

# **Control Systems**

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## **PREFACE**

Control system engineering is an exciting and challenging field and is a multidisciplinary subject. This book is designed and organized around the concepts of control system engineering as they have been developed in the frequency and time domain for an introductory undergraduate course in the control system for engineering student of all disciplines.

The author has tried to set out the material in simple and readily available form. A scope of knowledge of higher mathematics necessary for understanding the content corresponds to the syllabus for earlier stages of tuition at higher technical education institutions. This book includes the coverage of classical methods of control systems engineering: Laplace transforms and transfer functions, block diagrams and spectral signal representation, root locus analysis and design, Routh-Hurwitz stability analysis, frequency response method of analysis including Bode, Nyquist, steady-state error analysis, second order systems, phase and gain margin and bandwidth, phase plane method and dynamic systems phase portraits, harmonic linearization method.

Book content provides the foundations which help solve problems quickly and effectively find practice tasks. The author focused on engineer calculation methods using applied program tools as Mathcad, Electronics Workbench, MATLAB - Simulink. Most programs used in the book are designed by the author, though standard application programs as MATLAB - Simulink are also used in some cases.

The book contains plenty of explanatory examples. Tasks given in appendix help fasten the understanding of the material. This book is addressed to the wide range of readers: students and specialists resuming studies in the field of control theory because of expanding range of problems concerned with automatic processes, students and specialists who want to refresh their knowledge by studying a part of the book that has not been include into engineering specialities curriculum

I would appreciate being informed of errors, or receiving comments about the book. Please send to the authors' e-mail to *Isaev\_yusup@mail.ru*

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## **CHAPTER 1. LINEAR SYSTEMS**

### **1.1. Linear systems description and analysis with the help of differential equations**

#### **1.1.1. General notions on automatic control**

Automatic control theory and automatic control system are parts of engineering cybernetics.

Engineering cybernetics studies general principles of technological and manufacturing processes dynamic system control.

Automatic regulation is defined as the process of keeping up the required value of some physical variable in technical devices without human involvement with the help of regulators.

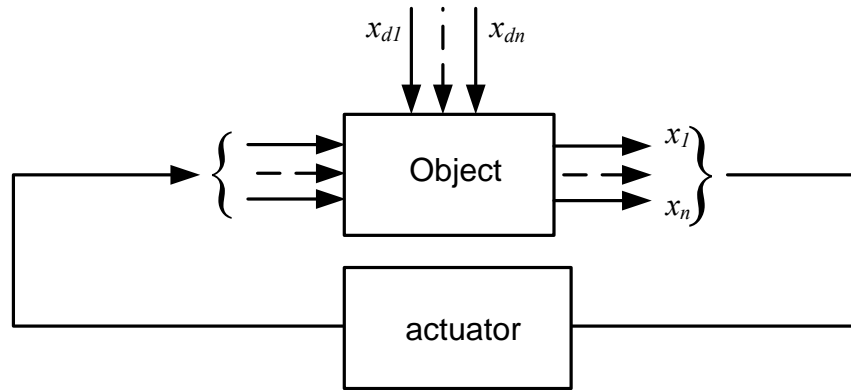
Automatic control is changing process according to some law some physical variable in technical devices without human.

Technological process of energy production, transmission and distribution is referred as the ordered interaction of power engineering objects: turbo and hydrogenerators, transmission lines, transformers and other machinery. Directional influence on these objects is called control process. Cooperating electric power objects which are exposed to intended influences are called control objects.

Control systems are based on feedback principle which is so that the state of controlled object is characterized by the same parameters  $x_1, x_2, \dots, x_n$  (Figure 1) which is compared with desired values and if current state of parameters is different from the desired one then the system establishes actuating signals  $x_{p1}, x_{p2}, \dots, x_{pm}$ , which return the state of system to the desired one.

Feedback mechanisms can help today's increasingly complex computer systems adapt to changes in workloads or operating conditions. Control theory offers a principled way for designing feedback loops to deal with unpredictable changes, uncertainties, and disturbances in systems.

For power plants disturbances occur as unexpected change of fuel quantity, station capacity give to electric system by other power plants and capacity assumed by consumers etc



**Figure 1**

Because of the fact that object characteristics are given parameters than the necessary requirements to control system should be provided by the corresponding controller characteristics selection.

Control by disturbance means that if there is a certain correlation between the disturbance actions and regulated variable than the object can be influenced to prevent the expected controlled variable deviation.

### **1.1.2 Differential equations. Linearization**

At CS investigations two problems are solved

1. Analysis problem – CS structure and parameters are given, its properties should be determined.
2. Design problem – CS should be constructed according to given properties.



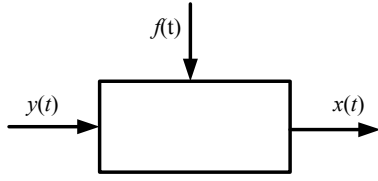


Figure 2

Both problems are not simple and they can be solved with the help of mathematics. CS is divided into subsystems “bricks”. Each component is described by non-linear equation, differential in most cases.

Let's analyze one of these blocks which is described by non-linear equation:

$$F(y, \dot{y}, x, \dot{x}, \ddot{x}) = f, \text{ where } \dot{x} = \frac{d}{dt}x, \ddot{x} = \frac{d^2}{dt^2}x. - \text{dynamics equation}$$

$y(t)$  – input value,  $x(t)$  – output value,  $f(t)$  – disturbance.

Equation always has equilibrium state with coordinates :

$$x = x_0, y = y_0, f = f_0, \dot{x} = \ddot{x} = \dot{y} = 0.$$

$$\rightarrow F(y, 0, x, 0, 0) = f, - \text{statics equation.}$$

Investigation of this equation is not an easy task. So it is reduced – i.e. linearized to neighbourhood of equilibrium point, decomposing in Taylor series.

$$F(y_0, 0, x_0, 0, 0) + \left(\frac{\partial F}{\partial x}\right)_0 \Delta x + \left(\frac{\partial F}{\partial \dot{x}}\right)_0 \Delta \dot{x} + \left(\frac{\partial F}{\partial \ddot{x}}\right)_0 \Delta \ddot{x} + \left(\frac{\partial F}{\partial y}\right)_0 \Delta y + \left(\frac{\partial F}{\partial \dot{y}}\right)_0 \Delta \dot{y} + \dots = f$$

$$\Delta x = x(t) - x_0, \Delta y = y(t) - y_0, \Delta \dot{x} = \dot{x}, \dots$$

Ignoring the components of higher order of vanishing subtract from the obtained equation statics equation and obtain:

$$\left(\frac{\partial F}{\partial x}\right)_0 \Delta x + \left(\frac{\partial F}{\partial \dot{x}}\right)_0 \Delta \dot{x} + \left(\frac{\partial F}{\partial \ddot{x}}\right)_0 \Delta \ddot{x} + \left(\frac{\partial F}{\partial y}\right)_0 \Delta y + \left(\frac{\partial F}{\partial \dot{y}}\right)_0 \Delta \dot{y} = f - f_0 = \Delta f$$

Divide linearized equation by invariable  $\left(\frac{\partial F}{\partial x}\right)_0$  and obtain:

$$\Delta x + \frac{\left(\frac{\partial F}{\partial \dot{x}}\right)_0}{\left(\frac{\partial F}{\partial x}\right)_0} \Delta \dot{x} + \frac{\left(\frac{\partial F}{\partial \ddot{x}}\right)_0}{\left(\frac{\partial F}{\partial x}\right)_0} \Delta \ddot{x} + \frac{\left(\frac{\partial F}{\partial y}\right)_0}{\left(\frac{\partial F}{\partial x}\right)_0} \Delta y + \frac{\left(\frac{\partial F}{\partial \dot{y}}\right)_0}{\left(\frac{\partial F}{\partial x}\right)_0} \Delta \dot{y} = \frac{\Delta f}{\left(\frac{\partial F}{\partial x}\right)_0}$$

$$T_2^2 \Delta \ddot{x} + T_1 \Delta \dot{x} + \Delta x = k_1 \Delta y + k_2 \Delta \dot{y} + k_3 \Delta f \quad \text{где} \quad \left\{ \begin{array}{l} T_2^2 = \frac{\left( \frac{\partial F}{\partial \ddot{x}} \right)_0}{\left( \frac{\partial F}{\partial x} \right)_0}, T_1 = \frac{\left( \frac{\partial F}{\partial \dot{x}} \right)_0}{\left( \frac{\partial F}{\partial x} \right)_0} \\ k_1 = -\frac{\left( \frac{\partial F}{\partial y} \right)_0}{\left( \frac{\partial F}{\partial x} \right)_0}, k_2 = -\frac{\left( \frac{\partial F}{\partial \dot{y}} \right)_0}{\left( \frac{\partial F}{\partial x} \right)_0}, \\ k_3 = \frac{1}{\left( \frac{\partial F}{\partial x} \right)_0} \end{array} \right.$$

If derivation procedure is set down by  $p = \frac{d}{dt}$  then the equation can be rewritten in the form of:

$$(T_2^2 p^2 + T_1 p + 1) \Delta x = (k_1 + k_2 p) \Delta y + k_3 \Delta f$$

from which the required variable can be found

$$\Delta x = \frac{(k_1 + k_2 p)}{(T_2^2 p^2 + T_1 p + 1)} \Delta y + \frac{k_3}{(T_2^2 p^2 + T_1 p + 1)} \Delta f = W(p) \Delta y + W_f(p) \Delta f$$

Formulae  $W(p) = \frac{(k_1 + k_2 p)}{(T_2^2 p^2 + T_1 p + 1)}$ ,  $W_f(p) = \frac{k_3}{(T_2^2 p^2 + T_1 p + 1)}$  are called transfer functions of deviation of defined variable and of disturbance

The last reduction is to divide all the components into basic variables  $x = \Delta x / x^0$ ,  $y = \Delta y / y^0$ ,  $f = \Delta f / f^0$

$$(T_2^2 p^2 + T_1 p + 1) x = (\tilde{k}_1 + \tilde{k}_2 p) y + \tilde{k}_3 f,$$

where  $\tilde{k}_1 = \frac{y^0}{x^0} k_1$ ,  $\tilde{k}_2 = \frac{y^0}{x^0} k_2$ ,  $\tilde{k}_3 = \frac{f^0}{x^0} k_3$

### 1.1.3 Control system component definition and mathematic description of linear component properties

To analyze control system component properties their block diagram is used, in which separate elements are depicted as blocks connected to each other according



**Figure 3**

to substitute elements connection.

Separate component is characterized by a definite mathematical connection between input ( $x_{in}$ ) and output ( $x_{out}$ ) variables (Figure 3).

System components obtain the property of action direction, i.e. ability to one way signal passing from input to output.

Connection between output and input variables of linear component (or system) is depicted by linear differential equation with constant coefficients.

$$\begin{aligned} a_0 \cdot \frac{d^n x}{dt^n} + a_1 \cdot \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_{n-2} \cdot \frac{d^2 x}{dt^2} + a_{n-1} \cdot \frac{dx}{dt} + a_n \cdot x = \\ = b_0 \cdot \frac{d^m y}{dt^m} + b_1 \cdot \frac{d^{m-1} y}{dt^{m-1}} + \dots + b_{m-2} \cdot \frac{d^2 y}{dt^2} + b_{m-1} \cdot \frac{dy}{dt} + b_m \cdot y \end{aligned} \quad (1)$$

Linear differential equations have solution, which consist of two components

$$x = x_c + x_\infty \quad (2)$$

First component ( $x_c$ ) is a complementary solution of homogeneous differential equation, which is obtained from (1) by making its right part equal zero.

In general homogeneous differential equation has the following solution

$$x_c = C_1 \cdot e^{p_1 t} + C_2 \cdot e^{p_2 t} + \dots + C_n \cdot e^{p_n t} \quad (3)$$

$p_1, p_2, \dots, p_n$  are characteristic equation roots which are given by

$$a_0 \cdot p^n + a_1 \cdot p^{n-1} + \dots + a_{n-2} \cdot p^2 + a_{n-1} \cdot p + a_n = 0. \quad (4)$$

Coefficients  $C_1, C_2, \dots, C_n$  are determined by initial conditions.

Forced component ( $x_\infty$ ) is a particular solution of inhomogeneous equation (1) when the right part of equation is not zero.

If in steady state  $y$  is not a function of time then forced component can be calculated by making all the derivatives equal zero, the following result is obtained

$$a_n \cdot x_\infty = b_m \cdot y_\infty;$$

$$x_\infty = \frac{b_m}{a_n} \cdot y_\infty.$$

## 1.2 Systems description and analysis with the help of impulse response functions

### 1.2.1 Block transfer function

The transfer function is based on application of operator method, namely the Laplace transform. The formula of the Laplace transform for a function of time  $x(t)$  is given by

$$X(p) = \int_0^\infty e^{-pt} \cdot x(t) \cdot dt = L[x(t)], \quad x(t) = L^{-1}[X(p)]. \quad (6)$$

where  $p$  is a complex variable, and  $L$  is Laplace operator

The transfer function is defined as the ratio of the Laplace transform of the output  $x_{\text{out}}(p)$  to the Laplace transform of the input  $x_{\text{in}}(p)$  under the assumption that all initial conditions are zero

$$W(p) = \frac{X_{\text{out}}(p)}{X_{\text{in}}(p)}. \quad (7)$$

The transfer function can be obtained from differential equation if the Laplace transform is applied to it

$$\begin{aligned} a_0 \cdot p^n \cdot X(p) + a_1 \cdot p^{n-1} \cdot X(p) + \dots + a_{n-1} \cdot p \cdot X(p) + a_n \cdot X(p) = \\ = b_0 \cdot p^m \cdot Y(p) + b_1 \cdot p^{m-1} \cdot Y(p) + \dots + b_{m-1} \cdot p \cdot Y(p) + b_m \cdot Y(p). \end{aligned} \quad (8)$$

From (8) it is easy to define that

$$W(p) = \frac{b_0 \cdot p^m + b_1 \cdot p^{m-1} + \dots + b_{m-1} \cdot p + b_m}{a_0 \cdot p^n + a_1 \cdot p^{n-1} + \dots + a_{n-1} \cdot p + a_n} = \frac{A(p)}{B(p)}. \quad (9)$$

It should be noted that denominator of transfer function coincides with the left part of characteristic equation mentioned above.

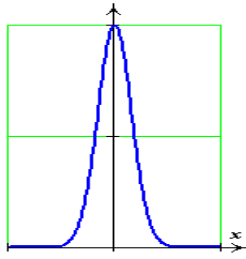
Reverse to the Laplace transfer function is Green's function (weight function – impulse response function)

$$w(t) = L^{-1}[W(p)] \rightarrow W(p) = L[w(t)]$$

### 1.2.2 Impulse response characteristic (impulse response function)

Let's consider two important functions:

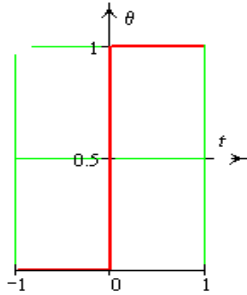
the Dirac delta function  $\delta(x - x_0)$  and  $\theta(x - x_0)$  - The Heaviside step function



The Dirac delta function property:

$$\delta(t - t_0) = \begin{cases} \infty & \text{at } t = t_0 \\ 0 & \text{at } t \neq t_0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt = f(t_0)$$

The Heaviside step function property:

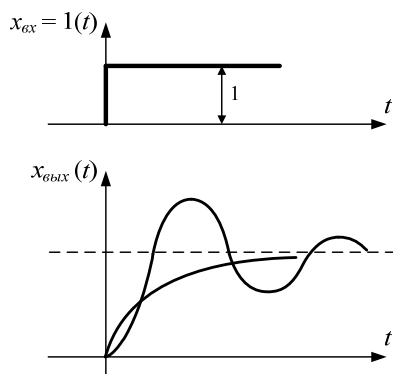


$$\theta(t - t_0) = \begin{cases} 0 & \text{at } t \leq t_0 \\ 1 & \text{at } t > t_0 \end{cases} \rightarrow \theta'(t) = \delta(t)$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) dx = \theta(x - x_0)$$

The Laplace transform of these functions

$$L[\delta(t)] = 1, \quad L[\theta(t)] = \frac{1}{p}$$



**Figure 4**

Differential equation solution with unit step function  $\theta(t)$  in the right part is impulse response function  $h(t)$ :

$$\frac{dx(t)}{dt} + x(t) = \theta(t) \rightarrow \text{equation solution is } x(t) = h(t)$$

then by external change  $f(t)$  we obtain:

$$\frac{dx(t)}{dt} + x(t) = f(t) \rightarrow \text{equation solution is } x(t) = h(t)f(0) + \int_0^t h'(\tau)f(t-\tau)d\tau$$

At zero initial conditions:

$$x(t) = \int_0^t h'(\tau)f(t-\tau)d\tau$$

Differential equation solution with impulse right part  $\delta(t)$  is Green's function or weight function:

$$\frac{dx(t)}{dt} + x(t) = \delta(t) \rightarrow \text{equation solution is } x(t) = w(t).$$

**Impulse response function  $w(t)$  describes the reaction of the output in response to input change in the form of delta function.**

By arbitrary external change  $f(t)$  we obtain:

$$\frac{dx(t)}{dt} + x(t) = f(t) \rightarrow \text{equation solution is } x(t) = h(t)f(0) + \int_0^t w(\tau)f(t-\tau)d\tau$$

At zero initial conditions:

$$x(t) = \int_0^t w(\tau)f(t-\tau)d\tau$$

### 1.2.3. Impulse response function (weight function-Green's function)

Relation between transfer and impulse response functions can be seen using the following correlations:

$$\begin{aligned} \delta(t) &= \theta'(t) \rightarrow w(t) = h'(t) \\ \theta(t) &= \int_0^t \delta(t)dt \rightarrow h(t) = \int_0^t w(t)dt \end{aligned}$$

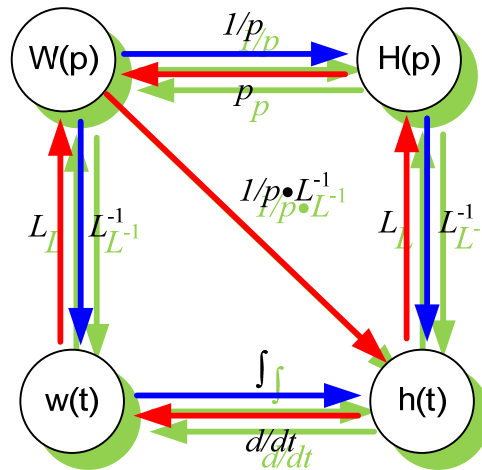
Remember that the Laplace transform of integral and derivative has the form of:

$$L[f(t)] = F(p) \quad L\left[\int_0^t f(t)dt\right] \rightarrow \frac{F(p)}{p}, \quad L[f'(t)] \rightarrow pF(p)$$

Hence the impulse response function has the form of

$$H(p) = L[h(t)], W(p) = L\left[\frac{dh(t)}{dt}\right], H(p) = \frac{1}{p} \cdot W(p) \quad (11)$$

Relation between dynamic characteristics can be seen in diagram shown in Figure 5



**Figure 5**

**Example 1:** Differential equation is given

$$5 \frac{d^2 x}{dt^2} + 4 \frac{dx}{dt} + 3x = y(t) + 2 \frac{dy(t)}{dt}, \quad x(0) = y(0) = 0, \quad \frac{dx(0)}{dt} = \frac{dy(0)}{dt} = 0.$$

Determine the transfer function of differential equation

1. Write the equation in operator form :

$$5p^2 X + 4pX + 3X = Y + 2pY \rightarrow (5p^2 + 4p + 3)X = (1 + 2p)Y$$

2. Calculate the transfer function:  $W(p) = \frac{(1 + 2p)}{(5p^2 + 4p + 3)}$

**Example 2:** Differential equation is given

$$0,1 \frac{d^2 x}{dt^2} + 10 \frac{dx}{dt} + 100x = f(t), \quad x(0) = 0, \quad \frac{dx(0)}{dt} = 0.$$

Calculate the equation using **MathCAD**.

