Professional English
(Kitayeva Anna)

Probability and Statistics Pre-course Survey

The purpose of this survey is to indicate what you already know and think about probability and statistics.
If you are unsure of what you are being asked to do, please consult the teacher.

Part I
There are a series of statements concerning beliefs or attitudes about probability, statistics and mathematics. Following each statement is an "agreement" scale which ranges from 1 to 5, as shown below.

1	2	3	4	5
Strongly Disagree	Disagree	Neither Agree, nor Disagree	Agree	Strongly Agree

If you strongly agree with a particular statement, circle the number 5 on the scale. If you strongly disagree with the statement, circle the number 1.

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1. I often use statistical information in forming my opinions or making decisions.
1 2 3 4 5

2. To be an intelligent consumer, it is necessary to know something about statistics.
1 2 3 4 5

3. Because it is easy to lie with statistics, I don't trust them at all.
1 2 3 4 5

4. Understanding probability and statistics is becoming increasingly important in our society, and may become as essential as being able to add and subtract.
1 2 3 4 5

5. Given the chance, I would like to learn more about probability and statistics.
1 2 3 4 5

6. You must be good at mathematics to understand basic statistical concepts.
1 2 3 4 5

7. When buying a new car, asking a few friends about problems they have had with their cars is preferable to consulting an owner satisfaction survey in a consumer magazine.
1 2 3 4 5

8. Statements about probability (such as what the odds are of winning a lottery) seem very clear to me.
1 2 3 4 5

9. I can understand almost all of the statistical terms that I encounter in newspapers or on television.
1 2 3 4 5

10. I could easily explain how an opinion poll works.
1 2 3 4 5

Part II

1. A small object was weighed on the same scale separately by nine students in a science class. The weights (in grams) recorded by each student are shown below.

6.2 6.0 6.0 15.3 6.1 6.3 6.2 6.15 6.2

The students want to determine as accurately as they can the actual weight of this object. Of the following methods, which would you recommend they use?

_____ a. Use the most common number, which is 6.2.
_____ b. Use the 6.15 since it is the most accurate weighing.
_____ c. Add up the 9 numbers and divide by 9.
_____ d. Throw out the 15.3, add up the other 8 numbers and divide by 8.

2. A marketing research company was asked to determine how much money teenagers (ages 13 -19) spend on recorded music (cassette tapes, CDs and records). The company randomly selected 80 malls located around the country. A field researcher stood in a central location in the mall and asked passers-by who appeared to be the appropriate age to fill out a questionnaire. A total of 2,050 questionnaires were completed by teenagers. On the basis of this survey, the research company reported that the average teenager in this country spends $155 each year on recorded music.

Listed below are several statements concerning this survey. Place a check by every statement that you agree with.

_____ a. The average is based on teenagers' estimates of what they spend and therefore could be quite different from what teenagers actually spend.
_____ b. They should have done the survey at more than 80 malls if they wanted an average based on teenagers throughout the country.
_____ c. The sample of 2,050 teenagers is too small to permit drawing conclusions about the entire country.
_____ d. They should have asked teenagers coming out of music stores.
_____ e. The average could be a poor estimate of the spending of all teenagers given that teenagers were not randomly chosen to fill out the questionnaire.
_____ f. The average could be a poor estimate of the spending of all teenagers given that only teenagers in malls were sampled.
_____ g. Calculating an average in this case is inappropriate since there is a lot of variation in how much teenagers spend.
_____ h. I don't agree with any of these statements.

3. Which of the following sequences is most likely to result from flipping a fair coin 5 times?

_____ a. H H H T T
_____ b. T H H T H
_____ c. T H T T T
_____ d. H T H T H
_____ e. All four sequences are equally likely.

4. Select the alternative below that is the best explanation for the answer you gave for the item above.

_____ a. Since the coin is fair, you ought to get roughly equal numbers of heads and tails.
_____ b. Since coin flipping is random, the coin ought to alternate frequently between landing heads and tails.
_____ c. Any of the sequences could occur.
_____ d. If you repeatedly flipped a coin five times, each of these sequences would occur about as often as any other sequence.
_____ e. If you get a couple of heads in a row, the probability of a tails on the next flip increases.
_____ f. Every sequence of five flips has exactly the same probability of occurring.

5. Listed below are the same sequences of Hs and Ts that were listed in Item 3. Which of the sequences is least likely to result from flipping a fair coin 5 times?
_____ a. H H H T T
_____ b. T H H T H
_____ c. T H T T T
_____ d. H T H T H
_____ e. All four sequences are equally unlikely.

6. The Caldwells want to buy a new car, and they have narrowed their choices to a Buick or a Oldsmobile. They first consulted an issue of Consumer Reports, which compared rates of repairs for various cars. Records of repairs done on 400 cars of each type showed somewhat fewer mechanical problems with the Buick than with the Oldsmobile.
The Caldwells then talked to three friends, two Oldsmobile owners, and one former Buick owner. Both Oldsmobile owners reported having a few mechanical problems, but nothing major.
The Buick owner, however, exploded when asked how he like his car: "First, the fuel injection went out - $250 bucks. Next, I started having trouble with the rear end and had to replace it. I finally decided to sell it after the transmission went. I'd never buy another Buick."
The Caldwells want to buy the car that is less likely to require major repair work. Given what they currently know, which car would you recommend that they buy?
_____ a. I would recommend that they buy the Oldsmobile, primarily because of all the trouble their friend had with his Buick. Since they haven't heard similar horror stories about the Oldsmobile, they should go with it.
_____ b. I would recommend that they buy the Buick in spite of their friend's bad experience. That is just one case, while the information reported in Consumer Reports is based on many cases. And according to that data, the Buick is somewhat less likely to require repairs.
_____ c. I would tell them that it didn't matter which car they bought. Even though one of the models might be more likely than the other to require repairs, they could still, just by chance, get stuck with a particular car that would need a lot of repairs. They may as well toss a coin to decide.

7. Half of all newborns are girls and half are boys. Hospital A records an average of 50 births a day. Hospital B records an average of 10 births a day. On a particular day, which hospital is more likely to record 80% or more female births?
_____ a. Hospital A (with 50 births a day)
_____ b. Hospital B (with 10 births a day)
_____ c. The two hospitals are equally likely to record such an event.

8. "Megabucks" is a weekly lottery played in many states. The numbers 1 through 36 are placed into a container. Six numbers are randomly drawn out, without replacement. In order to win, a player must correctly predict all 6 numbers. The drawing is conducted once a week, each time beginning with the numbers 1 through 36.
The following question about the lottery appeared in The New York Times (May 22, 1990).
Are your odds of winning the lottery better if you play the same numbers week after week or if you change the numbers every week?
What do you think?
______a. I think the odds are better if you play the same numbers week after week.
______b. I think the odds are better if you change the numbers every week.
______c. I think the odds are the same for each strategy.

9. For one month, 500 elementary students kept a daily record of the hours they spent watching television. The average number of hours per week spent watching television was 28. The researchers conducting the study also obtained report cards for each of the students. They found that the students who did well in school spent less time watching television than those

students who did poorly. Listed below are several possible statements concerning the results of this research. Place a check by every statement that you agree with.

_____ a. The sample of 500 is too small to permit drawing conclusions.

_____ b. If a student decreased the amount of time spent watching television, his or her performance in school would improve.

_____ c. Even though students who did well watched less television, this doesn't necessarily mean that watching television hurts school performance.

_____ d. One month is not a long enough period of time to estimate how many hours the students really spend watching television.

_____ e. The research demonstrates that watching television causes poorer performance in school.

_____ f. I don't agree with any of these statements.

10. An experiment is conducted to test the efficiency of a new drug on curing a disease. The experiment is designed so that the number of patients who are cured using the new drug is compared to the number of patients who are cured using the current treatment. The percentage of patients who are cured using the current treatment is 50% and 65% are cured who have used the new drug. A P-value of 5% (.05) is given as an indication of the statistical significance of these results. The P-value tells you:

______a. There is a 5% chance that the new drug is more effective than the current treatment.

______b. If the current treatment and the new drug were equally effective, then 5% of the times we conducted the experiment we would observe a difference as big or bigger than the 15% we observed here.

______c. There is a 5% chance that the new drug is better than the current treatment by at least 15%.

11. Gallup* reports the results of a poll that shows that 58% of a random sample of adult Americans approve of President Clinton's performance as president. The report says that the margin of error is 3%. What does this margin of error mean?

______a. One can be 95% "confident" that between 55% and 61% of all adult Americans approve of the President's performance.

______b. One can be sure that between 55% and 61% of all adult Americans approve of the President's performance.

______c. The sample percentage of 58% could be off by 3% in either direction due to inaccuracies in the survey process.

______d. There is a 3% chance that the percentage of 58 is an inaccurate estimate of the population of all Americans who approve of President Clinton's performance as president.

*Gallup Poll - a sampling by the American Institute of Public Opinion or its British counterpart of the views of a representative cross section of the population, used as a means of forecasting voting. Etymology: named after George Horace Gallop (1901-84), US statistician.

Gallup Poll - опрос Гэллапа, опрос общественного мнения (анкетный опрос населения по различным вопросам, политическим и социальным. Проводится с 1938 Британским институтом общественного мнения [British Institute of Public Opinion], а с 1952 институтом "Социальные исследования (опросы Гэллапа)" [Social Surveys (Gallup Polls) Ltd]) назван по имени основателя американского Института общественного мнения Дж. Гэллапа [George Horace Gallup, 1901-84].

General conditions for succeeding in your professional life:

- Work hard: Hard work is the best investment a man can make. If the power to do hard work is not talent, it is the best possible substitute for it. Ambition by itself never gets anywhere until it forms a partnership with work.
- Study hard: Knowledge enables a man to work more intelligently and effectively.
- Love your work: Then you will find pleasure in mastering it.
- Have initiative: Ruts often deepen into graves.
- Be exact: Slipshod methods bring slipshod results.
- Have the spirit of conquest: Thus you can successfully battle and overcome difficulties.
- Cultivate personality: Personality is to a man what perfume is to the flower.
- Be fair: Unless you feel right towards your fellow men you can never be a successful leader.
- Help and share with others: The real test of life greatness lies in giving opportunity to others.
- In all things do your best: The man who has done his best has done everything. The man who has done less than his best has done nothing. If you do the best you can then you will find that you have done as well as or better than anyone else.

To Educe means to bring out a potential existence. Education, therefore, is a process of intellectual growth. Education is essential to change, for education creates both new wants and the ability to satisfy them. A student must be curious, open-minded, reflective, strategic, skeptical, and must learn for deep understanding. Education is a progressive discovery of our own ignorance. To discover your potentiality, think for yourself.

