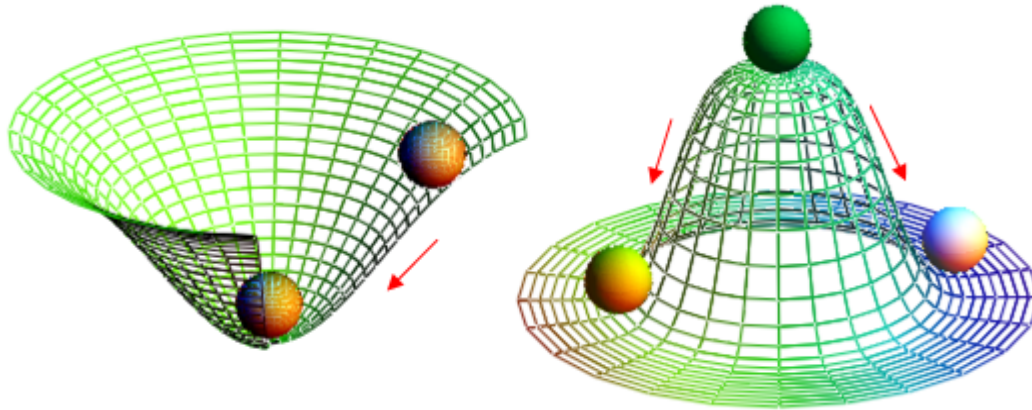


## CHAPTER 2. NONLINEAR DYNAMIC SYSTEMS

### 2.1. Phase space and dynamic systems phase portraits



**Figure 1**      Stable system

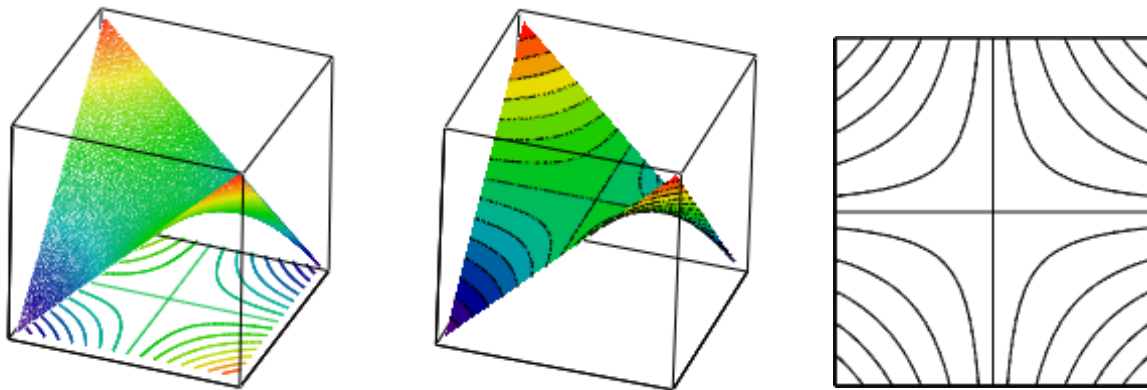
Unstable system

Any electromechanic system is a dynamic system. Elements forming the system can be nonlinear, hence differential equations describing dynamic systems are nonlinear.

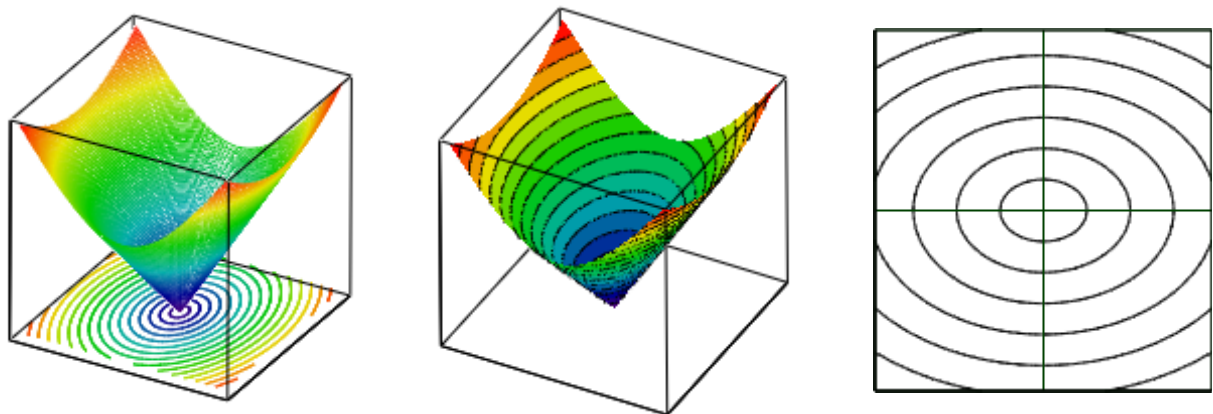
To investigate nonlinear systems and make the understanding of complex dynamic taking place in them *phase space* is used in which one can construct *phase portraits* (see Figure 2). Each dynamic system has its phase portrait.

On phase portrait there are *special points* – *equilibrium points*, which help foretell dynamic system behavior without solving differential equations. These equilibrium points can be stable and unstable. If dynamic system is in the range of stable equilibrium point, then small disturbances will not make the system unstable (see Figure 1). If equilibrium point is not stable, then disturbances persist and make the system "unstable" (see Figure 1).

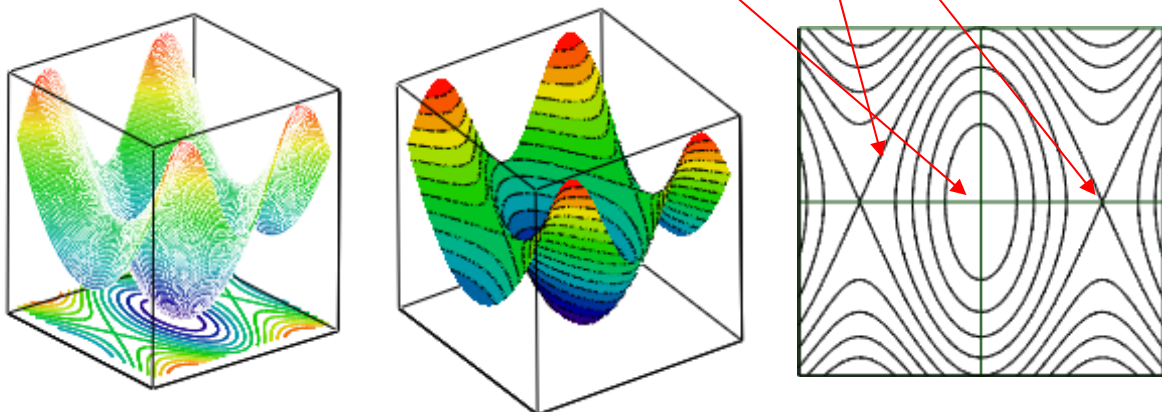
equilibrium point of "saddle" type



equilibrium point of "centre" type



equilibrium points of "centre" and "saddle" type,  
the line separating solutions—"separatrix"



**Figure 2.** Dynamic system phase portraits

Let's analyze phase space method relating to dynamic system of the second order.

$$\begin{cases} \frac{dx}{dt} = F_1(x, y) \\ \frac{dy}{dt} = F_2(x, y) \end{cases}, \quad (1)$$

where  $F_1(x, y)$  and  $F_2(x, y)$  are nonlinear functions of their own arguments.

To get phase trajectory time need to be eliminated. To do this divide the second equation by the first one

$$\frac{dy}{dx} = \frac{F_2(x, y)}{F_1(x, y)} \quad (2)$$

At equilibrium points time derivative turns zero, then we obtain the expression  $0/0$  which has no defined value and is called an indeterminate form:

$$\frac{dy}{dx} = \frac{F_2(x, y)}{F_1(x, y)} = \frac{0}{0} \quad (3)$$

The points at which there are uncertainties are called singular points. In singular points on phase plane solutions can bifurcate. Let us describe equilibrium state points search algorithm.

At equilibrium state the given variables  $x$  and  $y$  do not change, hence derivatives equal zero:

$$\begin{cases} 0 = F_1(x, y) \\ 0 = F_2(x, y) \end{cases} \quad (4)$$

By solving the given nonlinear equations system we find equilibrium state points  $x_0, y_0$ . After determining equilibrium state point coordinates one need to determine point type. To determine point type expand  $F_1(x, y)$  and  $F_2(x, y)$  functions in

equilibrium state points neighborhood  $x_0, y_0$ , and analyze the first three expansion terms:

$$\begin{cases} F_1(x, y) = F_1(x_0, y_0) + \frac{\partial F_1}{\partial x}(x - x_0) + \frac{\partial F_1}{\partial y}(y - y_0) + \dots \\ F_2(x, y) = F_2(x_0, y_0) + \frac{\partial F_2}{\partial x}(x - x_0) + \frac{\partial F_2}{\partial y}(y - y_0) + \dots \end{cases} \quad (5)$$

Put down coefficients at linear terms and obtain Jacobian matrix:

$$\mathbf{A}(x_0, y_0) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad (6)$$

Jacobian matrix elements  $a_{i,j}$  are constant variables.

Next step is to determine point character .

To do this one should find eigenvalues of Jacobian matrix  $\lambda$  by solving characteristic equation.

$$|\mathbf{I} - \lambda \mathbf{A}| = 0. \quad (7)$$

where  $\mathbf{I}$  is identity matrix.

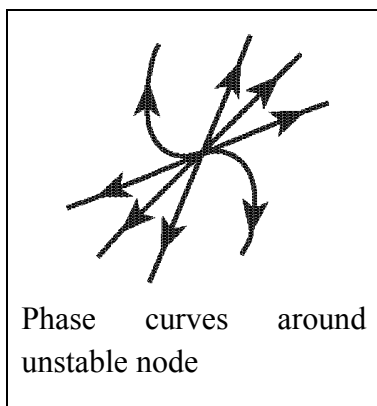
It is easy to determine eigenvalues by using **MathCAD** **eigenvalues(A)** function

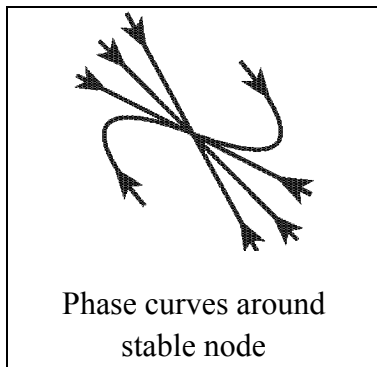
application, several variants are acceptable.

1. Both eigenvalues are real and positive:

$$\lambda_1 > 0, \lambda_2 > 0$$

Then, equilibrium state point with coordinates  $x_0, y_0$  is called an ***unstable node***

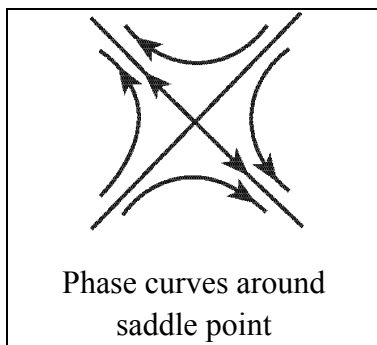




2. Both eigenvalues are real and negative:

$$\lambda_1 < 0, \lambda_2 < 0$$

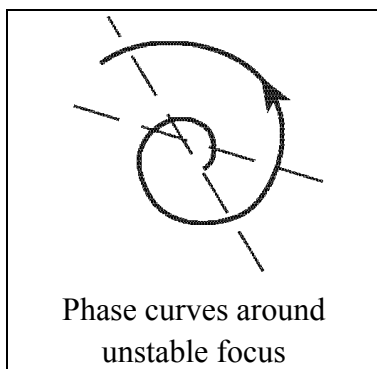
Then equilibrium state point with coordinates  $x_0, y_0$  is called a **stable node**.