## Determine the transfer function of differential equation

### And $w(t)$ -Green's function (weight function)

$$Z(p) := 0.1 \cdot p^2 + 10 \cdot p + 10^2$$

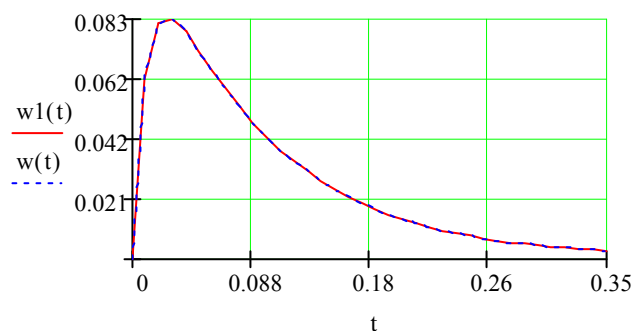
$$\underline{w}(p) := \frac{1}{Z(p)}$$

$$w_1(t) := \frac{1}{0.1 \cdot p^2 + 10 \cdot p + 10^2} \left| \begin{array}{l} \text{invlaplace, } p \\ \text{float, } 4 \end{array} \right. \rightarrow .2582 e^{(-50.) \cdot t} \cdot \sinh(38.73 t)$$

$$p := Z(p) \text{ solve, } p \rightarrow \begin{pmatrix} -88.729833462074168853 \\ -11.270166537925831148 \end{pmatrix} \quad \tau := \frac{1}{|p_1|}$$

$$z(p) := \frac{d}{dp} Z(p) \quad w(t) := \frac{1}{z(p_1)} \cdot e^{p_1 \cdot t} + \frac{1}{z(p_0)} \cdot e^{p_0 \cdot t} \quad w(t) \text{ float, } 4 \rightarrow .1291 e^{(-11.27) \cdot t} - .1291 e^{(-88.73) \cdot t}$$

$$t := 0, 0.01.. 4 \cdot \tau$$



$$h_1(t) := \frac{1}{(0.1 \cdot x^2 + 10 \cdot x + 10^2) \cdot x} \left| \begin{array}{l} \text{invlaplace, } x \\ \text{float, } 4 \end{array} \right. \rightarrow .1000 e^{-1} - .1000 e^{-1} e^{(-50.) \cdot t} \cdot \cosh(38.73 t) - .1291 e^{-1} e^{(-50.) \cdot t} \cdot \sinh(38.73 t)$$

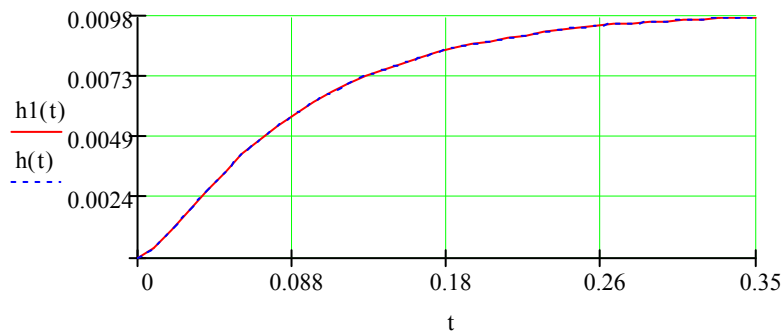
$$\underline{z}(p) := \frac{d}{dp} Z(p) \quad h(t) := \frac{1}{Z(0)} + \frac{1}{p_1 \cdot z(p_1)} \cdot e^{p_1 \cdot t} + \frac{1}{p_0 \cdot z(p_0)} \cdot e^{p_0 \cdot t}$$

$$h(t_1) \text{ float, } 2 \rightarrow .10 e^{-1} - .11 e^{-1} e^{(-11.) \cdot t_1} + .15 e^{-2} e^{(-89.) \cdot t_1}$$

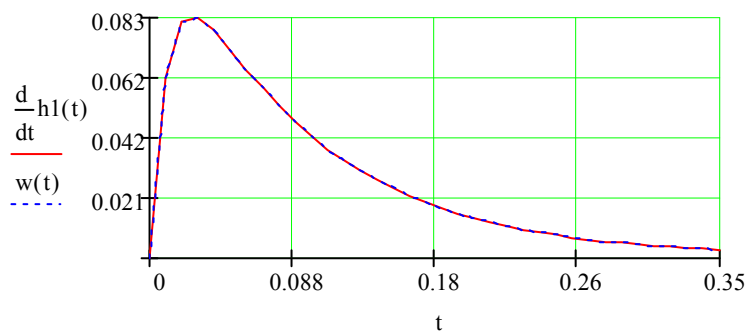
$$t := 0, 0.01.. 4 \cdot \tau$$

## Determine $h(t)$ impulse response function of differential equation





**Check  $w(t) = h'(t)$**



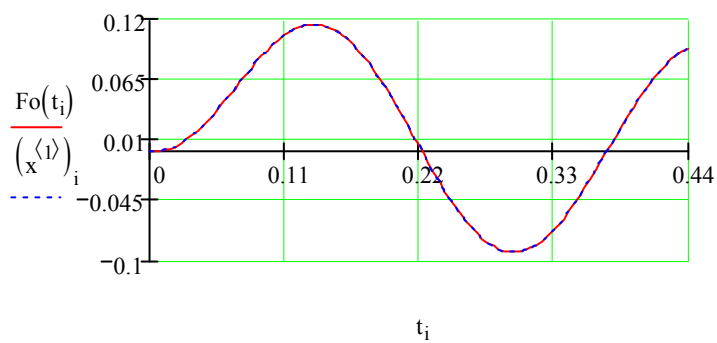
**Calculate the transient process by external influence  $f(t)=20\sin(\omega t)$**

$$\omega := 20 \quad f(t) := 20 \cdot \sin(\omega \cdot t)$$

$$Fo(t) := \int_0^t w(t - \tau) \cdot f(\tau) d\tau$$

$$0.1 \cdot \frac{d^2}{dt^2} x(t) + 10 \cdot \frac{d}{dt} x(t) + 10^2 \cdot x = f(t) \quad D(t, x) := \begin{pmatrix} x_1 \\ \frac{-10}{0.1} \cdot x_1 - \frac{10^2}{0.1} \cdot x_0 + \frac{f(t)}{0.1} \end{pmatrix} \quad \underline{N} := 100$$

$$x := \text{rkfixed} \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0, 5 \cdot \tau, N, D \right] \quad i := 0..N \quad t := x^{(0)}$$



### 1.3 Spectral signal representation

Any time function can be depicted as sum of harmonic functions.

$$X(t) = \frac{a_0}{2} + \sum_{k=1}^N a_k \cos(k\omega t) + b_k \sin(k\omega t) = \sum_{k=0}^N A_k \cos(k\omega t + \phi_k),$$

$\omega = 2\pi / T$ ,  $T$  is function period. Expansion coefficients are determined by formulae:

$$a_k = \frac{2}{T} \int_{T/2}^{T/2} X(t) \cos\left(\frac{k2\pi t}{T}\right) dt, \quad b_k = \frac{2}{T} \int_{T/2}^{T/2} X(t) \sin\left(\frac{k2\pi t}{T}\right) dt,$$

$$A_k = \sqrt{a_k^2 + b_k^2}, \quad \phi_k = \arctg\left(\frac{b_k}{a_k}\right) \pm \begin{cases} 0 & \text{npu } a < 0 \\ \pi & \text{npu } a > 0 \\ \pi / 2 & \text{npu } a = 0 \end{cases}$$

If it is written using Euler's formula

$$e^{jt} = \cos(t) + j \sin(t), \quad e^{-jt} = \cos(t) - j \sin(t),$$

then expression for series is the following :

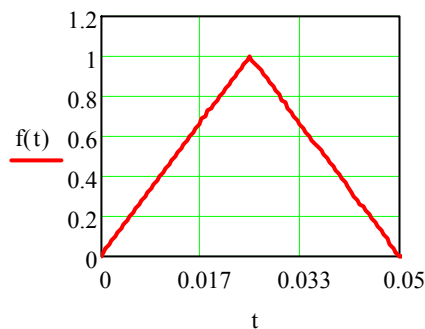
$$X(t) = \frac{a_0}{2} + \sum_{k=1}^N a_k \cos(k\omega t) + b_k \sin(k\omega t) = \sum_{k=-N}^N A_k \exp(jk\omega t), \quad A_k = \frac{1}{2}(a_k - jb_k)$$

Function expansion into a Fourier series with **MathCAD** help

**Example 1:** Expansion of a triangular pulse into a Fourier series

$$T := 0.05 \quad t := 0, 0.01 \cdot T .. 2 \cdot T \quad \omega := \frac{2 \cdot \pi}{T} \quad f := \frac{1}{T}$$

$$f(t) := \begin{cases} \frac{t \cdot 2}{T} & \text{if } 0 \leq t \leq \frac{T}{2} \\ \left(1 - \frac{t}{T}\right) \cdot 2 & \text{if } \frac{T}{2} \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

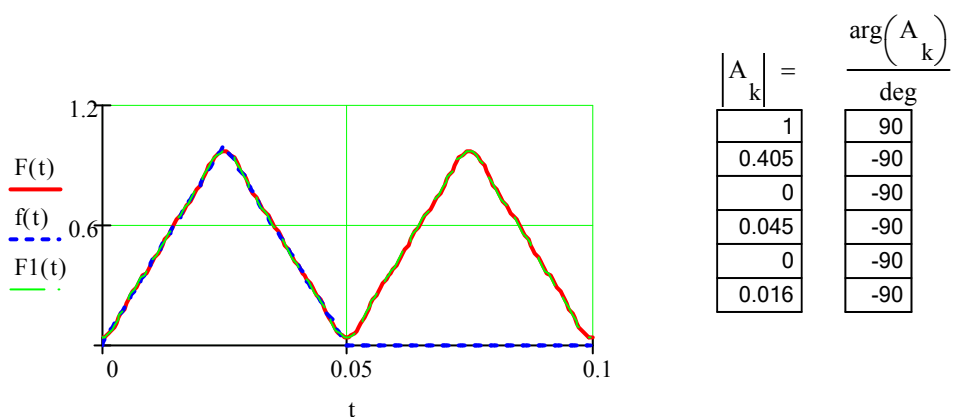


$$M := 5 \quad k := 0..M \quad a_k := \frac{2}{T} \cdot \int_0^T f(t) \cdot \cos(\omega \cdot k \cdot t) dt \quad a = \begin{pmatrix} 1 \\ -0.405 \\ 0 \\ -0.045 \\ 0 \\ -0.016 \end{pmatrix} \quad A_k := i \cdot a_k$$

Two forms of expansion

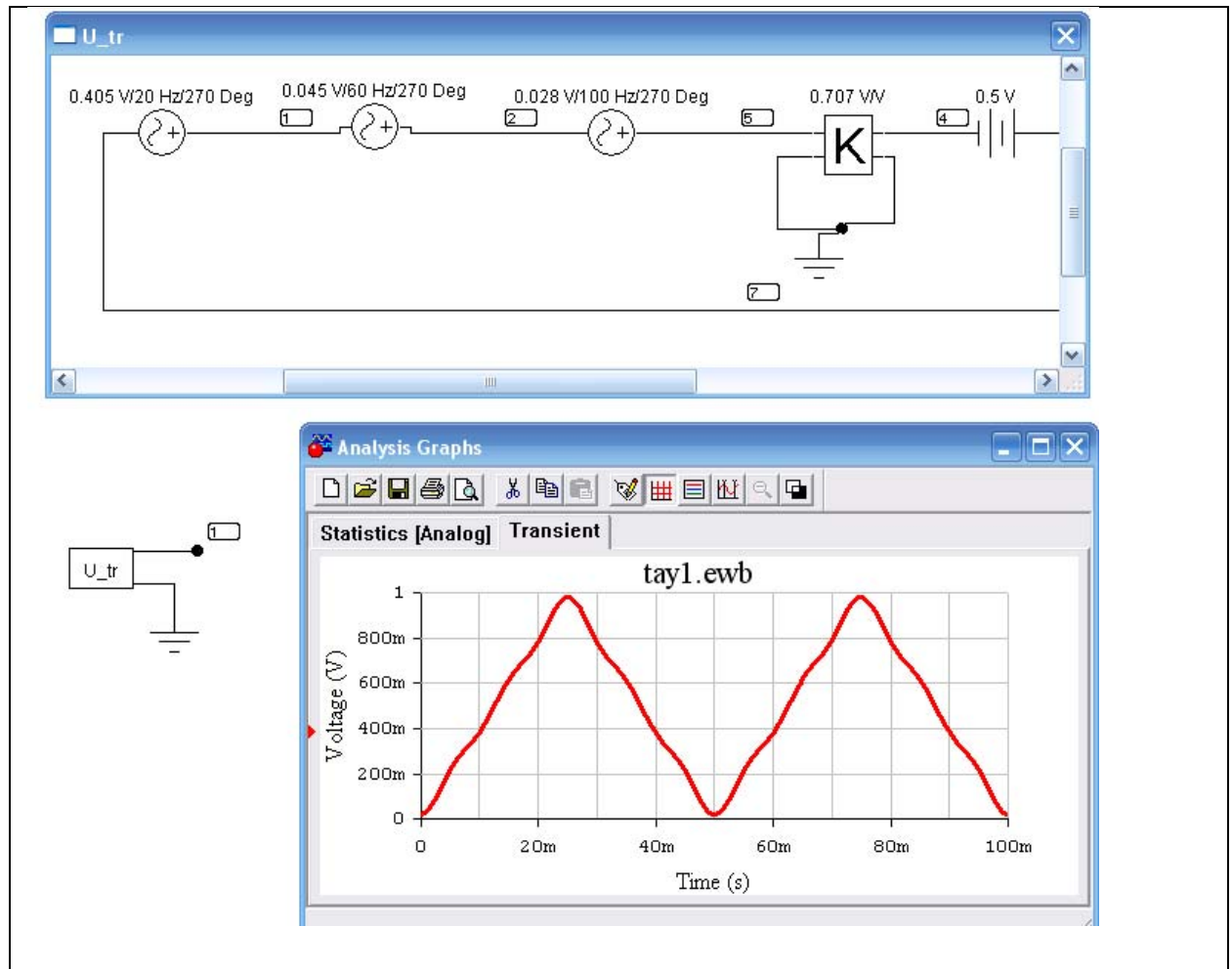
$$F(t) := \frac{a_0}{2} + \sum_{k=1}^M \left( a_k \cdot \cos(\omega \cdot k \cdot t) \right) \quad F1(t) := \frac{|A_0|}{2} + \sum_{k=1}^M \left( |A_k| \cdot \sin(\omega \cdot k \cdot t + \arg(A_k)) \right)$$

Expansion check



Generator assembly which gives triangular alternating impulses in program **Electronics Workbench**. The required impulse form we assembly as

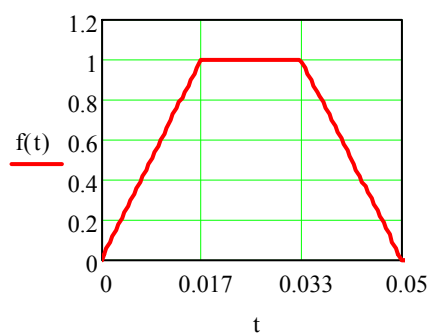
superposition of sinusoidal sources with corresponding frequencies, amplitudes and phase shifts.

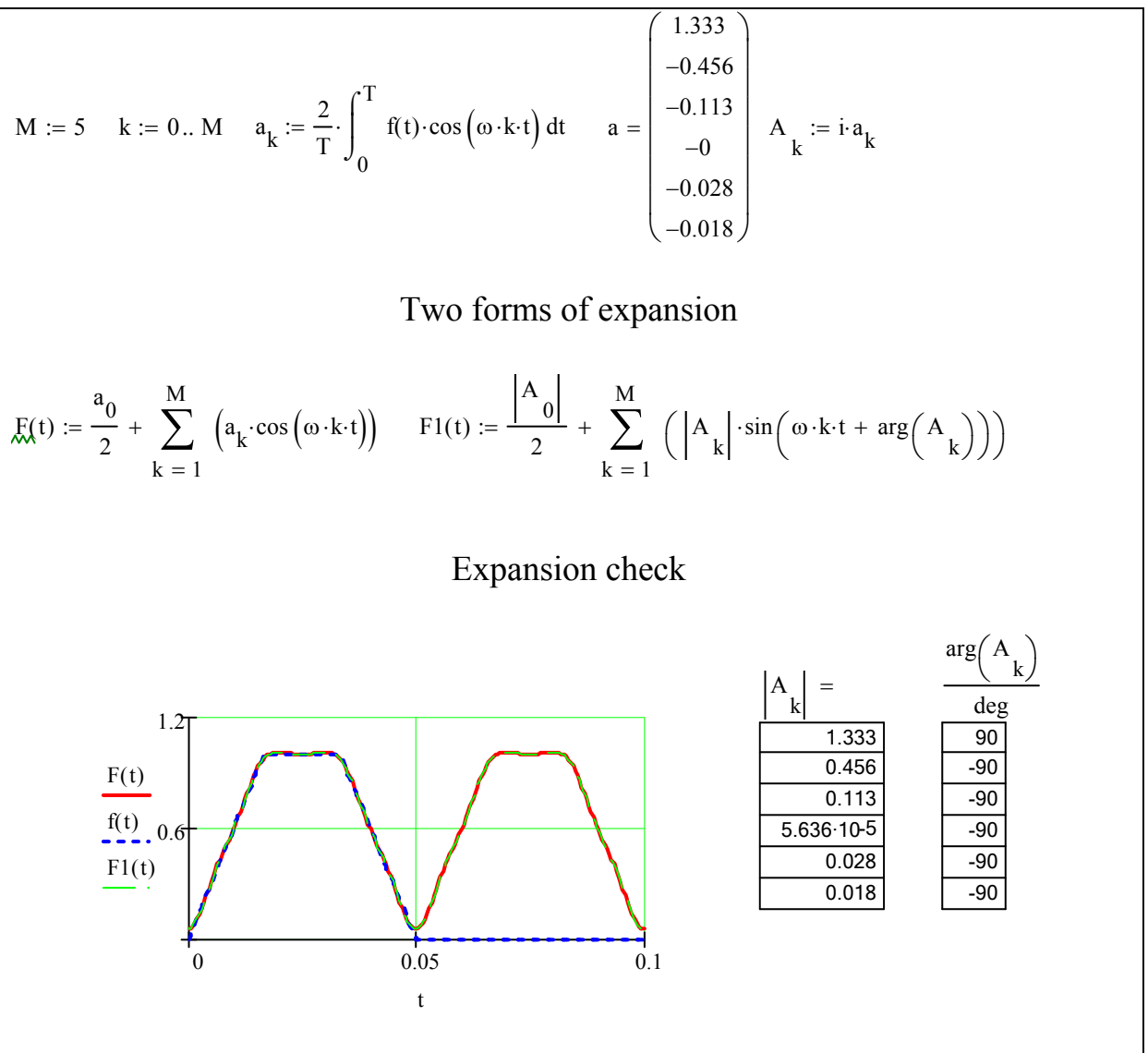


## Example 2: Trapezoidal impulse expansion in a Fourier series

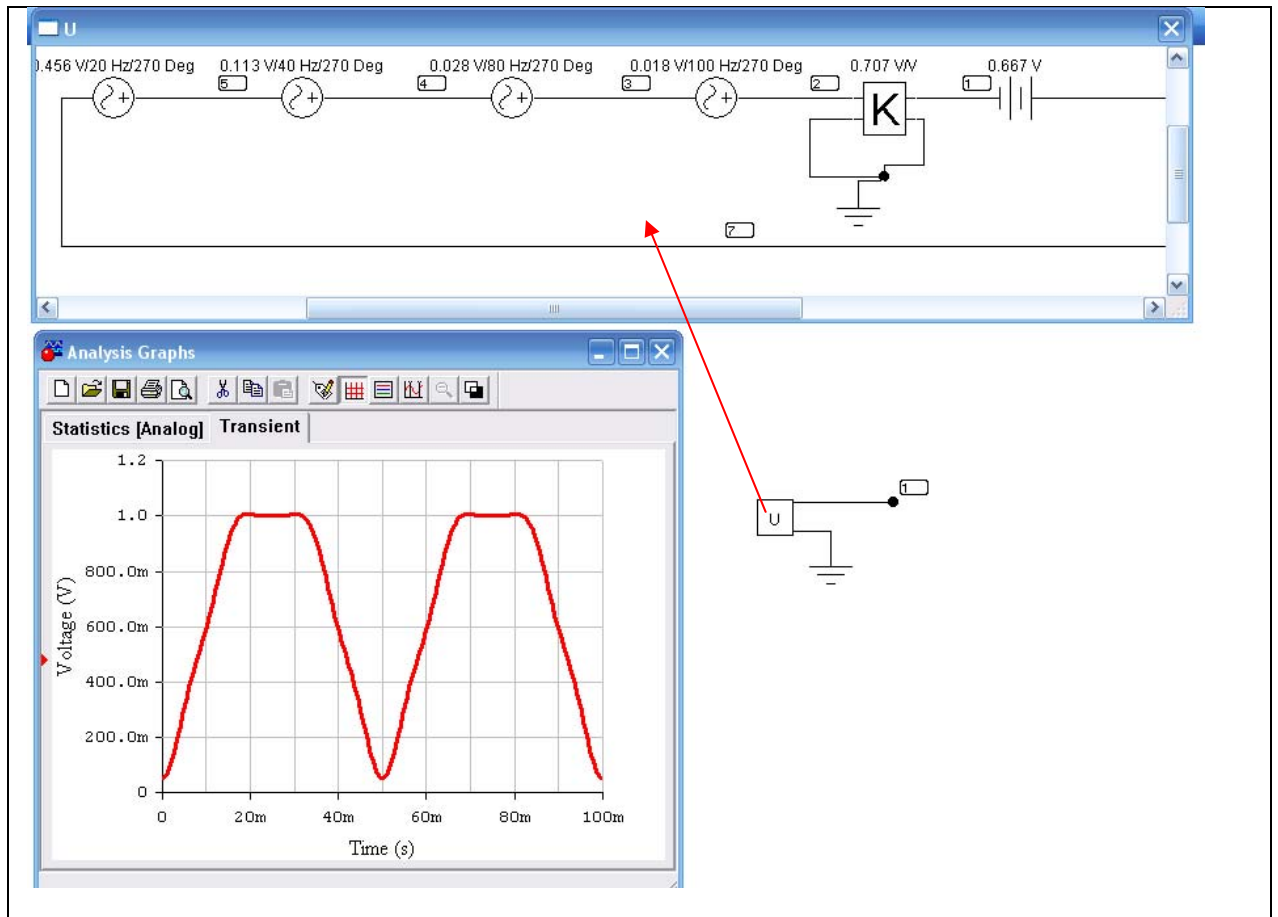
$$T := 0.05 \quad t := 0, 0.01 \cdot T \dots 2 \cdot T \quad \omega := \frac{2 \cdot \pi}{T} \quad f := \frac{1}{T}$$

$$f(t) := \begin{cases} \frac{t \cdot 3}{T} & \text{if } 0 \leq t \leq \frac{T}{3} \\ 1 & \text{if } \frac{T}{3} \leq t \leq \frac{2}{3} \cdot T \\ \left(1 - \frac{t}{T}\right) \cdot 3 & \text{if } \frac{T}{3} \cdot 2 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$





Generator assembly which gives trapezoidal alternating impulses in program **Electronics Workbench**. The required impulse form we assembly as superposition of sinusoidal sources with corresponding frequencies, amplitudes and phase shifts.