Formal education must teach you how to think. The hard facts you learn are secondary to that. Education is not how much you have committed to memory, or even how much you know. It is being able to differentiate between what you know and what you do not. Teaching should be such that what is offered by the teacher will be perceived as a valuable gift and not as a hard duty.

The big thing you take away from school is how to induct and deduct in a constructive way. The object of education is to prepare the students to educate themselves throughout their lives. Critical thinking, self-examination, and the questioning of assumptions are all widely genuflected to as part of any good education. … students are active participants creating knowledge and developing skills rather than passive recipients of information. Students appreciate a teacher who gives them something to take home to think about besides homework. One often forgets what one was taught. However, one only can make use of what one has learnt.

All the interest of education should come together to make students responsible decision makers. Our highest endeavor must be to develop free human beings who, of themselves, are able to give purpose and direction to their lives. It is the ability to decide for yourself and the responsibility for making a self for yourself; the educator is merely a mid-wife in this process. A professor should never measure students intelligence with his/her own, otherwise there is no progress for the new generation. Moreover, there is no such thing as being intelligent in everything and all the time.

Problem Solving in Mathematics: When we think of having the ability 'to do' math, we think numbers. However, math is more than computations and numbers; it's about being able to solve problems. Applying what you know to the situation presented. Here are a few tips on becoming successful at problem solving.

The main reason for learning all about math is to become better problem solvers in all aspects of life. Many problems are multi step and require some type of systematic approach. Most of all, there are a couple of things you need to do when solving problems. Ask yourself exactly what type of information is being asked for. Then determine all the information that is being given to you in the question. When you clearly understand the answers to those two questions, you are then ready to devise your plan. Some key questions as you approach the problem may be:

What are my key words?

Do I need a diagram? List? Table?

Is there a formula or equation that I'll need? Which one?

Will I use a calculator? Is there a pattern I can use and or follow?

Remember: Read the problem carefully, decide on a method to solve the problem, solve the problem.

"And happiness too swiftly flies
Thought would destroy their paradise
No more; where ignorance is bliss, 'tis folly to be wise."
('Ode on a Distant Prospect of Eton College')

About style

"I realized another thing, that in this world fast runners do not always win the race, and the brave do not always win the battle.
Wise men do not always earn a living, intelligent man do not always get rich, and capable men do not always rise to high positions. Bad luck happens to everyone."

"I returned and saw under the sun that the race is not to the swift, nor the battle to the strong, neither yet bread to the wise, nor yet riches to men of understanding, not yet favor to men of skill; but time and chance happeneth to them all."

"Objective consideration of contemporary phenomena compels the conclusion that success or failure in competitive activities exhibits no tendency to be commensurate with innate capacity, but that a considerable element of the unpredictable must invariably be taken into account."

What are the names of the operands in common operations?

- addend + addend = sum (*also*: term + term = sum)
- minuend - subtrahend = difference
- multiplier × multiplicand = product (*also*: factor × factor = product)
- dividend / divisor = quotient
- base$^{\text{exponent}}$ = power
- $^{\text{index}}\sqrt{}$radicand = root

(*multiplier* and *multiplicand* can be used for *noncommutative* multiplications.)

Note the consistent use of the suffixes, which are of Latin origin:

- Divid<u>end</u>= "That which <u>is to be</u> divided" (the orator Cato ended all of its speeches with the famous quote "Carthago del<u>end</u>a est": Catharge is to be destroyed). In a nonmathematical context, dividends are profits that are to be divided among all shareholders.
- Divis<u>or</u> = "That which <u>is to do</u> the dividing". A direct<u>or</u> does the directing, an advis<u>or</u> does the advising, etc. In ancient Rome, the Emperor was the "Imperat<u>or</u>", the one supposed to issue the orders (Latin: "Imperare")

How native speakers pronounce formulas in English?

Here are the basics:

- x+y : "x plus y".
- x-y : "x minus y".
- x y : "x times y" or "x into y" (the latter is more idiomatic).
- x/y : "x over y".
  - If x and y are both integers, y is pronounced as an ordinal: 3/4 = "three fourth".
- x^y : "x to the power of y".
  - Longer version (elementary level): "x raised to the power of y".
  - Shorter version (when x is easy to pronounce): "x to the y".
  - If y is an integer, "x to the y^{th} [power]" ("power" is often dropped).
  - If y is 2 or 3, "x square[d]" or "x cube[d]" are most common.

There's also the issue of indicating parentheses and groupings when pronouncing expressions. If the expression is *simple enough*, a parenthesis is adequately pronounced by marking a short pause. For example, x (y+z) / t could be spoken out "x times ... y plus z ... over t" (pronouncing "y plus z" very quickly). When dictating more complex expressions involving parentheses, it's best to indicate *open parent'* and *close parent'*, whenever needed.

Sample space

is a collection of all possible outcomes of an experiment. The individual (complete) outcomes are called simple events.

A few examples:

1. Rolling a die
2. Rolling 2 (*n* in general) dice (or, equivalently, one die twice, or *n* times)
3. Selecting 2 people out of 4 (k objects out of *n* in general)
4. Flipping a coin until a head appears
5. Rotating a wheel with a pointer
6. Flipping a tack $(\perp)$.

For the examples above, we get:

1. The outcomes can be uniquely represented by the number of dots shown on the top face. The sample space is thus the following set of six elements: $\{1, 2, 3, 4, 5, 6\}$.

2. With two dice, we have a decision to make: do we want to consider the dice as indistinguishable (to us, they usually are) and have the sample space consist of unordered pairs of numbers, or should we mark the dice (red and green say) and consider an ordered pair of numbers as an outcome of the experiment (the first number for red, the second one for green die)? The choice is ours; we are allowed to consider as much or as little detail about the experiment as we need, but there two constraints:

(a) We have to make sure that our sample space has enough information to answer the questions at hand (if the question is: what is the probability that the red die shows a higher number than the green die, we obviously need the ordered pairs).

(b) Subsequently, we learn how to assign probabilities to individual outcomes of a sample space. This task can quite often be greatly simplified by a convenient design of the sample space. It just happens that, when rolling two dice, the simple events (pairs of numbers) of the sample space have the same simple probability of 1/36 when they are ordered; assigning correct probabilities to the unordered list would be extremely difficult.

That is why, for this kind of experiment (rolling a die any fixed number of times), we always choose the sample space to consist of an ordered set of numbers (whether the question requires it or not).

In the case of two dice, we will thus use the following (conveniently organized) sample space:

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

and correspondingly for more than 2 dice (we will no longer be able to write it down explicitly, but we should be able to visualize the result). Note that a single simple event consists of two (or more) numbers. As explained earlier, we will never try to simplify this sample space by removing the order; there is one simplification one can make though, if the question is concerned only with sixes versus non-sixes: we can reduce the sample space of 36 simple events to: {66, 6O, O6, OO} where O stands for any other number but 6.

Assigning probabilities will be a touch more difficult now, but it will prove to be manageable.

3. Selecting 2 (distinct) people out of 4. Here (unless the question demands it), we can ignore the order of the selection, and simplify the sample space to: {AB, AC, AD, BC, BD, CD} [unordered pairs], with $\binom{4}{2} = 6$ equally likely outcomes (simple events). Selecting k out of n objects will similarly result in $\binom{n}{k}$ equally likely possibilities. Another typical experiment of this kind is dealing 5 cards out of 52.

4. The new feature of this example (waiting for the first head) is that the sample space is infinite: {H, T H, T T H, T T T H, T T T T H, T T T T T H, ...}. Eventually, we must learn to differentiate between the discrete (countable) infinity, where the individual simple events can be labeled 1st, 2nd, 3rd, 4th, 5th, ... in an exhaustive manner, and the continuous infinity (real numbers in any interval). The current example is obviously a case of discrete infinity, which implies that the simple events cannot be equally likely (they would all have the probability of $1/\infty = 0$, implying that their sum is 0, an obvious contradiction). But we can easily manage to assign correct and meaningful probabilities even in this case (as discussed at the lecture).

5. The rotating wheel has also an infinite sample space (an outcome is identified with the final position − angle − of the pointer, measured from some fixed direction), this time being represented by all real numbers from the interval $[0, 2\pi)$ [assuming that angles are measured in radians]. This infinity of simple events is of the continuous type, with some interesting consequences. Firstly, from the symmetry of the experiment, all of its outcomes must be equally likely. But this implies that the probability of each single outcome is zero!

Isn't this a contradiction as well? The answer is no; in this case the number of outcomes is no longer countable, and therefore the infinite sum (actually, an integral) of their zero probabilities can become nonzero (we need them to add up to 1). The final puzzle is: how do we put all these zero probabilities together to answer a simple question such as: what is the probability that the pointer will stop in the $[0, \pi/2]$ interval? This will require introducing a new concept of the so called probability density (probability of an interval, divided by the length of the interval). We will postpone this until the second part of this course (random variables).

6. What exactly is new about the tack and its two simple outcomes: $\{\perp, \curlywedge\}$.

Here, for the first time, we will not be able to introduce probabilities based on any symmetry argument, these will have to be established empirically by flipping the tack many times, finding the proportion of times it lands in the $(\perp)$ position and calling this the probability of $(\perp)$ (to be quite correct, the exact probability of $(\perp)$ is the limit of these experiments, when their number approaches infinity). That effectively implies that the probability of any such event can never be known exactly; we deal with this by replacing it by a parameter p, which we substitute for the exact probability in all our formulas. Eventually we may learn (if we manage to reach those chapters) how to test hypotheses concerning the value of p (such as, for example, $p = 0.7$).

Give a possible sample space for each of the following experiments:
(a) An election decides between two candidates A and B.
(b) A two-sided coin is tossed.
(c) A student is asked for the month of the year and the day of the week on which her birthday falls.
(d) A student is chosen at random from a class of ten students.
(e) You receive a grade in this course.

For which of the cases would it be reasonable to assign the uniform distribution function?

PARADOXES

The main reason for learning all about math is to become better problem solvers in all aspects of life.

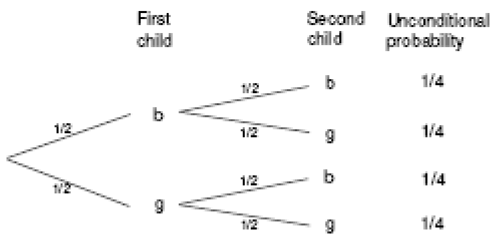
One must be very careful in dealing with problems involving conditional probability.

