad \Rightarrow \qquad \ln|v| = -\int P(x)dx \qquad \Rightarrow$$

$$v = e^{-\int P(x)dx} . \tag{18}$$

A constant of integration is chosen to be equal to zero because it is enough to have one function only, which obeys condition (17).

In view of (18), equation (16) is reduced to the directly integrable equation of the form

$$u' = Q(x)e^{f(x)}, \tag{19}$$

where $f(x) = \int P(x)dx$ is one of primitives of P.

Therefore,

$$u(x) = \int Q(x) e^{f(x)} dx + C. \quad (20)$$

Thus, equation (14) has the following general solution:

$$y(x) = e^{-f(x)} (\int Q(x) e^{f(x)} dx + C). \quad (21)$$

Example: Find the general solution of the equation

$$y' = 3y/x + x. \quad (22)$$

Solution: Let $y = uv$. Then $y' = u'v + uv'$.
Substituting these expressions into the original equation, we obtain

$$u'v + v'u = 3uv/x + x \quad \Rightarrow$$

$$u'v + u(v' - 3v/x) = x. \quad (23)$$

Then we find the function $v(x)$ by solving of the equation

$$v' - 3v/x = 0.$$

The variables are easily separated and we have

$$\int \frac{dv}{v} = 3\int \frac{dx}{x} \quad \Rightarrow \quad \ln|v| = 3\ln|x| \quad \Rightarrow \quad v = x^3.$$

Now we come back to (23), which is reduced to the separable equation

$$u'x^3 = x.$$

Therefore, $$u = \int \frac{dx}{x^2} + C = -\frac{1}{x} + C.$$

Finally, we obtain

$$y = u\,v = (-\frac{1}{x} + C)x^3 = -x^2 + Cx^3.$$

7.6. The Bernoulli Equations

The Bernoulli Equation is an equation of the form

$$y'(x) + P(x)y = Q(x)y^n, \quad (24)$$

where n is any rational number except 0 and 1.
The technique of solving the Bernoulli equations is just the same as for linear equations: A new dependent variable $u(x)$ is introduced by means of the equality

$$y = u(x)\,v(x). \quad (25)$$

This variable satisfies the equation

$$u'v + u(v' + P(x)\,v) = Q(x)\,u^n v^n, \quad (26)$$

where the function $v(x)$ is a partial solution of the equation

$$v' + P(x)v = 0 \qquad (27)$$

and hence,

$$v = e^{-\int P(x)dx}. \qquad (28)$$

Therefore, equation (26) is transformed to the form

$$u'v = Q(x)u^n v^n$$

and can be rewritten as a separable equation:

$$u^{-n}du = Q(x)v^{n-1}dx.$$

By integrating, we obtain

$$\frac{1}{-n+1}u^{-n+1} = \int Q(x)v^{n-1}dx + C. \qquad (29)$$

Thus,

$$u(x) = \left((1-n)\int Q(x)v^{n-1}dx + C\right)^{\frac{1}{1-n}}. \qquad (30)$$

The general solution of (24) is $y(x) = u(x)v(x)$.

Example: Find the general solution of the equation

$$y' + 4xy = 2xe^{-x^2}\sqrt{y}. \qquad (31)$$

Solution: Let $y = uv$. Since the derivative of y is $y' = u'v + uv'$, then (31) can be transformed to the equation with respect to the variable $u(x)$:

$$u'v + v'u + 4xuv = 2xe^{-x^2}\sqrt{uv} \quad \Rightarrow$$

$$u'v + u(v' + 4xv) = 2xe^{-x^2}\sqrt{uv}. \qquad (32)$$

To find the function $v(x)$, we solve the equation

$$v' + 4vx = 0.$$

This is the separable equation, and its partial solution is

$$v = e^{-2x^2}. \qquad (33)$$

From (32) we have

$$u'e^{-2x^2} = 2xe^{-x^2}\sqrt{ue^{-2x^2}} \quad \Rightarrow \quad u' = 2x\sqrt{u} \quad \Rightarrow$$

$$\int \frac{du}{\sqrt{u}} = \int 2xdx + C \quad \Rightarrow \quad 2\sqrt{u} = x^2 + C \quad \Rightarrow$$