### 1.3.1. Bode plot

Bode plot is the ratio between complex representation of output variable and complex representation of input variable which is being changed according to potential law.

$$y(t) = A_y \cdot \sin \omega t. \quad (16)$$

By  $y(t)$  change according to (16) at linear component output in steady state the output variable will change according to potential law with the same frequency, but different amplitude and phase.

$$x(t) = A_x \cdot \sin(\omega t + \varphi). \quad (17)$$

Complex representation (16) and (17)

$$\underline{y} = A_y \cdot e^{j\omega t}; \quad \underline{x} = A_x \cdot e^{j(\omega t + \varphi)}. \quad (18)$$

Hence,  $W(j\omega)$  which stands for Bode plot equals

$$W(j\omega) = \frac{\underline{x}}{\underline{y}} = \frac{A_y e^{j\omega t}}{A_x e^{j\omega t}} \cdot e^{j\varphi} = A(\omega) \cdot e^{j\varphi(\omega)}. \quad (19)$$

In (19)  $A(\omega)$  and  $\varphi(\omega)$  functions are Bode magnitude plot and Bode phase plot.

Bode plot (19) which is a complex variable has the algebraic form of

$$W(j\omega) = U(\omega) + jV(\omega). \quad (20)$$

In (20)  $U(\omega)$  and  $V(\omega)$  are real and imaginary frequency characteristics.

Bode plot is interconnected to the Transfer function.

Mind that formula for derivative of n-order from  $\underline{y}$  and  $\underline{x}$  are equal

$$\begin{aligned} \frac{d^k \underline{y}}{dt^k} &= (j\omega)^k \cdot A_y \cdot e^{j\omega t} = (j\omega)^k \cdot \underline{y}; \\ \frac{d^k \underline{x}}{dt^k} &= (j\omega)^k \cdot A_x \cdot e^{j\omega t} = (j\omega)^k \cdot \underline{x}, \end{aligned} \quad (21)$$

By substituting (18) in the differential equation (9) we obtain

$$W(j\omega) = \frac{b_0 \cdot (j\omega)^m + b_1 \cdot (j\omega)^{m-1} + \dots + b_{m-1} \cdot (j\omega) + b_m}{a_0 \cdot (j\omega)^n + a_1 \cdot (j\omega)^{n-1} + \dots + a_{n-1} \cdot (j\omega) + a_n} = \frac{R(j\omega)}{Q(j\omega)} \quad (22)$$

Bode plot has its real and imaginary parts.

$$W(j\omega) = A(\omega)e^{j\phi(\omega)} = U(\omega) + jV(\omega)$$

$$U(\omega) = A(\omega)\cos(\phi(\omega)), \quad V(\omega) = A(\omega)\sin(\phi(\omega))$$

$$A(\omega) = \sqrt{U(\omega)^2 + V(\omega)^2}, \quad \phi(\omega) = \arctg\left(\frac{V(\omega)}{U(\omega)}\right) \pm \begin{cases} 0 & \text{npu } U(\omega) \geq 0 \\ \pi & \text{npu } U(\omega) < 0 \end{cases}$$

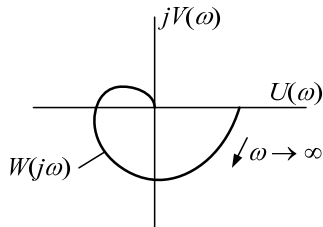


Figure 6

As shown in correlation (9) and (22) that formula for Bode plot is obtained from substituting in transfer function  $p = j\omega$ .

To solve practical problems of control systems properties Bode plot is plotted on complex planes  $U(\omega)$  and  $V(\omega)$  as locus which is a geometric part of vector end  $W(j\omega)$  at frequency change from 0 to  $\infty$  (Figure 3).

### 1.3.2. Correlation between component input and output variables in steady state.

Correlation in steady state can be obtained if we assume all the derivatives equal zero in transfer function  $p = 0$ .

Hence

$$x_{\infty} = y_{\infty} \cdot W(0) = \frac{b_m}{a_n} = k. \quad (23)$$

Coefficient  $k = \frac{b_m}{a_n}$  is called component coefficient of amplification

### 1.3.3. Steady and minimal phase components frequency characteristics peculiarities.

The system is described by  $A(\omega)$  and  $e^{j\phi(\omega)}$  or  $U(\omega)$  and  $V(\omega)$  variables. But for some types of components (steady component) there exists the correlation between the functions which formed these pairs and one of such functions can be used to describe the system. To determine the weight function for instance the following ratio should be used:

$$w(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} U(\omega) \cos(\omega t) dt \quad \text{or} \quad w(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} V(\omega) \sin(\omega t) dt$$

On integrating these formulae we obtain the formulae for transfer function:

$$h(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{U(\omega)}{\omega} \sin(\omega t) dt \quad \text{or} \quad h(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{V(\omega)}{\omega} \cos(\omega t) dt.$$

These correlations are for steady components in which denominator poles of transfer function have negative real parts.



**Minimal phase component** is a component which have poles and zeros with negative real parts. That means they have minimal phase shift at frequency  $\omega$  compared to other components.

**Example:**

$$W(p) = \frac{k}{Tp + 1}$$

Calculate Bode plot

$$W(j\omega) = \frac{k}{jT\omega + 1}$$

Calculate its real and imaginary parts

$$W(j\omega) = U(\omega) + jV(\omega) = \frac{k(1 - jT\omega)}{(1 + jT\omega)(1 - jT\omega)} = \frac{k - jkT\omega}{1 + T^2\omega^2} = \frac{k}{1 + T^2\omega^2} - j\frac{kT\omega}{1 + T^2\omega^2}$$

$$U(\omega) = \frac{k}{1 + T^2\omega^2}, \quad V(\omega) = -\frac{kT\omega}{1 + T^2\omega^2},$$

$$A(\omega) = \sqrt{U(\omega)^2 + V(\omega)^2} = \sqrt{\frac{k^2 + k^2T^2\omega^2}{(1 + T^2\omega^2)^2}} = k \sqrt{\frac{\cancel{1 + T^2\omega^2}}{(1 + T^2\omega^2)^2}} = \frac{k}{\sqrt{1 + T^2\omega^2}}$$

$$\phi(\omega) = \arctg\left(\frac{V(\omega)}{U(\omega)}\right) = \arctg(-T\omega) = -\arctg(T\omega)$$

When  $\omega$  is changed,  $\phi$  is changing from 0 to  $-\frac{\pi}{2}$ .

Let's analyze the block with transfer function

$$W(p) = \frac{k}{Tp - 1}$$

Do the same procedure

Calculate its real and imaginary parts:

$$W(j\omega) = U(\omega) + jV(\omega) = -\frac{k(1+jT\omega)}{(1+jT\omega)(1-jT\omega)} = \frac{k + jkT\omega}{1+T^2\omega^2} = -\frac{k}{1+T^2\omega^2} - j\frac{kT\omega}{1+T^2\omega^2}$$

$$U(\omega) = -\frac{k}{1+T^2\omega^2}, \quad V(\omega) = -\frac{kT\omega}{1+T^2\omega^2},$$

$$A(\omega) = \sqrt{U(\omega)^2 + V(\omega)^2} = \sqrt{\frac{k^2 + k^2T^2\omega^2}{(1+T^2\omega^2)^2}} = k \sqrt{\frac{1+T^2\omega^2}{(1+T^2\omega^2)^2}} = \frac{k}{\sqrt{1+T^2\omega^2}}$$

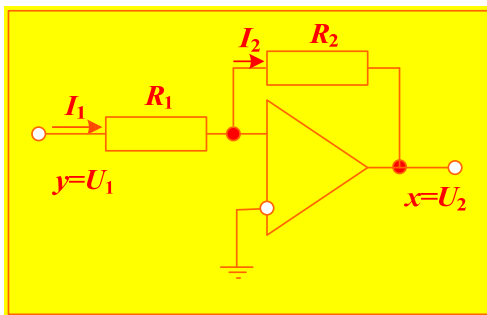
$$\phi(\omega) = \arctg\left(\frac{V(\omega)}{U(\omega)}\right) - \pi = \arctg(T\omega) - \pi, \quad \arctg(T\omega) \in (0, \pi/2)$$

When  $\omega$  is changed,  $\phi$  is changing from  $-\pi$  до  $-\frac{\pi}{2}$ . Thus the second component makes greater delay.

## 1.4. Some typical components of control systems

### 1.4.1. Amplifying component

Let's compute transfer function of inertia-free component

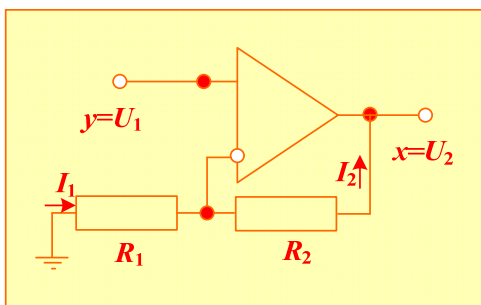


$$I_1 = \frac{U_1}{R_1}, I_2 = \frac{-U_2}{R_2}, \quad I_1 = I_2 \rightarrow \frac{U_1}{R_1} = -\frac{U_2}{R_2}$$

$$\rightarrow \frac{U_2}{U_1} = -\frac{R_2}{R_1}$$

Component transfer function:

$$W(p) = \frac{U_2(p)}{U_1(p)} = -\frac{R_2}{R_1} = k$$



$$I_1 = -\frac{U_1}{R_1}, I_2 = \frac{U_1 - U_2}{R_2}, \quad I_1 = I_2 \rightarrow -\frac{U_1}{R_1} = \frac{U_1 - U_2}{R_2}$$

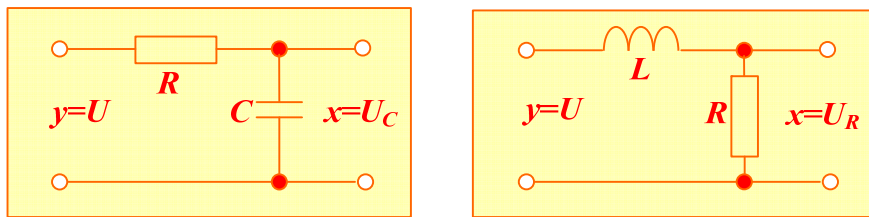
$$\frac{U_2}{R_2} = \frac{U_1}{R_2} + \frac{U_1}{R_1}, \quad \frac{U_2}{R_2} = \left( \frac{1}{R_2} + \frac{1}{R_1} \right) U_1 \quad \text{Component transfer function:}$$

$$\frac{U_2}{R_2} = \left( \frac{1}{R_2} + \frac{1}{R_1} \right) U_1,$$

$$W(p) = \frac{U_2(p)}{U_1(p)} = \left( \frac{1}{R_2} + \frac{1}{R_1} \right) R_2 = \left( 1 + \frac{R_2}{R_1} \right) = k,$$

#### 1.4.2. Aperiodic inertial component of the first period

Consider the example of aperiodic inertial component. Electrotechnical diagram shown below can be such components.



Develop the formula for inertial component with capacity:

$$i_C = C \frac{dU_C}{dt}, \quad U_C + i_C R = U \rightarrow U_C + RC \frac{dU_C}{dt} = U$$

Let's represent for time invariable  $T = RC$ :

$$U_C + T \frac{dU_C}{dt} = U$$

⇓

$$U_C + TpU_C = U \rightarrow (1 + Tp)U_C = U$$

In our case:

$$(1 + Tp)X = Y \rightarrow W = \frac{X}{Y} = \frac{1}{1 + Tp}$$

We can obtain the same formula through diagram resistance having complex representation:

$$Z(\omega) = R + \frac{1}{j\omega C},$$

Find current  $I(\omega) = U / Z(\omega)$ , voltage on the required element  $U_c = I(\omega) \frac{1}{j\omega C}$ ,

then substitute all the values and obtain:

$$U_c = I(\omega) \frac{1}{j\omega C} = \frac{U}{Z(\omega)} \frac{1}{j\omega C} = \frac{U}{\left(R + \frac{1}{j\omega C}\right)} \frac{1}{j\omega C} = \frac{U}{(j\omega CR + 1)},$$

$$\frac{U_c}{U} = W(j\omega) = \frac{1}{(j\omega CR + 1)} \rightarrow W(p) = \frac{1}{(pT + 1)}$$

Thus transfer function of inertial component is:

$$W(p) = \frac{k}{(pT + 1)}$$

And its weight function (Green's function):  $w(t) = \frac{1}{T} k e^{-t/T}$

To calculate its transfer function  $h(t)$  its impulse response function  $W(p)$  with  $1/p$  multiplier can be used

$$\frac{W(p)}{p} = \frac{1}{p} \frac{k}{(pT + 1)} \rightarrow \frac{A(0)}{B(0)} + \frac{A(p)}{pB'(p)} e^{pt} = k - \frac{k}{\frac{1}{T}} e^{t/T} = k(1 - e^{-t/T})$$

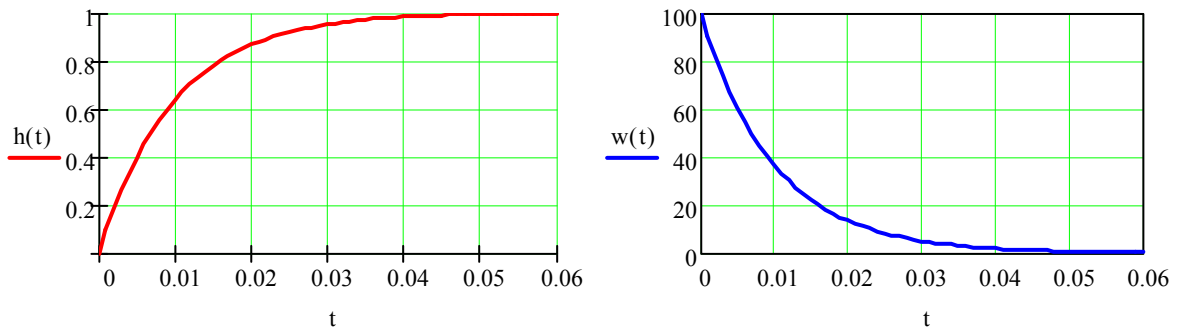
Check

$$w(t) = h'(t) = \frac{k}{T} e^{-t/T}$$

For the second example do the same calculations taking into consideration time constant equals  $T = L / R$  :

$$\frac{U_R}{U} = W(j\omega) = \frac{1}{(j\omega L / R + 1)} \rightarrow W(p) = \frac{1}{(pT + 1)}$$

Plot graphic responses dependences:



Bode plot is described by formulae which we have already analyzed:

$$W(j\omega) = U(\omega) + jV(\omega) = \frac{k(1 - jT\omega)}{(1 + jT\omega)(1 - jT\omega)} = \frac{k - jkT\omega}{1 + T^2\omega^2} = \frac{k}{1 + T^2\omega^2} - j \frac{kT\omega}{1 + T^2\omega^2}$$

$$U(\omega) = \frac{k}{1 + T^2\omega^2}, \quad V(\omega) = -\frac{kT\omega}{1 + T^2\omega^2},$$

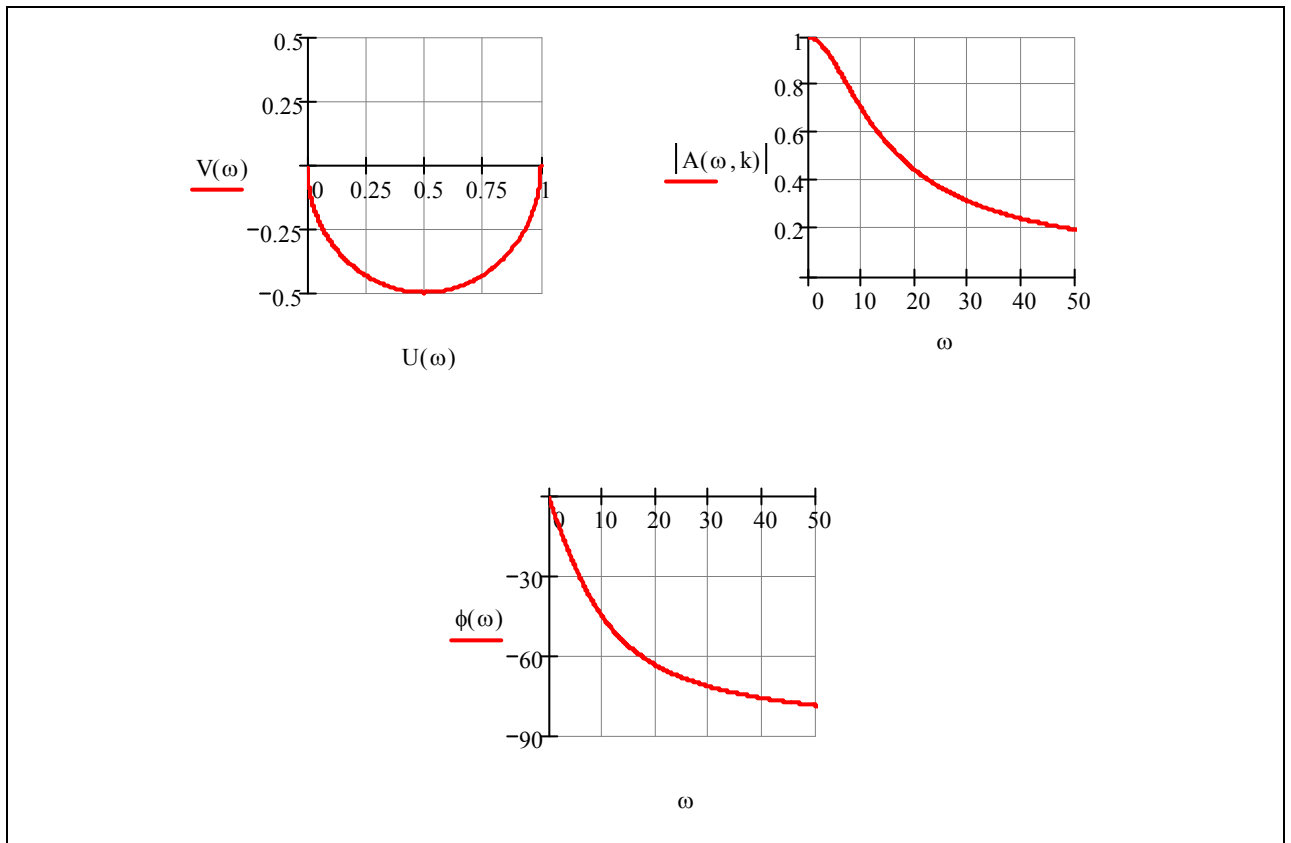
$$A(\omega) = \sqrt{U(\omega)^2 + V(\omega)^2} = \sqrt{\frac{k^2 + k^2T^2\omega^2}{(1 + T^2\omega^2)^2}} = k \sqrt{\frac{1 + T^2\omega^2}{(1 + T^2\omega^2)^2}} = \frac{k}{\sqrt{1 + T^2\omega^2}}$$

$$\phi(\omega) = \arctg\left(\frac{V(\omega)}{U(\omega)}\right) = \arctg(-T\omega) = -\arctg(T\omega)$$

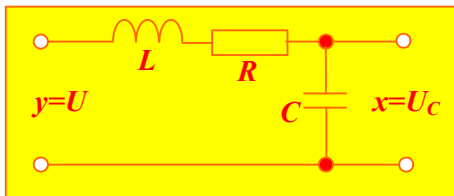
When  $\omega$  is changed,  $\phi$  is changing from 0 до  $-\frac{\pi}{2}$ .

