RECURSIVE KERNEL ESTIMATION OF THE INTENSITY FUNCTION OF AN INHOMOGENEOUS POISSON PROCESS

Anna Kitayeva[1], and Mihail Kolupaev[2]

[1]Tomsk Polytechnic University, Tomsk, Russia, kit1157@yandex.ru
[2]Tomsk State University, Tomsk, Russia, al13n@sibmail.com

In this paper we give an extension of the results considered in [1] to the recursive algorithms. Recursive estimation is particularly useful in large sample size since the result can be easily updated with each additional observation. The structure of the estimate is similar to the recursive kernel estimate of density function introduced in [2, 3], and thoroughly examined in [4]. The estimate is constructed on a single realization of a Poisson process on fixed interval $[0,T]$. Specificity of the statistic (1) is due to a random sample size.

Let $\{t_i,\quad i=\overline{1,N},\quad 0\le t_i\le T\}$ be a realization of a Poisson point process having unknown intensity function $\lambda(\cdot)$ on $[0,T]$, where N is the number of points falling into $[0,T]$, denote $\Lambda(a,b)=\int_a^b \lambda(t)dt$. Consider the following expression as an estimate of the function $\lambda(\cdot)/\Lambda(0,T)$ at a point t

$$S_N=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{h_i}K\left(\frac{t-t_i}{h_i}\right)=\left(1-\frac{1}{N}\right)S_{N-1}+\frac{1}{Nh_N}K\left(\frac{t-t_N}{h_N}\right) \tag{1}$$

where $(h_n)\downarrow 0$ is a sequence of real numbers, $nh_n\to\infty$, $K(\cdot)$ is a kernel function; let $S_N=0$ if $N=0$.

Consider an asymptotic behavior of statistic (1) under following scheme of series: let series of observations are done on $[0,T]$ with the intensity of the process in n-th trial equals to $\lambda_n(\cdot)=n\lambda(\cdot)$. Denote the value of the statistic (1) in n-th trial

$$S_n=\frac{1}{N_n}\sum_{i=1}^{N_n}\frac{1}{h_{in}}K\left(\frac{t-t_{in}}{h_{in}}\right), \tag{2}$$

where N_n and (t_{in}) – respectively the number of observations and the realization of the process in n-th trial.

Theorem 1 (asymptotic unbiasedness). Let the kernel $K(\cdot)$ is a compact bounded real valued Borel function on $[-T,T]$ and $\left|\int_{-T}^{T}K(u)u^m du\right|\le M\ \forall m=1,2,...;$ the intensity function $\lambda(\cdot)$ is continuous at the point $t\in(0,T)$, $\Lambda(t,T)\ne 0;$ (h_{in}) is monotonically no increasing with i sequence, $(h_{in})\to 0$ as $i\to\infty$, and $\left|\sum_{i=0}^{m}(-1)^i C_m^i h_{(i+N)n}^m\right|\le Ch_{Nn}^m\ \forall m$ with some constant C. Then the estimate (1) is asymptotically unbiased: $\lim_{n\to\infty}E(S_n)-\lambda(t)/\Lambda(0,T)=0.$

Proof. Let $p_i(x/n)$ be the conditional density function t_i given $N=n\ge 1$, $p_n=P(N=n)=\Lambda(0,T)^n e^{-\Lambda(0,T)}/n!,\ n\ge 0$. Then the expected value

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$$E(S_N)=\sum_{n=1}^{\infty}p_n\frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_i}\int_0^T K\left(\frac{t-x}{h_i}\right)p_i(x/n)dx=$$

$$=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_i}\int_0^T K\left(\frac{t-x}{h_i}\right)\frac{\Lambda(0,x)^{i-1}}{(i-1)!}\frac{\Lambda(x,T)^{n-i}}{(n-i)!}\frac{\lambda(x)}{\Lambda(0,T)}dx\frac{e^{-\Lambda(0,T)}}{1-e^{-\Lambda(0,T)}}. \qquad (3)$$

