Task 1 (COUNTING)

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Mathematics is not primarily a matter of plugging numbers into formulas and performing rote computations. It is a way of questioning and thinking that may be unfamiliar to many of us, but is available to almost all of us.
John Allen Paulos, *A Mathematician Reads the Newspaper.*

1. A college team plays a series of 10 games which they can either win (W), lose (L) or tie (T).
    (a) How many possible outcomes can the series have (differentiating between WL and LW, i.e. order is important).
    (b) How many of these have exactly 5 wins, 4 losses and 1 tie?
    (c) Same as (a) if we don't care about the order of wins, losses and ties?

2. A student has to answer 20 true-false questions.
    (a) In how many distinct ways can this be done?
    (b) How many of these will have exactly 7 correct answers?
    (c) At least 17 correct answers?
    (d) Fewer than 3? (excludes 3).

3. In how many ways can 3 Americans, 4 Frenchmen, 4 Danes and 2 Canadians be seated (here we are particular about nationalities, but not about individuals)
    (a) in a row.
    (b) In how many of these will people of the same nationality sit together?
    (c) Repeat (a) with circular arrangement.
    (d) Repeat (b) with circular arrangement.

4. Four couples (Mr&Mrs. A, Mr&Mrs. B, ...) are to be seated at a round table.
    (a) In how many ways can this be done?
    (b) How many of these have all spouses sit next to each other?
    (c) How many of these have the men and women alternate?
    (d) How many of these have the men (and women) sit together?

5. In how many ways can we put 12 books into 3 shelves?
   Remark. This question is somehow ambiguous: do we want to treat the books as distinct or identical, and if we do treat them as distinct, do we care about the order in which they are placed within a shelf? The choice is ours, let's try it each way (the shelves are obviously distinct, and large enough to accommodate all 12 books if necessary).

6. Twelve men can be seated in a row in 12! = 479001600 number of ways (trivial).

(a) How many of these will have Mr. A and Mr. B sit next to each other?

(b) How many of the original arrangements will have Mr. A and Mr. B sit apart?

(c) How many of the original arrangements will have exactly 4 people sit between Mr. A and Mr. B?

7. Security council of the UN has 15 permanent members, US, Russia, GB, France and China among them. These can be seated in a row in 15! possible arrangements. How many of these have France and GB sit together but (at the same time) US and Russia sit apart?

8. Consider the standard deck of 52 cards (4 suits: hearts, diamonds, spades and clubs, 13 ’values’: 2, 3, 4...10, Jack, Queen, King, Ace). Deal 5 cards from this deck. This can be done in 2598960 distinct ways (trivial).

(a) How many of these will have exactly 3 diamonds?

(b) Exactly 2 aces?

(c) Exactly 2 aces and 2 diamonds?

9. In how many ways can we deal 5 cards each to 4 players?

(b) So that each gets exactly one ace?

(c) None gets any ace.

(d) Mr. A gets 2 aces, the rest get none.

(e) (Any) one player gets 2 aces, the other players get none.

(f) Mr. A gets 2 aces.

(g) Mr. C gets 2 aces.

(h) At least one player gets 2 aces (regardless of what the others get).

10. (Game of Poker): 5 cards are dealt from an ordinary deck of 52. The total number of possible outcomes (5-card hands) is 2598960 (trivial). How many of these contain exactly:

(a) One pair, i.e. two identical values (and no other duplication of values).

(b) Two pairs.

(c) A triplet.

(d) Full house (a pair and a triplet).

(f) A straight (five consecutive values − ace can be considered both as the highest and the lowest value, i.e. Ace, 2, 3, 4, 5 is a straight).

(g) Flush (five cards of the same suit).

(h) We should note that a hand can be both a straight and a flush (a first overlap encountered so far).

(i) None of the above.

11. Roll a die five times. The number of possible (ordered) outcomes is 7776 (trivial). How many of these will have:

(a) One pair of identical values(and no other duplicates).

(b) Two pairs.
(d) ’Full house’ (a triplet and a pair).
(e) ’Four of a kind’.
(g) Nothing.

12. Let us try the same thing with 15 rolls of a die (4.7018×10^{11} outcomes in total). How many of these will have:
(a) A quadruplet, 2 triplets, 2 pairs and 1 singlet.
(b) 3 triplets and 3 pairs.

Answers 1

1. (a) Answer: $3^{10} = 59049$.
(b) Answer: $\binom{10}{5,4,1} = 1260$.
(c) Answer: $\binom{12}{2} = 66$ (only one of these will have 5 wins, 4 losses and 1 tie).

2. (a) Answer: $2^{20} = 1048576$.
(b) Answer: $\binom{20}{7} = 77520$.
(c) (Here, these is no ’shortcut’ formula, we have to do this individually, one by one, adding the results): $\binom{20}{17}+\binom{20}{18}+\binom{20}{19}+\binom{20}{20}=\binom{20}{3}+\binom{20}{2}+\binom{20}{1}+\binom{20}{0} = 1351$.
(d) $\binom{20}{2}+\binom{20}{1}+\binom{20}{0} = 211$.

