2 NUMERICAL INTEGRATION

Let function f (x) is defined and continuous on an interval [a, b]. It is required to calculate definite integral

$$J = \int_a^b f(x)dx$$

According to Newton-Leibnitz formula

$$J = \int_a^b f(x)dx = F(b) - F(a),$$

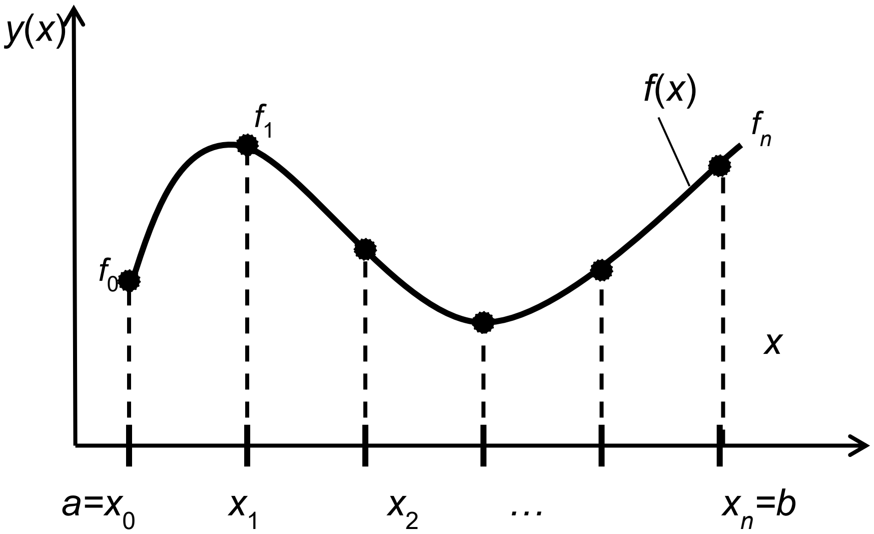
where f(x)- integrand,

F(x) - primitive of the integrand f (x).

Sometimes the primitive F(x) can be difficult for obtaining. Often function f(x) can be given numerically in the form of separate table values (counting or measurements).

In this case, the task of the numerical (approximate) integration can be set.

For its solution integral is calculated as the sum of values of the integrand $f(x_i)$, i=1,2,…n.



$$I \approx \sum_{i=1}^{n} A_i \cdot f(x_i), \qquad (2)$$

where A_i - numerical coefficients;

x_i – grid (mesh) nodes (knots, points) in which values of the function f(x) is calculated;

$x_i = x_{i-1} + h$; h=(b-a)/n – grid step;

$x_i = a + i \cdot h$;

n – the number of nodes in the interval (number of subintervals) [a, b].

The most popular is formulas of numerical integration for ***equidistant nodes***

Choice (number and location) integration nodes on a range [a, b] can be controlled accuracy (precision) of calculation.

Expression (1) is known quadrature formula. It does not allow to evaluate the integral, because the coefficient A_i are not known generally. However, by replacing on the each subinterval $[x_i, x_{i-1}]$ the integrand f (x) by the polynomial of some degree, you can get a simple formula for numerical integration.

The value of the integral is approximately equal to the area under the diagram of integrand function.

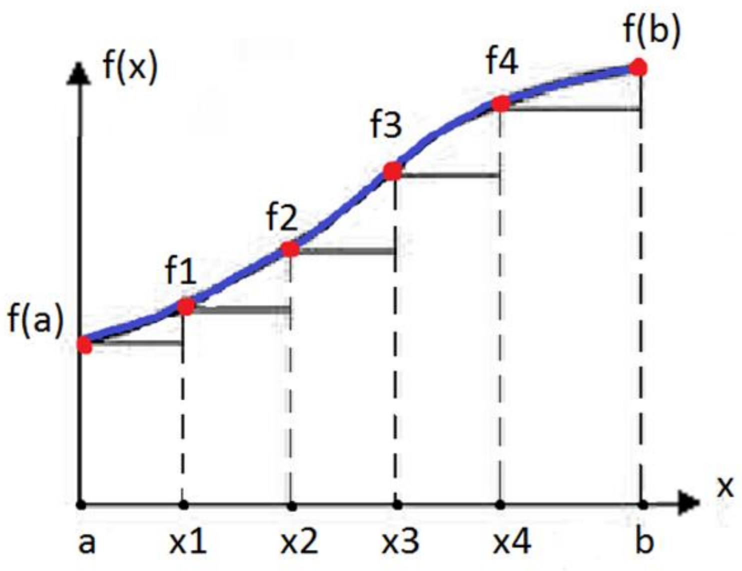
RECTANGULAR INTEGRATION

1. Let's divide the integration interval **[a, b]** into **n** equal parts – subintervals (partial intervals).

2. Within each of the i-th subintervals $[x_i, x_{i-1}]$, the integrand function f (x) is replaced by a polynomial of degree 0th P (x) (i.e. a direct (straigt) line, a parallel to the abscissa axis, x-axis).