Taking into account $\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\Lambda(0,x)^{i-1}}{(i-1)!}\frac{\Lambda(x,T)^{n-i}}{(n-i)!}=\sum_{n=1}^{\infty}\frac{\Lambda(0,T)^{n-1}}{n!}=\frac{e^{\Lambda(0,T)}-1}{\Lambda(0,T)}$ and boundedness of the kernel we know that series (3) convergences absolutely. Let $h_i<\min(t/T,1-t/T)\,\forall i,$ and since $K(\cdot)$ is a compact function on $[-T,T]$ we have

$$\frac{1}{h_i}\int_0^T K\left(\frac{t-x}{h_i}\right)\frac{\Lambda(0,x)^{i-1}}{(i-1)!}\frac{\Lambda(x,T)^{n-i}}{(n-i)!}\lambda(x)dx=\int_{-T}^{T}K(x)\frac{\Lambda(0,t-h_ix)^{i-1}}{(i-1)!}\frac{\Lambda(t-h_ix,T)^{n-i}}{(n-i)!}\lambda(t-h_ix)dx.$$

Note that $\lambda(t-h_{in}x)=\lambda(t)+\widetilde{C}h_{in}x$ for sufficiently small h_{in}, then

$$E(S_n)=\frac{e^{-n\Lambda(0,T)}}{1-e^{-n\Lambda(0,T)}}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{i=1}^{k}n^k\int_{-T}^{T}K(x)\frac{\Lambda(0,t-h_{in}x)^{i-1}}{(i-1)!}\frac{\Lambda(t-h_{in}x,T)^{k-i}}{(k-i)!}\lambda(t-h_{in}x)dx=$$

$$=\frac{e^{-n\Lambda(0,T)}}{1-e^{-n\Lambda(0,T)}}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{i=1}^{k}n^k\int_{-T}^{T}K(x)\frac{\Lambda(0,t-h_{in}x)^{i-1}}{(i-1)!}\frac{\Lambda(t-h_{in}x,T)^{k-i}}{(k-i)!}\left(\lambda(t)+\widetilde{C}h_{in}x\right)dx=\lambda(t)L_n+L_n^o.$$

For the proof of the theorem we establish the convergence $\lim_{n\to\infty}L_n=\Lambda(0,T)^{-1}$.

Using binomial expansion and the linearization $\Lambda(t-h_{in}x,t)=\int_{t-h_{in}x}^{t}\lambda(t)dt=rh_{in}x,$ where $r>0$ – some constant, we obtain

$$\sum_{k=1}^{\infty}\frac{1}{k}\sum_{i=1}^{k}n^k\int_{-T}^{T}K(x)\frac{\Lambda(0,t-h_{in}x)^{i-1}}{(i-1)!}\frac{\Lambda(t-h_{in}x,T)^{k-i}}{(k-i)!}dx=$$

$$=\sum_{k=1}^{\infty}\frac{1}{k}\sum_{i=1}^{k}n^k\int_{-T}^{T}K(x)\sum_{l=0}^{i-1}(-1)^{i-1-l}\frac{\Lambda(0,t)^l}{l!}\frac{(rh_{in}x)^{i-1-l}}{(i-1-l)!}\sum_{m=0}^{k-i}\frac{\Lambda(t,T)^m}{m!}\frac{(rh_{in}x)^{k-i-m}}{(k-i-m)!}dx=$$

$$=\sum_{k=0}^{\infty}\frac{n}{k+1}\sum_{l=0}^{k}\frac{\Lambda(0,t)^l}{l!}n^k\sum_{i=0}^{k-l}(-1)^i\int_{-T}^{T}K(x)\frac{(rh_{in}x)^i}{i!}\sum_{m=0}^{k-i-l}\frac{\Lambda(t,T)^m}{m!}\frac{(rh_{in}x)^{k-i-l-m}}{(k-i-l-m)!}dx=$$

$$=\sum_{k=0}^{\infty}\frac{n}{k+1}\sum_{l=0}^{k}\frac{\Lambda(0,t)^ln^l}{l!}\sum_{m=0}^{k-l}\frac{\Lambda(t,T)^mn^m}{m!}n^{k-l-m}\sum_{i=0}^{k-l-m}(-1)^i\int_{-T}^{T}K(x)\frac{(rh_{in}x)^{k-l-m}}{i!(k-i-l-m)!}dx= \qquad (4)$$