Plot the locus for Bode plot at  $k=1$  and  $T=0.1$  using **MathCAD**

$$\begin{aligned} T &:= 0.1 \quad k := 1 \quad W(p, k) := \frac{k}{T \cdot p + 1} \quad A(\omega, k) := W(j \cdot \omega, k) \quad U(\omega) := \operatorname{Re}(A(\omega, k)) \quad V(\omega) := \operatorname{Im}(A(\omega, k)) \\ \phi(\omega) &:= \frac{\arg(A(\omega, k))}{\deg} \quad \omega := 0..10^3 \end{aligned}$$



### 1.4.3. Oscillatory, aperiodic and conservative components



Parallel series RLC circuit can be the oscillatory component. Let's calculate the voltage of capacitance

$$i_c = C \frac{dU_c}{dt}, U_L = L \frac{di_L}{dt}, i_c = i_L, U_c + i_c R + U_L = U$$

$$\rightarrow L \frac{d^2 U_c}{dt^2} + RC \frac{dU_c}{dt} + U_c = U$$

Let's write down the formula for this equation

$$LC \frac{d^2 U_c}{dt^2} + RC \frac{dU_c}{dt} + U_c = U \rightarrow (LCp^2 + RCp + 1)U_c = U$$

Calculate transfer function:

$$W(p) = \frac{U_c(p)}{U(p)} = \frac{1}{(LCp^2 + RCp + 1)} = \frac{1}{(T_2^2 p^2 + T_1 p + 1)}, T_2 = \sqrt{LC}, T_1 = RC$$

Or through resistance:

$$Z(p) = R + pL + 1/Cp, I(p) = U(p) / Z(p)$$

$$U_c(p) = I(p) \frac{1}{pC} = \frac{U(p)}{R + pL + \frac{1}{pC}} \frac{1}{pC} = U(p) / (LCp^2 + RCp + 1)$$

$$W(p) = U_c(p) / U(p) = 1 / (LCp^2 + RCp + 1)$$

Calculate Green's function – weight function:

$$W(p) = A(p) / B(p) = 1 / (LCp^2 + RCp + 1),$$

$$B(p) = 0, \rightarrow p_{1,2} = \frac{-RC \pm \sqrt{R^2 C^2 - 4LC}}{2LC} =, \frac{-R}{2L} \pm \sqrt{\left(\frac{RC}{2L}\right)^2 - \frac{1}{L^2}}$$

$$\beta = \frac{R}{2L}, \omega = \sqrt{\frac{1}{L^2} - \left(\frac{RC}{2L}\right)^2}, p_{1,2} = -\beta \pm j\omega$$

Then the weight function equals:

$$w(t) = 2 \operatorname{Re} \left( \frac{A(p_1)}{B'(p_1)} e^{p_1 t} \right)$$

Now transfer function can be calculated:

$$\frac{W(p)}{p} = A(p) / pB(p) = \frac{1}{p(LCp^2 + RCp + 1)}$$

$$h(t) = \frac{A(0)}{B(0)} + 2 \operatorname{Re} \left( \frac{A(p_1)}{B'(p_1)} e^{p_1 t} \right) = 1 + \frac{\sqrt{\omega^2 + \beta^2}}{\omega} e^{-\beta t} \sin \left( \omega t + \arctg \left( \frac{\omega}{\beta} \right) \right)$$

Take into consideration the second order component behaviour in **MathCAD**.

In program determine the circuit impedance and transfer function after that. Both impulse response function and Green's function are determined with the help of Laplace transform.

$$\underline{L} := 0.1 \quad \underline{C} := 100 \cdot 10^{-6} \quad \underline{R} := 10$$

$$Z(p) := R + p \cdot L + \frac{1}{C \cdot p} \quad \underline{W}(p) := \frac{1}{Z(p) \cdot C \cdot p}$$

$$p := Z(p) \left| \begin{array}{l} \text{solve, p} \\ \text{float, 4} \end{array} \right. \rightarrow \begin{bmatrix} (-50.) - 312.2i \\ (-50.) + 312.2i \end{bmatrix} \quad W(s) \text{ simplify} \rightarrow \frac{100000.}{100 \cdot s + s^2 + 100000}$$

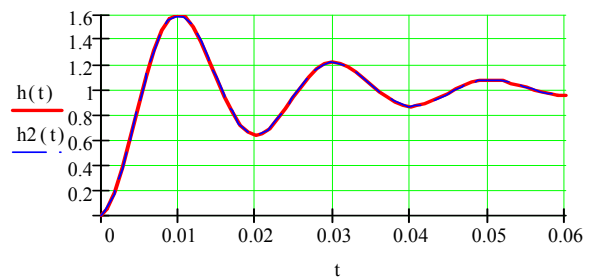
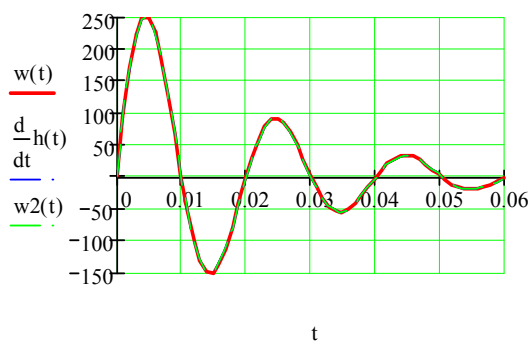
$$B(s) := 100 \cdot s + s^2 + 100000 \quad \underline{A}(p) := 100000 \quad B'(p) := \frac{d}{dp} B(p)$$

$$\underline{w}(t) := W(x) \left| \begin{array}{l} \text{invlaplace, x} \\ \text{float, 4} \end{array} \right. \rightarrow 320.3 e^{(-50.) \cdot t} \cdot \sin(312.2 t)$$

$$\underline{h}(t) := \frac{W(x)}{x} \left| \begin{array}{l} \text{invlaplace, x} \\ \text{float, 4} \end{array} \right. \rightarrow 1. - 1. \cdot e^{(-50.) \cdot t} \cdot \cos(312.2 t) - .1601 e^{(-50.) \cdot t} \cdot \sin(312.2 t)$$

Determine the same functions through theory analysis

$$w_2(t) := 2 \cdot \text{Re} \left( \frac{A(p_2)}{B'(p_2)} \cdot e^{p_2 \cdot t} \right) \quad h_2(t) := 2 \cdot \text{Re} \left( \frac{A(p_2)}{B'(p_2) \cdot p_2} \cdot e^{p_2 \cdot t} \right) + \frac{A(0)}{B(0)}$$



In general oscillatory component equation has the form of:

$$a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_0 y.$$



or

$$\frac{a_2}{a_0} \frac{d^2 x}{dt^2} + \frac{a_1}{a_0} \frac{dx}{dt} + x = \frac{b_0}{a_0} y$$

$$\alpha \frac{d^2 x}{dt^2} + 2\xi \frac{dx}{dt} + x = ky$$

At static mode we have :

$$x = \frac{b_0}{a_0} y \rightarrow x = ky \rightarrow W(0) = \frac{x}{y} = k$$

Characteristic equation:

$$\alpha \frac{d^2 x}{dt^2} + 2\xi \frac{dx}{dt} + x = ky$$

$$(\alpha p^2 + 2\xi p + 1)X = kY \rightarrow W(p) = \frac{X}{Y} = \frac{k}{(\alpha p^2 + 2\xi p + 1)} = \frac{A(p)}{B(p)}$$

Denominator fraction  $B(p)$  has two roots which can be joined(united) complex,  $p_{1,2} = -\beta \pm j\omega$ , or right half plane(RHP) or left half plane  $p_1, p_2 < 0$ . If the characteristic equation which is a differential equation describing the second order component has left half plane roots then the component is called ***aperiodic component of the second power***.

Calculate Bode plot of oscillatory component:

$$W(\omega) = U(\omega) + jV(\omega) = \frac{k}{(j2\xi\omega + 1 - \alpha\omega^2)}$$

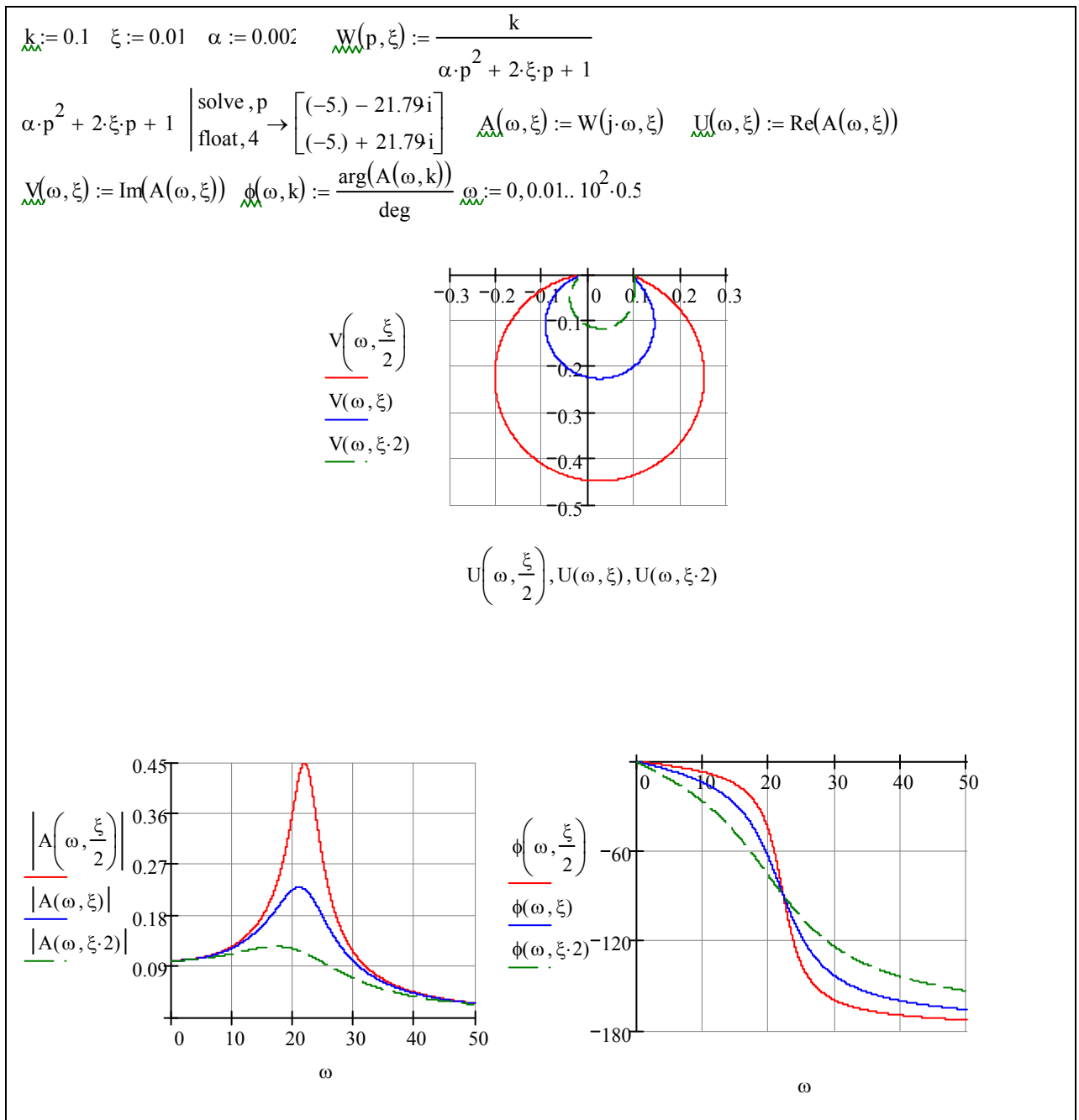
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$$\frac{k(1 - \alpha\omega^2)}{4\xi^2\omega^2 + (1 - \alpha\omega^2)^2} - j \frac{k2\xi\omega}{4\xi^2\omega^2 + (1 - \alpha\omega^2)^2},$$

$$A(\omega) = \sqrt{U(\omega)^2 + V(\omega)^2} = \frac{k\sqrt{4\xi^2\omega^2 + (1 - \alpha\omega^2)^2}}{4\xi^2\omega^2 + (1 - \alpha\omega^2)^2} = \frac{k}{\sqrt{4\xi^2\omega^2 + (1 - \alpha\omega^2)^2}}$$

Let's consider the example of frequency characteristics of locus assembly with different values of damp coefficient  $\xi$  through transfer function with **MathCAD** help.



One more type of second order component is conservative (self-vibrating) component:

$$\alpha \frac{d^2 x}{dt^2} + x = ky$$

$$(\alpha p^2 + 1)X = kY \rightarrow W(p) = \frac{X}{Y} = \frac{k}{(\alpha p^2 + 1)}$$

All mentioned above components are static components, because at substituting in transfer function  $W(p)$ ,  $p=0$  we obtain a certain final number and formula is the same for all the components:

**1. Amplifying (noninertial, static) component:**  $W(p) = k$ ,  $W(0) = k$

**2. Aperiodic(Inertial)component:**  $W(p) = \frac{k}{Tp + 1}$ ,  $W(0) = k$

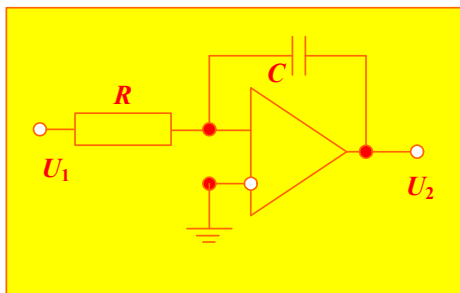
**3. Oscillatory component:**  $W(p) = \frac{k}{\alpha p^2 + 2\beta p + 1}$ ,  $W(0) = k$

**4. Aperiodic component of the second order:**

$$W(p) = \frac{k}{\alpha p^2 + 2\beta p + 1}, W(0) = k$$

**5. Conservative (self-vibrating) component:**  $W(p) = \frac{k}{\alpha p^2 + 1}$ ,  $W(0) = k$

#### 1.4.4. Astatic integrated component



Let's analyze the example of integrated component:

$$i_1 = U_1 / R, i_C = -C \frac{dU_2}{dt}, i_1 = i_C$$

$$-C \frac{dU_2}{dt} = U_1 / R \rightarrow U_2(t) = -\frac{1}{RC} \int_0^t U_1(t) dt$$

$$\frac{dx}{dt} = ky \rightarrow pX = kY \rightarrow W(p) = k / p \rightarrow x = k \int_0^t y dt$$

Green's function (weight function):

$$W(p) = k / p, \quad w(t) = L^{-1}[W(p)] = k$$

Transfer function:

$$W(p) = k / p, \quad h(t) = L^{-1}\left[\frac{W(p)}{p}\right] = L^{-1}\left[\frac{1}{p^2}\right] = kt$$

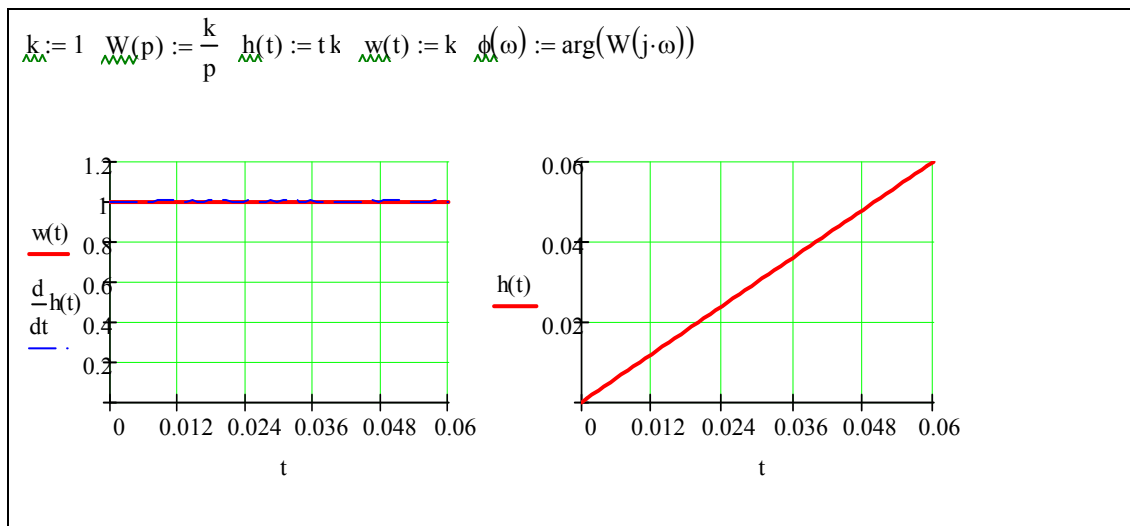
Check:

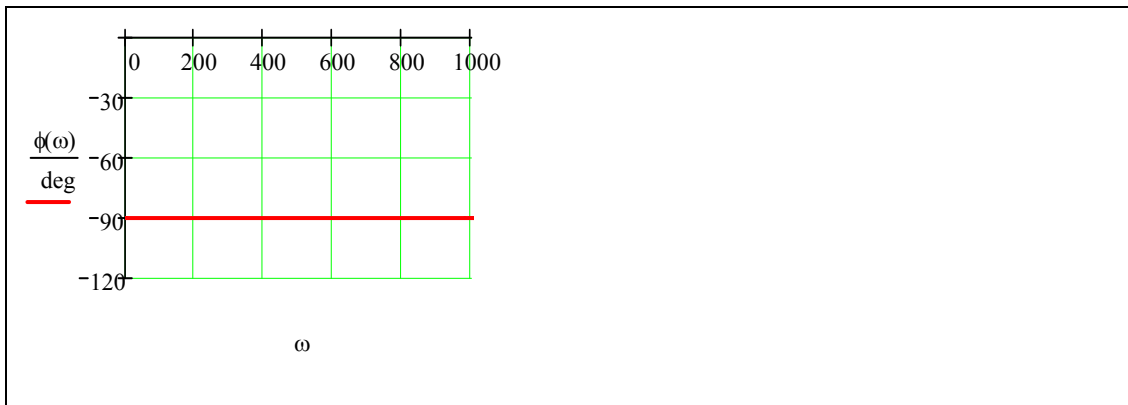
$$w(t) = h'(t) = kt' = k$$

Determine frequency response and bode plot :

$$W(j\omega) = k / j\omega, \quad W(j\omega) = U(\omega) + jV(\omega) = -jk / \omega, \quad U(\omega) = 0, \quad V(\omega) = -k / \omega$$

$$A(\omega) = |W(j\omega)| = k / \omega, \quad \phi(\omega) = -\pi / 2 = 90^\circ$$





7. **Real integrated component** possess some inertia

$$(Tp + 1)pX = kY \rightarrow W(p) = \frac{k}{(Tp + 1)p}$$

The examined component is not simple because it has two blocks/components in series - integrated and inertial(aperiodic).

$$W(p) = \frac{k}{(Tp + 1)p} = \frac{k}{p} \frac{1}{(Tp + 1)} = W_1(p)W_2(p)$$

#### 1.4.5. Differential component.

Impulse response function of differential component has the form of:

$$x = k \frac{dy}{dt}, X = kpY \rightarrow W(p) = kp$$

Let's find transfer function:

$$h(t) = L^{-1} \left[ \frac{W(p)}{p} \right] = L^{-1} \left[ \frac{\cancel{p}k}{\cancel{p}} \right] = k\delta(t)$$

Now Green's function can be obtained:

$$w(t) = h'(t) = k\delta'(t)$$

At random external influence we obtain:

$$x(t) = \int_{-\infty}^{\infty} f(\tau)h'(t-\tau)d\tau = kf'(t)$$

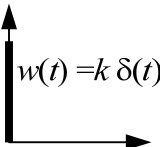
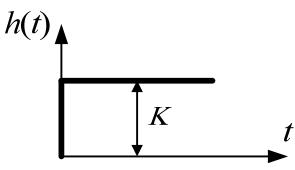
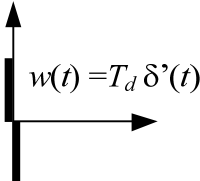
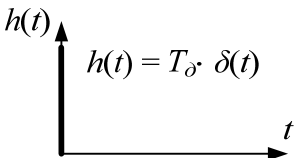
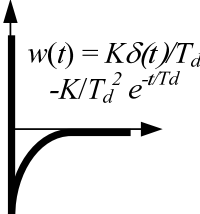
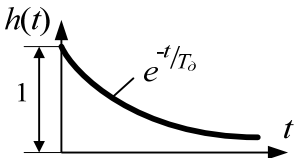
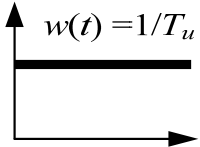
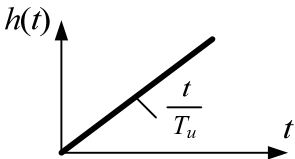
Bode plot:

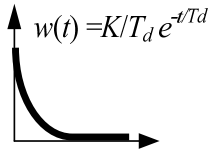
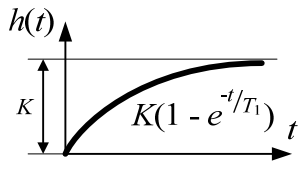
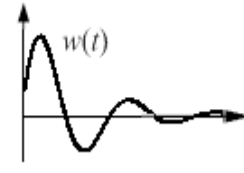
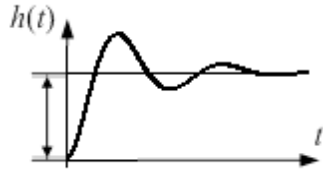
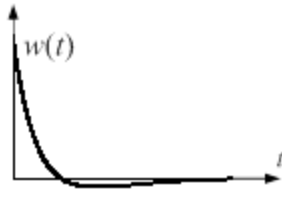
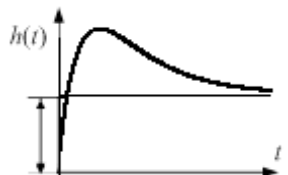
$$W(j\omega) = jk\omega = U(\omega) + jV(\omega), \quad U(\omega) = 0, V(\omega) = k\omega$$

$$A(\omega) = |W(j\omega)| = k\omega, \quad \phi(\omega) = \pi / 2$$

Let's tabulate all transfer function and their properties in Table 1

**Table 1**

Component type	The impulse response function	Weight function	Transfer function
1. Inertia-free (amplifying component)	$W(p) = k$		
2. Ideal differential component	$W(p) = T_d \cdot p$		
3. Real differential component	$W(p) = \frac{T_d \cdot p}{T_d \cdot p + 1}$		
4. Integral component	$W_{int}(p) = \frac{1}{T_{int} \cdot p}$		

5. Inertial component of the first degree	$W(p) = \frac{K}{T_1 \cdot p + 1}$		
6. Inertial component of the second power (oscillatory)	$W(p) = \frac{K}{T_1 \cdot p^2 + T_2 \cdot p + 1}$		Characteristic equation roots $p_1 = \alpha + j\omega$ ; $p_2 = \alpha - j\omega$ 
7. Inertial component of the second degree (aperiodic)	$W(p) = \frac{K}{T_1 \cdot p^2 + T_2 \cdot p + 1}$		Characteristic equation roots $p_1 = -\alpha_1$ ; $p_2 = -\alpha_2$ 

### Things to remember:

1. All the components can be divided into two groups which differ a lot from each other.
  - Static components are components which in steady state (at  $p=0$ ) have a certain connection between output and input variables. These components have stable equilibrium. All components given in Table 1 are static components but integral one.