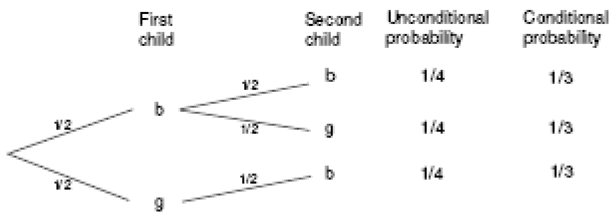
You will recall that in the Monty Hall problem, if the contestant chooses the door with the car behind it, then Monty has a choice of doors to open. We made an assumption that in this case, he will choose each door with probability 1/2. We then noted that if this assumption is changed, the answer to the original question changes. In this section, we will study other examples of the same phenomenon.

Example 1. Consider a family with two children. Given that one of the children is a boy, what is the probability that both children are boys?

One way to approach this problem is to say that the other child is equally likely to be a boy or a girl, so the probability that both children are boys is 1/2.

The "textbook" solution would be to draw the tree diagram and then form the conditional tree by deleting paths to leave only those paths that are consistent with the given information. The result is shown in the Figure.





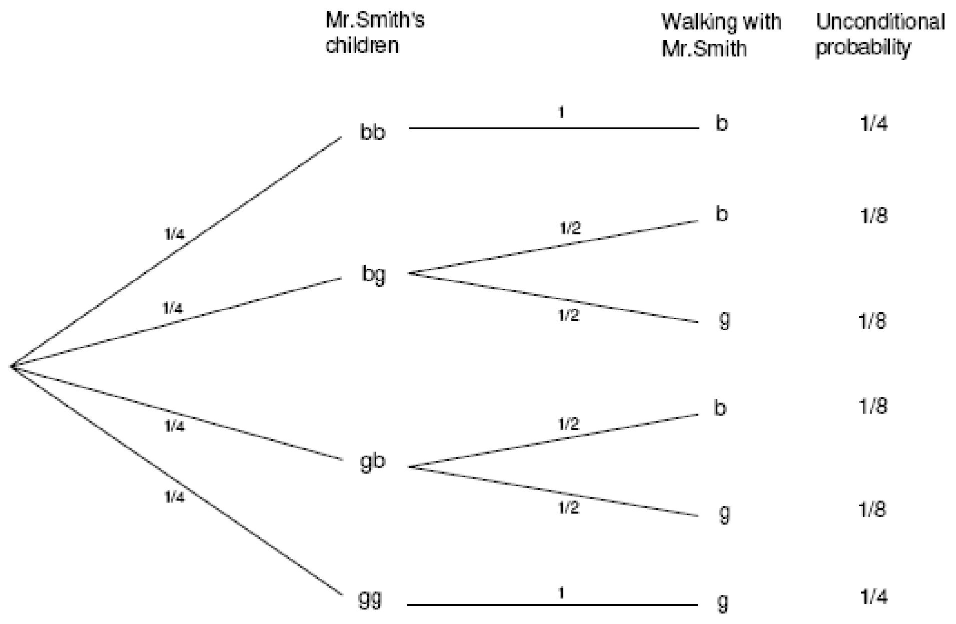
We see that the probability of two boys given a boy in the family is not 1/2 but rather 1/3.

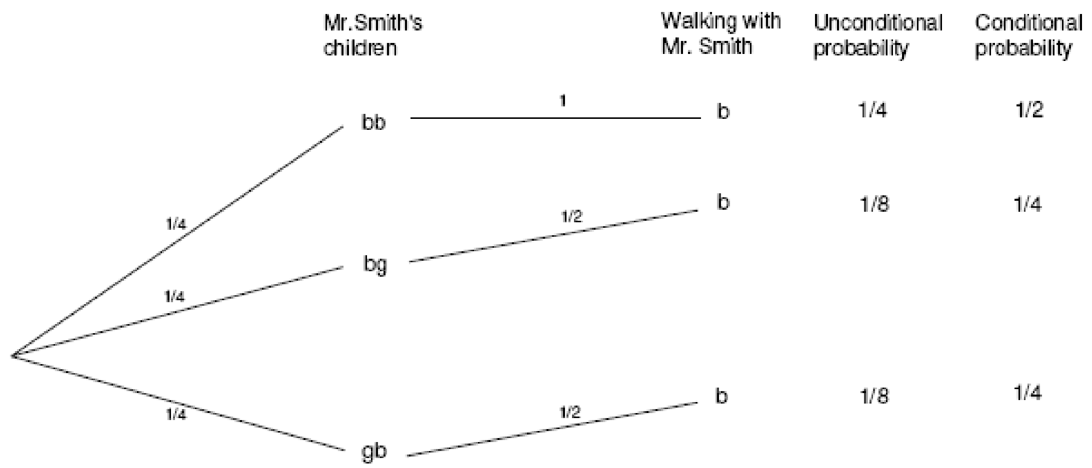
This problem and others like it are discussed in Bar-Hillel and Falk.[1] These authors stress that the answer to conditional probabilities of this kind can change depending upon how the information given was actually obtained. For example, they show that 1/2 is the correct answer for the following scenario.

Example 2. Mr. Smith is the father of two. We meet him walking along the street with a young boy whom he proudly introduces as his son. What is the probability that Mr. Smith's other child is also a boy?

As usual we have to make some additional assumptions. For example, we will assume that if Mr. Smith has a boy and a girl, he is equally likely to choose either one to accompany him on his walk. In the Figure we show the tree analysis of this problem and we see that 1/2 is, indeed, the correct answer.

[1]M. Bar-Hillel and R. Falk, "Some teasers concerning conditional probabilities," Cognition, vol. 11 (1982), pgs. 109-122.





Example 3. It is not so easy to think of reasonable scenarios that would lead to the classical 1/3 answer. An attempt was made by Stephen Geller in proposing this problem to Marilyn vos Savant.[2] Geller's problem is as follows: A shopkeeper says she has two new baby beagles to show you, but she doesn't know whether they're both male, both female, or one of each sex. You tell her that you want only a male, and she telephones the fellow who's giving them a bath. "Is at least one a male?" she asks. "Yes," she informs you with a smile. What is the probability that the other one is male? The reader is asked to decide whether the model which gives an answer of 1/3 is a reasonable one to use in this case.

In these examples, the apparent paradoxes could easily be resolved by clearly stating the model that is being used and the assumptions that are being made.

[2]M. vos Savant, "Ask Marilyn," Parade Magazine, 9 September; 2 December; 17 February 1990, reprinted in Marilyn vos Savant, Ask Marilyn, St. Martins, New York, 1992.

The Monty Hall problem

Conditional probabilities are the sources of many "paradoxes" in probability.

We consider now a problem called the Monty Hall problem. This has long been a favorite problem but was revived by a letter from Craig Whitaker to Marilyn vos Savant for consideration in her column in Parade Magazine. Craig wrote:

Suppose you're on Monty Hall's Let's Make a Deal! You are given the choice of three doors, behind one door is a car, the others, goats. You pick a door, say 1, Monty opens another door, say 3, which has a goat. Monty says to you "Do you want to pick door 2?" Is it to your advantage to switch your choice of doors?

Marilyn gave a solution concluding that you should switch, and if you do, your probability of winning is 2/3. Several irate readers, some of whom identified themselves as having a PhD in mathematics, said that this is absurd since after Monty has ruled out one door there are only two possible doors and they should still each have the same probability 1/2 so there is no advantage to switching. Marilyn stuck to her solution and encouraged her readers to simulate the game and draw their own conclusions from this. Other readers complained that Marilyn had not described the problem completely.

In particular, the way in which certain decisions were made during a play of the game were not specified. We will assume that the car was put behind a door by rolling a three-sided die which made all three choices equally likely. Monty knows where the car is, and always opens a door with a goat behind it. Finally, we assume that if Monty has a choice of doors (i.e., the contestant has picked the door with the car behind it), he chooses each door with probability 1/2. Marilyn clearly expected her readers to assume that the game was played in this manner. As is the case with most apparent paradoxes, this one can be resolved through careful analysis.

We begin by describing a simpler, related question. We say that a contestant is using the "stay" strategy if he picks a door, and, if offered a chance to switch to another door, declines to do so (i.e., he stays with his original choice). Similarly, we say that the contestant is using the "switch" strategy if he picks a door, and, if offered a chance to switch to another door, takes the offer.

Now suppose that a contestant decides in advance to play the "stay" strategy. His only action in this case is to pick a door (and decline an invitation to switch, if one is offered).

What is the probability that he wins a car? The same question can be asked about the "switch" strategy.

Using the "stay" strategy, a contestant will win the car with probability 1/3, since 1/3 of the time the door he picks will have the car behind it. On the other hand, if a contestant plays the "switch" strategy, then he will win whenever the door he originally picked does not have the car behind it, which happens 2/3 of the time.

This very simple analysis, though correct, does not quite solve the problem that Craig posed. Craig asked for the conditional probability that you win if you switch, given that you have chosen door 1 and that Monty has chosen door 3.

To solve this problem, we set up the problem before getting this information and then compute the conditional probability given this information.