3. Both eigenvalues are real and have different signs:

$$\lambda_1 < 0, \lambda_2 > 0 \quad \text{или} \quad \lambda_2 < 0, \lambda_1 > 0$$

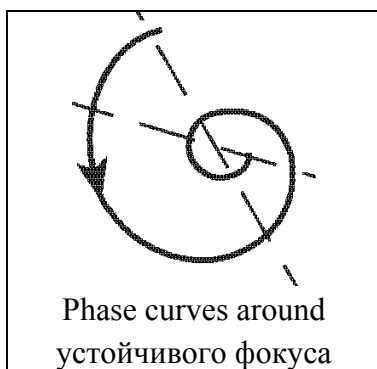
Then equilibrium state point  $x_0, y_0$  is called a **saddle or a saddle node**.



4. Eigenvalues are complex conjugated numbers which have positive real parts:

$$\lambda_1 = \beta + j\omega, \lambda_2 = \beta - j\omega, \quad \beta > 0$$

Then equilibrium state point  $x_0, y_0$  is called an **unstable focus**

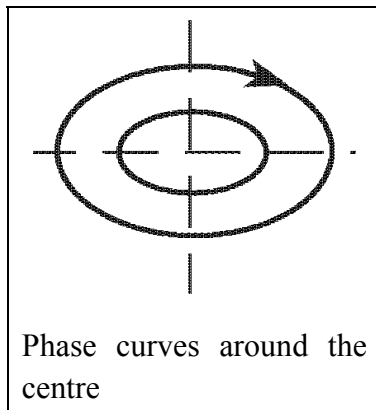


5. Eigenvalues are complex conjugated numbers which have negative real parts:

$$\lambda_1 = \beta + j\omega, \lambda_2 = \beta - j\omega, \quad \beta < 0$$

Then equilibrium state point  $x_0, y_0$  is called a **stable focus**

6. Eigenvalues are imaginary conjugated numbers:



$$\lambda_1 = j\omega, \lambda_2 = -j\omega$$

Then equilibrium state point with coordinates  $x_0, y_0$  is called a **centre**.

Let's analyze some examples of equilibrium state points determination and determine their type using **MathCAD**.

ORIGIN:= 1

**Example 1. Dynamic system is given**

$$\frac{d}{dt}x = x^2 + y^2 - 17$$

$$\frac{d}{dt}y = x \cdot y + 4$$

Find equilibrium points of dynamic system

$$x^2 + y^2 - 17 = 0$$

$$x \cdot y + 4 = 0$$

$$z := \begin{pmatrix} x^2 + y^2 - 17 \\ x \cdot y + 4 \end{pmatrix} \text{ solve, } \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -4 & 1 \\ -1 & 4 \\ 1 & -4 \\ 4 & -1 \end{pmatrix}$$

$$\text{yo} := z^{(2)} \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Expand nonlinear functions in the neighborhood of equilibrium state points.

$$x^2 + y^2 \text{ series, } x = -4, y = 1, 2 \rightarrow (-17) - 8 \cdot x + 2 \cdot y$$

$$a := \begin{pmatrix} -8 & 2 \\ 1 & -4 \end{pmatrix}$$

$\text{eigenvals}(a) = \begin{pmatrix} -8.449 \\ -3.551 \end{pmatrix}$  point with coordinates  $x=-4$  и  $y=1$  is a *stable node*

$x \cdot y$  series,  $x = -4, y = 1, 2 \rightarrow 4 - 4 \cdot y + x$

$$|a - \lambda \cdot I| \rightarrow 30 + 12 \cdot \lambda + \lambda^2$$

$x^2 + y^2$  series,  $x = -1, y = 4, 2 \rightarrow (-17) - 2 \cdot x + 8 \cdot y$

$$a := \begin{pmatrix} -2 & 8 \\ 4 & -1 \end{pmatrix}$$

$\text{eigenvals}(a) = \begin{pmatrix} -7.179 \\ 4.179 \end{pmatrix}$  point with coordinates  $x = -1$  и  $y = 4$  is called a *saddle node*.

$x \cdot y$  series,  $x = -1, y = 4, 2 \rightarrow 4 - y + 4 \cdot x$

$$|a - \lambda \cdot I| \rightarrow (-30) + 3 \cdot \lambda + \lambda^2$$

$x^2 + y^2$  series,  $x = 1, y = -4, 2 \rightarrow (-17) + 2 \cdot x - 8 \cdot y$

$$a := \begin{pmatrix} 2 & -8 \\ -4 & 1 \end{pmatrix}$$

$\text{eigenvals}(a) = \begin{pmatrix} 7.179 \\ -4.179 \end{pmatrix}$  point with coordinates  $x = 1$  и  $y = -4$  is called a *saddle*

$x \cdot y$  series,  $x = 1, y = -4, 2 \rightarrow 4 + y - 4 \cdot x$

$$|a - \lambda \cdot I| \rightarrow (-30) - 3 \cdot \lambda + \lambda^2$$

$x^2 + y^2$  series,  $x = 4, y = -1, 2 \rightarrow (-17) + 8 \cdot x - 2 \cdot y$

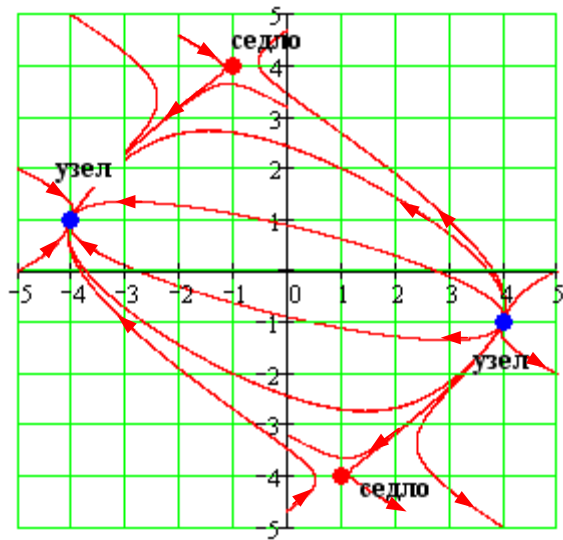
$x \cdot y$  series,  $x = 4, y = -1, 2 \rightarrow 4 + 4 \cdot y - x$

$$a := \begin{pmatrix} 8 & -2 \\ -1 & 4 \end{pmatrix}$$

$\text{eigenvals}(a) = \begin{pmatrix} 8.449 \\ 3.551 \end{pmatrix}$  point with coordinates  $x = 4$  и  $y = -1$  is called *an unstable node*.

$$|a - \lambda \cdot I| \rightarrow 30 - 12 \cdot \lambda + \lambda^2$$

## Phase portrait of dynamic system



### Example 2. Dynamic system is given

$$\frac{d}{dt}x = x^2 - y^2 - 5$$

$$\frac{d}{dt}y = x^2 + y^2 - 13$$

Find equilibrium points of dynamic system

$$x^2 - y^2 - 5 = 0$$

$$x^2 + y^2 - 13 = 0$$

$$z := \begin{pmatrix} x^2 - y^2 - 5 \\ x^2 + y^2 - 13 \end{pmatrix} \text{ solve, } \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 \\ -3 & 2 \\ 3 & -2 \\ -3 & -2 \end{pmatrix}$$

$$x_0 := z^{(1)} \quad y_0 := z^{(2)} \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Expand nonlinear functions in the neighbourhood of equilibrium state points

$$x^2 - y^2 \text{ series, } x=3, y=2,2 \rightarrow (-5) + 6 \cdot x - 4 \cdot y$$

$$x^2 + y^2 \text{ series, } x=3, y=2,2 \rightarrow (-13) + 6 \cdot x + 4 \cdot y$$



$$a := \begin{pmatrix} 6 & -4 \\ 6 & 4 \end{pmatrix} \quad |a - \lambda \cdot I| \rightarrow 48 - 10\lambda + \lambda^2$$

$\text{eigenvals}(a) = \begin{pmatrix} 5 + 4.796i \\ 5 - 4.796i \end{pmatrix}$  point with coordinates  $x=3$  и  $y=2$  is called an *unstable focus*

$$x^2 - y^2 \text{ series, } x = -3, y = 2, 2 \rightarrow (-5) - 6 \cdot x - 4 \cdot y$$

$$x^2 + y^2 \text{ series, } x = -3, y = 2, 2 \rightarrow (-13) - 6 \cdot x + 4 \cdot y$$

$$a := \begin{pmatrix} -6 & -4 \\ -6 & 4 \end{pmatrix} \quad |a - \lambda \cdot I| \rightarrow (-48) + 2\lambda + \lambda^2$$

$\text{eigenvals}(a) = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$  point with coordinates  $x=-3$  и  $y=2$  is called a *saddle*.

$$x^2 - y^2 \text{ series, } x = 3, y = -2, 2 \rightarrow (-5) + 6 \cdot x + 4 \cdot y$$

$$x^2 + y^2 \text{ series, } x = 3, y = -2, 2 \rightarrow (-13) + 6 \cdot x - 4 \cdot y$$

$$a := \begin{pmatrix} 6 & 4 \\ 6 & -4 \end{pmatrix} \quad |a - \lambda \cdot I| \rightarrow (-48) - 2\lambda + \lambda^2$$

$\text{eigenvals}(a) = \begin{pmatrix} 8 \\ -6 \end{pmatrix}$  point with coordinates  $x=3$  и  $y=-2$  is called a *saddle*.

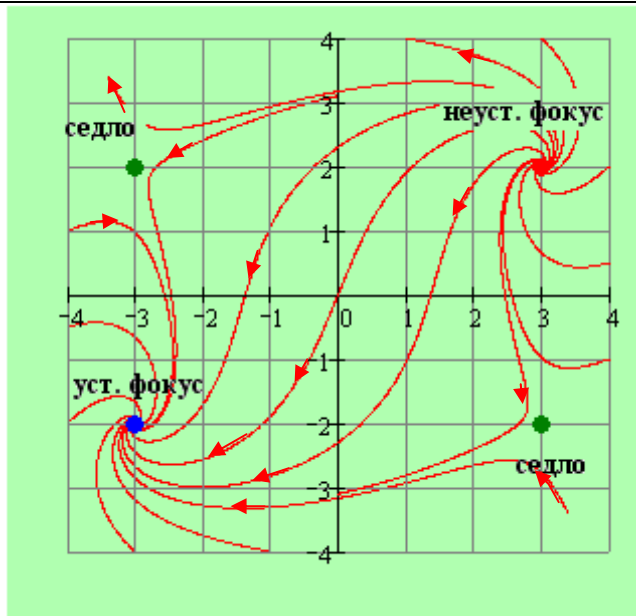
$$x^2 - y^2 \text{ series, } x = -3, y = -2, 2 \rightarrow (-5) - 6 \cdot x + 4 \cdot y$$

$$x^2 + y^2 \text{ series, } x = -3, y = -2, 2 \rightarrow (-13) - 6 \cdot x - 4 \cdot y$$

$$a := \begin{pmatrix} -6 & 4 \\ -6 & -4 \end{pmatrix} \quad |a - \lambda \cdot I| \rightarrow 48 + 10\lambda + \lambda^2$$

$\text{eigenvals}(a) = \begin{pmatrix} -5 + 4.796i \\ -5 - 4.796i \end{pmatrix}$  point with coordinates  $x=3$  и  $y=2$  is called a *stable focus*.