3. (a) Answer (same as the number of permutations of AAAF F F F DDDDCC): $\binom{13}{3,4,4,2} = 900900$.
(b) Answer: We just have to arrange the four nationalities, say A, F, D and C: 4! = 24.
(c) Answer (13 of the original arrangements are duplicates now, as AAAF F F F DDDDCC, AAF F F F DDDDCCA,...,CAAAF F F F DDDDC are identical): 900900\13= 69300.
(d) Answer (circular arrangement of nationalities): 3! = 6.

4. (a) Answer: 7! = 5040.
(b) Solution: First we have to arrange the families with respect to each other. This can be done in 3! ways. Then, having two seats reserved for each couple, we have to decide the mutual position of every wife and husband ($2 \times 2 \times 2 \times 2$).
Answer: $3!\times2^4 = 96$. (Later on, our main task will be converting these to probabilities. If the seating is done randomly, the probability of keeping the spouses together will be then 96/5040 = 1.905%).
(c) Solution: Place Mr. A into one chair, then select his right-hand neighbor (must be a woman) in 4 ways, select her extra neighbor (3 ways), and so on.
Answer: $4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 144$ (corresponds to 2.86%).

(d) Solution: This is analogous to (b). We have to arrange the two groups (men and women) with respect to each other first. But, in the circular arrangement, this can be done in one way only! Then, we have to take care of arranging the 4 men and four women within the four chairs allocated to them. This can be done in 4! ways each.
Answer: $(4!)^2 = 576$ (correspond to 11.43%).

5. (a) If the books are treated as 12 identical copies of the same novel, then the only decision to be made is: how many books do we place on each shelf.
The answer follows from (Unordered Selection, Allowing Duplication) with n = 3 and r = 12 - for each book we have to select a shelf, but the order does not matter (1, 3 and 3, 1 puts one book each on Shelf 1 and 3), and duplication is allowed: $\binom{14}{2} = 91$.

(b) If we treat the books as distinct and their order within each shelf important, we solve this in two stages: First we decide how many books we place in each shelf, which was done in (a), then we choose a book to fill, one by one, each allocated slot (here we have 12 × 11 × 10 × .... × 2 × 1 choices).
Answer: 91 × 12! = 43, 589, 145, 600.

(c) Finally, if the books are considered all distinct, but their arrangement within each shelf is irrelevant, we simply have to decide which shelf will each book go to [similar to (a), order important now].
This can be done in $3\times3\times3\times....\times3 = 3^{12} = 531441$ number of ways.

6. (a) Solution: Mr. A and Mr. B have to be first treated as a single item, for a total of 11 items. These can be permuted in 11! number of ways. Secondly, we have to place Mr. A and Mr. B in the two chairs already allocate to them, in 2! ways.
Answer: 2 × 11! = 79833600.

(b) This set consists of those which did not have them sit next to each other, i.e. 12! − 2 × 11! = 399168000.

(c) Solution: First, we allocate two chairs for Mr. A and Mr. B, thus: ¥¤¤¤¤¥¤¤¤¤¤¤,¤¥¤¤¤¤¥¤¤¤¤¤,....,¤¤¤¤¤¤¥¤¤¤¤¥, altogether in 7 possible ways (here we count using our fingers − no fancy formula). Secondly, we seat the people. We have 10! choices for filling the ¤¤...¤ chairs, times 2 choices for how to fill ¥ and ¥.
Answer: 7 × 2 × 10! = 50803200.

7. Solution: We break the problem into two parts:
i. France and GB sit together in 2 × 14! = 174, 356, 582, 400 of the original 15! arrangements (we understand the logic of this answer from the previous question).
ii. France and GB sit together and (at the same time) US and Russia sit together in 2×2×13! = 24, 908, 083, 200 arrangements (similar logic: first we create two groups of two, one for France/GB, the other for US/Russia and permute the resulting 13 items, then we seat the individual people).
The answer is obviously the difference between the two: $2 \times 14! - 2^2 \times 13! = 149, 448, 499, 200$.
(To make the answer more meaningful, convert it to probability).

8. (a) Solution: First select 3 diamonds and two 'non-diamonds', then combine these together, in $\binom{13}{3}\binom{39}{2} = 211926$ number of ways.

(b) Same logic: $\binom{4}{2}\binom{48}{3} = 103776$.

(c) This is slightly more complicated because there is a card which is both an ace and a diamond. The deck must be first divided into four parts, the ace of diamonds (1 card) the rest of the aces (3), the rest of the diamonds (12), the rest of the cards (36). We then consider two cases, either the ace of diamonds is included, or not. The two individual answers are added, since they are mutually incompatible (no 'overlap'): $\binom{1}{1}\binom{3}{1}\binom{12}{1}\binom{36}{2}+\binom{1}{0}\binom{3}{2}\binom{12}{2}\binom{36}{1}$=29808.

9. (a) Answer: $\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5}= 1.4783 \times 10^{24}$.