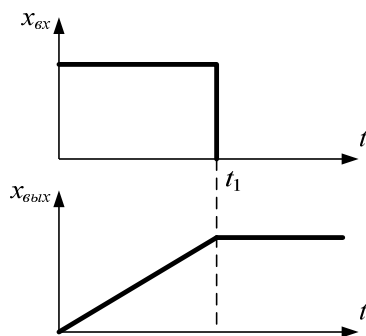


Figure 7

- Astatic components have neutral equilibrium

Integral component is astatic in Table 1. At  $p = 0$ :  $W_u(0) \rightarrow \infty$ . This means that at  $x_{ex} = 1(t)$  theoretically is  $x_{gbx} \rightarrow \infty$ . In input influence is applied on limited time then output variable remains constant, i.e. equals the integral of input variable, for this time (Figure 7).

2. There is an interconnection between the transfer function  $W(p)$  and impulse response function  $h(t)$  capable of transient process  $h(t)$ . When substituting

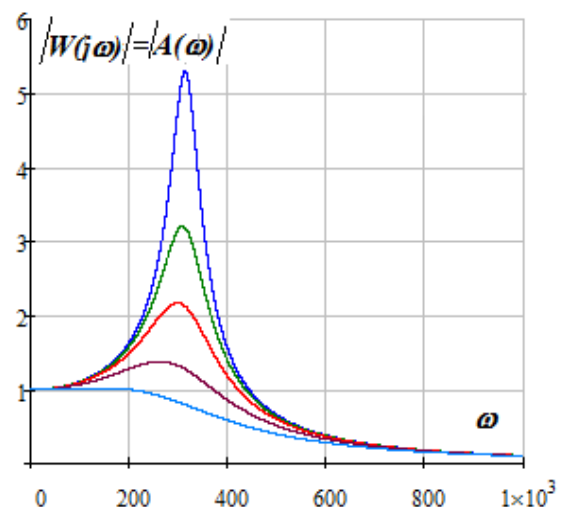
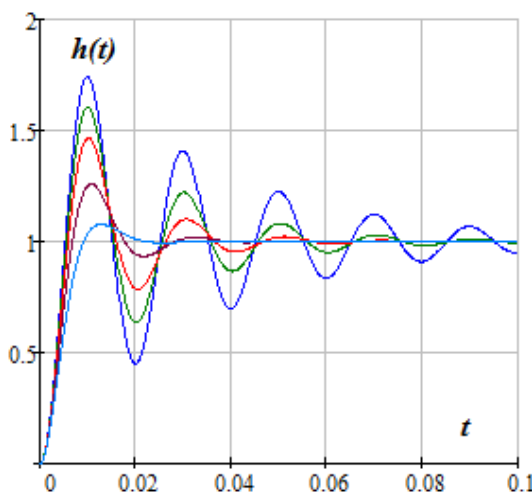
$p = 0$  in impulse response function  $W(0) \rightarrow h(\infty)$  we obtain steady state in time domain. When substituting  $p = \infty$  in impulse response function  $W(\infty) \rightarrow h(0)$  we obtain transient process beginning.

3. Frequency response locus always starts from real axis because real part of frequency response  $U(\omega)$  is an even function, and imaginary part  $V(\omega)$  is an odd function:  $W(j\omega) = U(\omega) + jV(\omega) \rightarrow W(0) = U(0), V(0) = 0$ .

4. Frequency characteristics are determined by the following factors:

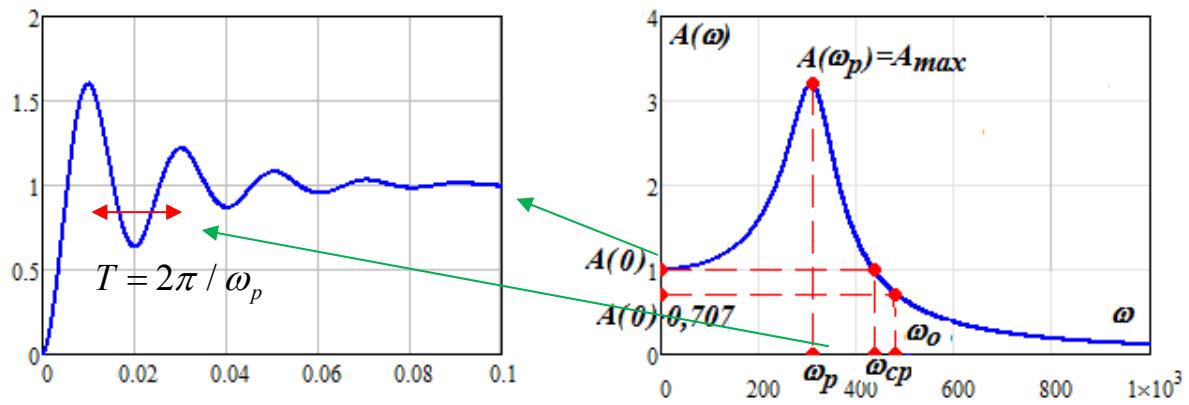
- oscillation indicator  $M = A_{\max}(\omega) / A(0)$  – characterises system tendency to oscillations: the higher it is the less qualitative is the system ( $1.1 < M < 1.5$ )
- Resonance frequency is the frequency at which magnitude frequency response is at its high (at this frequency oscillations have maximum amplification)
- System bandwidth – interval from  $\omega = 0$  to  $\omega_0$  at which the following condition is fulfilled  $A(\omega_0) \leq 0,707 A(0)$
- Frequency cutoff  $A(\omega_{cp}) = A(0)$  indirectly characterizes the duration of transient process, the following correlation is true  $T_y = (1 \div 2) 2\pi / \omega_{cp}$

More



The greater the peak of resonance frequency the more intensive are the oscillations





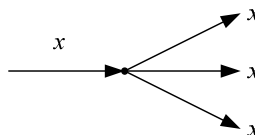
## 1.5. Block diagrams

### 1.5.1. Control system transfer function calculation by blocks transfer functions

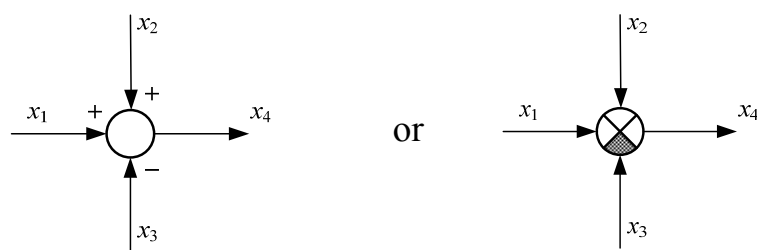
A control system consists of a number of components connected to perform a desired function. Once a component (system or subsystem) is reduced to a mathematical model, it can be represented as a block, with the component operation described by the mathematical function. It is convenient and useful to represent the elements of a control system by blocks. The properties of the block are contained in the transfer function represented by the Laplace transform. Many control systems involve several blocks representing the controller actions, the plant, feedback elements, and other functions. A block diagram is a pictorial representation of an entire control system in terms of all the elements and their transfer functions.

When constructing structural diagrams of control systems the following symbols illustrated in diagrams below are helpful as well

- Signal branching sign

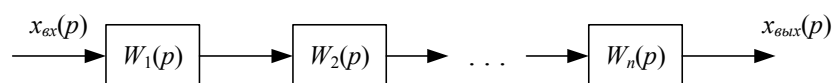


- The summing point also known as a summing junction is the block used to represent the addition/subtraction of signal as shown in Figure below



$$x_4 = x_1 + x_2 - x_3.$$

### 1.5.2. Blocks in series

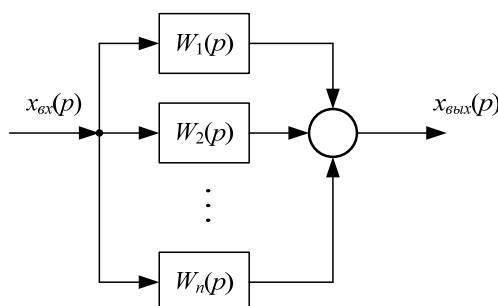


**Figure 8**

From transfer function definition it follows that

$$W_{\text{экв}}(p) = W_1(p) \cdot W_2(p) \cdot \dots \cdot W_n(p). \quad (24)$$

### 1.5.3. Blocks in parallel



**Figure 9**

Parallel subsystems have a common input and output formed by the algebraic sum of the outputs from all the subsystems.

Hence equivalent transfer function will be equal to

$$W_{\text{экв}}(p) = W_1(p) + W_2(p) + \dots + W_n(p) \quad (25)$$

### 1.5.4. Feedback control system

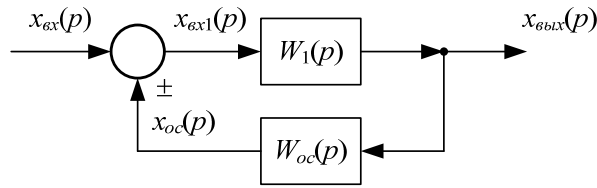


Figure 10

Closed loop control systems are divided into two categories, based upon the nature of the feedback signal: negative-feedback closed loop and positive-feedback closed loop systems. The system consists of one block in the forward path and one block in the feedback path. Hence we can write:

$$\begin{aligned} x_{6x1}(p) &= x_{6x}(p) \pm x_{oc}(p), \\ \text{where, } x_{oc}(p) &= W_{oc}(p) \cdot x_{6blx}(p). \end{aligned} \quad (26)$$

In (26) sign "+" corresponds to positive feedback, and sign "-" corresponds to negative feedback.

The transfer function of reduced form of a closed loop control system can be obtained from

$$x_{6blx}(p) = W_1(p) \cdot [x_{6x}(p) \pm W_{oc}(p) \cdot x_{6blx}(p)]. \quad (27)$$

From (27) calculate the required transfer function

$$W_{\text{ред}}(p) = \frac{x_{6blx}(p)}{x_{6x}(p)} = \frac{W_1(p)}{1 \mp W_1(p) \cdot W_{oc}(p)}. \quad (28)$$

Notice that when obtaining the formula (28) signs corresponding to positive and negative feedback are changed into the opposite ones. In (28) "-" sign corresponds to positive feedback and "+" sign is negative.

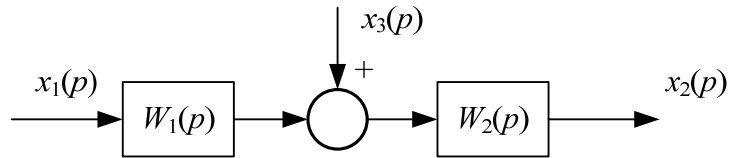
### 1.5.5. Block diagram reduction

Block diagram reduction help reduce separate elements of block diagram to three block types described above.

Block diagram reduction is quite simple and illustrated below.

### 1.5.6. Summing point moving

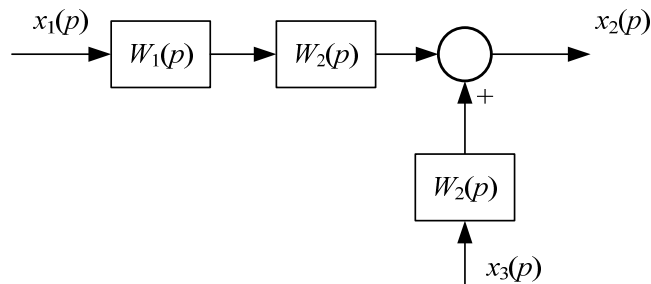
Original diagram



$$\begin{aligned} x_2(p) &= [x_1(p) \cdot W_1(p) + x_3(p)] \cdot W_2(p) \\ &= x_1(p) \cdot W_1(p)W_2(p) + x_3(p)W_2(p) \end{aligned}$$

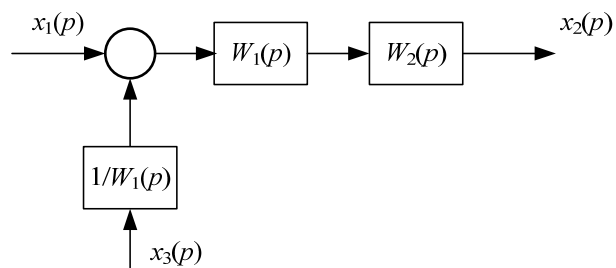
**a**

Moving a summing point to the output



**b**

Moving a summing point to the input



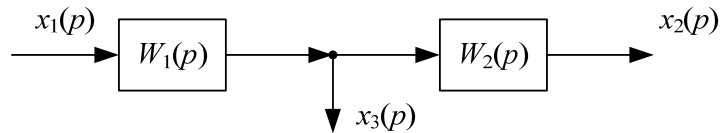
**c**

It is clear that the relation between  $x_2(p)$  and  $x_1(p)$  in diagrams **b** and **c** is the same as in **a** diagram.

### 1.5.7. Branching point moving

Equivalence at branching point moving is to keep the relation between input value  $x_1(p)$  and branching moving point value.

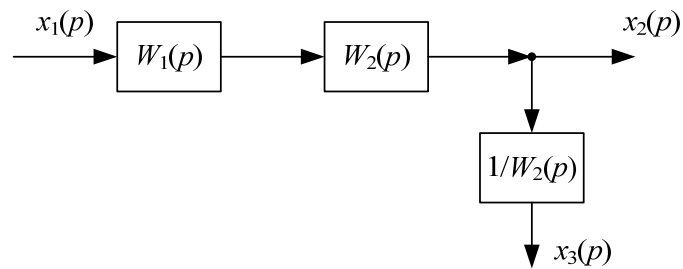
Original diagram



$$x_3(p) = W_1(p) \cdot x_1(p) .$$

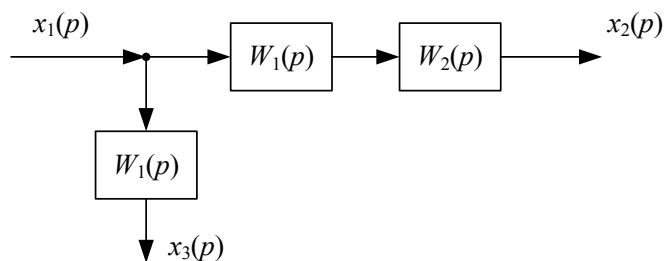
**a**

Moving a summing point to the output



**b**

Moving a summing point to the input



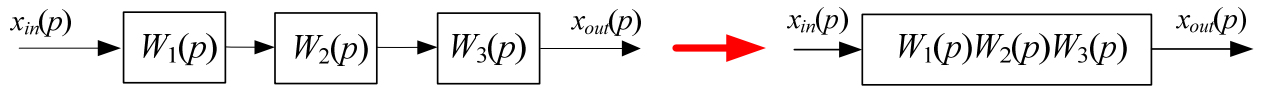
**c**

It is clear that relation in **b** and **c** diagrams is the same as in **a** diagram

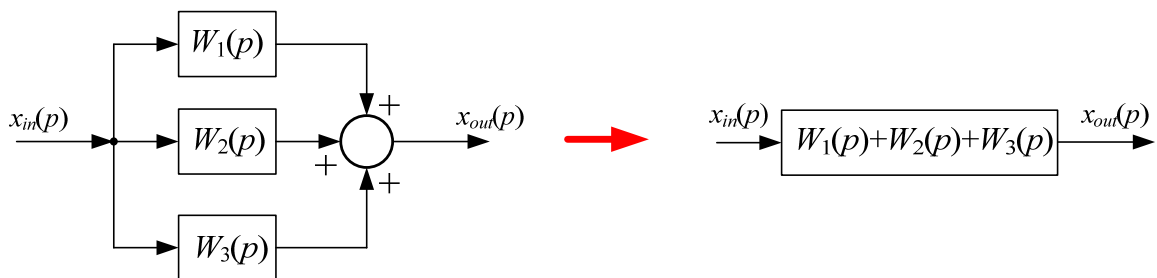
$$x_3(p) = W_1(p) \cdot x_1(p).$$

### 1.5.8. Examples of diagrams reduction rules

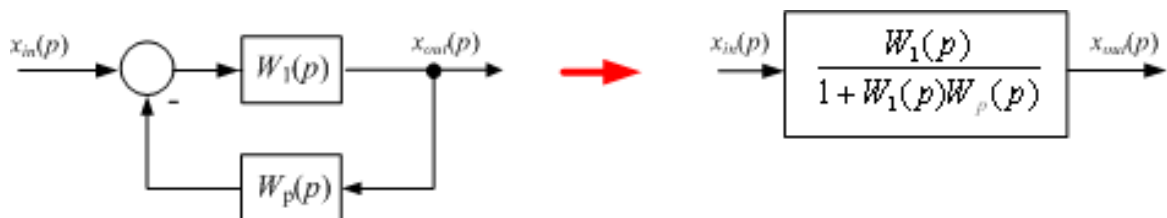
1. Blocks in series:



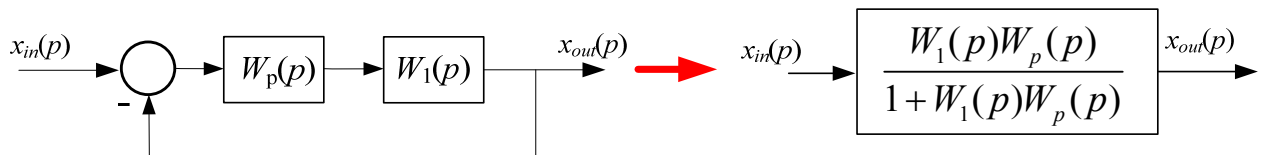
2. Blocks in parallel:



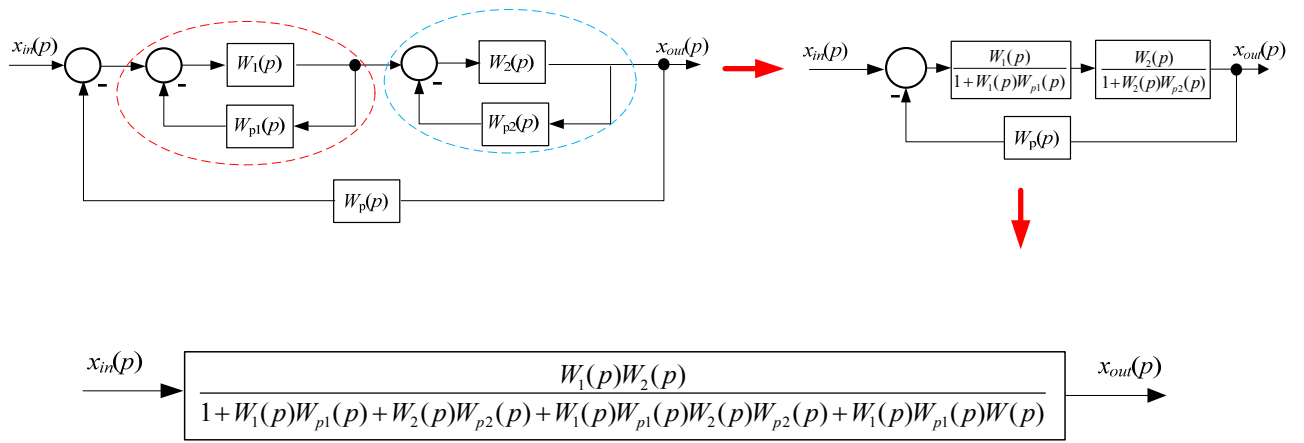
3. Closed loop system reduction with negative feedback:



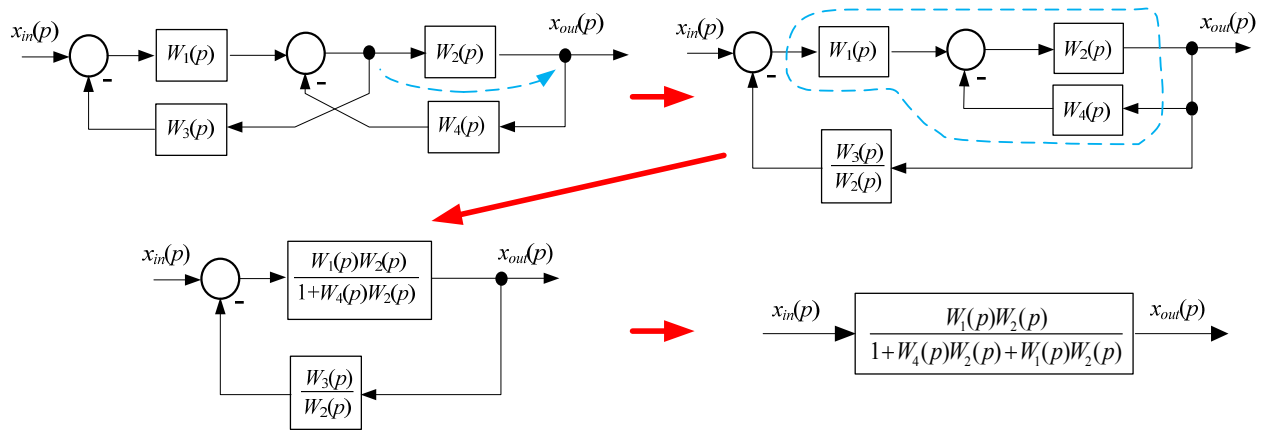
4. Closed loop system reduction with unit negative feedback:



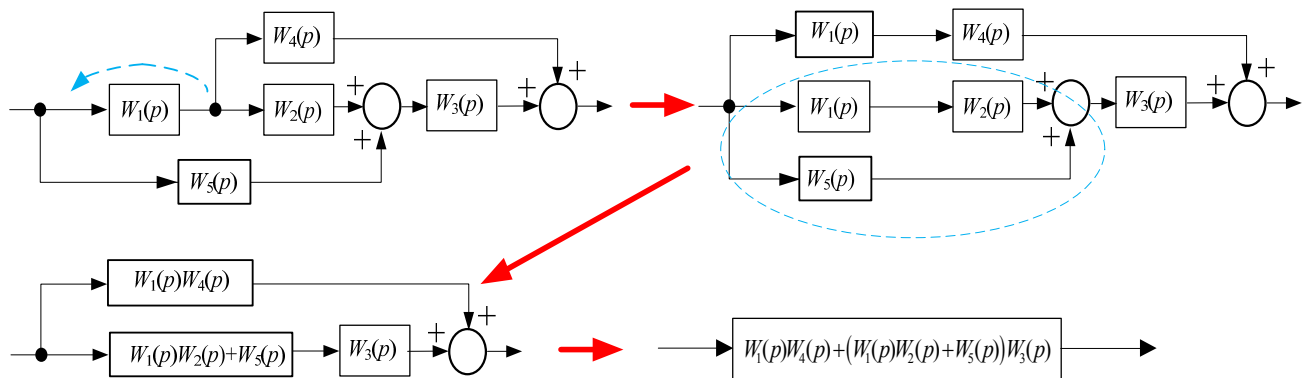
5. Application of block diagrams reduction rules:



## 6. Block reduction with takeoff point moving ahead of a block:

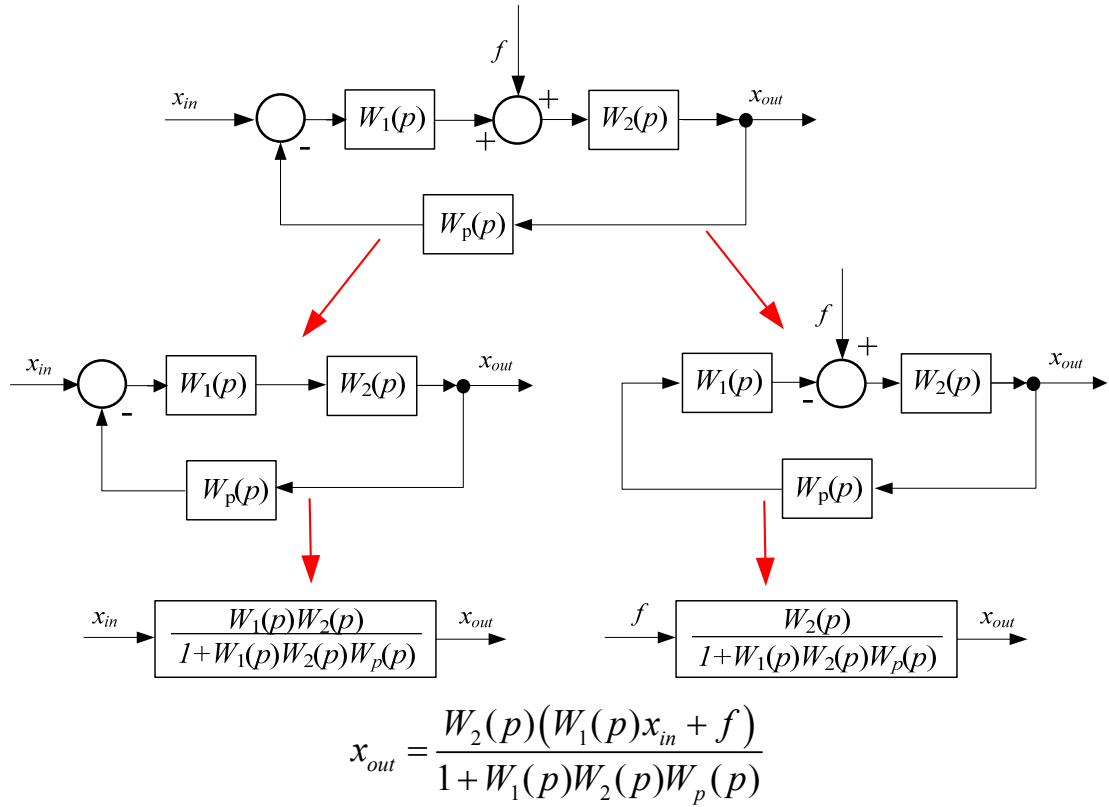


## 7. Block reduction with takeoff point moving:



## 8. Application of superposition principle

Find the relation between input  $x_{in}$  and output value  $x_{out}$ , and relation between disturbance  $f$  and output value  $x_{out}$ :



### 1.6. Control system generalized equivalent circuit

Applying block diagram reduction rules any block diagram can be reduced to two blocks diagram (Figure 6). Consider one of the blocks to be equivalent controlled object with transfer function  $W_o(p)$  and the second block to be equivalent controller with transfer function  $W_c(p)$ .

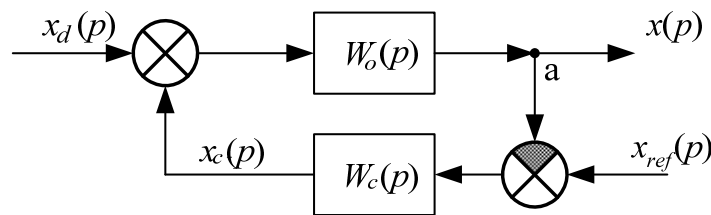


Figure 11

In shown in Figure 8:

$x$  - controlled variable,

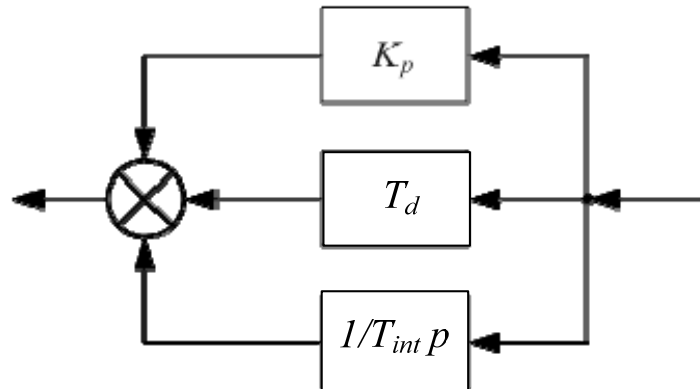
$x_{ref}$  - reference input signal,



$x_d$  - disturbance,

$x_c$  - control response

To figure out how the control signal influences control system behavior let us represent controller block diagram as three blocks in parallel: inertialess, ideal differentiated and integral(Figure 9).



**Figure 12**

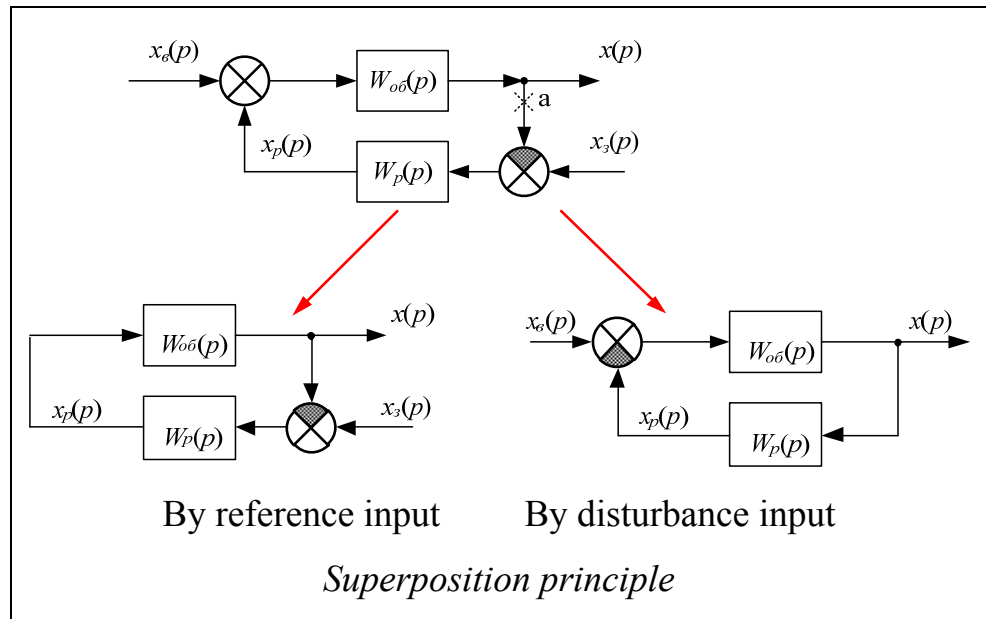
Controller transfer function according to diagram in Figure 9

$$W_c(p) = K_p + T_d \cdot p + \frac{1}{T_{int} \cdot p}. \quad (29)$$

The obtained equivalent diagram shows the influence of object parameters and controller on control system properties in steady state and transient processes.

System properties are determined by the dependence of control response on input – reference or disturbance.

As all system blocks are taken to be linear, then to obtain the required dependences we use ***superposition principle***. Hence the following transfer functions appear.



1. Transfer function by reference signal

$$W_{ref}(p) = \frac{x(p)}{x_{ref}(p)}; \quad \text{at } x_d(p) = 0.$$

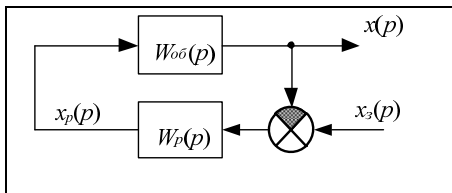
2. Transfer function by disturbance action channel

$$W_d(p) = \frac{x(p)}{x_d(p)}; \quad \text{at } x_{ref}(p) = 0.$$

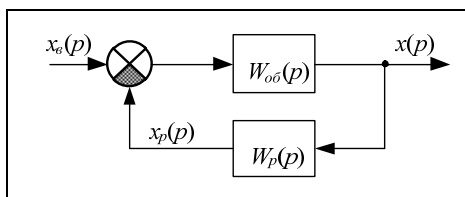
For analysis use the transfer function for open loop system. To determine the transfer function break the system at "a" point(Figure 8) let  $x_d(p) = 0$ . Then

$$W_{pa3}(p) = \frac{x(p)}{x_{ref}(p)}.$$

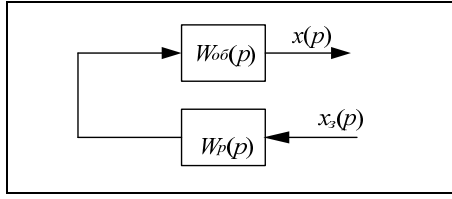
Based on block diagrams reduction we obtain



$$W_{ref}(p) = \frac{W_o(p) \cdot W_c(p)}{1 + W_o(p) \cdot W_c(p)}; \quad (30)$$



$$W_d(p) = \frac{W_o(p)}{1 + W_o(p) \cdot W_c(p)}; \quad (31)$$



$$W_{pa3}(p) = W_o(p) \cdot W_c(p). \quad (32)$$

According to superposition principle

$$x(p) = W_3(p) \cdot x_3(p) + W_e(p) \cdot x_e(p). \quad (33)$$

Analyze the influence of controller transfer function. Consider that transfer function of the object is stable and is a static component, at  $p = 0 \rightarrow W_{o\sigma}(0) = K_{o\sigma}$  (in steady state).

### 1.6.1. Control system behavior in steady state

The relation between controlled variable and perturbation action in steady state can be obtained (33) at  $p = 0$

$$x_\infty = W_3(0) \cdot x_{3\infty} + W_e(0) \cdot x_{e\infty} \quad (34)$$

Let us consider steady state at using controllers of different types

### 1.6.2. Proportional controller (P controller)

The transfer function of controller in this case is given by

$$W_p(p) = K_p. \quad (35)$$

Hence (24) let  $T_\delta = 0$ ;  $T_u \rightarrow \infty$ .

Consider that  $W_{o\sigma}(0) = K_{o\sigma}$  then (33) obtain at  $p = 0$

$$x_\infty = \frac{K_{o\sigma} \cdot K_p}{1 + K_{o\sigma} \cdot K_p} \cdot x_{3\infty} \pm \frac{K_{o\sigma}}{1 + K_{o\sigma} \cdot K_p} \cdot x_{e\infty}. \quad (36)$$

Double sign (36) is used because disturbance in some cases can increase or decrease the controlled variable.

As(36)shows, when P controller is used for the object described by static unit we have a mistake in reproduction of reference input signal which equals

$$\Delta x_3 = (x - x_{3\infty}) = \left( \frac{K_{o\delta} \cdot K_p}{1 + K_{o\delta} \cdot K_p} - 1 \right) \cdot x_{3\infty} . \quad (37)$$

Residual is the difference of the obtained value  $x$  from the desired value  $x_3$  (reference value).

Some influence on the controlled variable of disturbance remains which is equal to

$$\Delta x_6 = \pm \frac{K_{o\delta}}{1 + K_{o\delta} \cdot K_p} \cdot \Delta x_{6\infty} . \quad (38)$$

From (37) and (38) it follows that, the bigger the gain factor is, the more precisely is the reproduction of disturbance and the less is the influence of disturbance.

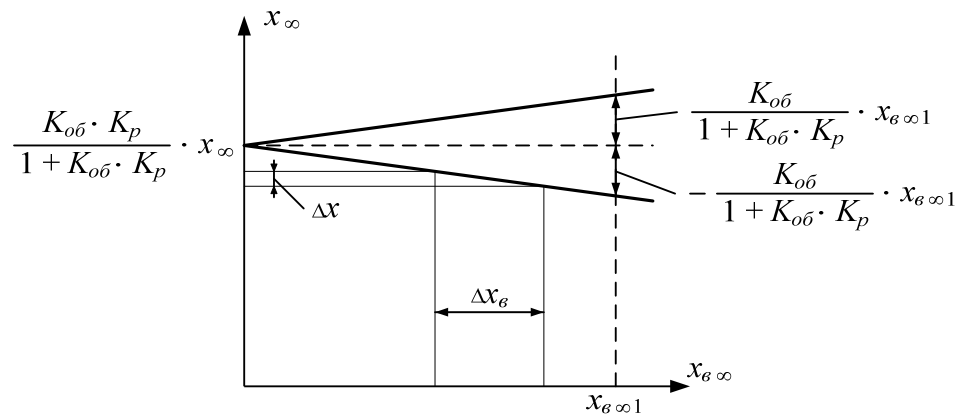
Control systems which possess the after-effect of disturbance are called static systems. Dependence of controlled variable from disturbance action is shown in Figure 10.

In some engineering fields the dependence  $x_\infty(x_{6\infty})$  is called control characteristic, and slope coefficient of this characteristic is called statism coefficient, which is determined as follows(Figure 10)

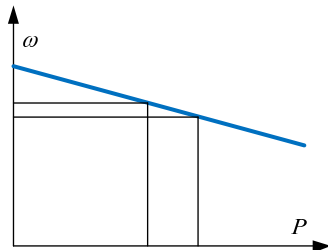
$$K_c = \frac{\Delta x}{\Delta x_6} .$$

By letting  $x$  and  $x_6$  some basic values  $x_0$  and  $x_{60}$  statism coefficient can be determined in relative units

$$K_{c*} = \frac{\Delta x}{x_0} \cdot \frac{x_{60}}{\Delta x_6} . \quad (39)$$



**Figure 13.** Controller output characteristic and external characteristic



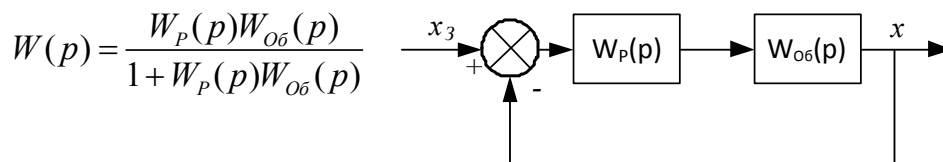
In power supplies there is an analog dependence – static characteristics of turbine controller (controller external characteristics). Such a characteristic help control frequency change when load rise occurs.

$$s = \frac{\Delta \omega}{\Delta P} \frac{P_0}{\omega_0}, \text{ for dependence } \omega(P)$$

It is more often used as  $1/s$  for dependence  $P(\omega)$ .

**Example:** The object with transfer function is given  $W_{o\delta}(p) = \frac{K_{o\delta}}{T_{o\delta} \cdot p + 1}$ ,

$T_{o\delta} = 0,1$   $K_{o\delta} = 0,5$   $x_3 = 1$ . Calculate transfer function for closed loop at P controller presence.



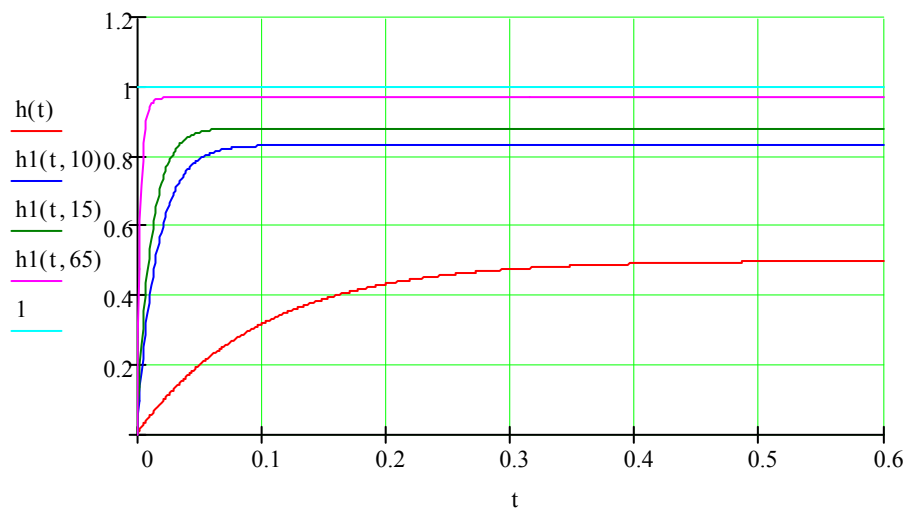
Let's solve the example using **MathCAD**

$$T := 0.1 \quad W(p) := \frac{0.5}{p \cdot T + 1}$$

$$W1(k,p) := \frac{W(p) \cdot k}{1 + W(p) \cdot k} \quad h(t) := \frac{W(p)}{p} \left| \text{invlaplace}, p \rightarrow (-.50000) \cdot e^{(-10.) \cdot t} + .50000 \right.$$

$$h1(t,k) := \frac{W1(k,p)}{p} \left| \text{invlaplace}, p \rightarrow .50000k \cdot \left[ \frac{10.}{10. + 5 \cdot k} - \frac{10.}{10. + 5 \cdot k} \cdot e^{(-1.) \cdot (10. + 5 \cdot k) \cdot t} \right] \right.$$

t := 0, .001.. 6·T



Proportional controller minimizes the static error and decreases transient process time

### 1.6.3. Proportional derivative controller(PD controller)

Controller of this type has transfer function

$$W_c(p) = K_p + T_d \cdot p. \quad (40)$$

As at  $p = 0 \rightarrow W_c(0) = K_p$ , then the qualities of system with PD controller in steady state regime coincide with P controller system.

The component of transfer function which proportional to the derivative term influences much the qualities of control system in transient process.

### 1.6.4. Integral controller ( I controller)

I controller transfer function

$$W_p(p) = \frac{1}{T_u \cdot p}. \quad (41)$$

Transfer functions on reference signal and disturbance in system with I controller

$$W_3(p) = \frac{W_{o\delta}(p) \cdot W_p(p)}{1 + W_{o\delta}(p) \cdot W_p(p)}; \quad W_6(p) = \frac{W_{o\delta}(p)}{1 + W_{o\delta}(p) \cdot W_p(p)}$$

$$W_3(p) = \frac{W_{o\delta}(p)}{T_u \cdot p + W_{o\delta}(p)}, \quad (42)$$

$$W_6(p) = \frac{W_{o\delta}(p) \cdot T_u \cdot p}{T_u \cdot p + W_{o\delta}(p)}. \quad (43)$$

At  $p = 0$   $W_3(0) = 1$ ;  $W_6(0) = 0$ . Hence at given equivalent block diagram the application of integral controller provides the precise reference signal reproducing. It also eliminates the influence of perturbation action of controlled variable.

***Automatic system control at which the regulated component does not depend on perturbation action is called astatic.***

#### 1.6.5. Proportional integral controller (PI controller)

Controller transfer function is given as

$$W_p(p) = K_p + \frac{1}{T_u \cdot p} \quad (44)$$

Transfer functions by reference and disturbance signals

$$W_3(p) = \frac{W_{o\delta}(p) \cdot (K_p \cdot T_u \cdot p + 1)}{T_u \cdot p + W_{o\delta}(p) \cdot (K_p \cdot T_u \cdot p + 1)}. \quad (45)$$

$$W_6(p) = \frac{W_{o\delta}(p) \cdot T_u \cdot p}{T_u \cdot p + W_{o\delta}(p) \cdot (K_p \cdot T_u \cdot p + 1)}. \quad (46)$$

The same as in system with I controller at  $p = 0$   $W_3(0) = 1$ ;  $W_6(0) = 0$ .