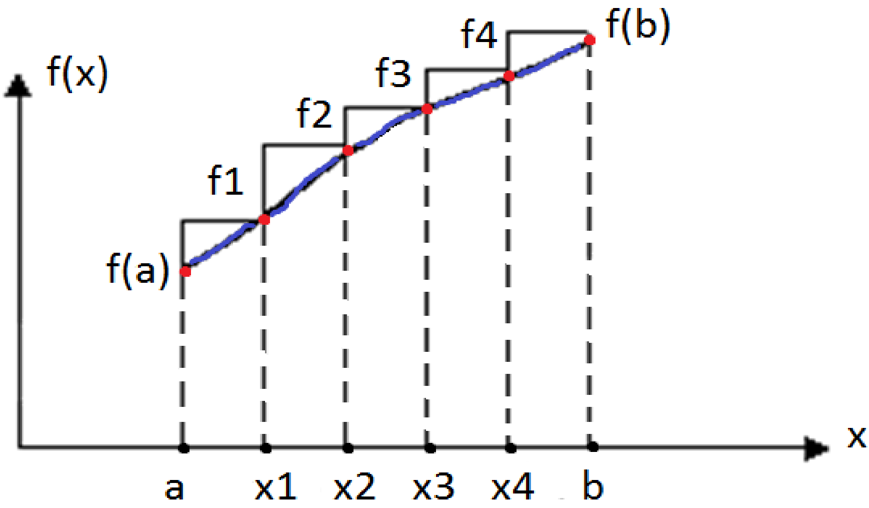
A straight line can be drawn through the left boundary of the interval $f(x_{i-1})$, through the right boundary - $f(x_{i-1})$ or mid point - $f\left(\frac{x_{i-1}+x_i}{2}\right)$. Methods for calculating the integrals named respectively - left, right and middle rectangles.

Method of left rectangles



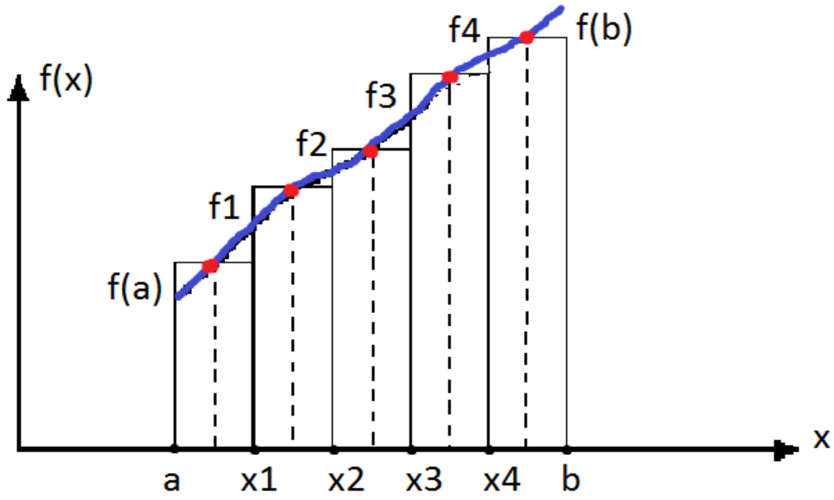
$$I \approx \frac{b-a}{n} \cdot \sum_{i=1}^{n} f(x_{i-1}). \qquad (2)$$

Method of left rectangles



$$I \approx \frac{b-a}{n} \cdot \sum_{i=1}^{n} f(x_i). \qquad (3)$$

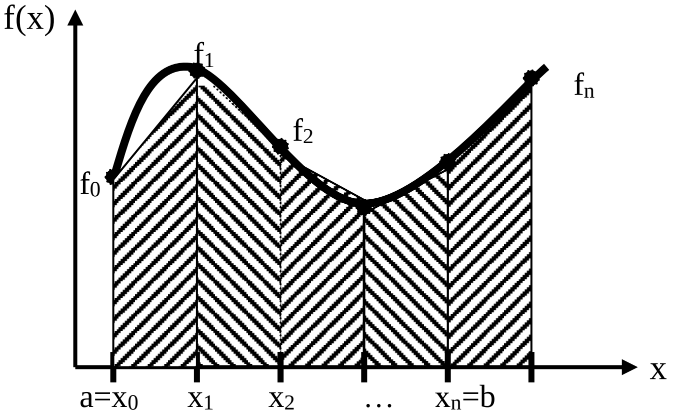
Method of average rectangles



$$I \approx \frac{b-a}{n} \cdot \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right). \qquad (4)$$

Geometrically, this means that the area of a curvilinear trapezoid approximately replaced by the sum of areas of rectangles with base h and height f(x).

TRAPEZOIDAL INTEGRATION



If value of integrand function is replaced on each subinterval $[x_{i-1}, x_i]$ a polynomial of 1st degree $\mathbf{P_1(x)}$, then the quadrature formula will become:

$$I \approx \frac{b-a}{2\cdot n}\cdot[f(x_0)+2\cdot\sum_{i=1}^{n-1} f(x_i)+f(x_n)] \qquad (5)$$

In this method the area of a curvilinear trapezoid is replaced by the sum of squares rectilinear trapezoidals.