(b) Answer (consider dealing the aces and the non-aces separately): $4!\binom{48}{4}\binom{44}{4}\binom{40}{4}\binom{36}{4}= 3.4127 \times 10^{21}$.

(c) Answer: $\binom{48}{5}\binom{43}{5}\binom{38}{5}\binom{33}{5}= 1.9636 \times 10^{23}$.

(d) Answer: $\binom{4}{2}\binom{48}{3}\binom{45}{5}\binom{40}{5}\binom{35}{5}= 2.7084 \times 10^{22}$.

(e) Solution: The previous answer is correct whether it is Mr. A, B, C or D who gets the 2 aces (due to symmetry), all we have to do is to add the four (identical) numbers, because the four corresponding sets cannot overlap, i.e. are mutually incompatible or exclusive). Answer: $4 \times 2.7084 \times 10^{22}= 1.0834 \times 10^{23}$

(f) Answer: $\binom{4}{2}\binom{48}{3}\binom{47}{5}\binom{42}{5}\binom{37}{5}= 5.9027\times10^{22}$. Note that when computing the probability of this happening, the $\binom{47}{5}\binom{42}{5}\binom{37}{5}$ part cancels out (we can effectively deal 5 cards to him and stop).

(g) Solution: If he is the third player to be dealt his cards, we can either do this to long and impractical way (taking into account how many aces have been dealt to Mr. A and Mr. B), thus: $=5.9027\times 10^{22}$, or be smart and argue that, due to the symmetry of the experiment, the answer must be the same as for Mr. A.

(h) This is quite a bit more difficult, to extend that we must postpone solving it.

10. (a) Solution: This is done in two stages, first we select the value to be represented by a pair and three distinct values to be represented by a singlet each: $\binom{13}{1}\binom{12}{3}$, then we select two individual cards from four suits: $\binom{4}{2}$ and the other 3 cards each from four suits: 4^3.

Answer: $\binom{13}{1}\binom{12}{3}\binom{4}{2}4^3= 1098240$.

(b) Following the same logic: $\binom{13}{2}\binom{11}{1}\binom{4}{2}^2 4= 123552$.

(c) $\binom{13}{1}\binom{12}{2}\binom{4}{3}4^2 = 54912$.

(d) $\binom{13}{1}\binom{12}{1}\binom{4}{3}\binom{4}{2}= 3744$.

(e) $\binom{13}{1}\binom{12}{1}\binom{4}{4}\binom{4}{1}= 624$.

(f) Solution: There are 10 possibilities as to the sequence of values (starting from Ace...5, up to 10...Ace), once this is chosen, one has to select the individual cards: $4 \times 4 \times 4 \times 4 \times 4$.
Answer: $10 \times 4^5 = 10240$.

(g) Solution: 4 ways of selecting the suit, $\binom{13}{5}$ ways of selecting the individual cards from it.
Answer: $4\binom{13}{5} = 5148$.

(h) We should note that a hand can be both a straight and a flush (a first overlap encountered so far).
We have again 10 possibilities for the values, but only 4 ways of selecting the cards (they must be of the same suit). The number of these hands is thus $10 \times 4 = 40$.

(i) Solution: First we select five distinct values, disallowing the 10 cases resulting in a straight: $\binom{13}{5} - 10$, then we select one card of each chosen value, disallowing a flush, which happens in only 4 of these cases: $4^5 - 4$.
Answer: $\left(\binom{13}{5} - 10\right)(4^5 - 4) = 1302540$.

One can verify that adding all these answers, except for (h) which needs to be subtracted (why?), results in the correct total of 2598960 (check).

11. (a) Solution: First we choose the value which should be represented twice and the three values to go as singles: $\binom{6}{1}\binom{5}{3}$, then we decide how to place the 5 selected numbers in the five blank boxes, which can be done in $\binom{5}{2,1,1,1}$ ways (equal to the number of aabcd permutations).
Answer: $\binom{6}{1}\binom{5}{3}\binom{5}{2,1,1,1} = 3600$.

(b) The same logic gives: $\binom{6}{2}\binom{4}{1}\binom{5}{2,2,1} = 1800$.

(c) $\binom{6}{1}\binom{5}{2}\binom{5}{3,1,1} = 1200$.

(d) $\binom{6}{1}\binom{5}{1}\binom{5}{3,2} = 300$.

(e) $\binom{6}{1}\binom{5}{1}\binom{5}{4,1} = 150$.

(f) $\binom{6}{1}\binom{5}{5} = 6$.

(g) Solution: We again fill the empty boxes, one by one, avoiding any duplication: $6 \times 5 \times 4 \times 3 \times 2 = 720$.
Note that all these answers properly add up to 7776 (check).

12. (a) $\binom{6}{1}\binom{5}{2}\binom{3}{2}\binom{1}{1}\times\binom{15}{4,3,3.2,2,1} = 6.8108 \times 10^{10}$.

(b) $\binom{6}{3}\binom{3}{3}\binom{15}{3,3,3.2,2,2} = 1.5135 \times 10^{10}$.

We will not try to complete this exercise; the full list would consist of 110 possibilities.