Phase portrait of dynamic system.



**Example 3. Determine equilibrium state point type of a linearized dynamic system.**

$$\frac{d}{dt}x = -x - 4y$$

$$\frac{d}{dt}y = 2x + 5y$$

$$\underline{\underline{A}} := \begin{pmatrix} -1 & -4 \\ 2 & 5 \end{pmatrix} \quad \lambda := \text{eigenvals}(A) \quad \lambda = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{Point type } x=0, y=0 \text{ } \textit{unstable node}$$

**Example 4. Determine equilibrium state point type of a linearized dynamic system.**

$$\frac{d}{dt}x = -x - y$$

$$\frac{d}{dt}y = x - 3y$$

$$A := \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix} \quad \underline{\underline{\lambda}} := \text{eigenvals}(A) \quad \lambda = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \quad \text{Point type } x=0, y=0 \text{ } \textit{is a stable node.}$$

### 2.1.1. Phase portraits construction with the help of surfaces.

Let's analyze some more examples of surface construction which help determine decomposition structure of phase plane into trajectories. First additional calculations should be done. Put down Newton's equation of motion with unit mass on which force  $F(x)$  influences:

$$\frac{d^2x}{dt^2} = F(x) \quad (8)$$

Taking into account that force equals potential function gradient with negative sign the equation can be presented as:

$$F(x) = -\frac{dU(x)}{dx} \rightarrow \frac{d^2x}{dt^2} = -\frac{dU(x)}{dx} \quad (9)$$

This equation can be rearranged by multiplying each part by velocity  $v = dx / dt$

$$v \frac{dv}{dt} = -\frac{dU}{dx} \frac{dx}{dt} \rightarrow \frac{1}{2} \frac{dv^2}{dt} + \frac{dU}{dt} = 0 \rightarrow \frac{d}{dt} \left( \frac{v^2}{2} + U \right) = 0$$

From the obtained formula it follows that formula in brackets equals some constant  $E$

$$\frac{v^2}{2} + U(x) = E(x, v) \quad \text{где} \quad U(x) = -\int_0^x F(x) dx \quad (10)$$

Now let's investigate the dynamic system having the form of:

$$\begin{cases} \frac{dx}{dt} = 2y \\ \frac{dy}{dt} = 4x - 4x^3 \end{cases}$$

Denote velocity in the first equation by  $2y$ , in the second equation denote the right part by force influencing the particle with unit mass, then auxiliary potential function

$U(x)$  has the following form :

$$U(x) = -\int_0^x (4x - 4x^3) dx = -2x^2 + x^4$$

In this case constant energy surface expression can be presented as:

$$U(x) + 2y^2 = E \rightarrow (-2x^2 + x^4 + 2y^2) = E$$

Now we can plot the phase portrait of constant energy surface

To solve this equation we use **MathCAD**

$$\frac{d}{dt}x = 2y$$

$$\frac{d}{dt}y = 4x - 4x^3$$

$$F_1(x, y) := 2 \cdot y \quad F_2(x, y) := 4 \cdot x - 4 \cdot x^3$$

Find equilibrium state points

$$z := \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix} \left| \begin{array}{l} \text{solve,} \begin{pmatrix} x \\ y \end{pmatrix} \\ \text{float, 5} \end{array} \right. \rightarrow \begin{pmatrix} 0 & 0 \\ 1. & 0 \\ -1. & 0 \end{pmatrix} \quad x := z^{\langle 0 \rangle} \quad y := z^{\langle 1 \rangle}$$

$$x = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Find Jacobian

$$A(x, y) := \begin{pmatrix} \frac{\partial}{\partial x} F_1(x, y) & \frac{\partial}{\partial y} F_1(x, y) \\ \frac{\partial}{\partial x} F_2(x, y) & \frac{\partial}{\partial y} F_2(x, y) \end{pmatrix} \quad A(x, y) \rightarrow \begin{pmatrix} 0 & 2 \\ 4 - 12 \cdot x^2 & 0 \end{pmatrix}$$

Determine point type with coordinates 0, 0

$$A_1 := A(x_0, y_0) \quad A_1 = \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix} \quad \lambda := \text{eigenvals}(A_1)$$

$$\lambda = \begin{pmatrix} 2.828 \\ -2.828 \end{pmatrix} \quad \textbf{Saddle point}$$

Determine point type with coordinates 1, 0

$$A_2 := A(x_1, y_1) \quad A_2 = \begin{pmatrix} 0 & 2 \\ -8 & 0 \end{pmatrix} \quad \lambda := \text{eigenvals}(A_2)$$

$$\lambda = \begin{pmatrix} 4i \\ -4i \end{pmatrix} \quad \textbf{Centre}$$

Determine point type with coordinates -1, 0

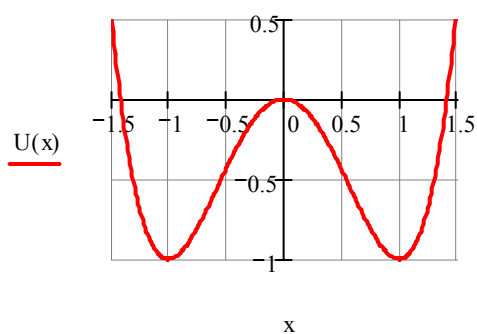
$$A_3 := A(x_2, y_2) \quad A_3 = \begin{pmatrix} 0 & 2 \\ -8 & 0 \end{pmatrix} \quad \lambda := \text{eigenvals}(A_3)$$

$$\lambda = \begin{pmatrix} 4i \\ -4i \end{pmatrix} \quad \textbf{Centre}$$

Auxiliary potential functions  $U(x)$  is written as

$$U(x) := - \int_0^x 4x - 4x^3 \, dx \rightarrow (-2) \cdot x^2 + x^4$$

$$x := -1.5, -1.5 + 0.01 \dots 1.5$$



Auxiliary energy function  $E(x, y)$  is written as

$$E(x, y) := y^2 \cdot 2 + U(x)$$

Plot phase portrait

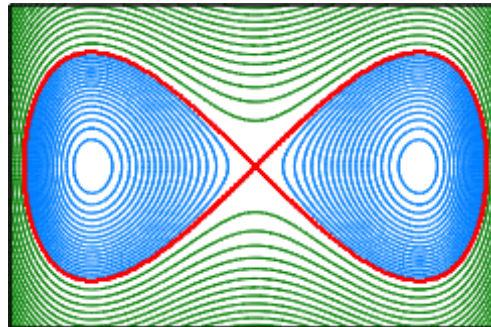
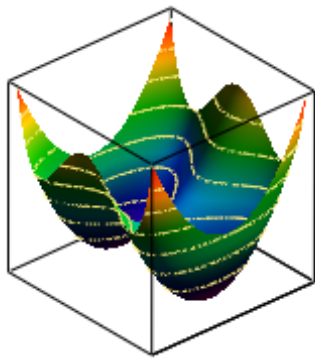
$$N := 251 \quad i := 0 \dots N \quad j := i \quad x_1 := -1.5 + \frac{3}{N} \cdot i \quad y_j := -1 + \frac{2}{N} \cdot j$$

Surface  $\tilde{E}_{i,j} := E(x_i, y_j)$

Surface limited by separatrix  $E1_{i,j} := \text{if}(E_{i,j} < 0, E_{i,j}, 0)$

Surface behind separatrix  $E2_{i,j} := \text{if}(E_{i,j} > 0, E_{i,j}, 0)$

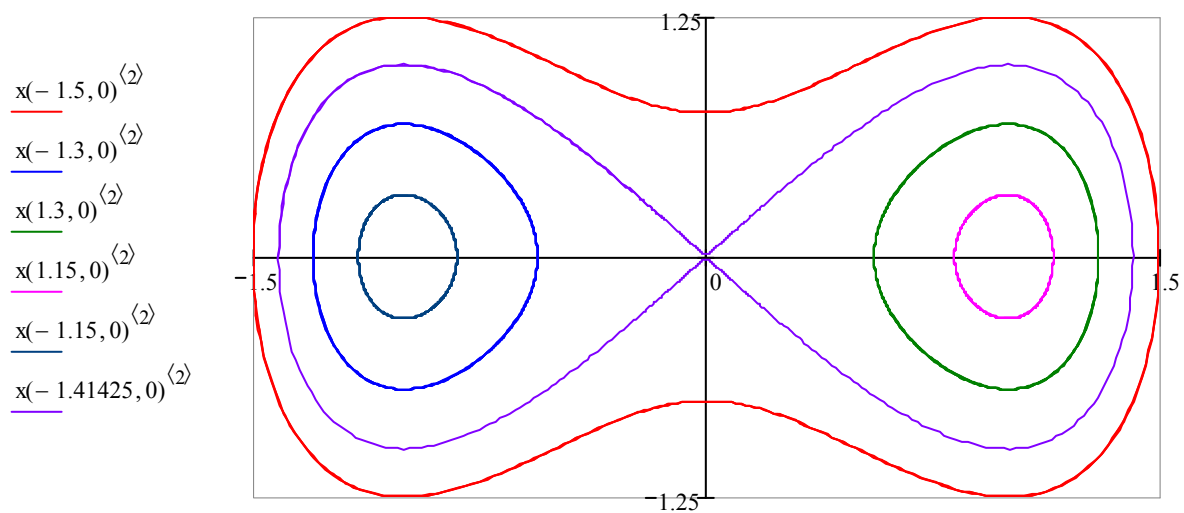
Separatrix  $\tilde{\delta} := 10^{-1}$   $E_{s,i,j} := \text{if}(E_{i,j} > 0, \delta, 0)$



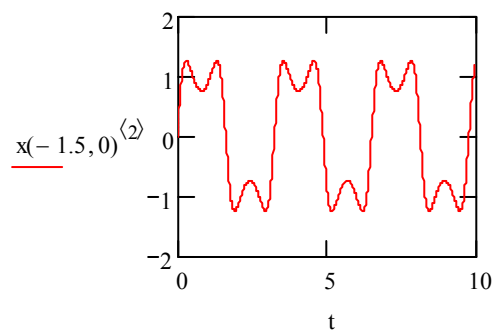
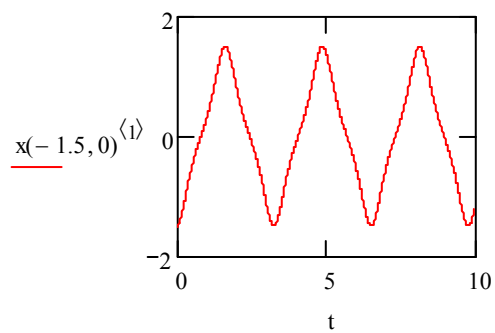
Direct differential equations solutions are given below.

All curves on phase plane are closed curves hence the solutions will be periodic.