$$=\sum_{k=0}^{\infty}\frac{n}{k+1}\sum_{l=0}^{k}\frac{\Lambda(0,t)^ln^l}{l!}\frac{\Lambda(t,T)^{k-l}n^{k-l}}{(k-l)!}+\Delta_n,$$

where

$$\Delta_n=\sum_{k=1}^{\infty}\frac{n}{k+1}\sum_{l=0}^{k-1}\frac{\Lambda(0,t)^ln^l}{l!}\sum_{m=0}^{k-l-1}\frac{\Lambda(t,T)^mn^m}{m!}n^{k-l-m}\sum_{i=0}^{k-l-m}(-1)^i\int_{-T}^{T}K(x)\frac{(rh_{in}x)^{k-l-m}}{i!(k-i-l-m)!}dx+$$

$$+n+\sum_{k=0}^{\infty}\frac{\Lambda(0,t)^kn^{k+1}}{(k+1)!}.$$

In the next step we prove the convergence $\lim_{n\to\infty}e^{-n\Lambda(0,T)}\Delta_n=0$:

$$\lim_{n\to\infty} e^{-n\Lambda(0,T)}\left|\sum_{k=1}^{\infty}\frac{n}{k+1}\sum_{l=0}^{k-1}\frac{\Lambda(0,t)^l n^l}{l!}\frac{\Lambda(t,T)^{k-l}n^{k-l}}{(k-l)!}\sum_{m=1}^{k-l}\frac{C_{k-l}^m r^m}{\Lambda(t,T)^m}\sum_{i=0}^{m}(-1)^i C_m^i h_{(i+l+1)n}^m\int_{-T}^{T}K(x)x^m dx\right| \le$$

$$\le MC\lim_{n\to\infty} e^{-n\Lambda(0,T)}\sum_{k=1}^{\infty}\frac{n}{k+1}\sum_{l=0}^{k-1}\frac{\Lambda(0,t)^l n^l}{l!}\frac{\Lambda(t,T)^{k-l}n^{k-l}}{(k-l)!}\sum_{m=1}^{k-l}\frac{C_{k-l}^m r^m h_{(l+1)n}^m}{\Lambda(t,T)^m} = \qquad (5)$$

$$= MC\lim_{n\to\infty} e^{-n\Lambda(0,T)}\sum_{k=1}^{\infty}\frac{n}{k+1}\sum_{l=0}^{k-1}\frac{\Lambda(0,t)^l n^l}{l!}\frac{\Lambda(t,T)^{k-l}n^{k-l}}{(k-l)!}\left(\left(1+\frac{rh_{(l+1)n}}{\Lambda(t,T)}\right)^{k-l}-1\right)=0.$$

It follows from

$$\lim_{n\to\infty} e^{-n\Lambda(0,T)}\sum_{k=1}^{\infty}\frac{n}{k+1}\sum_{l=0}^{k-1}\frac{\Lambda(0,t)^l n^l}{l!}\frac{\Lambda(t,T)^{k-l}n^{k-l}}{(k-l)!} = \lim_{n\to\infty} e^{-n\Lambda(0,T)}\sum_{k=1}^{\infty}\frac{n}{k+1}\sum_{l=0}^{k}\frac{\Lambda(0,t)^l n^l}{l!}\frac{\Lambda(t,T)^{k-l}n^{k-l}}{(k-l)!} =$$

$$=\lim_{n\to\infty} e^{-n\Lambda(0,T)}\sum_{k=1}^{\infty}\frac{n^{k+1}}{k+1}\frac{\Lambda(0,T)^k}{k!} = \lim_{n\to\infty} e^{-n\Lambda(0,T)}\frac{\left(e^{n\Lambda(0,T)}-1-n\Lambda(0,T)\right)}{\Lambda(0,T)} = \frac{1}{\Lambda(0,T)},$$

$$\frac{1}{\Lambda(0,T)} = \lim_{n\to\infty} e^{-n\Lambda(0,T)}\sum_{k=1}^{\infty}\frac{n}{k+1}\sum_{l=0}^{k-1}\frac{\Lambda(0,t)^l n^l}{l!}\frac{\Lambda(t,T)^{k-l}n^{k-l}}{(k-l)!} \le$$