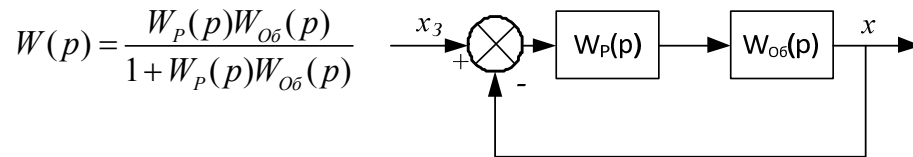
Thus, adding to proportional part of integral reference also provides astatism of control system.

**Example:** The object with transfer function  $W_{o\delta}(p) = \frac{K_{o\delta}}{T_{o\delta} \cdot p + 1}$  is given,

$T_{o\delta} = 0,1$   $K_{o\delta} = 1$ ,  $x_3 = 1$ . Find the transfer function for closed loop at presence

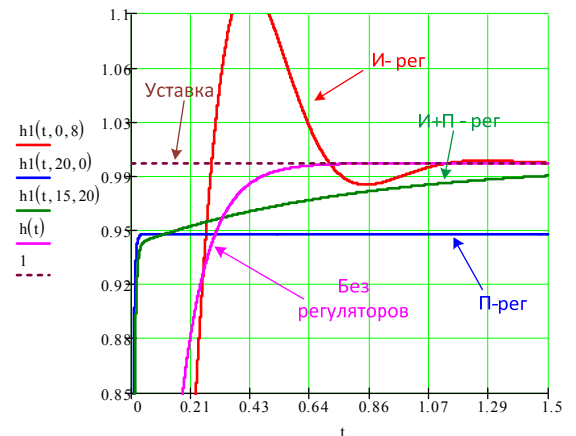
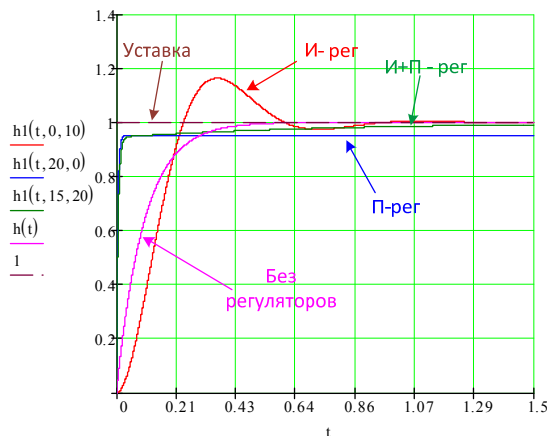
**P** ( $W_p(p) = k_n$ ), **I** ( $W_p(p) = k_u / p$ ) and **PI** ( $W_p(p) = k_n + k_u / p$ ) controllers.

Solution:



$$\text{P} \rightarrow W(p) = \frac{K_{o\delta} k_n}{T_{o\delta} p + (1 + K_{o\delta} k_n)}, \quad \text{P} \rightarrow W(p) = \frac{K_{o\delta} k_u}{T_{o\delta} p^2 + p + K_{o\delta} k_u}$$

$$\text{PI} \rightarrow W(p) = \frac{K_{o\delta} (k_n p + k_u)}{T_{o\delta} p^2 + p(1 + k_n K_{o\delta}) + K_{o\delta} k_u}$$



Увеличенный фрагмент начало процесса

### 1.6.6. Proportional integral derivative controller (PID controller)

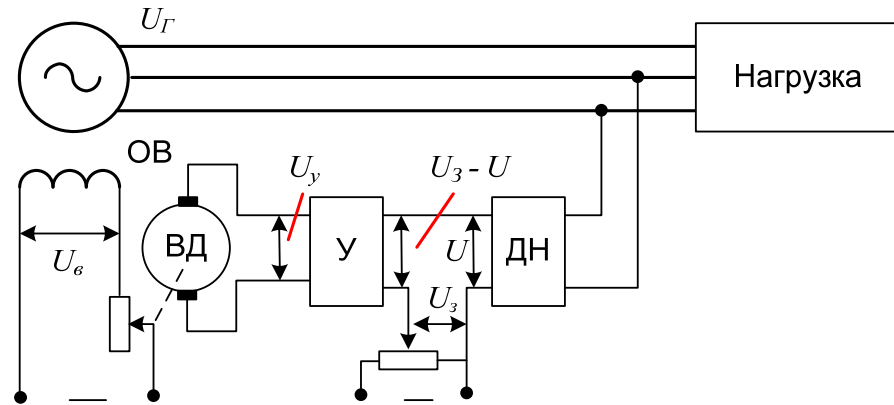
The transfer function is given as

$$W_p(p) = K_p + T_o \cdot p + \frac{1}{T_u \cdot p}. \quad (47)$$



It is evident that in steady state regime ( $p=0$ ) system with PID controller possesses the same qualities as the system with PI controller.

**Example:** Let's analyze voltage control circuit of synchronous generator.

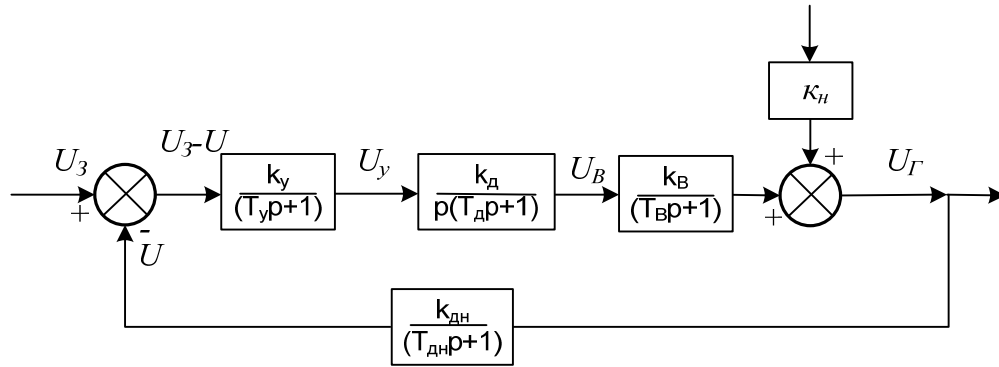


Generator excitation current changes at rheostat resistance change. Rheostat engine shift is performed by internal combustion engine. Engine is an integrated component, because turn of its shaft angle  $\delta = \int \omega dt$  is proportional to rotation speed integral  $\omega$ . Rotation speed  $\omega$  if proportional to voltage given to armature  $U_y$ . When mismatch between  $U_3$  and  $U$  an amplifier input signal occurs, which is transmitted to excitation winding through amplifier and integrated component. Excitation winding amplifies or reduces magnetic field till the mismatch disappears  $\Delta U = U_3 - U$

### **Block diagram**

$T_y, T_d, T_B, T_{дн}$  are amplifier, engine, excitation loop or divisor time constants

o,  $k_H$  is load coefficient.



## 1.7. Transient process calculation

### 1.7.1. Transient processes calculation in state-space

State space is one of the most effective methods of transient processes calculation. The method essence is to transform  $n$  order differential equation into the system of first order  $n$  differential equations written in Cauchy form. For example for RLC series circuit equations in Cauchy form can be written as:

$$\begin{aligned}\frac{di_L}{dt} &= -\frac{R}{L}i_L - \frac{1}{L}u_C + \frac{e(t)}{L}; \\ \frac{du_C}{dt} &= \frac{i_L}{C}\end{aligned}$$

If state vector is considered as  $\mathbf{x}(t) = \{i_L(t), u_C(t)\}$  and write out the coefficients at unknown variables and given influences in matrix form then the equations can be written as:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{V}(t) \quad \text{или} \quad \mathbf{D}(\mathbf{x}, t) = \mathbf{A} \cdot \mathbf{x} + \mathbf{F}(t),$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) - \text{coupling equation}$$

where  $\mathbf{A}$  is square state matrix of  $n \times n$  dimension;

$\mathbf{V}(t)$  is vector  $m$  of given influences;

$\mathbf{B}$  is matrix of  $n \times m$  dimension, elements of which are determined by circuit parameters and its structure.

For series  $RLC$ -circuit matrices are written down:

$$\mathbf{x}(t) = \begin{pmatrix} i_L(t) \\ u_C(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix}.$$

The most complicated thing in state space method is  $\mathbf{A}$  and  $\mathbf{B}$  state matrices formation. There exist several effective methods of state space equations formation, but they all are based on Kirchhoff's equation system with elimination of variables which are not state space variables. MathCAD has functions which help solve equations in symbolic form and this simplifies state space equations and respective matrices formation. Below we analyze the transient process calculation by using state space approach in MathCAD.

**Example 44.** Determine inductance current  $i_L(t)$  and voltage on circuit elements (Figure 34) for the given circuit:

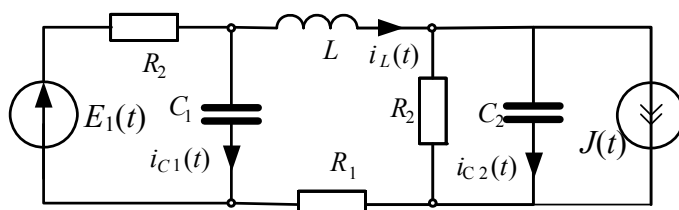


Рис. 34

$$u_{C1}(t), u_{C2}(t), u_{R2}(t), u_{R1}(t), \\ u_{R2}(t), u_L(t)$$

$$e(t) = E = 20 \text{ V}, J = 1 \text{ A}$$

$$\text{если } R_1 = 20 \, \Omega, \quad R_2 = 100 \, \Omega, \\ C_1 = 100 \, \mu\text{F}, \quad C_2 = 50 \, \mu\text{F}, \\ L = 0,1 \text{ H}.$$

**Solution.** To form state space equations let's write down circuit equations according to Kirchhoff's laws, then use computing unit **Given Find**. Let's perform differential equation integration numerically, using rkfixed function as shown in MathCAD file below.

$$i_L = C_2 \cdot U'_{C2} + \frac{U_{C2}}{R_2} + J$$

$$i_L \cdot R_1 + L \cdot i'_L - U_{C1} + U_{C2} = 0$$

Given

Circuit equation written according to Kirchhoff's laws

$$E = (i_L + C_1 \cdot U'_{C1}) \cdot R_2 + U_{C1}$$

$$A(U_{C1}, U_{C2}, i_L, E, J) := \text{Find}(U'_{C1}, U'_{C2}, i'_L) \rightarrow \begin{bmatrix} \frac{E - R_2 \cdot i_L - U_{C1}}{R_2 \cdot C_1} \\ \frac{-[(-R_2) \cdot i_L + U_{C2} + J \cdot R_2]}{C_2 \cdot R_2} \\ \frac{-(i_L \cdot R_1 - U_{C1} + U_{C2})}{L} \end{bmatrix}$$

Matrices formation

$$A := \text{augment}(\mathbf{A}(1, 0, 0, 0, 0), A(0, 1, 0, 0, 0), A(0, 0, 1, 0, 0)) \quad B := \mathbf{A}(0, 0, 0, E, J)$$

$$A \rightarrow \begin{pmatrix} \frac{-1}{R_2 \cdot C_1} & 0 & \frac{-1}{C_1} \\ 0 & \frac{-1}{C_2 \cdot R_2} & \frac{1}{C_2} \\ \frac{1}{L} & \frac{-1}{L} & \frac{-R_1}{L} \end{pmatrix} \quad B \rightarrow \begin{pmatrix} \frac{E}{R_2 \cdot C_1} \\ \frac{-J}{C_2} \\ 0 \end{pmatrix}$$

$$\underline{R_1} := 20 \quad \underline{R_2} := 100 \quad \underline{C_1} := 100 \cdot 10^{-6} \quad \underline{C_2} := 50 \cdot 10^{-6} \quad \underline{L} := 0.1 \quad \underline{E} := 20 \quad \underline{J} := 1$$

$$A = \begin{pmatrix} -100 & 0 & -1 \times 10^4 \\ 0 & -200 & 2 \times 10^4 \\ 10 & -10 & -200 \end{pmatrix} \quad B = \begin{pmatrix} 2000 \\ -20000 \\ 0 \end{pmatrix}$$

State space matrix eigenvalues determination which should equal roots of characteristic equations

$$\lambda := \text{eigenvals}(A) \quad \lambda = \begin{pmatrix} -132.84 \\ -183.58 + 545.458i \\ -183.58 - 545.458i \end{pmatrix}$$

$$Z(p) := \frac{1}{\left(\frac{1}{R_2} + C_1 \cdot p\right)} + R_1 + L \cdot p + \frac{1}{\left(\frac{1}{R_2} + C_2 \cdot p\right)}$$

$$p := Z(p) \left| \begin{array}{l} \text{solve, } p \\ \text{float, } 5 \end{array} \right. \rightarrow \begin{bmatrix} (-183.58) - 545.46i \\ (-183.58) + 545.46i \\ -132.84 \end{bmatrix}$$

Roots of characteristic equation and eigenvalues coincide, that means **A matrix** is formed correctly. To check **B** matrix let's determine forced components of state variables which should satisfy the equality  $-\mathbf{A}^{-1} \mathbf{B}$ .

$$-\mathbf{A}^{-1} \cdot \mathbf{B} = \begin{pmatrix} -34.545 \\ -45.455 \\ 0.545 \end{pmatrix} \quad U_{C1np} := - \left( \frac{E \cdot R_2}{R_2 + R_2 + R_1} + \frac{J \cdot R_2}{R_2 + R_2 + R_1} \cdot R_2 \right) + E$$

$$U_{C1np} = -34.545$$

$$U_{C2np} := \frac{E}{R_2 + R_2 + R_1} \cdot R_2 - \frac{J \cdot (R_2 + R_1)}{R_2 + R_2 + R_1} \cdot R_2 \quad U_{C2np} = -45.455$$

$$i_{Lnp} := \frac{E}{R_2 + R_2 + R_1} + \frac{J \cdot R_2}{R_2 + R_2 + R_1} \quad i_{Lnp} = 0.545$$

To determine additional values form the connection matrix C:

$$U_{R2} = E - U_{C1}$$

$$U_{R1} = R_1 \cdot i_L$$

$$U_L = U_{C2} - U_{C1} - R_1 \cdot i_L$$

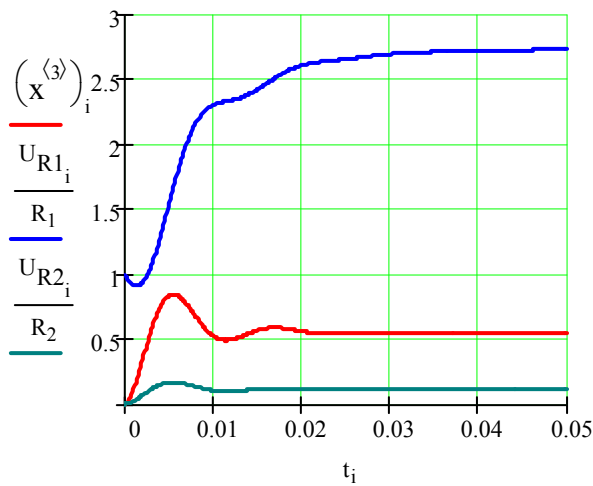
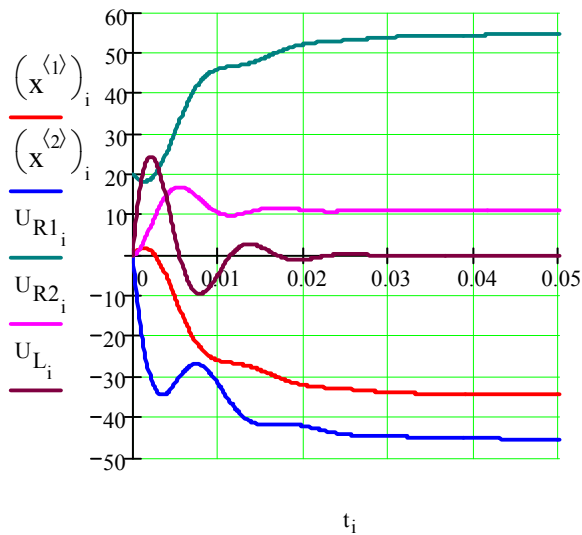
$$\mathbf{C} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & R_1 \\ 1 & -1 & -R_1 \end{pmatrix}$$

Solve differential equation system by Rounge-Kutta method:

$$\mathbf{D}(t, \mathbf{x}) := \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \quad \mathbf{x}_0 := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \underline{N} := 400 \quad \underline{T} := 0.05 \quad i := 0..N \quad \mathbf{x} := \text{rkfixed}(\mathbf{x}_0, 0, T, N, \mathbf{D}) \quad t := \mathbf{x}^{\langle 0 \rangle}$$

$$\begin{pmatrix} U_{R1_i} \\ U_{R2_i} \\ U_{L_i} \end{pmatrix} := \mathbf{C} \cdot \begin{bmatrix} \left( \mathbf{x}^{\langle 1 \rangle} \right)_i \\ \left( \mathbf{x}^{\langle 2 \rangle} \right)_i \\ \left( \mathbf{x}^{\langle 3 \rangle} \right)_i \end{bmatrix} + \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}$$

$$t := \mathbf{x}^{\langle 0 \rangle} \quad i := 0..N$$



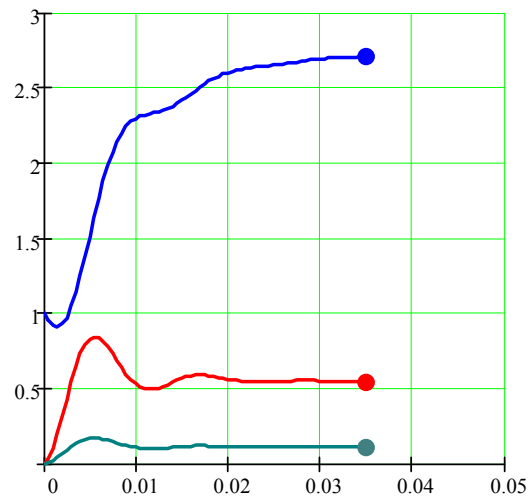
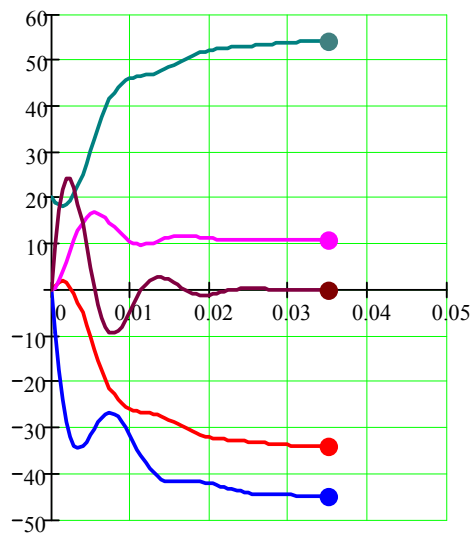
To determine state space components derivatives the extended state space matrix should be multiplied by diagonal matrix of capacitances and inductance

$$\begin{pmatrix} iC1_i \\ iC2_i \\ U_{L_i} \end{pmatrix} := \begin{pmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L \end{pmatrix} D_{t_i} \begin{bmatrix} \begin{pmatrix} x^{<1>} \end{pmatrix}_i \\ \begin{pmatrix} x^{<2>} \end{pmatrix}_i \\ \begin{pmatrix} x^{<3>} \end{pmatrix}_i \end{bmatrix}$$

To animate the transient process it is enough to substitute the number of iterations on time interval N for **FRAME** variable. The in diagram editing window in позиции **Traces** activate **Hide arguments** to hide arguments at axis coordinates. If you wish you can end diagram by black point. Keep in mind that by animating arguments margins at diagram axis should be fixed.