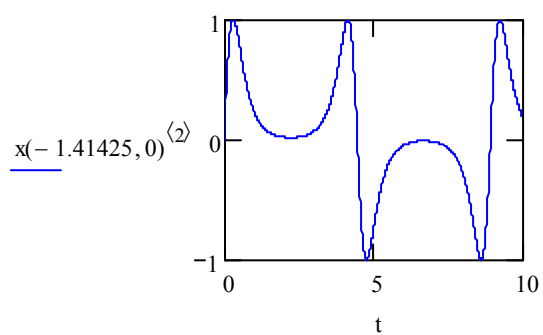
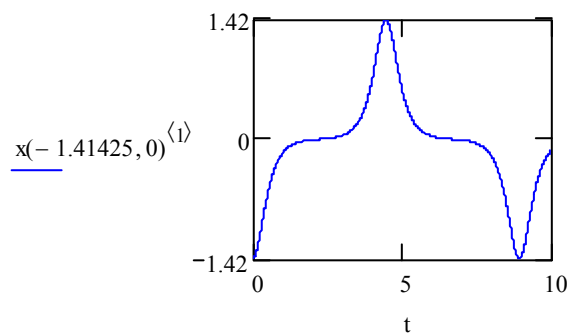
$$D(t, x) := \begin{bmatrix} 2 \cdot x_1 \\ 4 \cdot x_0 - 4 \cdot (x_0)^3 \end{bmatrix} \quad \tilde{N} := 10^3 \quad \tilde{T} := 10 \quad x(x_0, y_0) := \text{rkfixed} \left( \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, 0, T, N, D \right) \quad t := x(0, 0)^{\langle 0 \rangle}$$



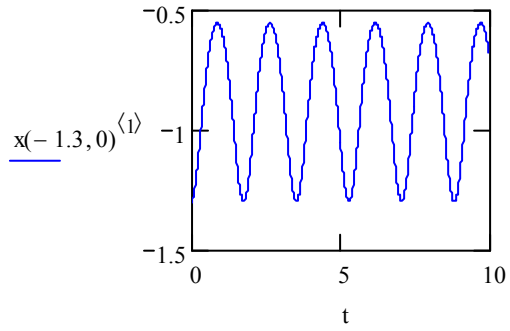
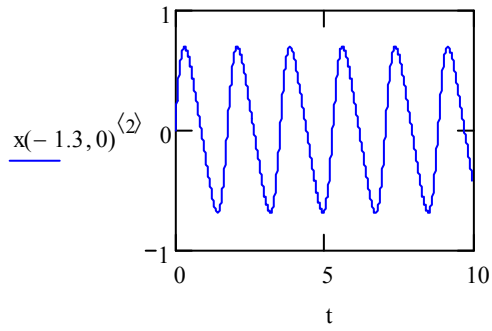
$$x(-1.5, 0)^{(1)}, x(-1.3, 0)^{(1)}, x(1.3, 0)^{(1)}, x(1.15, 0)^{(1)}, x(-1.15, 0)^{(1)}, x(-1.41425, 0)^{(1)}$$



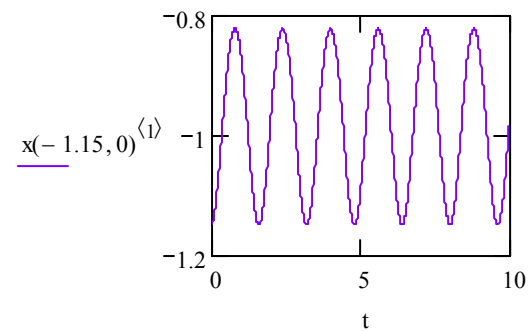
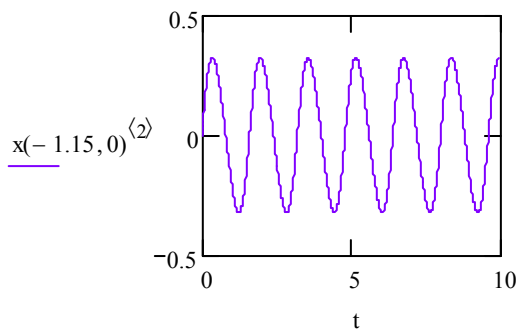
Solution behind the separatrix



Solution on the separatrix



Solution within the separatrix



Solution within the separatrix

The more is the field covered by phase curve the longer is oscillation period

Motion equation of generator rotor for small frequency change can be written as:

$$\tau \frac{d^2 \delta}{dt^2} = P_T - P_r \quad (11)$$

Rewrite the equation as the system of first order equations. Normalize the presented equations by  $\tau$  and taking into account that  $P_r = P_m \sin(\delta)$ :

$$\begin{aligned} \frac{d\omega}{dt} &= P_T - P_m \sin(\delta) \\ \frac{d\delta}{dt} &= \omega \end{aligned} \quad (12)$$

Find equilibrium state points of dynamic system and plot phase portrait using

**MathCAD**



$$P_T := 0.7 \quad P_m := 1$$

$$\left| \begin{array}{l} \frac{d}{dt} \delta = \omega \\ \frac{d}{dt} \omega = P_T - P_m \cdot \sin(\delta) \end{array} \right. \quad F_1(x, y) := y \quad F_2(x, y) := 0.7 - \sin(x)$$

Find equilibrium state points

$$z := \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix} \left| \begin{array}{l} \text{solve,} \begin{pmatrix} x \\ y \end{pmatrix} \\ \text{float, 5} \end{array} \right. \rightarrow (.77540 \ 0) \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} := z^T \quad x_1 := \pi - x_0 \quad y_1 := y_0 \quad x = \begin{pmatrix} 0.775 \\ 2.366 \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Find Jacobian

$$\underset{\text{Jacobian}}{A}(x, y) := \begin{pmatrix} \frac{\partial}{\partial x} F_1(x, y) & \frac{\partial}{\partial y} F_1(x, y) \\ \frac{\partial}{\partial x} F_2(x, y) & \frac{\partial}{\partial y} F_2(x, y) \end{pmatrix} \quad A(x, y) \rightarrow \begin{pmatrix} 0 & 1 \\ -\cos(x) & 0 \end{pmatrix}$$

Determine point type with coordinates 0.775 , 0

$$A_1 := A(x_0, y_0) \quad A_1 = \begin{pmatrix} 0 & 1 \\ -0.714 & 0 \end{pmatrix} \quad \lambda := \text{eigenvals}(A_1)$$

$$\lambda = \begin{pmatrix} 0.845i \\ -0.845i \end{pmatrix} \quad \textbf{Centre}$$

Determine point type with coordinates 2.366, 0

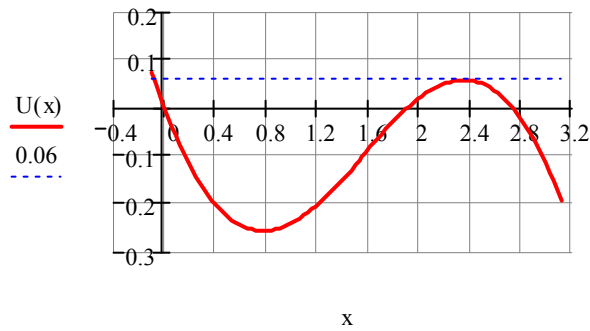
$$A_2 := A(x_1, y_1) \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0.714 & 0 \end{pmatrix} \quad \underset{\text{Jacobian}}{\lambda} := \text{eigenvals}(A_2)$$

$$\lambda = \begin{pmatrix} 0.845 \\ -0.845 \end{pmatrix} \quad \textbf{Saddle point}$$

Auxiliary potential function U(x) is written as :

$$U(x) := - \int_0^x (0.7 - \sin(x)) \, dx \text{ float, 5} \rightarrow (-.70000) \cdot x - 1 \cdot \cos(x) + 1.$$

$$x := -0.1, -0.1 + 0.01 \cdot \pi .. \pi$$



Energy auxiliary function  $E(x,y)$  is written as:

$$E(x,y) := y^2 + U(x)$$

Plot phase portrait

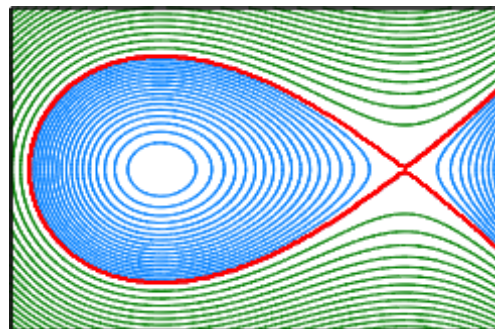
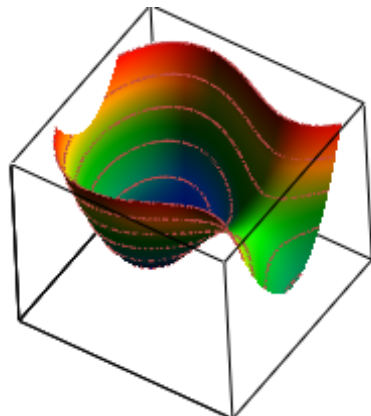
$$\underline{N} := 251 \quad i := 0..N \quad j := i \quad x_i := -0.2 + \frac{3.2}{N} \cdot i \quad y_j := -0.8 + \frac{1.6}{N} \cdot j$$

Surface  $\underline{E}_{i,j} := E(x_i, y_j)$

Surface within the separatrix  $E1_{i,j} := \text{if}(E_{i,j} < 0.058, E_{i,j}, 0.058)$

Surface behind the separatrix  $E2_{i,j} := \text{if}(E_{i,j} \geq 0.06, E_{i,j}, 0.06)$

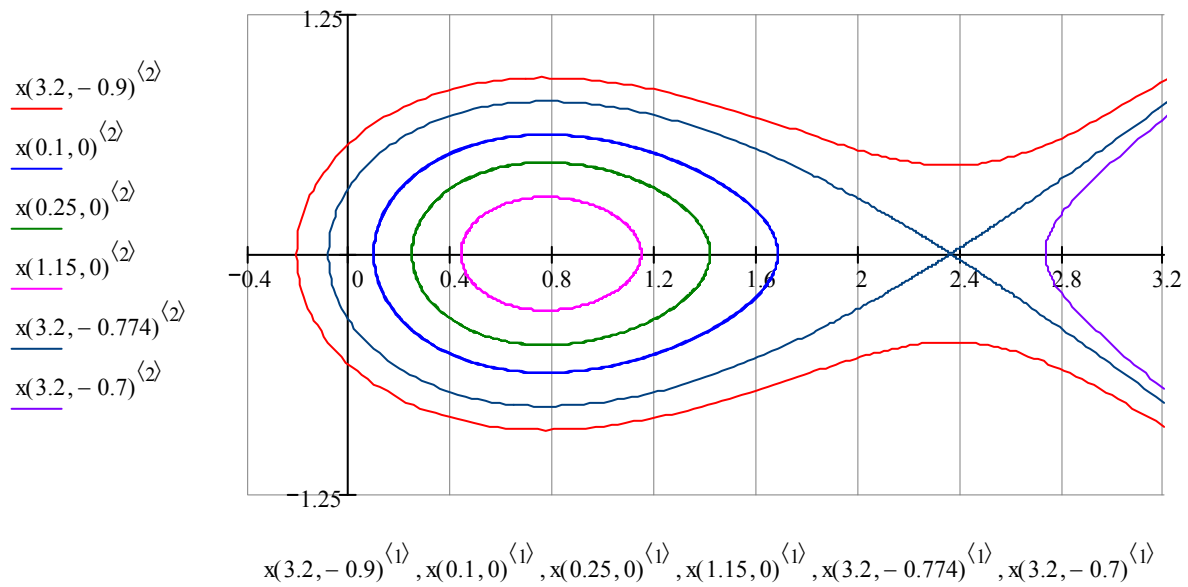
The separatrix  $\underline{\delta} := 10^{-3} \quad E_{s,i,j} := \text{if}(E_{i,j} \geq 0.058, \delta, 0)$