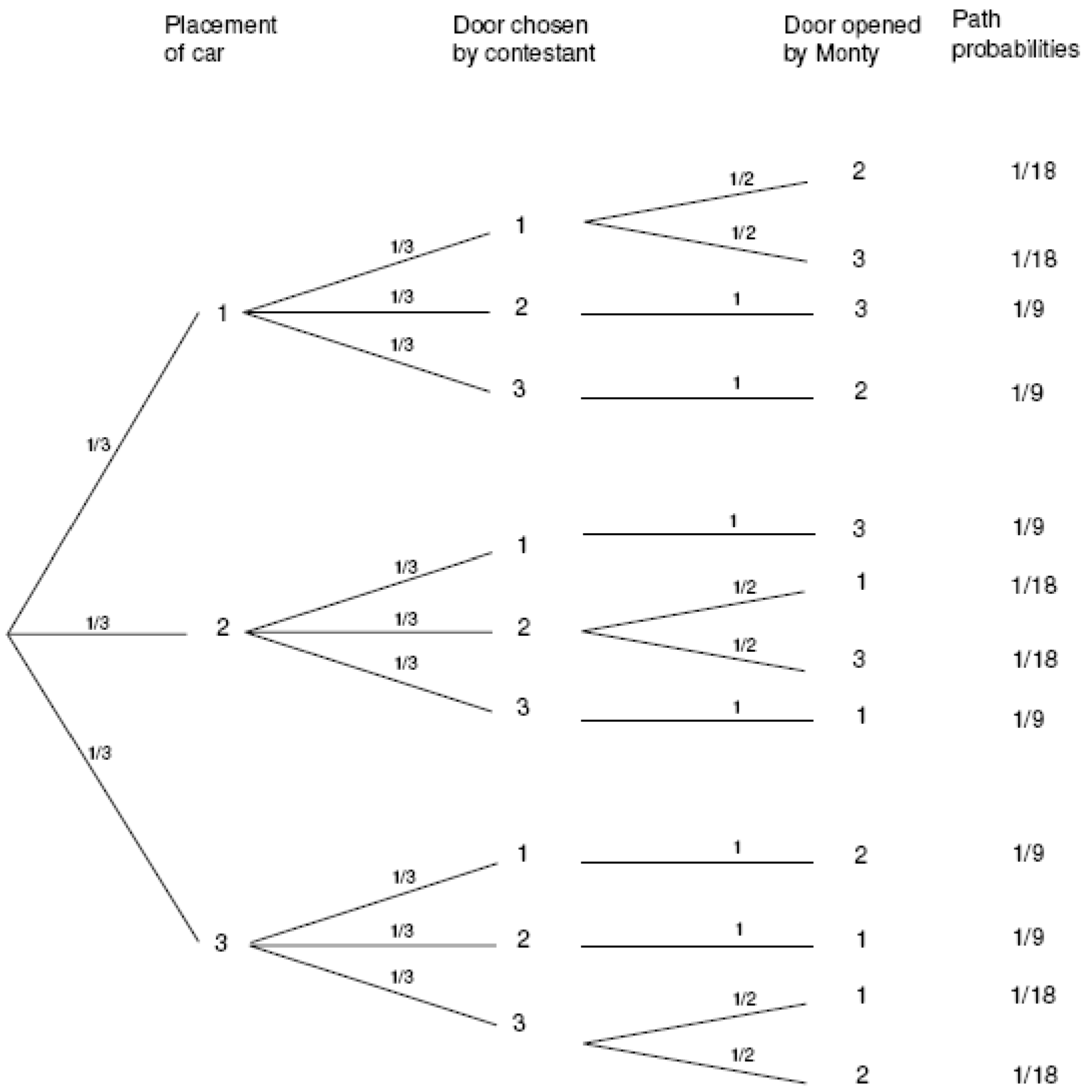
This is a process that takes place in several stages; the car is put behind a door, the contestant picks a door, and finally Monty opens a door.

Thus it is natural to analyze this using a tree measure. Here we make an additional assumption that if Monty has a choice of doors (i.e., the contestant has picked the door with the car behind it) then he picks each door with probability 1/2. The assumptions we have made determine the branch probabilities and these in turn determine the tree measure.

The resulting tree and tree measure are shown in the Figure.

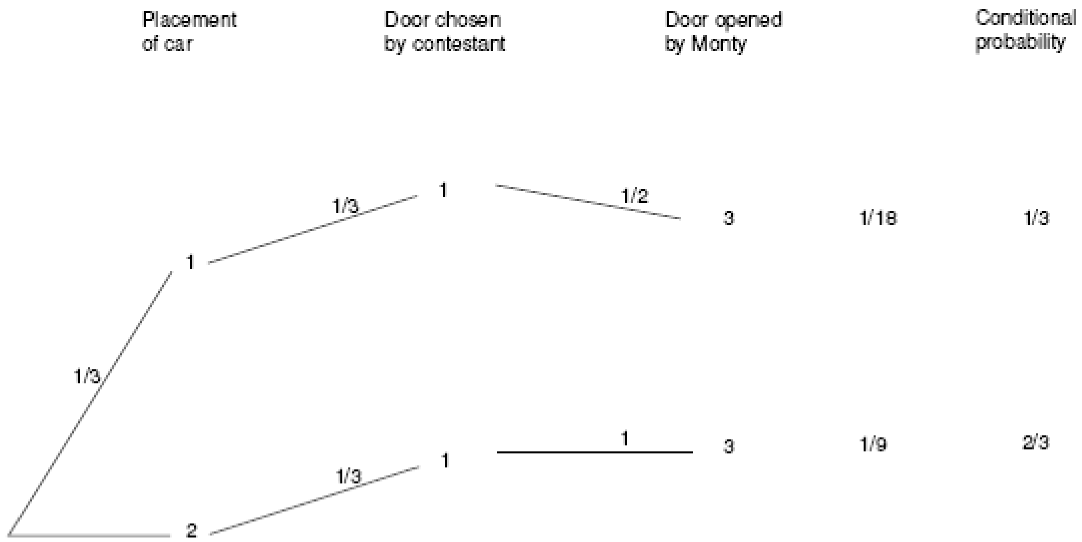
It is tempting to reduce the tree's size by making certain assumptions such as: "Without loss of generality, we will assume that the contestant always picks door 1." We have chosen not to make any such assumptions, in the interest of clarity.

The Monty Hall problem



Now the given information, namely that the contestant chose door 1 and Monty chose door 3, means only two paths through the tree are possible (see the next Figure).

Conditional probabilities for the Monty Hall problem



For one of these paths, the car is behind door 1 and for the other it is behind door 2. The path with the car behind door 2 is twice as likely as the one with the car behind door 1. Thus the conditional probability is 2/3 that the car is behind door 2 and 1/3 that it is behind door 1, so if you switch you have a 2/3 chance of winning the car, as Marilyn claimed.

Two court cases

1. A cab was involved in a hit and run accident at night. Two cab companies, the Green and the Blue, operate in the city. Here is some data: a) Although the two companies are equal in size, 85% of cab accidents in the city involve Green cabs and 15% involve Blue cabs. b) A witness identified the cab in this particular accident as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colors 80% of the time and failed 20% of the time.
   What is the probability that the cab involved in the accident was Blue rather than Green? If it looks like an obvious problem in statistics, then consider the following argument: The probability that the color of the cab was Blue is 80%! After all, the witness is correct 80% of the time, and this time he said it was Blue! What else need be considered? Nothing, right? If we look at Bayes theorem (pretty basic statistical theorem) we should get a much lower probability. But why should we consider statistical theorems when the problem appears so clear cut? Should we just accept the 80% figure as correct?
2. **People vs. Collins.** In 1964 the purse of an elderly woman shopping in Los Angeles was snatched by a young white female with a blond ponytail. The thief fled on foot but soon after was seen getting into a yellow car driven by

a black man who had a mustache and beard. A police investigation subsequently turned up a suspect who was blond, wore a ponytail, and lived with a black man who had a mustache, beard, and a yellow car. None of the eyewitnesses were able to identify the suspects so the police turned to probability

Characteristic Probability

Yellow car 0.1

Man with mustache 0.25

Black man with beard 0.01

Woman with ponytail 0.1

Woman with blond hair 0.33

Interracial couple in car 0.01

Multiplying the probabilities as if they were independent gives an overall probability of 1/12 000 000. This impressed the jury who handed down a verdict of second degree robbery. The Supreme Court of California later disagreed and reversed the conclusion based on the fact that…

Let's Make a Deal!

Here are two variations of the Monty Hall problem.

(a) Suppose that everything is the same except that Monty forgot to find out in advance which door has the car behind it. In the spirit of "the show must go on," he makes a guess at which of the two doors to open and gets lucky, opening a door behind which stands a goat. Now should the contestant switch?

(b) You have observed the show for a long time and found that the car is put behind door A 45% of the time, behind door B 40% of the time and behind door C 15% of the time. Assume that everything else about the show is the same. Again you pick door A. Monty opens a door with a goat and offers to let you switch. Should you? Suppose you knew in advance that Monty was going to give you a chance to switch. Should you have initially chosen door A?

You solve this problems (writing form, in detail; Russian or English – it doesn't matter) – I take it into consideration (at least you haven't to do Задачи 1)

The National Lottery (*Frank Duckworth.* The Royal Statistical Society Schools Lecture 2004)

What advice can I give you as a statistician?

Let us get one thing quite clear. There is nothing that I or anyone else can do to help you to increase your chances of picking the right numbers. So take no notice at all of any claims to help you pick the winning numbers.

But there *is* something I can do for you. I can't help you to win, but I *can* help you to win more *if* you win.

The key to this lies in people's selection of random numbers. If everyone who bought a ticket picked their six numbers from the numbers 1 to 49 completely at random, then there would be about two or three jackpot winners every draw, the number of winners would hardly ever exceed 15 and there should have been no more than about 40 roll-overs since the lottery started about 10 years ago [when there are no jackpot winners, the prize money rolls over to the next draw]. In fact, there have been over 130 roll-overs, there have been several instances of more than 20 winners, and on one occasion during the lottery's first year there were no fewer than 133 winners.

What we find is that in most draws there are either many fewer winners or many more winners than would be expected if every selection were made purely at random. So most people who win a share in the jackpot are having to share it with many others. The reason is that people in general are not very good at picking random numbers.

Item 1. There are four numbers written in a square: 1, 2, 3 and 4. Will you please select one of these numbers at random, 1, 2, 3 or 4, and put a cross over the number you have selected.

Item 2. Below this you have an empty square and I want you to pick a single-digit figure at random.

Item 3. I want you to make a random National Lottery selection of six different numbers between 1 and 49; put them in order and write them in the spaces provided.

What I have to tell you is that with truly random selections there will be two consecutive numbers in very nearly *one-half* of the draws. Indeed, from the 845 draws since the lottery started in November 1994 until the end of January 2004, there were consecutive numbers in 402 of them, which is about 48% of the draws. Examination of the numbers of winners tells us that when there are consecutive numbers there are fewer people sharing the jackpot. So there is the first piece of advice I can give you; make sure you have two consecutive numbers.

What you want to do to make sure you win a lot, *if* you win, is to pick numbers or combinations of numbers that few others pick. Camelot, who runs the UK National Lottery, does not reveal any information on the numbers people select, so we cannot look for the most popular numbers or combinations of numbers. But there are other identical lotteries in the world, notably in Switzerland and Canada, and these do make the details of selections available, and they tell us some very interesting things. First of all, the most popular number is 7; next comes 11. Odd numbers are more popular than even, and fewer than expected pick numbers over 40.

There are also interesting results from looking at combinations of numbers. Did anyone here pick the numbers 1, 2, 3, 4, 5 and 6? What I have to tell you is that if you had picked these numbers and they had come up, the indications are that you would probably have had to share your jackpot winnings with about 10,000 others!

So here is my advice. First of all, you must pick your six numbers at random. The best way of doing this is to measure out a 7-inch by 7-inch square, rule this into 49 squares, write the numbers 1 to 49 in the squares, cut them out separately and fold each one twice. Then put them in a hat, mix them thoroughly and draw out six numbers. But here is the trick. Put them in ascending order and examine them. To accept them they must satisfy each of the following four criteria:

1. there must be at least one instance of two or more consecutive numbers
2. there must *not* be a 7 or an 11
3. there must be no more than three odd numbers
4. there must be at least one number of 41 or over.