$$u = (x^2 + C)^2 / 4. \qquad (34)$$

Therefore, the general solution of the given equation is

$$y(x) = \frac{1}{4}(x^2 + C)^2 e^{-2x^2}. \qquad (35)$$

7.1. Exact Differential Equations

An exact differential equation has the following form

$$P(x,y)dx + Q(x,y)dy = 0, \qquad (36)$$

where the partial derivatives of $P(x,y)$ and $Q(x,y)$ obey the condition

$$P'_y = Q'_x. \qquad (37)$$

Due to condition (37), the expression on the left-hand side of (36) is the total differential of some function $u(x,y)$ by the theorem of a total differential (See Chapter 2, page 35.):

$$du(x,y) = P(x,y)dx + Q(x,y)dy = 0.$$

Therefore,

$$\frac{\partial u(x,y)}{\partial x} = P(x,y), \qquad (38)$$

$$\frac{\partial u(x,y)}{\partial y} = Q(x,y). \qquad (39)$$

If we hold fixed y, then by integrating of (38) with respect to x we obtain

$$u(x,y) = \int P(x,y)dx + \varphi(y). \qquad (40)$$

Note that a constant of integration may be a function of y because y is fixed during integration.

To find the function $\varphi(y)$, we substitute this expression for $u(x,y)$ into (39):

$$\frac{\partial}{\partial y}\int P(x,y)dx + \varphi'(y) = Q(x,y) \quad \Rightarrow$$

$$\varphi'(y) = Q(x,y) - \frac{\partial}{\partial y}\int P(x,y)dx. \qquad (41)$$

This is an ordinary differential equation for the function $\varphi(y)$. Note also that the expression on the right-hand side is a function of y only. Otherwise, the equation (36) is not an exact differential equation.

By solving equation (41), we find a partial solution $\varphi(y)$ and hence, the general solution: $u(x,y) = C$.

Example: Find the general solution of the equation

$$(y + 2x^{-2})dx + (x - 3y^{-2})dy = 0. \qquad (42)$$

Solution: Here $P(x,y) = y + 2x^{-2}$ and $Q(x,y) = x - 3y^{-2}$.

Let us check whether $P'_y(x,y) = Q'_x(x,y)$.

$$P'_y(x,y) = \frac{\partial}{\partial y}\left(y + \frac{2}{x^2}\right) = 1 \quad \text{and} \quad Q'_x(x,y) = \frac{\partial}{\partial x}\left(x - \frac{3}{y^2}\right) = 1.$$

Therefore, equation (42) is the exact differential equation of the form $du(x,y)=0$.

The general solution of this equation is $u(x,y)=C$.

All we need to write the answer is the function $u(x,y)$.

By formula (40),

$$u(x,y)=\int(y+\frac{2}{x^2})dx+\varphi(y)=yx-\frac{2}{x}+\varphi(y). \qquad (43)$$

In view of (41), we have

$$\varphi'(y)=x-\frac{3}{y^2}-\frac{\partial}{\partial y}(yx-\frac{2}{x}) \Rightarrow$$

$$\varphi'(y)=x-3y^{-2}-x=-3y^{-2}.$$

This is the directly integrable equation with a partial solution $\varphi(y)=3/y$.

Thus, by formula (44), we find the required function $u(x,y)$:

$$u(x,y)=yx-\frac{2}{x}+\frac{3}{y}.$$

Therefore, equation (42) has the following general integral:

$$yx-\frac{2}{x}+\frac{3}{y}=C.$$

In conclusion, let us note that any equation (36) can be transformed to the exact differential equation by multiplying both sides by some integrating factor $\mu(x,y)$. It is known that such a factor exists, however there is no general rule to find this factor but the following two cases:

1) If the expression $(Q'_x-P'_y)/P$ depends on the variable y only, then the integrating factor is also a function of y only, which obeys the equation

$$\frac{d\ln\mu(y)}{dy}=\frac{1}{P(x,y)}(Q'_x-P'_y).$$

2) If the expression $(Q'_x-P'_y)/Q$ depends on the variable x only, then the integrating factor is also a function of x only, which obeys the equation

$$\frac{d\ln\mu(x)}{dx}=-\frac{1}{Q(x,y)}(Q'_x-P'_y).$$

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