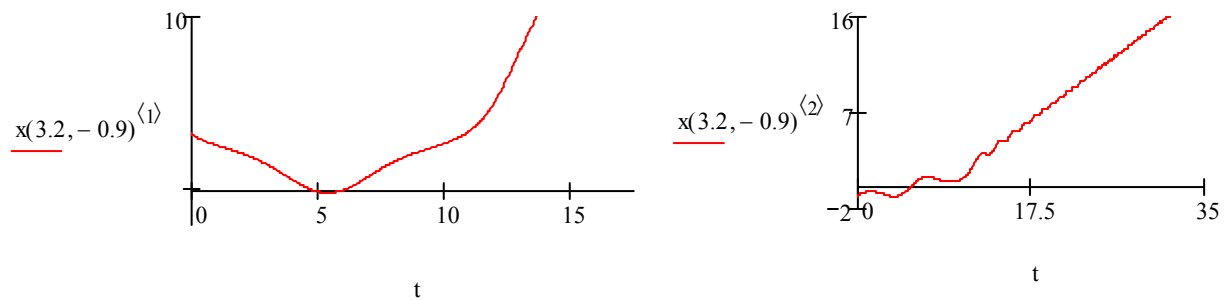
$$D(t,x) := \begin{pmatrix} x_1 \\ 0.7 - \sin(x_0) \end{pmatrix} \quad \underline{N} := 10^3 \quad \underline{T} := 35 \quad x(x_0, y_0) := \text{rkfixed} \left[ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, 0, T, N, D \right] \quad t := x(0,0) \langle 0 \rangle$$

Direct differential equations solutions are given below

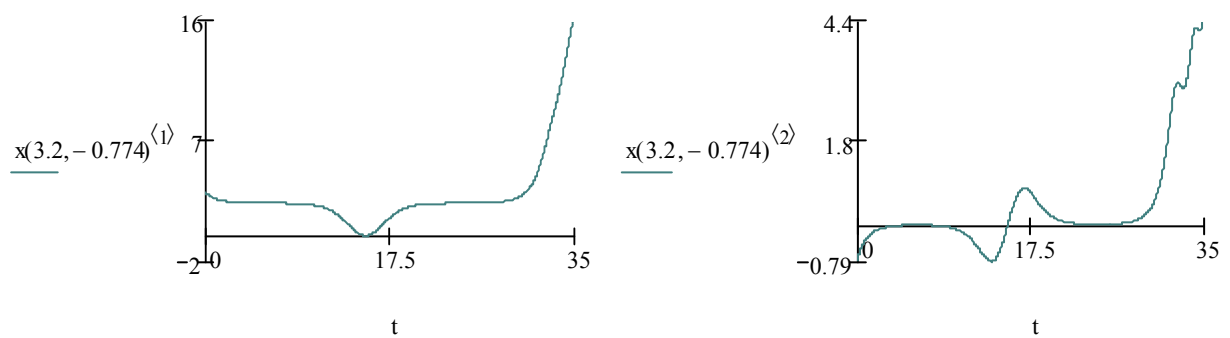
Closed curves on phase plane are periodic solutions, open curves are unstable solutions.



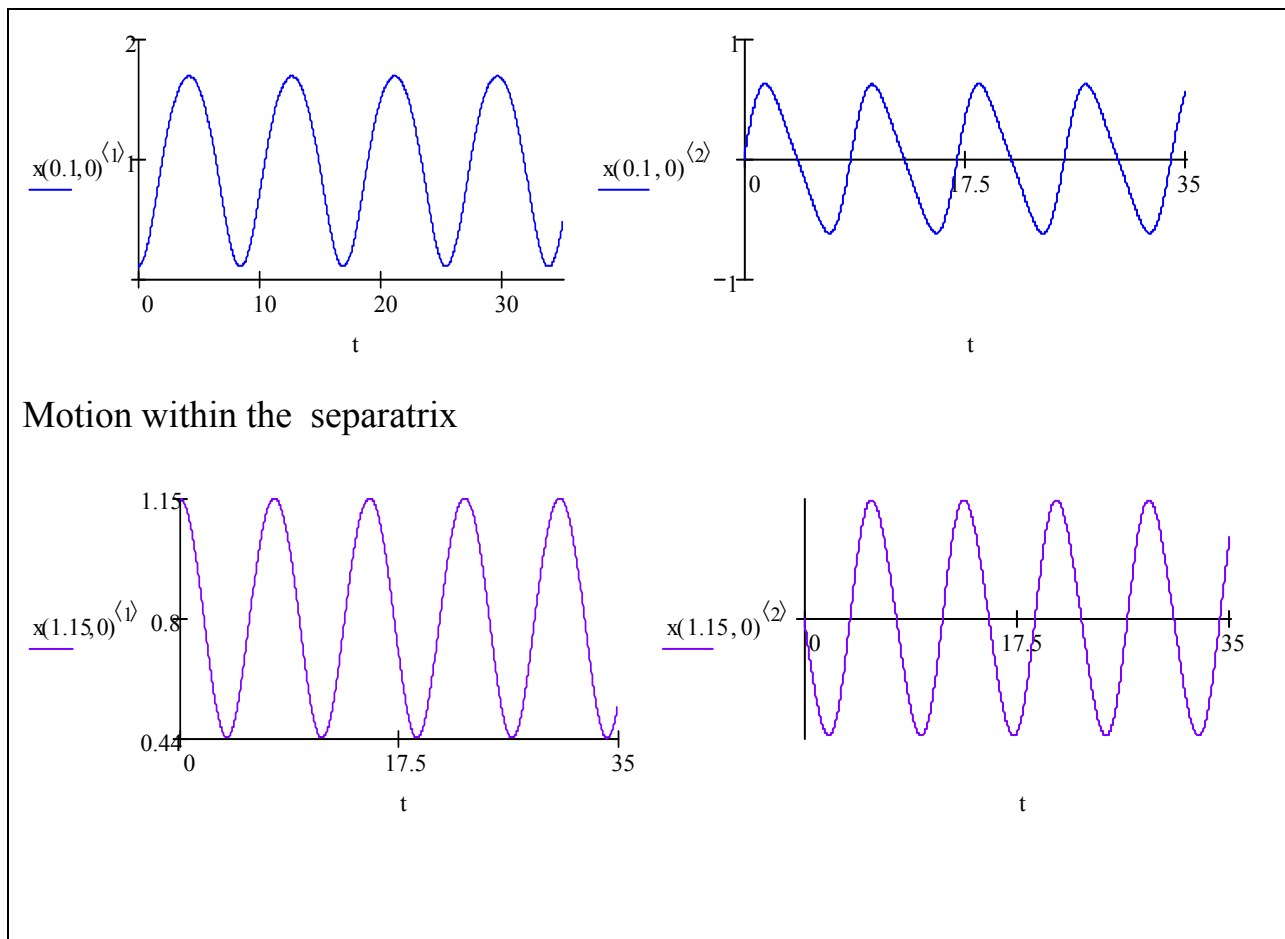
### Unstable motion behind the separatrix



### Motion on the separatrix



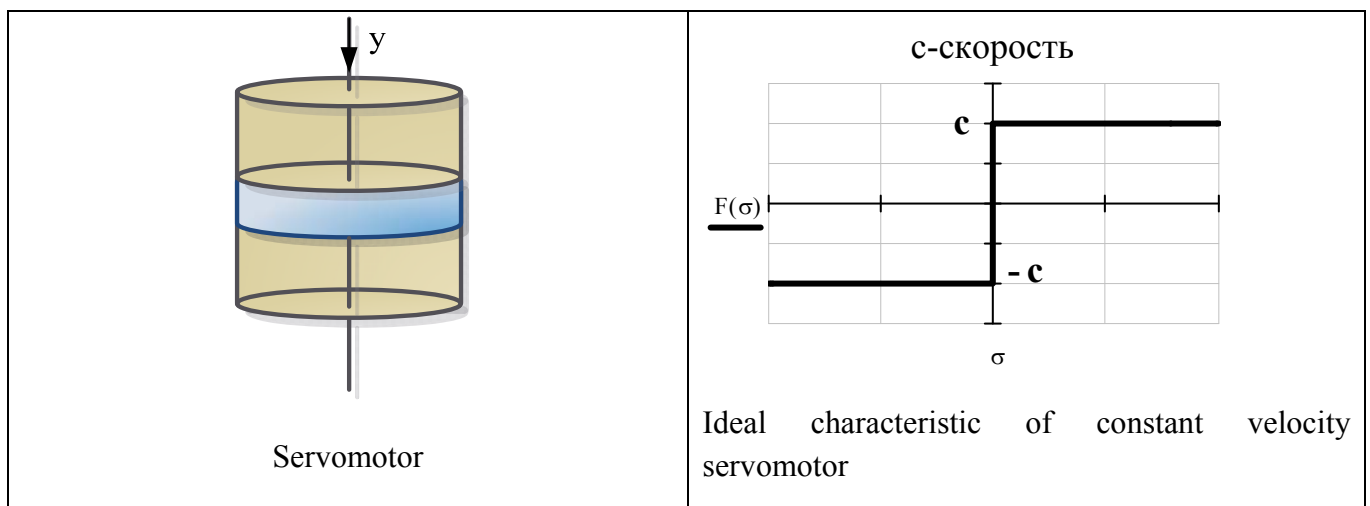
### Motion within the separatrix



## 2.2. Phase plane technique

### 2.2.1. Servomotor equation with ideal relay characteristic

Let us apply phase plane technique to detect important features of the process taking place in nonlinear system of turbine rotation frequency regulation by constant velocity servomotor described by equations [5]:



rotor

$$T_{\phi} \frac{d\phi}{dt} = \mu; \quad (13)$$

control element

$$\sigma = -\phi - \mu; \quad (14)$$

constant velocity servomotor

$$\frac{d\mu}{dt} = F(\sigma). \quad (15)$$

Where  $F(\sigma)$ :

$$F(\sigma) = c \cdot \text{sign}(\sigma). \quad (16)$$

As phase plane coordinates  $(x, y)$  choose

$$x = \phi, \quad y = \frac{d\phi}{dt}.$$

Dynamic system will be described by equations system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = \frac{F(\sigma)}{T_x} \end{cases} \quad \text{где} \quad F(\sigma) = c \cdot \text{sign}(-x - yT_x), \quad \sigma = -x - yT_x \quad (17)$$

Find equation of servomotor switching line (vertical segment on plot)

$$\sigma = -x - yT_x = 0, \quad y = -x / T_x \quad (18)$$

Find phase portrait lines equation

In area  $\sigma < 0$  (negative values  $F$  in plot)

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{c}{T_x} \end{cases} \rightarrow \frac{dy}{dx} = -\frac{c}{T_x y} \rightarrow y^2 = -\frac{2cx}{T_x} + C_1 \quad (19)$$

In area  $\sigma > 0$  (positive values  $F$  in plot)

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = \frac{c}{T_x} \end{cases} \rightarrow \frac{dy}{dx} = \frac{c}{T_x y} \rightarrow y^2 = \frac{2cx}{T_x} + C_2 \quad (20)$$