Otherwise, put all six numbers back, mix them thoroughly again, and redraw; and keep doing this until all four conditions are satisfied.

This will not make any difference to your chances of winning. You are still facing odds of nearly 14 million to one against, but in the very unlikely event that you do win, at least you probably won't have to share your winnings with very many others.

Fair or unfair games - "Expectation"

One of the most valuable uses to which a gambler can put his knowledge of probabilities is to decide whether a game or proposition is fair, or equitable. To do this a gambler must calculate his 'expectation'. A gambler's expectation is the amount he stands to win multiplied by the probability of his winning it. A game is a fair game if the gambler's expectation equals his stake. If a gambler is offered 10 units each time he tosses a head with a true coin, and pays 5 units for each toss is this a fair game? The gambler's expectation is 10 units multiplied by the probability of throwing a head, which is 1/2. His expectation is 10 units = 5 units, which is what he stakes on the game, so the game is fair.

Suppose a gambler stakes 2 units on the throw of a die. On throws of 1, 2, 3 and 6 he is paid the number of units shown, on the die. If he throws 4 or 5 he loses. Is this fair? This can be calculated as above. The probability of throwing any number is 1/6, so his expectation is 6/6 on number 6, 3/6 on number 3, 2/6 on number 2 and 1/6 on number 1.

His total expectation is therefore 6/6+3/6+2/6+1/6, which equals 2, the stake for a throw, so the game is fair.

Prospects of ruination

When we talk of fair games, there is another aspect to the question which experienced gamblers know about, and which all gamblers must remember. If a game is fair by the above criterion, it nevertheless is still true that if it is played until one player loses all his money, then the player who started with most money has the better chance of winning. The richer man's advantage can be calculated.

The mathematics required to arrive at the full formula are complex, but it can be shown that if one player's capital is c and another player's is C, then the probability that the player who began with c is ruined in a fair game played to a conclusion is $C/(c + C)$ and therefore that the probability that he will ruin his opponent is $c/(C + c)$.

Suppose player X, with 10 units, plays another player, Y, with 1,000 units. A coin is tossed and for each head player X pays player Y one unit, and for each tail player Y pays player X one unit. The probability of player X ruining player Y is 10/(1000+10) or 1/101. Player Y has a probability of 100/101 of ruining his opponent. An advantage is over 99%.

This overwhelming advantage to the player with the larger capital is based on a fair game. When a player tries to break the bank at a casino, he is fighting the house edge as well as an opponent with much larger resources. Even a small percentage in favour of the casino reduces the probability of the player ruining the casino to such an extent that his chance of profiting is infinitesimal. If you are anticipating playing a zero sum game then make sure you are aware of the prospect of ruin.

The law of large numbers / "The law of averages"

The theory of probability becomes of enhanced value to gamblers when it is used with the law of large numbers. The law of large numbers states that:

“If the probability of a given outcome to an event is P and the event is repeated N times, then the larger N becomes, so the likelihood increases that the closer, in proportion, will be the occurrence of the given outcome to N*P.”

For example:-

If the probability of throwing a double-6 with two dice is 1/36, then the more times we throw the dice, the closer, in proportion, will be the number of double-6s thrown to of the total number of throws. This is, of course, what in everyday language is known as the law of averages. The overlooking of the vital words 'in proportion' in the above definition leads to much misunderstanding among gamblers. The 'gambler's fallacy' lies in the idea that “In the long run” chances will even out. Thus if a coin has been spun 100 times, and has landed 60 times head uppermost and 40 times tails, many gamblers will state that tails are now due for a run to get even. There are fancy names for this belief. The theory is called the maturity of chances, and the expected run of tails is known as a 'corrective', which will bring the total of tails eventually equal to the total of heads. The belief is that the 'law' of averages really is a law which states that in the longest of long runs the totals of both heads and tails will eventually become equal.

In fact, the opposite is really the case. As the number of tosses gets larger, the probability is that the percentage of heads or tails thrown gets nearer to 50%, but that the difference between the actual number of heads or tails thrown and the number representing 50% gets larger.

Let us return to our example of 60 heads and 40 tails in 100 spins, and imagine that the next 100 spins result in 56 heads and 44 tails. The 'corrective' has set in, as the percentage of heads has now dropped from 60 per cent to 58 per cent. But there are now 32 more heads than tails, where there were only 20 before. The 'law of averages' follower who backed tails is 12 more tosses to the bad. If the third hundred tosses result in 50 heads and 50 tails, the 'corrective' is still proceeding, as there are now 166 heads in 300 tosses, down to 55-33 per cent, but the tails backer is still 32 tosses behind.

Put another way, we would not be too surprised if after 100 tosses there were 60 per cent heads. We would be astonished if after a million tosses there were still 60 per cent heads, as we would expect the deviation from 50 per cent to be much smaller. Similarly, after 100 tosses, we are not too surprised that the difference between heads and tails is 20. After a million tosses we would be very surprised to find that the difference was not very much larger than 20.

A chance event is uninfluenced by the events which have gone before. If a true die has not shown 6 for 30 throws, the probability of a 6 is still 1/6 on the 31st throw. One wonders if this simple idea offends some human instinct, because it is not difficult to find gambling experts who will agree with all the above remarks, and will express them themselves in books and articles, only to advocate elsewhere the principle of 'stepping in when a corrective is due'.

It is interesting that despite significant statistical evidence and proof of all of the above people will go to extreme lengths to fulfill there belief in the fact that a corrective is due. The number 53 in an Italian lottery had failed to appear for some time and this lead to an obsession with the public to bet ever larger amounts on the number. People staked so much on this corrective that the failure of the number 53 to occur for two years was blamed for several deaths and bankruptcies. It seems that a large number of human minds are just simply unable to cope with the often seemingly contradictory laws of probability. If only they had listened to their maths teacher.

An understanding of the law of the large numbers leads to a realisation that what appear to be fantastic improbabilities are not remarkable at all but, merely to be expected.

Gambling

"No one can possibly win at roulette unless he steals money from the table while the croupier isn't looking."

Albert Einstein

"No betting system can convert a subfair game into a profitable enterprise..."

Patrick Billingsley ("Probability and Measure", page 94, second edition)

In Las Vegas, a roulette wheel has 38 slots numbered 0, 00, 1, 2,.., 36. The 0 and 00 slots are green and half of the remaining 36 slots are red and half are black. A croupier spins the wheel and throws in an ivory ball. If you bet 1 dollar on red, you win 1 dollar if the ball stops in a red slot and otherwise you lose 1 dollar.
Another form of bet for roulette is to bet that a specific number (say 17) will turn up. If the ball stops on your number, you get your dollar back plus 35 dollars. If not, you lose your dollar.

The **Labouchere system** for roulette is played as follows. Write down a list of numbers, usually 1, 2, 3, 4. Bet the sum of the first and last, 1 + 4 = 5, on red. If you win, delete the first and last numbers from your list. If you lose, add the amount that you last bet to the end of your list. Then use the new list and bet the sum of the first and last numbers (if there is only one number, bet that amount). Continue until your list becomes empty. Show that, if this happens, you win the sum, 1 + 2 + 3 + 4 = 10, of your original list. Simulate this system and see if you do always stop and, hence, always win. If so, why is this not a foolproof gambling system?

(This system is also called the 'Cancellation' system. There are many variations. In its simplest form, you write down a series or a set of numbers; say, 1 2 3 4 5 6. The series can be short or long and not necessarily sequential such as 1 1 1 3 3 5 7. The choice of a particular series depends on the type of game you want to apply it to and the odds of the bet. Each number represents the amount in units or chips to bet. You bet the first and last of these numbers. In this example 1 and 6, which totals 7 units.

If you win, you cross out the two numbers and bet the next two 'ends' (the outside numbers). In this instance 2 and 5. If you win again you bet on the next two remaining numbers 3 and 4, and if you win that too, you would have made a 'coup' or completed one game. Then you start all over again.

If you lose, then you add that one number to the end of the series. Say you lost your first bet of 7 units (1+ 6). Then you add number 7 to the end of the series to look like this: 1 2 3 4 5 6 7 and your next bet would be 8 units (1+ 7). If you won the first bet but lost the second 2 and 5, then the series of numbers would look like this: 2 3 4 5 7.

If you work it out, you will see that when the series is completed or when you make a 'coup', there is always a profit. The negative side of this system is that you could end up betting large sums of money even if your initial bet is small.)

Another well-known gambling system is the **martingale** doubling **system**. Suppose that you are betting on red to turn up in roulette. Every time you win, bet 1 dollar next time. Every time you lose, double your previous bet. Suppose that you use this system until you have won at least 5 dollars or you have lost more than 100 dollars. In his book “The Newcomes”, W. M. Thackeray remarks “You have not played as yet? Do not do so; above all avoid a martingale if you do.” Was this good advice?

http://www.fortunepalace.co.uk – may look for others gambling system.

Not only do betting systems fail to beat casino games with a house advantage, they can't even dent it. Roulette balls and dice simply have no memory. Every spin in roulette and every toss in craps is independent of all past events. In the short run you can fool yourself into thinking a betting system works, by risking a lot to win a little. However, in the long run no betting system can withstand the test of time. Furthermore, the longer you play, the ratio of money lost to money bet will get closer to the expectation for that game.

The Gambler's Fallacy

The biggest gambling myth is that an event that has not happened recently becomes overdue and more likely to occur. This is known as the "gambler's fallacy." Thousands of gamblers have devised betting systems that attempt to exploit the gambler's fallacy by betting the opposite way of recent outcomes. For example, waiting for three reds in roulette and then betting on black. Hucksters sell "guaranteed" get-rich-quick betting systems that are ultimately based on the gambler's fallacy. None of them work.

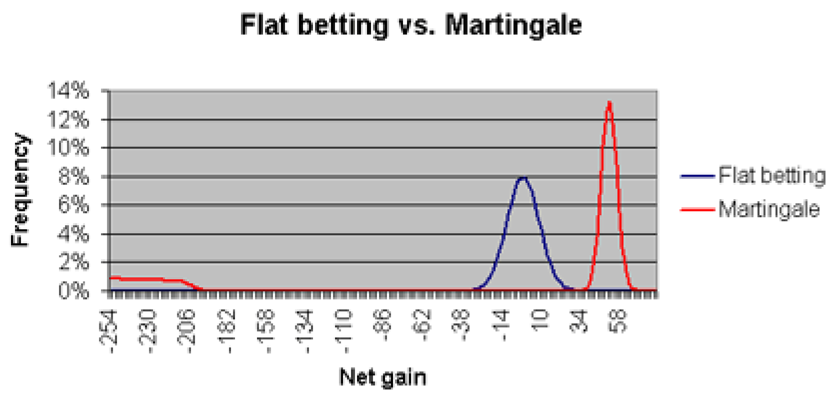
A common gamblers' fallacy called 'the doctrine of the maturity of the chances' (or 'Monte Carlo fallacy') falsely assumes that each play in a game of chance is not independent of the others and that a series of outcomes of one sort should be balanced in the short run by other possibilities. A number of 'systems' have been invented by gamblers based largely on this fallacy; casino operators are happy to encourage the use of such systems and to exploit any gambler's neglect of the strict rules of probability and independent plays. -- *Encyclopedia Britannica* (look under "gambling.")