$$\le \lim_{n\to\infty} e^{-n\Lambda(0,T)}\sum_{k=1}^{\infty}\frac{n}{k+1}\sum_{l=0}^{k-1}\frac{\Lambda(0,t)^l n^l}{l!}\frac{\Lambda(t,T)^{k-l}n^{k-l}}{(k-l)!}\left(1+\frac{rh_{(l+1)n}}{\Lambda(t,T)}\right)^{k-l} \le$$

$$\le \lim_{n\to\infty} e^{-n\Lambda(0,T)}\sum_{k=1}^{\infty}\frac{n}{k+1}\sum_{l=0}^{k-1}\frac{\Lambda(0,t)^l n^l}{l!}\frac{\Lambda(t,T)^{k-l}n^{k-l}}{(k-l)!}\left(1+\frac{rh_{2n}}{\Lambda(t,T)}\right)^{k-l} =$$

$$= \lim_{n\to\infty}\frac{\exp\left\{n\left[\Lambda(0,t)+\Lambda(t,T)\left(1+rh_{2n}/\Lambda(t,T)\right)\right]-n\Lambda(0,T)\right\}}{\Lambda(0,t)+\Lambda(t,T)\left(1+rh_{2n}/\Lambda(t,T)\right)} = \frac{1}{\Lambda(0,T)}. \qquad (6)$$

The theorem is proved.

Theorem 2 (the variance convergence). Let the assumptions of Theorem 1 hold, $\int_{-T}^{T}K^2(x)x^m dx < \widetilde{M} \quad \forall m=1,2,...;\ ih_i \to \infty$ as $i\to\infty$. Then $\lim_{n\to\infty} Var(S_n)=0$.

Proof. By $P(N_n \ge 2)\to 1$ as $n\to\infty$ the variance $Var(S_n)=$

$$= E\left\{\frac{1}{(N_n)^2}\left[\sum_{i=1}^{N_n}\frac{1}{h_{in}^2}K^2\left(\frac{t-t_{in}}{h_{in}}\right)+2\sum_{i>j}\frac{1}{h_{in}h_{jn}}K\left(\frac{t-t_{in}}{h_{in}}\right)K\left(\frac{t-t_{jn}}{h_{jn}}\right)\right]\right\}-\left\{E\left[\frac{1}{N_n}\sum_{i=1}^{N_n}\frac{1}{h_{in}}K\left(\frac{t-t_{in}}{h_{in}}\right)\right]\right\}^2 =$$

$$= C_1(n)\sum_{k=1}^{\infty}\frac{n}{k^2}\sum_{i=1}^{k}\frac{1}{h_{in}}\int_{-T}^{T}K^2(x)\lambda(t-h_{in}x)\frac{\Lambda(0,t-h_{in}x)^i}{(i-1)!}\frac{\Lambda(t-h_{in}x,T)^{k-i}}{(k-i)!}n^k dx e^{-n\Lambda(0,T)} +$$

$$+2C_2(n)\sum_{k=2}^{\infty}\frac{1}{k^2}\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\frac{1}{h_{in}h_{jn}}\int_0^T\int_x^T K\left(\frac{t-x}{h_{in}}\right)\lambda(x)K\left(\frac{t-y}{h_{jn}}\right)\lambda(y)\frac{\Lambda(0,x)^{i-1}}{(i-1)!}\frac{\Lambda(x,y)^{j-i-1}}{(j-i-1)!}\frac{\Lambda(y,T)^{k-j}}{(k-j)!}\times$$

$$\times n^k dy dx e^{-n\Lambda(0,T)} - C_3(n)\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_{in}}\int_0^T K\left(\frac{t-x}{h_{in}}\right)\frac{\Lambda(0,x)^{i-1}}{(i-1)!}\frac{\Lambda(x,T)^{n-i}}{(n-i)!}\lambda(x)dx e^{-n\Lambda(0,T)}\right), \qquad (7)$$

where $C_i(n)\to 1$ as $n\to\infty, i=1,2,3$. The second term in (7) convergences to $\left(\frac{\lambda(t)}{\Lambda(0,T)}\right)^2$ since according to Newton's polynom $\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\frac{\Lambda(0,x)^{i-1}}{(i-1)!}\frac{\Lambda(x,y)^{j-i-1}}{(j-i-1)!}\frac{\Lambda(y,T)^{k-j}}{(k-j)!}=$

$= \dfrac{\Lambda(0,T)^{k-2}}{(k-2)!}$ (the main part is finding similarly (4)).