Write the equation on vertical segment, equaling coefficients in equation of straight line and in phase plane

$$\frac{dy}{dx} = -\frac{c}{T_x y}, \quad y = -\frac{x}{T_x} \rightarrow -\frac{c}{T_x y} = -\frac{1}{T_x} \rightarrow y = c \quad (21)$$

Find differential equation on segment

$$\frac{dx}{dt} = y, \quad y = -\frac{x}{T_x} \rightarrow \frac{dx}{dt} = -\frac{x}{T_x} \rightarrow x(t) = C_3 e^{-t/T_x} \quad (22)$$

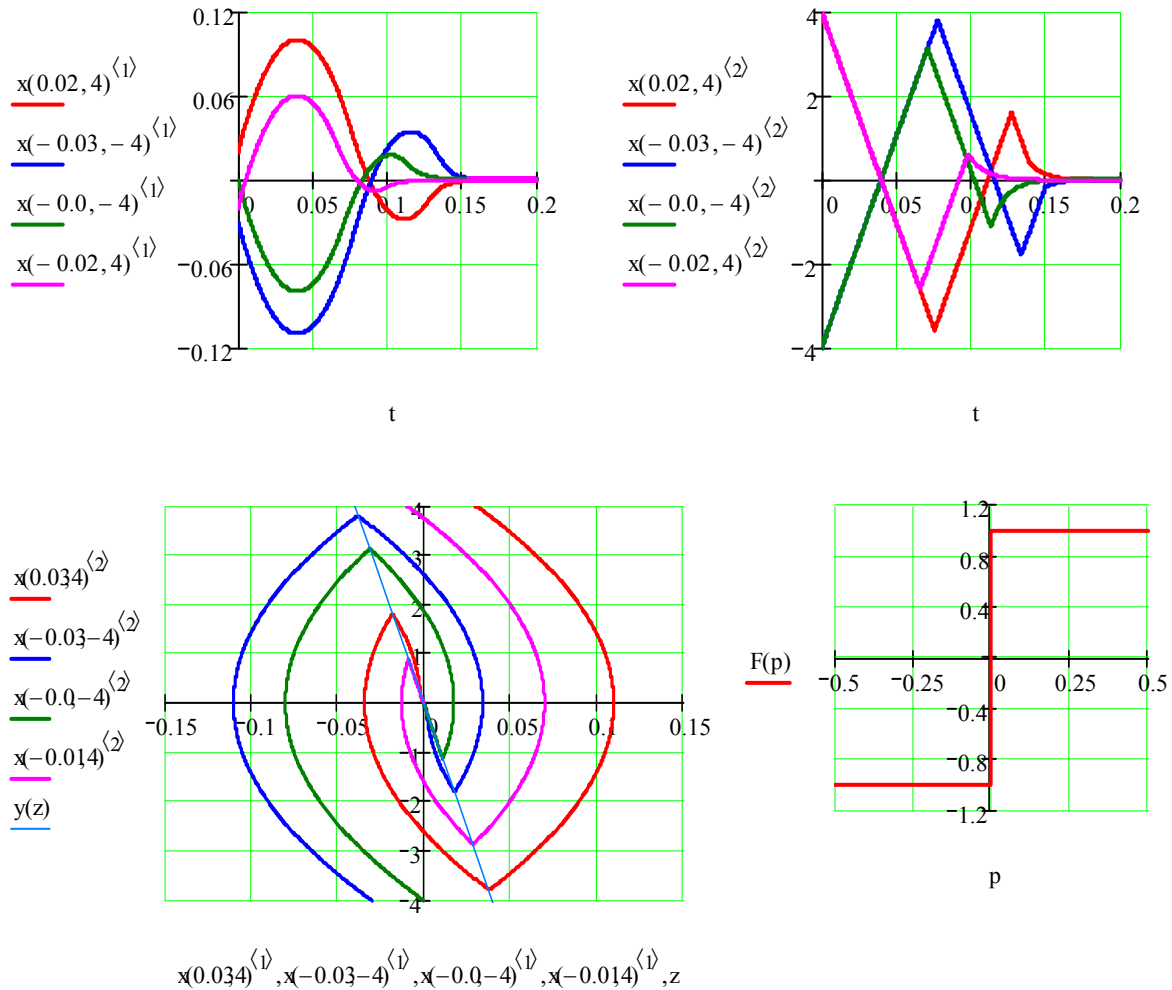
Sliding process

Let's analyze the solution with phase portrait plotting with use of **MathCAD**.

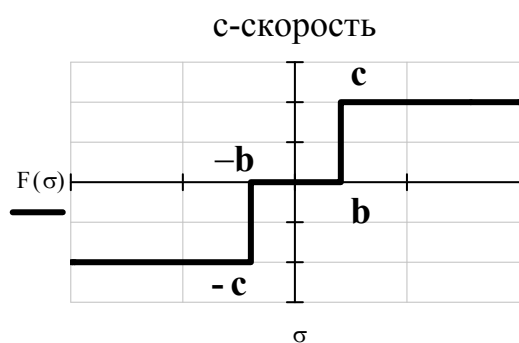
$$F(x) := \text{sign}(x) \cdot 1 \quad T1 := 0.01 \quad N := 10^3$$

$$D(t, x) := \begin{pmatrix} x_1 \\ \frac{F(-x_0 - T1 \cdot x_1)}{T1} \end{pmatrix} \quad T := 0.2 \quad x(x_0, x_1) := \text{rkfixed} \left[ \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, 0, T, N, D \right]$$

$$y(z) := -z \cdot \frac{1}{T1} \quad z := -0.05, -0.05 + 0.01 \dots 0.1 \quad p := -0.5, -0.5 + 0.001 \dots 0.5 \quad t := x(0, 0)^{(0)}$$



### 2.2.2. Servomotor equation with actual relay characteristic.



Two vertical segments correspond to servomotor switching.

$$\sigma = -x - yT_x = -b \rightarrow x = -yT_x + b$$

$$\sigma = -x - yT_x = b \rightarrow x = -yT_x - b$$

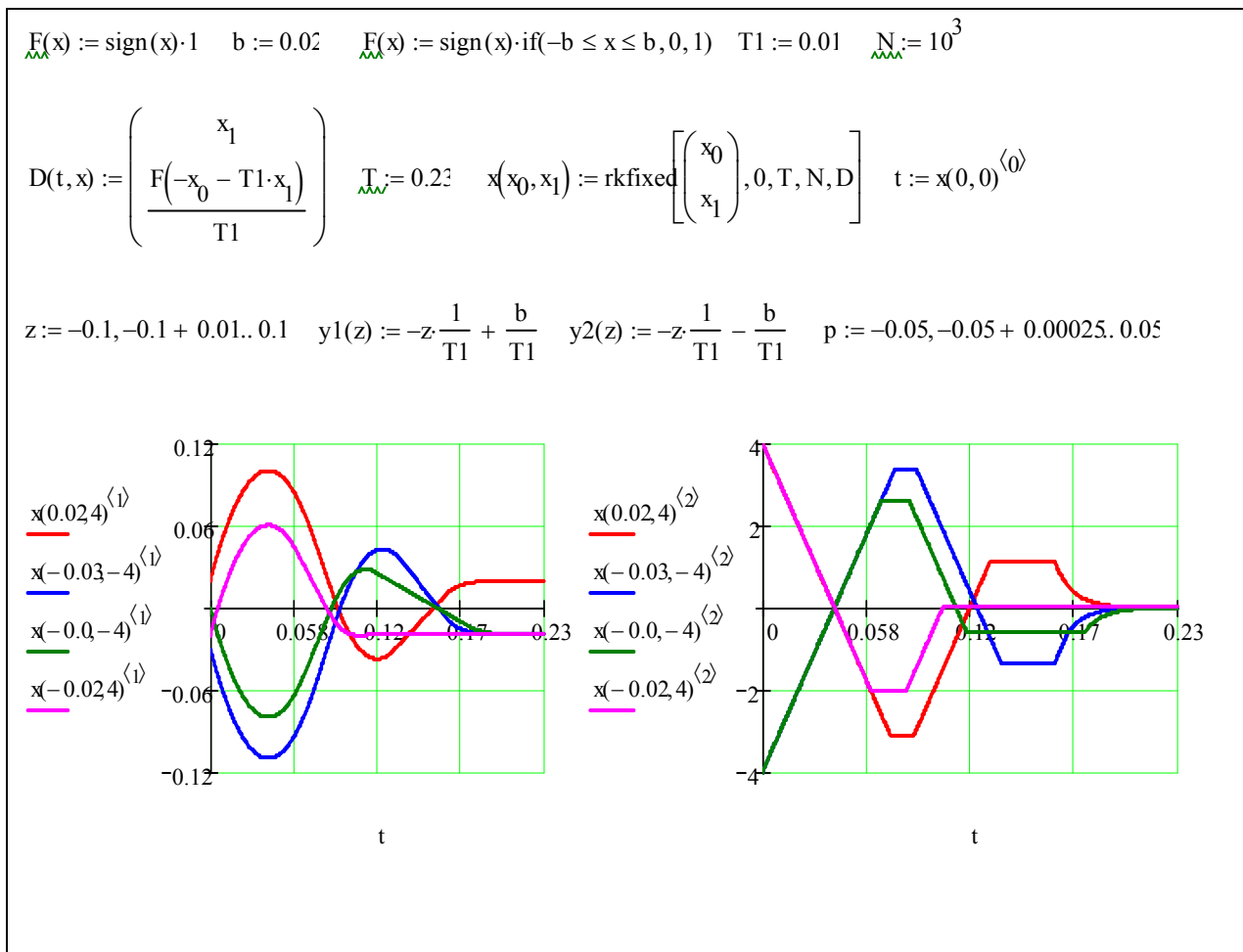
Plot its slopes on phase plane (see **MathCAD** document below)

In difference from the above ideal example in this case one more area appears

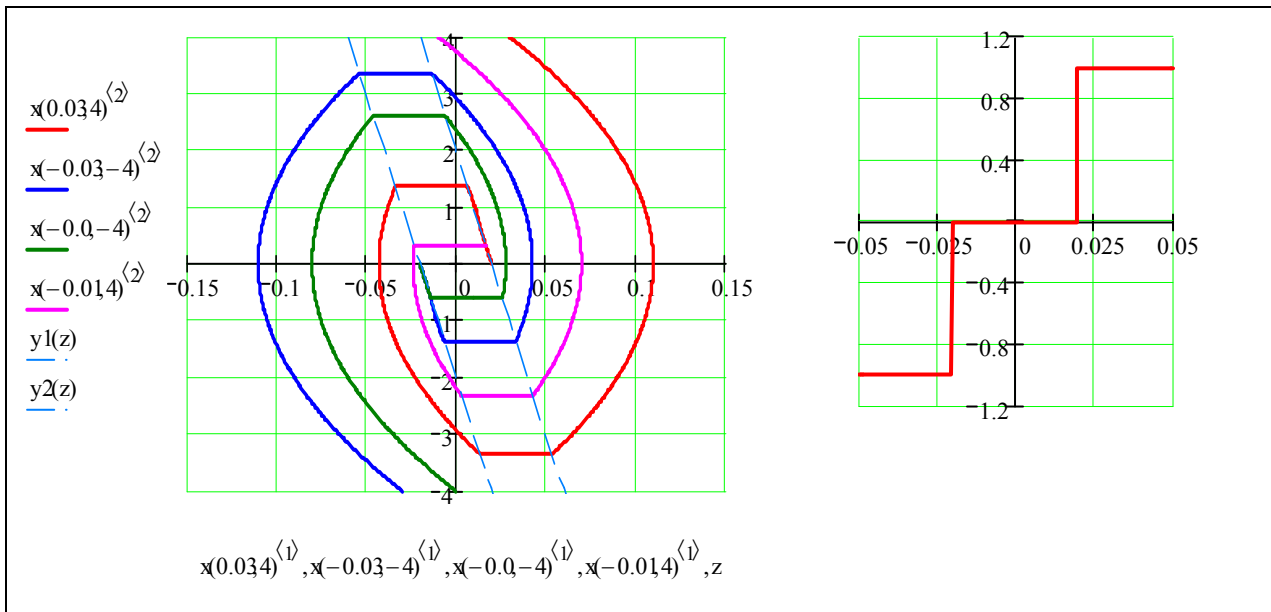
$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = 0 \end{cases} \rightarrow \frac{dy}{dx} = 0 \rightarrow y = \text{const}$$

Straight lines are parallel to axis  $x$ . Now equilibrium state is not one point but the segment.

$$y=0, \quad -b < x < b$$







### 2.3. Lyapunov straight method Прямой метод Ляпунова

Method is based on scalar functions applications having special properties on dynamic system solutions and being called Lyapunov functions. Lyapunov functions help estimate system stability and quality and construct control algorithms providing the required quality process properties.