The number of 'guaranteed' betting systems, the proliferation of myths and fallacies concerning such systems, and the countless people believing, propagating, venerating, protecting, and swearing by such systems are legion. Betting systems constitute one of the oldest delusions of gambling history. Betting systems votaries are spiritually akin to the proponents of perpetual motion machines, butting their heads against the second law of thermodynamics. -- *The Theory of Gambling and Statistical Logic* (page 53) by Richard A. Epstein

The Martingale

This system is generally played with an even money game such as the red/black bet in roulette or the pass/don't pass bet in craps and is known as the Martingale. The idea is that by doubling your bet after a loss, you would always win enough to cover all past losses plus one unit. For example if a player starts at $1 and loses four bets in a row, winning on the fifth, he will have lost $1+$2+$4+$8 = $15 on the four losing bets and won $16 on the fifth bet. The losses were covered and he had a profit of $1. The problem is that it is easier than you think to lose several bets in a row and run out of betting money after you've doubled it all away.

In order to prove this point a program was created that simulated two systems, the Martingale and flat betting, and applied each by betting on the pass line in craps (which has a 49.29% probability of winning). The Martingale bettor would always start with a $1 bet and start the session with $255 which is enough to cover 8 losses in a row. The flat bettor would bet $1 every time. The Martingale player would play for 100 bets, or until he couldn't cover the amount of a bet. In that case he would stop playing and leave with the money he had left. In the event his 100th bet was a loss, he would keep betting until he either won a bet or couldn't cover the next bet. The person flat betting would play 100 bets every time. This experiment was repeated for 1,000,000 sessions for both systems and tabulated the results. The graph below shows the results:



As you can see, the flat bettor has a bell curve with a peak at a loss of $1, and never strays very far from that peak. Usually the Martingale bettor would show a profit represented by the bell curve on the far right, peaking at $51; however, on the far left we see those times when he couldn't cover a bet and walked away with a substantial loss. That happened for 19.65% of the sessions. Many believers in the Martingale mistakenly believe that the many wins will more than cover the few loses. In this experiment the average session loss for the flat bettor was $1.12, but was $4.20 for the Martingale bettor. In both cases the ratio of money lost to money won was very close to 7/495, which is the house edge on the pass line bet in craps. This is not coincidental. No matter what system is used in the long run, this ratio will always approach the house edge. To prove this point consider the Martingale player on the pass line in craps who only desires to win $1, starts with a bet of $1, and has a bankroll of $2,047 to cover as many as 10 consecutive losses. The table below shows all possible outcomes with each probability, expected bet, and return.

Possible outcomes of Martingale up to ten losing bets							
Number of losses	Final outcome	Highest bet	Total bet	Net outcome	Probability	Expected bet	Expected return
0	Win	1	1	1	0.49292929	0.49292929	0.49292929
1	Win	2	3	1	0.24995001	0.74985002	0.24995001
2	Win	4	7	1	0.12674233	0.88719628	0.12674233
3	Win	8	15	1	0.06426732	0.96400981	0.06426732
4	Win	16	31	1	0.03258808	1.01023035	0.03258808
5	Win	32	63	1	0.01652446	1.04104089	0.01652446
6	Win	64	127	1	0.00837907	1.06414175	0.00837907
7	Win	128	255	1	0.00424878	1.08343900	0.00424878
8	Win	256	511	1	0.00215443	1.10091479	0.00215443
9	Win	512	1023	1	0.00109245	1.11757574	0.00109245
10	Win	1024	2047	1	0.00055395	1.13393379	0.00055395
10	Loss	1024	2047	-2047	0.00056984	1.16646467	-1.16646467
Total					1.00000000	11.81172639	-0.16703451

The expected bet is the product of the total bet and the probability. Likewise, the expected return is the product of the total return and the probability. The last row shows this Martingale bettor to have had an average total bet of 11.81172639 and an average loss of 0.16703451. Dividing the average loss by the average bet yields .01414141. We now divide 7 by 495 (the house edge on the pass line) and we again get 0.01414141! This shows that the Martingale is neither better nor worse than flat betting when measured by the ratio of expected loss to expected bet. All betting systems are equal to flat betting when compared this way, as they should be. In other words, **all betting systems are equally worthless**.

Don't waste your money

The Internet is full of people selling betting systems with promises of beating the casino at games of luck. Those who sell these systems are the present day equivalent of the 19th century snake oil salesmen. Under no circumstances should you waste one penny on any gambling system. Every time one has been put to a computer simulation it failed and showed the same ratio of losses to money bet as flat betting. If you ask a system salesman about this you likely will get a reply such as, "In real life nobody plays millions of trials in the casino." You're likely to also hear that his/her system works in real life, but not when used against a computer simulation. It is interesting that professionals use computers to model real life problems in just about every field of study, yet when it comes to betting systems computer analysis becomes "worthless and unreliable", as the salesman of one system put it. In any event, such an excuse misses the point; the computer runs billions of trials simply to prove that a system is unsound. If it won't work on a computer, it won't work in the casino.

Gambling systems have been around for as long as gambling has. No system has ever been proven to work. System salesmen go from selling one kind of system to another. It is a dirty business by which they steal ideas from each other, and are always attempting to rehash old systems as something new.

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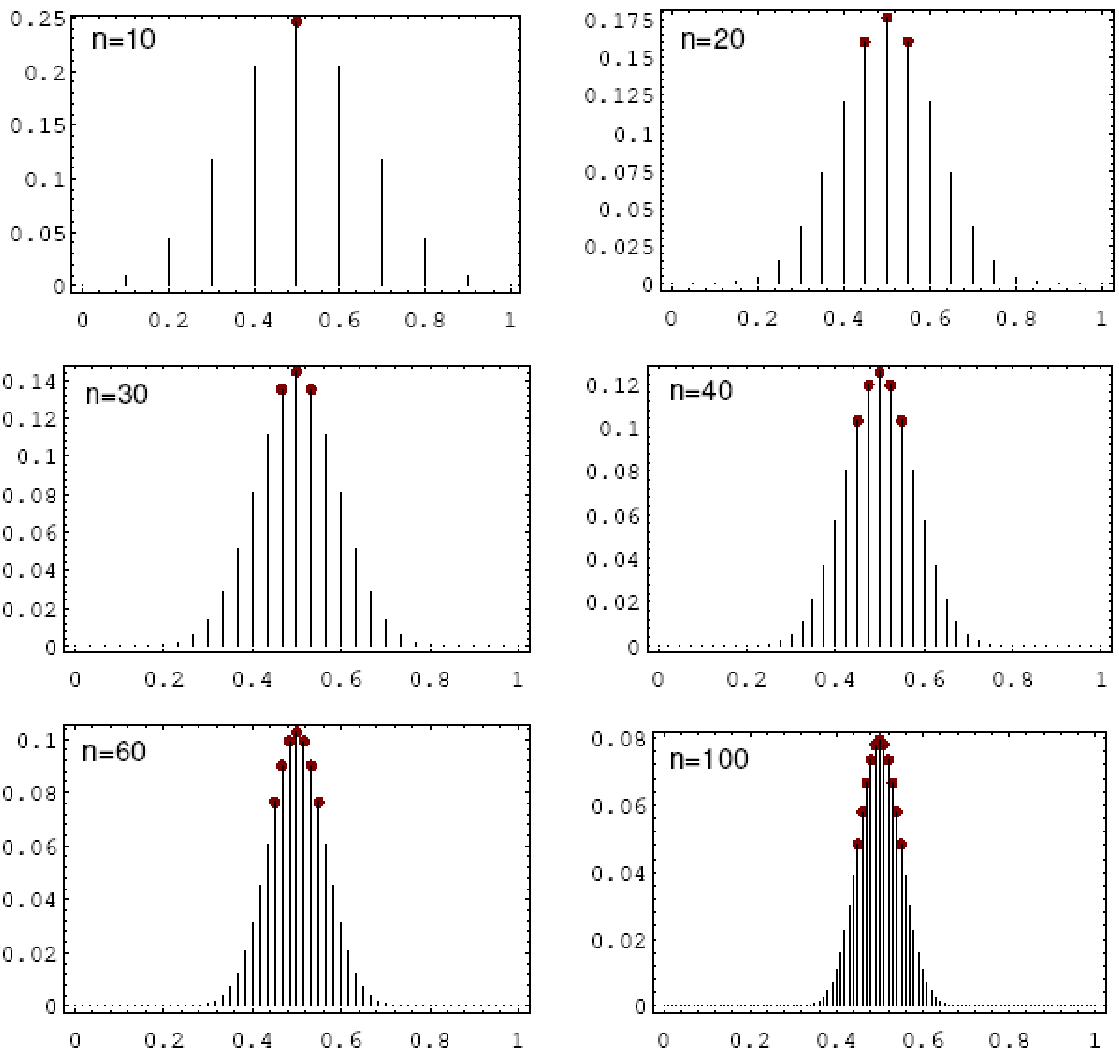
Law of Large Numbers (Law of Averages)

$\forall \varepsilon > 0$ (under some conditions)