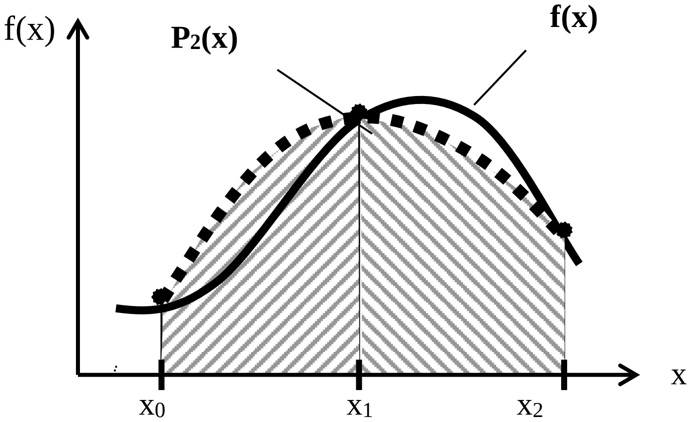
Way to reduce errors is to reduce the the integration step h.

SIMPSON (PARABOLAS) FORMULA

Simpson's formula (the formula of parabolas) are more accurate than previous methods.

The integrand function is replaced on each subinterval $[x_i, x_{i-1}]$ second-degree polynomial $\mathbf{P_2(x)}$ on three equidistant points.

Illustration for the single interval $[x_0, x_2]$.



$$S1 = \frac{h}{3}\cdot[f(x_0)+4\cdot f(x_1)+f(x_2)]$$

Summing all partial area, we will obtain Simpson's formula :

$$I=\frac{b-a}{3\cdot n}\cdot[f(x_0)+4\cdot\sum_{i=1,3}^{n-1}f(x_i)+2\cdot\sum_{i=2,4}^{n-1}f(x_i)+f(x_n)]. \quad (7)$$

Notes.

1. Simpson's formula is only defined for an even number of nodes n.
2. For odd n, we can add an additional node by any rule or make summing in the third item of a formula to the node (n-2).

ESTIMATES OF QUADRATURE FORMULAS

The assessment of an error of quadrature formulas can be done in two ways:

- on a formula of the remainder term,
- by Runge principle.

Runge principle allows us to estimate the integration error on the calculated values of the integral for step h and 2h (values Ih, I2h).

The general formula for any method

$$\mathbf{r} = \frac{\mathbf{I_h - I_{2h}}}{\mathbf{2^{k-1} - 1}}, \quad (8)$$

where k- order method,

k=3 - trapezoidal method;

k=5 – Simpson's method.

Based on the formula of the remainder the following estimates methods of integration are known.

Method of average rectangles:

$$|\mathbf{R_0(x)}| \le \frac{\mathbf{b-a}}{\mathbf{24}}\mathbf{h^2M_2}. \quad (9)$$

Trapezoidal method:

$$|\mathbf{R_1(x)}| \le \frac{\mathbf{b-a}}{\mathbf{12}}\mathbf{h^2M_2}. \quad (10)$$

Simpson's method:

$$|\mathbf{R_2(x)}| \le \frac{\mathbf{b-a}}{\mathbf{180}}\mathbf{h^4M_4}. \quad (11)$$

In this formulas of error estimates the value M2, M4 characterize the derivatives II, IV degree in the interval [a,b]

:

$$M_2 = \max|f''(x)|, \quad M_4 = \max|f^{IV}(x)|$$

$$x \in [a,b]$$
.

The formula of remainder term allows to find number **n** of points of subdivision of the interval [a, b] to evaluate the integral with the required accuracy ε.

Example.

Find the number **n** of points of subdivision of the interval [0, 3] to calculate the integral of the function **f(x)=sin(x)** by a method of trapezoids with accuracy **$\varepsilon=10^{-2}$.**

From a statement of the task the error does not exceed the given value of ε, that is :

$$|R_1(x)| \le \varepsilon.$$

For the method of trapezoids

$$|R_1(x)| \le \frac{b-a}{12} h^2 M_2$$

Replacing in this formula value of a step we $h = \dfrac{b-a}{n}$ will get

From this expression we find **n**:

$$R_1(x) \le \frac{b-a}{12} \cdot \frac{(b-a)^2}{n^2} \cdot M_2 = \frac{(b-a)^3}{12 \cdot n^2} \cdot M_2$$

$$n \ge \sqrt{\frac{(b-a)^3 \cdot M_2}{12 \cdot \varepsilon}}.$$

To calculate n we find the value M_2 in the interval [0, 3]:

$$f''(x) = -\sin(x); \quad \max|f''(x)| = 1$$

for any interval.

From here

$$n \ge \sqrt{\frac{3^3 \cdot 1}{12 \cdot 10^{-2}}} = 15.$$

So dividing the interval [0, 3] is not less than 15 parts, we can calculate the integral with an accuracy of 10^{-2}.