For system described by the system of differential equations

$$\begin{cases} \frac{dx}{dt} = f_1(x, y) \\ \frac{dy}{dt} = f_2(x, y) \end{cases} \quad (23)$$

where functions  $f_1, f_2$  are arbitrary and possess any kind of nonlinear nature but always satisfies the condition  $f_1 = f_2 = 0$ , при  $x_1 = x_2 = 0$ , as in steady state all variables deviations and their derivatives equal zero. A certain function of system all phase coordinates (23)  $V(x, y)$  can be entered where  $x, y$  are variables deviations from some steady values. The function can be presented in two-dimensional space. Then in each point of phase space  $V$  will possess the definite value and at the beginning of coordinates will equal zero.

The function  $V$  is said to be **definite sign function** in some area if in any point inside the area function  $V$  has definite sign and turns to zero at the beginning of coordinates.

Let us analyze the example of definite sign positive function of the second order system  $n = 2$

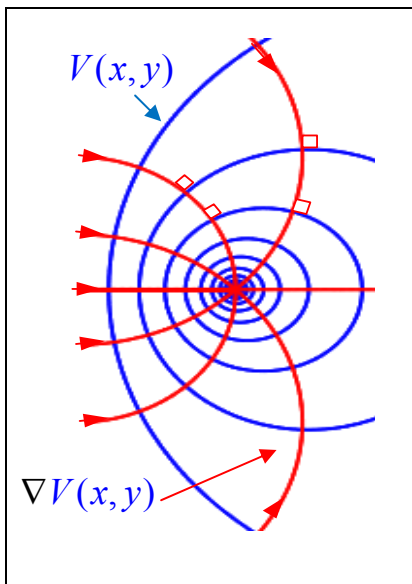
$$V(x, y) = x^2 + y^2 \quad (24)$$

it is clear that  $V > 0$ , и  $V = 0$ , only at  $x = y = 0$ .

Function  $V$  is called constant sign function, if it is of the same sign but it turns to zero not only at the beginning of the coordinates but in other points of the given area.

Function  $V$  is called variable sign function, if it in the given area around coordinates beginning can have different signs.

Arbitrary function  $V = V(x, y)$  which turns to zero only at  $x = y = 0$ , and where  $x, y$  are deviations in which the equation of system motion is written is called **Laypunov function**. Let's determine time derivative of function  $V$ . Recollect what gradient operator  $\nabla$  stands for:



$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} \rightarrow \nabla V(x, y) = \mathbf{grad} V(x, y)$$

– direction of fast change of function. Now time derivative of function  $f(x, y)$  can be written down, considering  $x, y$  depend on  $t$ :

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial y} \frac{dy}{dt} = \cancel{\frac{\partial V}{\partial t}} + \frac{\partial V}{\partial x} f_1(x, y) + \frac{\partial V}{\partial y} f_2(x, y)$$

$$\frac{dV}{dt} = (\nabla V, \mathbf{v}), \mathbf{v} = (f_1, f_2)$$

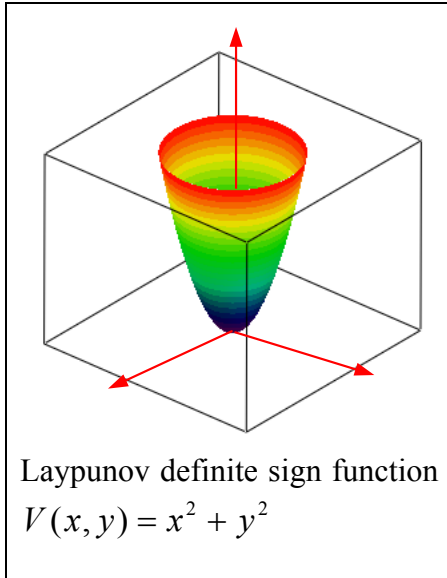
**Laypunov theorem about nonlinear systems stability:** if for equations of the system (23) one can choose the stable Laypunov function  $V(x, y)$ , such that its time derivative function  $dV / dt$  was also definite sign function (or constant sign function), but have the sign opposite to sign  $V$ , then the given system is stable.

**Example:** Let nonlinear system be described by equations

$$\begin{cases} \frac{dx}{dt} = -x - x^3 \\ \frac{dy}{dt} = -y^3 \end{cases}$$

Let us choose Lyapunov definite sign function of type:

$$V(x, y) = x^2 + y^2$$



Find time derivative of Laypunov function

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial y} \frac{dy}{dt} = \frac{\partial V}{\partial x} f_1(x, y) + \frac{\partial V}{\partial y} f_2(x, y)$$

$$\frac{dV}{dt} = 2x(-x - x^3) + 2y(-y^3) = -2(x^2 + x^4 + y^4)$$

Function  $dV / dt$  is definite sign function but it is opposite to function sign  $V(x, y)$ , hence the system is stable.

## 2.4. Harmonic linearization method

One of the main methods of higher order nonlinear systems at present time is approximative method of harmonic linearization. Let's analyze the example when the object with linear transfer function is controlled by the block with nonlinear characteristic

$$y = F(x). \quad (25)$$

Suppose harmonic signal enters nonlinear component input

$$x = A \sin(\omega t) \quad (26)$$

Output signal will be periodic hence it can be decomposed into

$$y = F(A \sin(\omega t)) = q(A) \sin(\psi) + q'(A) \cos(\psi) + \text{high harmonics}$$

Fourier series

$$\begin{aligned}
y = F(A \sin(\omega t)) &= \frac{1}{2\pi} \int_0^{2\pi} F(A \sin(\psi)) d\psi \\
&+ \left[ \frac{1}{\pi} \int_0^{2\pi} F(A \sin(\psi)) \sin(\psi) d\psi \right] \sin(\psi) + \left[ \frac{1}{\pi} \int_0^{2\pi} F(A \sin(\psi)) \cos(\psi) d\psi \right] \cos(\psi) + \\
&+ \text{high harmonics}
\end{aligned}$$

There will be no constant component for odd symmetry of nonlinear characteristic that is why we have

$$\frac{1}{2\pi} \int_0^{2\pi} F(A \sin(\psi)) d\psi = 0 \quad (27)$$

Then output signal can be written down

$$y = q(A)x + q'(A) \frac{\dot{x}}{\omega} + \text{high harmonics} \quad (28)$$

Linear part of closed CS вследствие инерционности является фильтром низких частот, т.е. высокие гармоники проходят ее со значительно большим ослаблением, чем первая:

$$\begin{aligned}
y &= q(A)a + q'(A) \frac{\dot{x}}{\omega} + \text{high harmonics} \\
&= \left[ q(A) + q'(A) \frac{p}{\omega} \right] x, \quad p \rightarrow \frac{d}{dt}
\end{aligned} \quad (29)$$

Such a representation is called harmonic linearization of non-linearity and variables are determined by formulae

$$q(A) = \frac{1}{\pi A} \int_0^{2\pi} F(A \sin(\psi)) \sin(\psi) d\psi, \quad q'(A) = \frac{1}{\pi A} \int_0^{2\pi} F(A \sin(\psi)) \cos(\psi) d\psi, \quad (30)$$

are called harmonic linearization coefficients.

Transfer function of linear component has the form of

$$y = \left[ q(A) + q'(A) \frac{p}{\omega} \right] x, \rightarrow W(A, p) = q(A) + q'(A) \frac{p}{\omega}, \quad p = j\omega \quad (31)$$

$$W(A, j\omega) = q(A) + q'(A) \frac{j\omega}{\omega} = q(A) + q'(A)j$$

Transfer function does not depend on frequency. It increases the magnitude of input signal and is called complex amplifying coefficient.

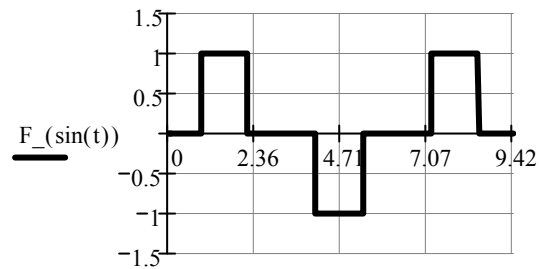
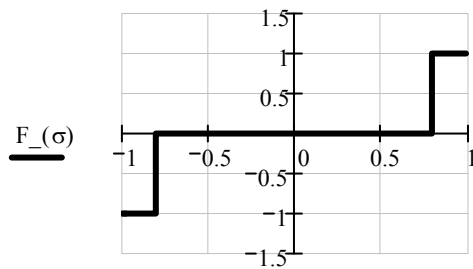
Thus nonlinear element can be substituted by linear element. Its frequency response depends on amplitude of input signal.