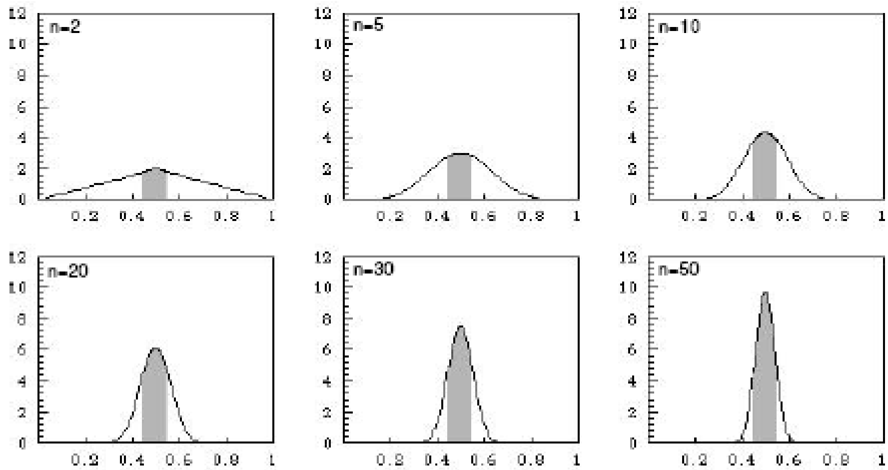
$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \to 0$$

$$P\left(\left|\frac{S_n}{n} - p\right| < \epsilon\right) \to 1$$

$$A_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

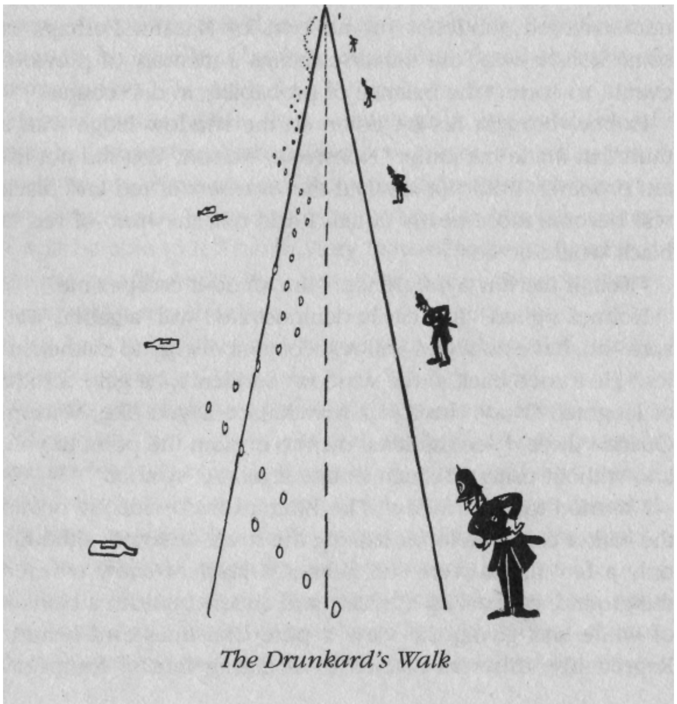


Bernoulli trials distributions (p = 1/2)



Uniform case

“Whence, finally, this one thing seems to follow: that if observations of all events were to be continued throughout all eternity, (and hence the ultimate probability would tend toward perfect certainty), everything in the world would be perceived to happen in fixed ratios and according to a constant law of alternation, so that even in the most accidental and fortuitous occurrences we would be bound to recognize, as it were, a certain necessity and, so to speak, a certain fate.
I do now know whether Plato wished to aim at this in his doctrine of the universal return of things, according to which he predicted that all things will return to their original state after countless ages have past.” (J. Bernoulli, The Art of Conjecturing IV, trans. Bing Sung, Technical Report No. 2, Dept. of Statistics, Harvard Univ., 1966)



The Drunkard's Walk

"That is just the name of a well-known technique for graphing a random series. Suppose that a man starts from a point in front of you and attempts to walk north, along the dotted line. However, after every step he tosses a coin and then takes an additional side step left or right, depending on whether the coin falls heads or tails. Thus, he progresses in a series of diagonals, like the man we saw below, and his track records the sequence of heads and tails. At a given moment, we can easily tell from his position whether the heads have exceeded the tails or vice versa. If he is standing on the dotted line, the number of heads has exactly equaled the number of tails at that point; if he is three paces to the left of the dotted line, there have been three more heads than tails."

Holmes's eyes narrowed. "Now, we shall assume the man cannot even see the line. There is no certainly no mystic force drawing him back toward it! As you might expect, the farther he walks north, the farther he is likely to stray from the line. Nevertheless, from our point of view, the farther he walks, the more nearly he will come to being exactly northward of our position. This is because the ratio between his distance north of us and the distance he has veered to east or west is likely to diminish. So by the time he is but a dot on the horizon, we will see him very close to due north."

"I think I grasp what you are saying, Holmes. But I am still not sure I understand why the ratio should diminish."

"Well, consider for example a moment when the man is ten steps to the west of the dotted line. When he takes the next step, will his distance to the east or west of you have increased or decreased, on the average?"

He is just as likely to veer east as west, so the average change will be none."

And yet his distance north of you will definitely have "creased by one pace. So—on the average—the ratio of his sideways distance to his northward distance from you will be just a little bit reduced. Of course there is one counter-example: if he is actually on the dotted line, a step either way will take him farther from it. But nevertheless, the farther he goes, the less the ratio of his total sideways to his total forward progress is likely to be, because the forward motion is always cumulative, and the sideways motion often cancels.

"Suppose that he starts out, by chance, with three consecutive paces all to the left. That starting bias will never be consciously corrected: a thousand paces later, he will still be six paces to the left of where he would have been, had those first paces been instead to the right, and the rest of the sequence the same. But by the time he has made a thousand paces, and is hull-down on the horizon, the difference those six paces make to the angle at which you see him will be negligible. In terms of direction, you will see him in the long term converge inevitably upon the northward."

I looked out of the window at the flakes still tumbling past the panes. Below, the deepening snow was steadily covering the features of the street. Already, you could not tell where the road ended and the pavement began.

"I am satisfied, Holmes," I said. "Any advantage of heads over tails, or of red over black on the roulette wheel, is smoothed out in the longer term, not by any kind of active cancellation, but by a more gentle and undirected happening, a kind of washing out. That is, the effect of any flukish run gradually becomes ever more negligible relative to the whole. At last I understand the matter, perfectly and in full!"

Central Limit Theorem for Bernoulli Trials

$$\lim_{n\to\infty} P\left(a \le \frac{S_n - np}{\sqrt{npq}} \le b\right) = \int_a^b \phi(x)\,dx\ .$$

In fact, our version of the Central Limit Theorem says that the distribution function of the random variable

$$S_n^* = \frac{S_n - np}{\sqrt{npq}}$$

is approximated by the standard normal density.

One frequently reads that a poll has been taken to estimate the proportion of people in a certain population who favor one candidate over another in a race with two candidates. Clearly, it is not possible for pollsters to ask everyone for their preference. What is done instead is to pick a subset of the population, called a sample, and ask everyone in the sample for their preference. Let p be the actual proportion of people in the population who are in favor of candidate A and let $q = 1-p$. If we choose a sample of size n from the population, the preferences of the people in the sample can be represented by random variables $X_1, X_2, \ldots, X_n$, where $X_i = 1$ if person i is in favor of candidate A, and $X_i = 0$ if person i is in favor of candidate B. Let $S_n = X_1 + X_2 + \cdots + X_n$. If each subset of size n is chosen with the same probability, then S_n is hypergeometrically distributed. If n is small relative to the size of the population (which is typically true in practice), then S_n is approximately binomially distributed, with parameters n and p.
The pollster wants to estimate the value p. An estimate for p is provided by the value $\hat{p} = S_n/n$, which is the proportion of people in the sample who favor candidate B. The Central Limit Theorem says that the random variable $\hat{p}$ is approximately normally distributed. (In fact, our version of the Central Limit Theorem says that the distribution function of the random variable

$$S_n^* = \frac{S_n - np}{\sqrt{npq}}$$

is approximated by the standard normal density.)

$$\bar{p}^* = \frac{\bar{p} - p}{\sqrt{pq/n}}$$

Since the distribution of the standardized version of $\hat{p}$ is approximated by the standard normal density, we know, for example, that 95% of its values will lie within two standard deviations of its mean, and the same is true of $\hat{p}$. So we have

$$P\left(p - 2\sqrt{\frac{pq}{n}} < \bar{p} < p + 2\sqrt{\frac{pq}{n}}\right) \approx .954 \ .$$

Now the pollster does not know p or q, but he can use $\hat{p}$ and $\hat{q} = 1 - \hat{p}$ in their place without too much danger. With this idea in mind, the above statement is equivalent to the statement

$$P\left(\bar{p} - 2\sqrt{\frac{\bar{p}\bar{q}}{n}} < p < \bar{p} + 2\sqrt{\frac{\bar{p}\bar{q}}{n}}\right) \approx .954 \ .$$

The resulting interval

$$\left(\bar{p} - \frac{2\sqrt{\bar{p}\bar{q}}}{\sqrt{n}},\ \bar{p} + \frac{2\sqrt{\bar{p}\bar{q}}}{\sqrt{n}}\right)$$

is called the 95 percent confidence interval for the unknown value of p. The name is suggested by the fact that if we use this method to estimate p in a large number of samples we should expect that in about 95 percent of the samples the true value of p is contained in the confidence interval obtained from the sample. The pollster has control over the value of n. Thus, if he wants to create a 95% confidence interval with length 6%, then he should choose a value of n so that

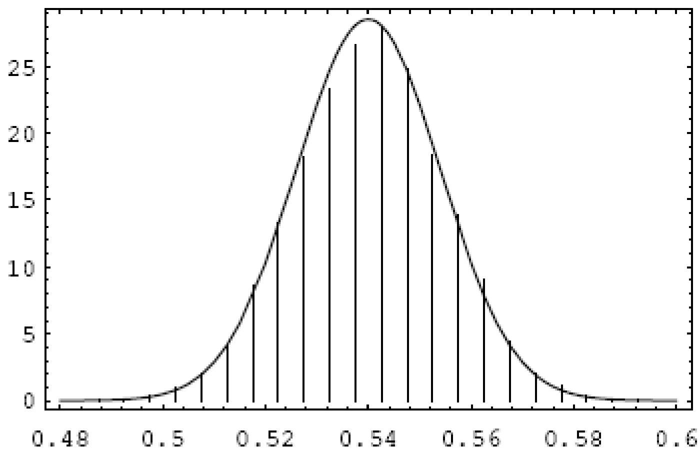
$$\frac{2\sqrt{\bar{p}\bar{q}}}{\sqrt{n}} \le .03 \ .$$

Using the fact that $\hat{p}\,\hat{q} \le 1/4$, no matter what the value of $\hat{p}$ is, it is easy to show that if he chooses a value of n so that

$$\frac{1}{\sqrt{n}} \le .03 \ ,$$

he will be safe. This is equivalent to choosing

$$n \ge 1111 \ .$$



Polling simulation

So if the pollster chooses n to be 1200, say, and calculates $\hat{p}$ using his sample of size 1200, then 19 times out of 20 (i.e., 95% of the time), his confidence interval, which is of length 6%, will contain the true value of p. This type of confidence interval is typically reported in the news as follows: this survey has a 3% margin of error.