To establish the statement of the theorem we must show $Q_n = \sum_{k=1}^{\infty} \frac{n}{k^2} \sum_{i=1}^{k} \frac{1}{h_{in}} \int_{-T}^{T} K^2(x) \lambda(t - h_{in}x) \frac{\Lambda(0, t - h_{in}x)^i}{(i-1)!} \frac{\Lambda(t - h_{in}x, T)^{k-i}}{(k-i)!} n^k dx =$

$= o\left(e^{-n\Lambda(0,T)}\right)$ as $n \to \infty$. Let us consider $Q_n = \sum_{k=0}^{\infty} \frac{n}{(k+1)^2} \sum_{l=0}^{k} \frac{\Lambda(0,t)^l n^l}{l!} \frac{\Lambda(t,T)^{k-l} n^{k-l}}{(k-l)!} +$

$$+ \sum_{k=1}^{\infty} \frac{n}{(k+1)^2} \sum_{l=0}^{k-1} \frac{\Lambda(0,t)^l n^l}{l!} \sum_{m=0}^{k-l-1} \frac{\Lambda(t,T)^m n^m}{m!} n^{k-l-m} \sum_{i=0}^{k-l-m} \frac{(-1)^i}{h_{in}} \int_{-T}^{T} K^2(x) \frac{(r h_{in} x)^{k-l-m}}{i!(k-i-l-m)!} dx =$$

$$= Q_n^{(1)} + Q_n^{(2)}, \; Q_n^{(1)} = \frac{1}{\Lambda(0,T)} \sum_{k=1}^{\infty} \frac{\Lambda(0,T)^k n^k}{k k!} = \frac{1}{\Lambda(0,T)} \left[\mathrm{Ei}(n\Lambda(0,T)) - \ln(n\Lambda(0,T)) - \gamma \right],$$

where $\mathrm{Ei}(n)$ – integral exponent, γ – Euler constant, $\lim_{n\to\infty} \mathrm{Ei}(n) e^{-n} = 0$ [5]. Thus we obtain $\lim_{n\to\infty} Q_n^{(1)} e^{-n} = 0.$ Similarly (5) we have

$$\left| Q_n^{(2)} \right| \le \widetilde{M}C \sum_{k=1}^{\infty} \frac{n}{(k+1)^2} \sum_{l=0}^{k-1} \frac{\Lambda(0,t)^l n^l}{l!} \frac{\Lambda(t,T)^{k-l} n^{k-l}}{(k-l)! h_{(l+1)n}} \left(\left(1 + \frac{r h_{(l+1)n}}{\Lambda(t,T)} \right)^{k-l} - 1 \right) + o(n) \le$$

$$\le \widetilde{M}C \sum_{k=1}^{\infty} \frac{n}{(k+1)} \sum_{l=0}^{k-1} \frac{\Lambda(0,t)^l n^l}{l!} \frac{\Lambda(t,T)^{k-l} n^{k-l}}{(k-l)!(l+1) h_{(l+1)n}} \left(\left(1 + \frac{r h_{(l+1)n}}{\Lambda(t,T)} \right)^{k-l} - 1 \right) + o(n) \le$$

$$\le \widetilde{M}C \sum_{k=1}^{\infty} \frac{n}{(k+1)} \sum_{l=0}^{k-1} \frac{\Lambda(0,t)^l n^l}{l!} \frac{\Lambda(t,T)^{k-l} n^{k-l}}{(k-l)!} \left(\left(1 + \frac{r h_{(l+1)n}}{\Lambda(t,T)} \right)^{k-l} - 1 \right) + o(n).$$

Taking (6) into account we have that $\lim_{n\to\infty} \left| Q_n^{(2)} \right| e^{-n} \le 0.$ The theorem is proved.

Literature

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