**Example.** Find amplification complex coefficient of given nonlinear component. We use **MathCAD**

$F(\sigma) := \text{if}(\sigma < 0, -1, 1) \quad b := 0.8$

$F_-(\sigma) := \text{if}(-b \leq \sigma \leq b, 0, F(\sigma))$  nonlinear component characteristic

$\sigma := -1, -1 + 10^{-4} \dots 1$



Nonlinear element characteristic and output signal

Find amplification coefficient **a** for different magnitudes, **b=0** as function is even

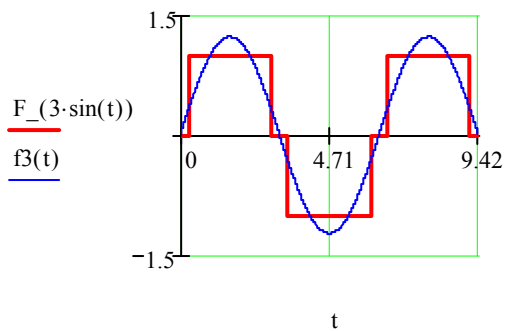
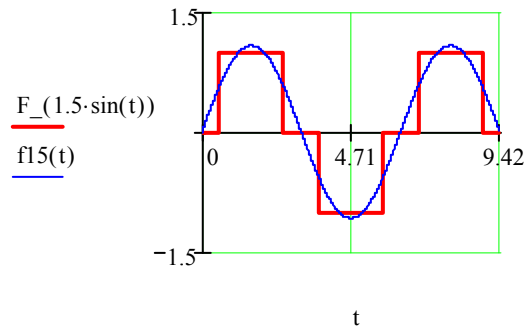
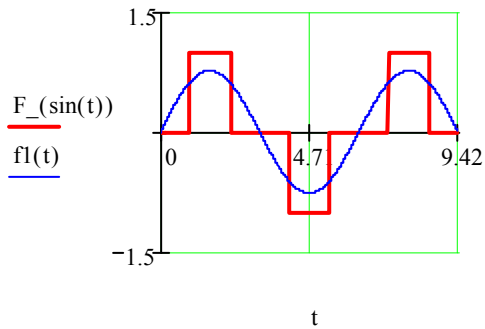
$$a(A) := \frac{1}{\pi \cdot A} \cdot \int_0^{2 \cdot \pi} F(A \cdot \sin(t)) \cdot \sin(t) \, dt$$

Find amplification coefficients and equivalent sinewaves at different magnitudes.

$$a(1) = 0.764 \quad f1(t) := a(1) \cdot \sin(t)$$

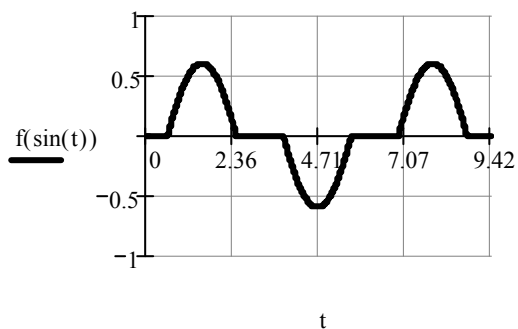
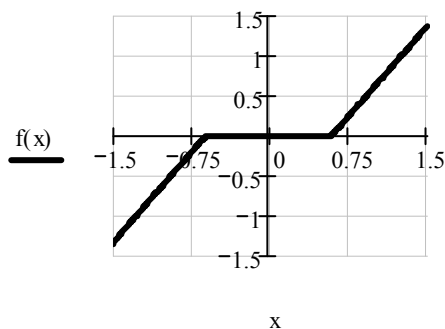
$$a(1.5) = 0.718 \quad f15(t) := a(1.5) \cdot 1.5 \cdot \sin(t)$$

$$a(3) = 0.409 \quad f_3(t) := a(3) \cdot 3 \cdot \sin(t)$$



$$A_- := 1.5 \quad f(x) := x \cdot A_- - B \quad x := -1.5, -1.5 + 0.01.. 1.5 \quad f(x) := \text{if}(x > 0, f(x), -f(-x))$$

$$f(x) := \text{if}(-bb < x < bb, 0, f(x))$$



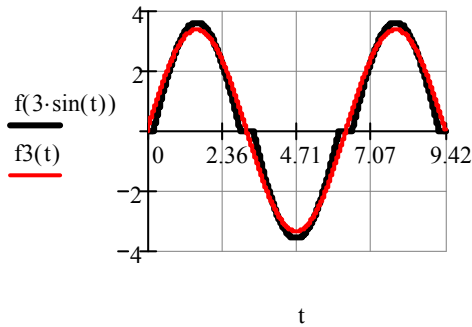
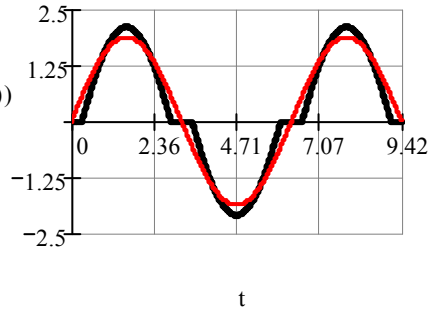
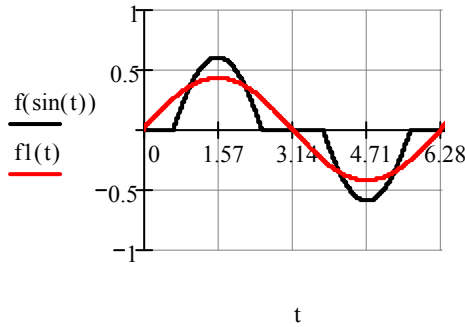
Nonlinear element characteristic and output signal

Find amplification coefficient **a** for different magnitudes, **b=0** as function is even.

$$a(A) := \frac{1}{\pi \cdot A} \cdot \int_0^{2 \cdot \pi} F(A \cdot \sin(t)) \cdot \sin(t) dt$$

Find amplification coefficients and equivalent sinewaves at different magnitudes

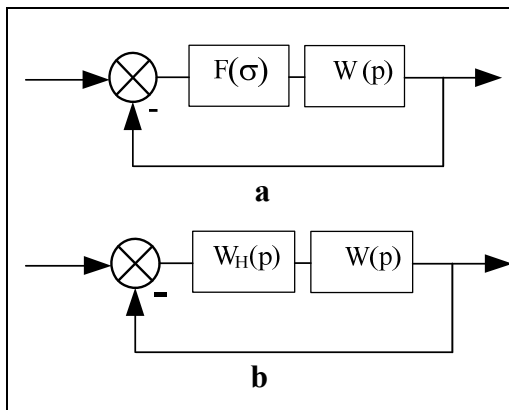
$$a(1) = 0.427 \quad f_1(t) := 0.427 \cdot \sin(t) \quad a(2) = 0.936 \quad f_2(t) := 0.936 \cdot 2 \cdot \sin(t) \quad a(3) = 1.121 \quad f_3(t) := 1.121 \cdot 3 \cdot \sin(t)$$



## 2.5. Self-exciting oscillations analysis algorithm.

With the help of nonlinear element harmonic linearization the closed loop system(Fig.a) equates to the system with linear equivalent element(Fig

b).Investigation of system with nonlinear element results in linear system investigation.



Find characteristic equation

$$W(p)W_H(A) = -1, \quad W(p) = -1 / W_H(A),$$

$$W(j\omega) = U(\omega) + jV(\omega), \quad W_H(A) = q(A) + jq'(A)$$

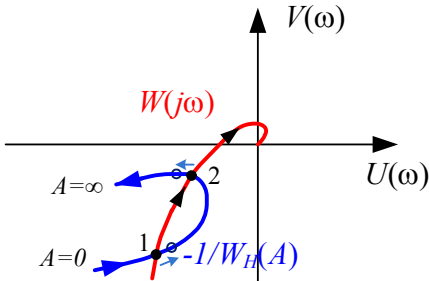
From the last equation it follows that:

$$U(\omega)q(A) - q'(A)V(\omega) = 1$$

$$U(\omega)q'(A) + q(A)U(\omega) = 0$$

Where  $q(A)$ ,  $q'(A)$  are harmonic linearization coefficients. Equation solution provides us equilibrium state points  $\omega_{II}$ ,  $A_{II}$ .

At solving the problem **Goldfarb** diagram is easy to use, its algorithm is given below.

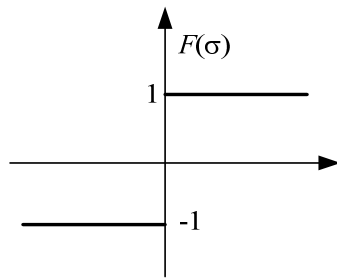
	<b>Goldfarb diagram</b>
	<ol style="list-style-type: none"> <li>1. Plot the locus <math>W(j\omega)</math></li> <li>2. Plot the locus <math>-1/W_H(A)</math></li> <li>3. Find <math>\omega_{II}</math>, <math>A_{II}</math> solving the equation : <math display="block">U(\omega)q(A) - q'(A)V(\omega) = 1</math> <math display="block">U(\omega)q'(A) + q(A)U(\omega) = 0</math> </li> <li>4. If at motion along locus <math>-1/W_H(A)</math> в сторону увеличения <math>A</math> the point is covered by locus <math>W(j\omega)</math>, then magnitude <math>A</math> will correspond unstable self-excited oscillations (point 1). If the point is not covered by locus <math>W(j\omega)</math>, then at magnitude <math>A</math> stable self-excited oscillations occur (point 2)</li> </ol>

Analyze the example with **MathCAD** use.

Investigate the system on self-exciting oscillations presence

$W(p) := \frac{3}{p^3 + 3 \cdot p^2 + p}$	$U(\omega) := \text{Re}(W(i \cdot \omega)) \quad V(\omega) := \text{Im}(W(i \cdot \omega))$
$U(\omega) := U(\omega) \left  \begin{array}{l} \text{complex} \\ \text{simplify} \rightarrow \\ \text{float, 5} \end{array} \right. \frac{-9}{7 \cdot \omega^2 + \omega^4 + 1}$	$V(\omega) := V(\omega) \left  \begin{array}{l} \text{complex} \\ \text{simplify} \rightarrow \\ \text{float, 5} \end{array} \right. \frac{3}{\omega} \cdot \frac{\omega^2 - 1}{7 \cdot \omega^2 + \omega^4 + 1}$





$$q(a) := \frac{4}{\pi \cdot a}$$

Harmonic linearization coefficient  
of nonlinear component

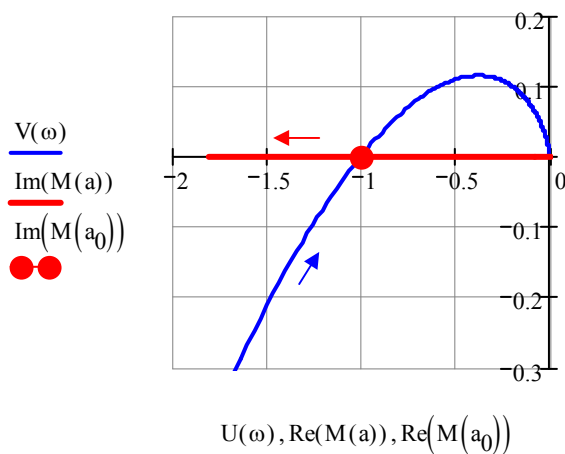
Find crossover point, frequency and magnitude

$$W_H(a) := q(a)$$

$$w := \left( \begin{array}{l} U(\omega) \cdot q(a) = -1 \\ V(\omega) \cdot q(a) = 0 \\ \omega > 0 \end{array} \right) \left| \begin{array}{l} \text{solve,} \left( \begin{array}{l} \omega \\ a \end{array} \right) \\ \text{float, 5} \end{array} \right. \rightarrow (1. \ 1.2732)$$

$$(\omega_0, a_0) := w \quad \omega_0 = 1 \quad a_0 = 1.273$$

$$\omega := 0, .01.. 10 \quad M(a) := \frac{-1}{W_H(a)} \quad U(1) = -1 \quad a := 0, 0.1.. 2.3$$



The obtained values of  
parameters  $\omega_0 = 1$  and  $a_0 = 1.273$   
correspond to stable self-excited  
oscillations because by point  
moving across the locus in the  
direction of increasing  
magnitude –  $a$ , locus does not  
cover the point

## **Literature**

1. Iy.I.Yurevich. Automatic Control Theory. StPb.:BHV- St. Petersburg 2007. 560p.
2. Rao V.Dukkipaty. Control System. Alpha Science International Ltd. Harrow,U.K. 2005. 698 p.
3. E.A.Nikulin. Backgrounds of Automatic Control Theory. StPb.:BHV- St. Petersburg 2007. 640p.