In fact, most of the surveys that one sees reported in the paper will have sample sizes around 1000. A somewhat surprising fact is that the size of the population has apparently no effect on the sample size needed to obtain a 95% confidence interval for p with a given margin of error. To see this, note that the value of n that was needed depended only on the number .03, which is the margin of error. In other words, whether the population is of size 100,000 or 100,000,000, the pollster needs only to choose a sample of size 1200 or so to get the same accuracy of estimate of p. (We did use the fact that the sample size was small relative to the population size in the statement that S_n is approximately binomially distributed.)

In Figure, we show the results of simulating the polling process. The population is of size 100,000, and for the population, $p = .54$. The sample size was chosen to be 1200. The spike graph shows the distribution of $\hat{p}$ for 10,000 randomly chosen samples. For this simulation, the program kept track of the number of samples for which $\hat{p}$ was within 3% of .54. This number was 9648, which is close to 95% of the number of samples used.

Another way to see what the idea of confidence intervals means is shown in Figure 2. In this figure, we show 100 confidence intervals, obtained by computing $\hat{p}$ for 100 different samples of size 1200 from the same population as before. You can see that most of these confidence intervals (96, to be exact) contain the true value of p.

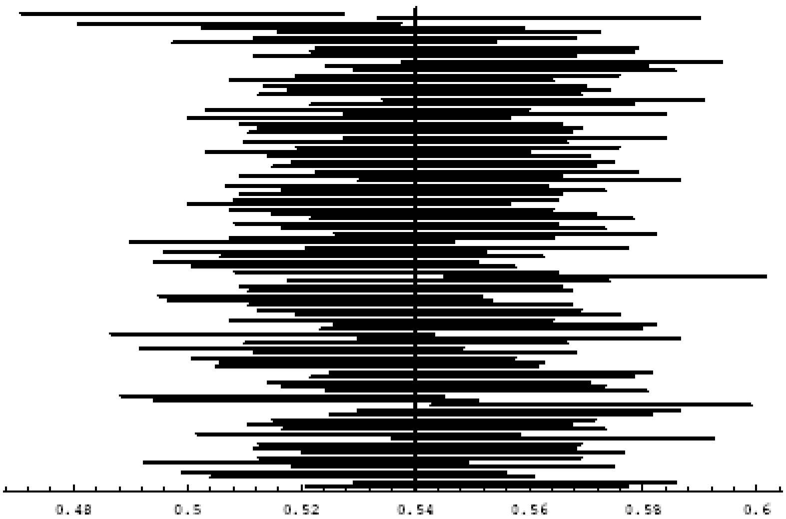


Figure 2. Confidence interval simulation.

The Gallup Poll has used these polling techniques in every Presidential election since 1936 (and in innumerable other elections as well). Table shows the results of their efforts.

Year	Winning Candidate	Gallup Final Survey	Election Result	Deviation
1936	Roosevelt	55.7%	62.5%	6.8%
1940	Roosevelt	52.0%	55.0%	3.0%
1944	Roosevelt	51.5%	53.3%	1.8%
1948	Truman	44.5%	49.9%	5.4%
1952	Eisenhower	51.0%	55.4%	4.4%
1956	Eisenhower	59.5%	57.8%	1.7%
1960	Kennedy	51.0%	50.1%	0.9%
1964	Johnson	64.0%	61.3%	2.7%
1968	Nixon	43.0%	43.5%	0.5%
1972	Nixon	62.0%	61.8%	0.2%
1976	Carter	48.0%	50.0%	2.0%
1980	Reagan	47.0%	50.8%	3.8%
1984	Reagan	59.0%	59.1%	0.1%
1988	Bush	56.0%	53.9%	2.1%
1992	Clinton	49.0%	43.2%	5.8%
1996	Clinton	52.0%	50.1%	1.9%

Gallup Poll accuracy record

You will note that most of the approximations to *p* are within 3% of the actual value of *p*. The sample sizes for these polls were typically around 1500. (In the table, both the predicted and actual percentages for the winning candidate refer to the percentage of the vote among the "major" political parties. In most elections, there were two major parties, but in several elections, there were three.)

This technique also plays an important role in the evaluation of the effectiveness of drugs in the medical profession. For example, it is sometimes desired to know what proportion of patients will be helped by a new drug. This proportion can be estimated by giving the drug to a subset of the patients, and determining the proportion of this sample who are helped by the drug.

Gallup reports the results of a poll that shows that 58% of a random sample of adult Americans approve of President Clinton's performance as president. The report says that the margin of error is 3%. What does this margin of error mean?

_______a. One can be 95% "confident" that between 55% and 61% of all adult Americans
approve of the President's performance.

_______b. One can be sure that between 55% and 61% of all adult Americans approve of the President's performance.

_______c. The sample percentage of 58% could be off by 3% in either direction due to inaccuracies in the survey process.

_______d. There is a 3% chance that the percentage of 58 is an inaccurate estimate of the
population of all Americans who approve of President Clinton's performance as president.

Gallup Poll - a sampling by the American Institute of Public Opinion or its British counterpart of the views of a representative cross section of the population, used as a means of forecasting voting. Etymology: named after George Horace Gallop (1901-84), US statistician.

Gallup Poll - опрос Гэллапа, опрос общественного мнения (анкетный опрос населения по различным вопросам, политическим и социальным). Проводится с 1938 Британским институтом общественного мнения [British Institute of Public Opinion], а с 1952 институтом "Социальные исследования (опросы Гэллапа)" [Social Surveys (Gallup Polls) Ltd]) назван по имени основателя американского Института общественного мнения Дж. Гэллапа [George Horace Gallup, 1901-84].

О счастье мы всегда лишь вспоминаем.
А счастье всюду. Может быть, оно
Вот этот сад осенний за сараем
И чистый воздух, льющийся в окно".

И. Бунин

The **Happy Planet Index (HPI)** is an index of human well-being and environmental impact, introduced by the NEF, in July 2006.

The Happy Planet Index: What it reveals

On a scale of 0 to 100 for the HPI, we have set a reasonable target for nations to aspire to of 83.5. This is based on attainable levels of life expectancy and well-being and a reasonably sized ecological footprint.

At this point in time, the highest HPI is only 68.2, scored by the Pacific archipelago of Vanuatu. The lowest, and perhaps less surprising than some other results, is Zimbabwe's at 16.6. No country achieves an overall high score and no country does well on all three indicators. Vanuatu, for example, has only a moderate level of life expectancy at 69 years.

The message is that when we measure the efficiency with which countries enable the fundamental inputs of natural resources to be turned into the ultimate ends of long and happy lives, all can do better.

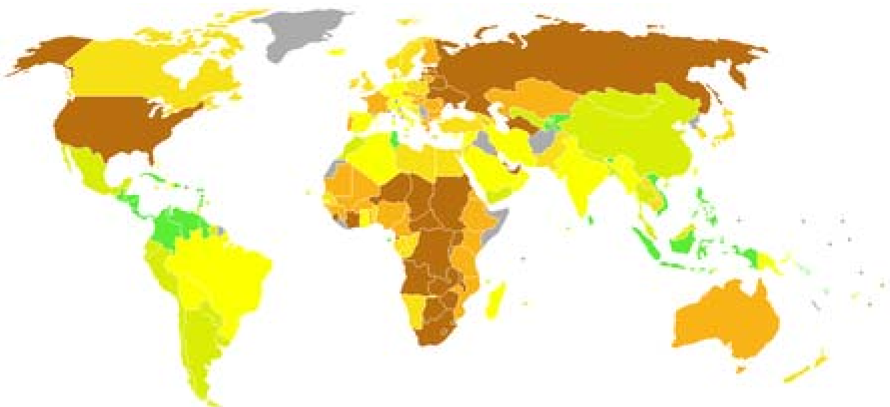
This conclusion is less surprising in the light of our argument that governments have been concentrating on the wrong indicators for too long. If you have the wrong map, you are unlikely to reach your destination.

Some of the most unexpected findings of the HPI concern the marked differences between nations, and the similarities among some groups of nations:

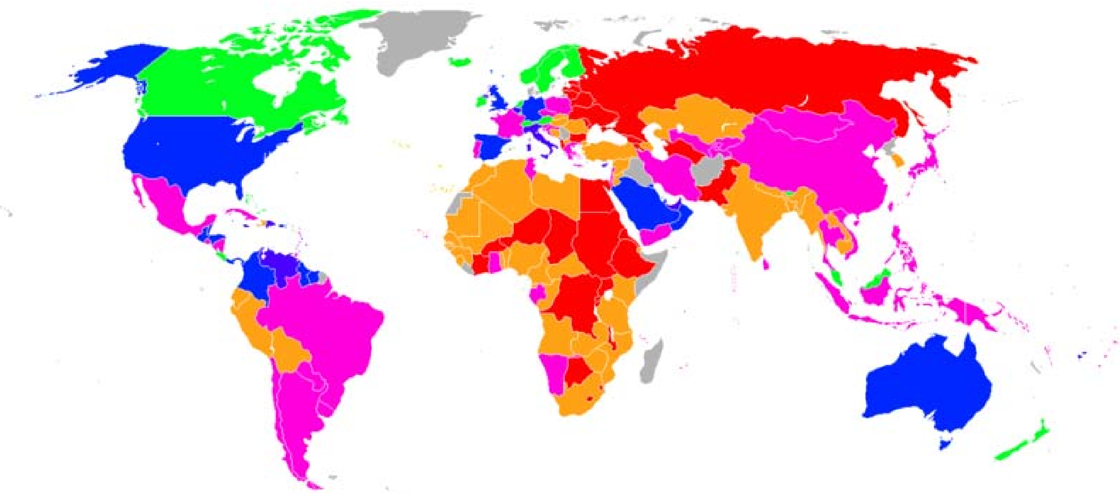
Island nations score well above average in the Index: They have higher life satisfaction, higher life expectancy and marginally lower Footprints than other states. Yet incomes (by GDP per capita) are roughly equal to the world average. Even within regions, islands do well. Malta tops the Western world with Cyprus in seventh place (out of 24); the top five HPI nations in

Africa are all islands; as well as two of the top four in Asia. Perhaps a more acute awareness of environmental limits has sometimes helped their societies to bond better and to adapt to get more from less. Combined with the enhanced well-being that stems from close contact with nature, the world as a whole stands to learn much from the experience of islands.

It is possible to live long, happy lives with a much smaller environmental impact: For example, in the United States and Germany people's sense of life satisfaction is almost identical and life expectancy is broadly similar. Yet Germany's Ecological footprint is only about half that of the USA. This means that Germany is around twice as efficient as the USA at generating happy long lives based on the resources that they consume.



Happy Planet Index, highest rank to lowest rank .



The Satisfaction with Life Index shows life satisfaction scores from the HPI without adjusting for ecological footprint and life expectancy.

green = most happy

blue

purple

orange

red = least happy