Diagram fragment for the 70<sup>th</sup> frame at **FRAME=70** and closed arguments.

k := FRAME i := 0..k



Below we give the example of transient process calculation in **MathCAD**, **Electronics Workbench** и **MATLAB** for dynamic system described by equation:

$$\ddot{x} + \dot{x} + x = 1, \quad x(0) = 0, \quad \dot{x}(0) = 0$$

### MathCAD

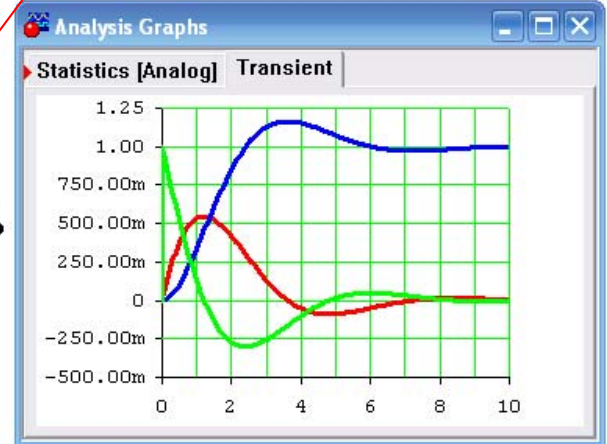
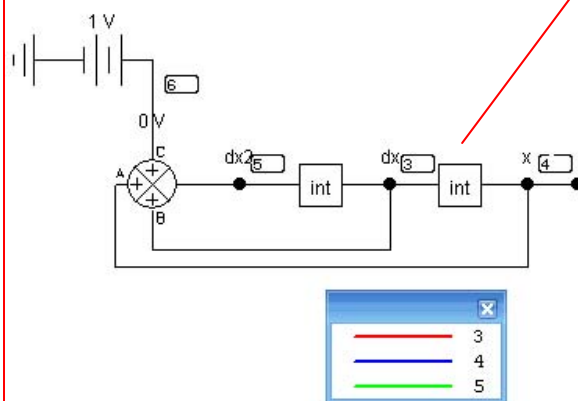
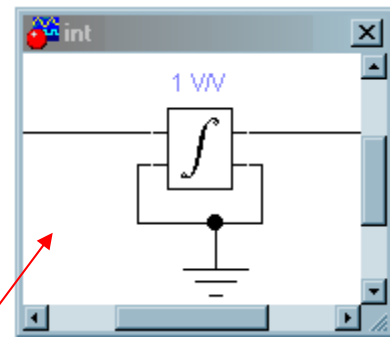
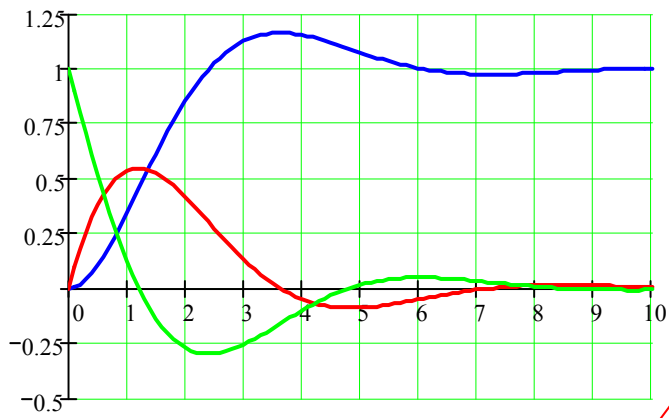
$$\begin{cases} \frac{d}{dt}x_1 = x_2 \\ \frac{d}{dt}x_2 = -x_1 - x_2 + 1 \end{cases} \quad A := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad B := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad -(A^{-1} \cdot B) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \lambda := \text{eigenvals}(A)$$

$$\lambda = \begin{pmatrix} -0.5 + 0.866i \\ -0.5 - 0.866i \end{pmatrix} \quad D(t, x) := A \cdot x + B \quad N := 100 \quad i := 0..N \quad x := \text{rkfixed}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0, 10, N, D\right]$$

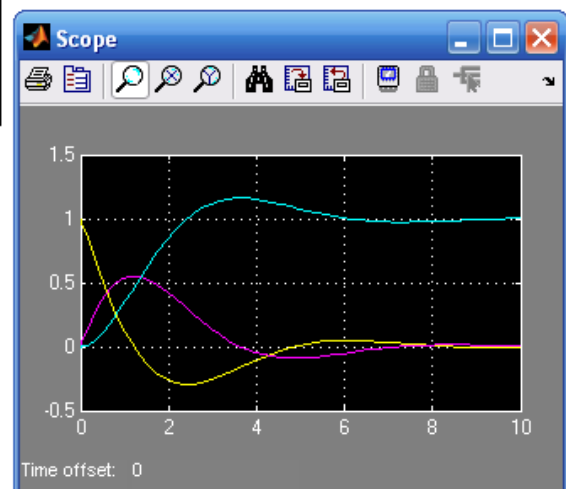
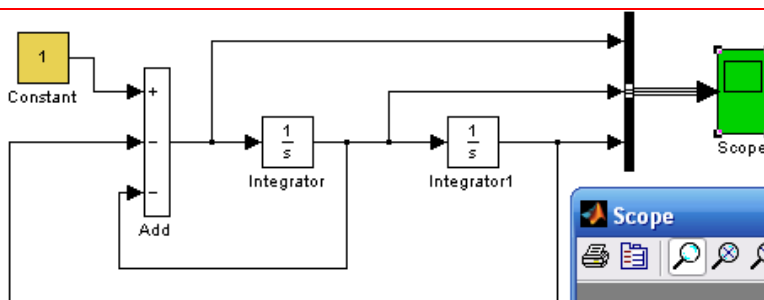
$$\begin{pmatrix} d2x_1 \\ d3x_1 \end{pmatrix} := D \left[ \begin{pmatrix} x^{(0)} \\ x^{(1)} \end{pmatrix}_i, \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix}_i \right]$$

### Electronics Workbench





## MATLAB



### 1.7.2. Convergence of matrix differential equation to scalar equations

It is evident that solution of first order differential equation with one variable more simple than several variables equation system. Hence it is interesting to calculate transient processes by substituting space state matrix by scalar equations. To change matrix equation into scalar equations system keep in mind that state matrix  $\mathbf{A}$  has eigenvectors  $\mathbf{\Lambda}$  and eigennumbers  $\lambda$  connected to each other by  $\mathbf{A}\mathbf{\Lambda} = \lambda\mathbf{\Lambda}$  correlation.

In matrix state space  $\frac{d\mathbf{x}(t)}{dt} = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{F}$  substitute the variables changing  $\mathbf{x} = \mathbf{\Lambda}\mathbf{y}$ . As a result we obtain:

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \mathbf{A} \cdot \mathbf{x} + \mathbf{F} \rightarrow \mathbf{\Lambda} \frac{d\mathbf{y}}{dt} = \mathbf{A} \cdot \mathbf{\Lambda} \cdot \mathbf{y} + \mathbf{F} \rightarrow \\ &\rightarrow \frac{d\mathbf{y}}{dt} = \mathbf{\Lambda}^{-1} \cdot \mathbf{A} \cdot \mathbf{\Lambda} \cdot \mathbf{y} + \mathbf{\Lambda}^{-1}\mathbf{F}.\end{aligned}$$

For eigenvectors the equality is true.

$$\mathbf{Q} = \mathbf{\Lambda}^{-1} \cdot \mathbf{A} \cdot \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{pmatrix},$$

That is why state space equation is written in the form of independent scalar first order equation  $\rightarrow \frac{d\mathbf{y}}{dt} = \boldsymbol{\lambda} \cdot \mathbf{y} + \mathbf{\Lambda}^{-1}\mathbf{F}$ .

As a result we obtain the system of independent differential equations

$$\frac{dy}{dt} = \lambda \cdot y + \Lambda^{-1} F \rightarrow \begin{cases} \frac{dy_1}{dt} = \lambda_1 y_1 + b_1 \\ \frac{dy_2}{dt} = \lambda_2 y_2 + b_2 \\ \dots\dots\dots \\ \frac{dy_n}{dt} = \lambda_n y_n + b_n \end{cases}.$$

At zero initial conditions these equations are solved as follows:

$$y_1(t) = -\frac{b_1}{\lambda_1}(1 - e^{\lambda_1 t}), \quad y_2(t) = -\frac{b_2}{\lambda_2}(1 - e^{\lambda_2 t}), \dots, \quad y_n(t) = -\frac{b_n}{\lambda_n}(1 - e^{\lambda_n t}).$$

After scalar equations one can go on to the initial variables

$$x = \Lambda y \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \Lambda \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_3 \end{pmatrix}.$$

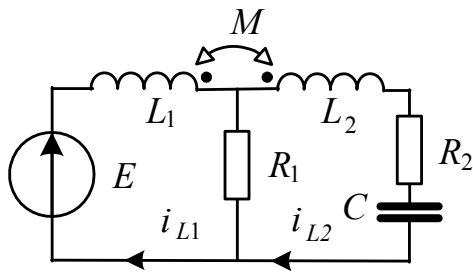


Рис. 38

**Example 47.** Let's analyze the diagram after switching with zero initial conditions

shown in Figure 38. Determine inductance currents and voltage on capacitance for the following numerical data:

$$E = 100 \text{ V}; \quad C = 40 \text{ mcF}; \quad R_1 = 70 \Omega;$$

$$R_2 = 50 \Omega; \quad L_1 = 0,2 \text{ H}; \quad L_2 = 0,1 \text{ H}; \quad M = 0,05 \text{ H}.$$

**Solution.** Write down equation system for instant current values and voltages according to Kirchoff's laws taking into account the mutual inductance:

$$L_1 \frac{d}{dt} i_{L1}(t) - M \frac{d}{dt} i_{L2}(t) + i_{L1}(t) \cdot R_1 - i_{L2}(t) \cdot R_1 = E;$$

$$L_2 \frac{d}{dt} i_{L2}(t) - M \frac{d}{dt} i_{L1}(t) + i_{L2}(t) \cdot R_2 + u_C(t) + (i_{L2}(t) - i_{L1}(t)) \cdot R_1 = E;$$

$$C \frac{d}{dt} u_C(t) = i_{L2}(t).$$

Using computing unit **Given Find**, find state space matrix **A** and independent influence **F**.

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & \frac{1}{C} \\ \frac{M}{-L_1 \cdot L_2 + M^2} & \frac{R_1 \cdot L_2 - R_1 \cdot M}{-L_1 \cdot L_2 + M^2} & \frac{-R_1 \cdot L_2 + R_2 \cdot M + R_1 \cdot M}{-L_1 \cdot L_2 + M^2} \\ \frac{L_1}{-L_1 \cdot L_2 + M^2} & \frac{-R_1 \cdot L_1 + R_1 \cdot M}{-L_1 \cdot L_2 + M^2} & \frac{R_2 \cdot L_1 + R_1 \cdot L_1 - R_1 \cdot M}{-L_1 \cdot L_2 + M^2} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 \\ \frac{-E \cdot L_2}{-L_1 \cdot L_2 + M^2} \\ \frac{-E \cdot M}{-L_1 \cdot L_2 + M^2} \end{pmatrix}$$

With the help of **eigenvals(A)** function calculate the  $\lambda$  eigenvalues of **A** state space matrix. Matrix **A** eigennumbers coincide with characteristic equation roots, that is why to check the matrix one should make up the formula for input equivalent circuit resistance (figure 39):

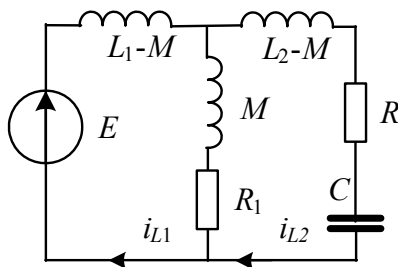


Рис. 39

$$Z(p) = \frac{(L_1 - M)p(R_1 + Mp)}{(L_1 - M)p + R_1 + Mp} + (L_2 - M)p + R_2 + \frac{1}{Cp} = 0$$

Further procedures and numerical calculations are given in **MathCAD**.

$$R_1 := 70 \quad R_2 := 50 \quad L_1 := 0.2 \quad L_2 := 0.1 \quad M := 0.05 \quad \underline{\underline{C}} := 40 \cdot 10^{-5} \\ E := 100$$

State space matrices and independent variables

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 10 \\ -2.857 & -200 & 57.15 \\ -11.43 & 600 & -1.171 \times 10^3 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 0 \\ 571.5 \\ 285.7 \end{pmatrix}$$

### Matrix A eigenvalues

$$\lambda := \text{eigenvals}(A) \quad \lambda = \begin{pmatrix} -79.294 \\ -106.354 \\ -1.186 \times 10^3 \end{pmatrix}$$

### Check A matrix

$$z(p) := \frac{(L_1 - M) \cdot p \cdot (R_1 + M \cdot p)}{(L_1 - M) \cdot p + R_1 + M \cdot p} + (L_2 - M) \cdot p + R_2 + \frac{1}{C \cdot p} \quad z(p) \left| \begin{array}{l} \text{solve, p} \\ \text{float, 4} \end{array} \right. \rightarrow \begin{pmatrix} -1186. \\ -106.4 \\ -79.29 \end{pmatrix}$$

### Matrix A eigenvectors

$$S := \text{eigenvecs}(A) \quad S = \begin{pmatrix} 0.9988 & 0.9975 & 0.9032 \\ -0.0386 & -0.0563 & 0.0275 \\ -0.0317 & -0.0424 & -0.4284 \end{pmatrix}$$

$$Q := S^{-1} \cdot A \cdot S \text{ float, 4} \rightarrow \begin{pmatrix} -79.29 & -.53\text{e-}15 & .578\text{e-}15 \\ .84\text{e-}17 & -106.4 & -.96\text{e-}16 \\ -.1\text{e-}16 & -.1\text{e-}15 & -1186. \end{pmatrix}$$

Find a new matrix of  $G \Rightarrow S^{-1}F$  sources for  $y$  variable:

$$\underline{G} := S^{-1} \cdot F \quad G = \begin{pmatrix} -141.978 \\ 3.155 \times 10^4 \\ -3.172 \times 10^4 \end{pmatrix} \quad \frac{d}{dt}y = Q \cdot y + G \quad \begin{pmatrix} u_C \\ i_{L1} \\ i_{L2} \end{pmatrix} = S \cdot y$$

Solution for a new  $y$  variable

$$y_1(t) := \frac{-G_0}{\lambda_0} (1 - e^{\lambda_0 \cdot t}) \quad y_2(t) := \frac{(-G)_1}{\lambda_1} (1 - e^{\lambda_1 \cdot t}) \quad y_3(t) := \frac{(-G)_2}{\lambda_2} (1 - e^{\lambda_2 \cdot t})$$

$$y_1(t) \text{ float, 4} \rightarrow 43.44 - 43.44 \cdot e^{(-79.29) \cdot t}$$

$$y_2(t) \text{ float, 4} \rightarrow (-5.066) + 5.066 \cdot e^{(-106.4) \cdot t}$$

$$y_3(t) \text{ float,4} \rightarrow (-.5006e-1) + .5006e-1 \cdot e^{(-1186.) \cdot t}$$

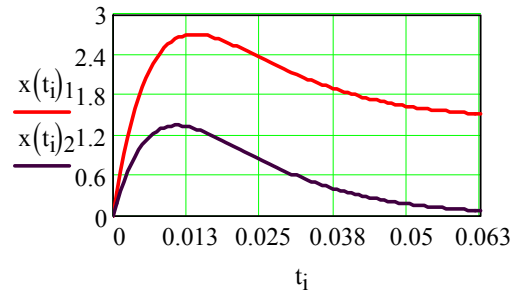
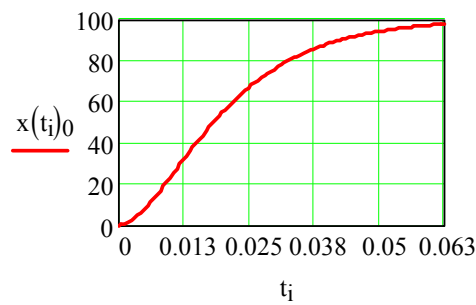
Get back to state space variables:

$$x(t) := S \cdot \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

$$x(t)_0 \text{ float,4} \rightarrow 100.0 - 397.4 \cdot e^{(-79.29) \cdot t} + 297.5 \cdot e^{(-106.4) \cdot t} - .1081 \cdot e^{(-1186.) \cdot t}$$

$$x(t)_1 \text{ float,4} \rightarrow 1.429 + 15.37 \cdot e^{(-79.29) \cdot t} - 16.80 \cdot e^{(-106.4) \cdot t} - .3287e-2 \cdot e^{(-1186.) \cdot t}$$

$$x(t)_2 \text{ float,4} \rightarrow .893e-18 + 12.60 \cdot e^{(-79.29) \cdot t} - 12.66 \cdot e^{(-106.4) \cdot t} + .5129e-1 \cdot e^{(-1186.) \cdot t}$$



$$\tau := \frac{1}{\min(|\lambda|)}$$

$$\tau = 0.013$$

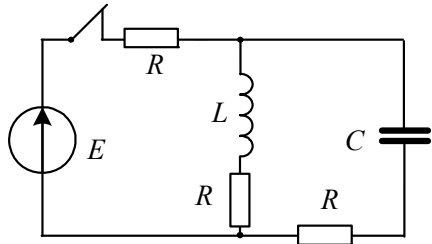
$$T := 5 \cdot \tau$$

$$T = 0.063$$

$$N := 100$$

$$i := 0..N \quad t_i := \frac{T}{N} \cdot i$$

**Example 48.** Determine inductance current  $i_L(t)$  and voltage on capacity  $u_C(t)$  when breaking the circuit at Figure 40 with given elements parameters :  $E=50$  V;  $R=10 \Omega$ ;  $L=0,05$  H;  $C=150$  mcF.



**Figure 40.**

**Solution.** Write down the equations for diagram after commutation:

$$u_L = -2Ri_L - u_C; \quad i_C = Cdu_C / dt.$$

It follows:

$$\frac{di_L(t)}{dt} = \frac{1}{L}(-2Ri_L - u_C); \quad \frac{du_C(t)}{dt} = \frac{1}{C}i_L;$$

$$\mathbf{A} = \begin{pmatrix} -\frac{2R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix}; \quad \mathbf{F} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Initial conditions:  $i_L(0) = \frac{E}{2R} = 2,5 \text{ A}$ ,  $u_C(0) = Ri_L(0) = 25 \text{ B}$ .

Further calculations are performed in MathCAD.

Given data and initial conditions

$$\underline{L} := 0.05 \quad \underline{C} := 150 \cdot 10^{-6} \quad \underline{R} := 10 \quad E := 50 \quad i_{L0} := \frac{E}{2 \cdot R} \quad u_{C0} := \frac{E}{2}$$

$$\underline{A} := \begin{pmatrix} -\frac{2 \cdot R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \quad A = \begin{pmatrix} -400 & -20 \\ 6.667 \times 10^3 & 0 \end{pmatrix} \text{ - state space matrix}$$

Determine eigenvalues and eigenvectors:

$$\lambda := \text{eigenvals}(A) \quad \lambda = \begin{pmatrix} -200 + 305.505i \\ -200 - 305.505i \end{pmatrix}$$

$$\Lambda := \text{eigenvecs}(A) \quad \Lambda = \begin{pmatrix} -0.03 + 0.046i & -0.03 - 0.046i \\ 0.999 & 0.999 \end{pmatrix}$$

Substitute the variables and obtain the scalar equation:

$$\mathbf{x} = \Lambda \cdot \mathbf{y} \rightarrow \mathbf{y} = \Lambda^{-1} \cdot \mathbf{x}$$

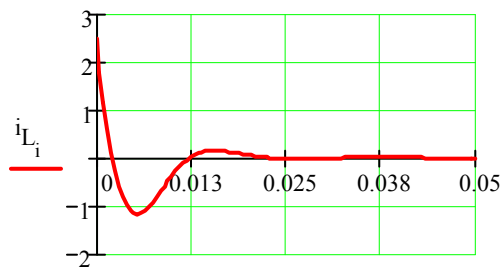
$$\frac{d\mathbf{y}}{dt} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} := \Lambda^{-1} \cdot \begin{pmatrix} i_{L0} \\ u_{C0} \end{pmatrix} \quad \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} 12.519 - 35.514i \\ 12.519 + 35.514i \end{pmatrix}$$

$$T := 0.05 \quad N := 100 \quad i := 0..N \quad t_i := \frac{T}{N} \cdot i$$

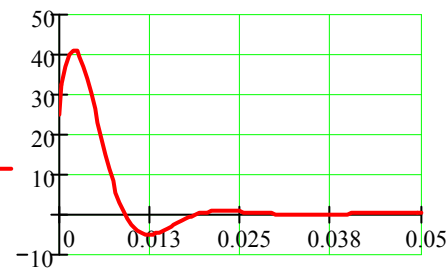
$$y_1(t) := y_{10} \cdot e^{\lambda_0 \cdot t} \quad y_2(t) := y_{20} \cdot e^{\lambda_1 \cdot t} \quad \begin{pmatrix} i_{L_i} \\ u_{C_i} \end{pmatrix} := \Lambda \cdot \begin{pmatrix} y_1(t_i) \\ y_2(t_i) \end{pmatrix}$$

Write down the final calculation and plot diagrams

$$\Lambda \cdot \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \begin{matrix} \text{float, 5} \\ \text{complex} \end{matrix} \rightarrow \begin{bmatrix} 2.5000 \cdot e^{(-200.00) \cdot t} \cdot \cos(305.51 \cdot t) + 3.2732 \cdot e^{(-200.00) \cdot t} \cdot \sin(305.51 \cdot t) \\ 25.000 \cdot e^{(-200.00) \cdot t} \cdot \cos(305.51 \cdot t) - 70.920 \cdot e^{(-200.00) \cdot t} \cdot \sin(305.51 \cdot t) \end{bmatrix}$$



$t_i$



$t_i$

### 1.7.3. Transient processes calculation with Impulse response function help

Let state space equation of arbitrary circuit has the form of

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{e}(t).$$

Laplace Transform will make its solution easier and let move up from differential equations to algebraic. At zero initial conditions we have



$$\mathbf{I}p\mathbf{X}(p) = \mathbf{A} \cdot \mathbf{X}(p) + \mathbf{B} \cdot \mathbf{E}(p) \rightarrow (\mathbf{I}p - \mathbf{A}) \cdot \mathbf{X}(p) = \mathbf{B} \cdot \mathbf{E}(p)$$

$$\mathbf{X}(p) = (\mathbf{I}p - \mathbf{A})^{-1} \mathbf{B} \cdot \mathbf{E}(p),$$

where  $\mathbf{I}$  is a unit matrix.

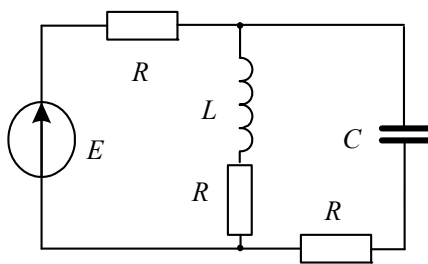
If we take external action as Dirac delta function, than for multidimensional transfer function we obtain:

$$\mathbf{W}(p) = \frac{\mathbf{X}(p)}{\mathbf{E}(p)} = (\mathbf{I}p - \mathbf{A})^{-1} \mathbf{B}.$$

To obtain multidimensional impulse response function  $h_{\delta}(t)$ , one should perform reverse Laplace transform for transfer function  $\mathbf{W}(p)$ . If at circuit input is emf source in the form of unit step(скачок) ( $E(p)=1/p$ ), then after reverse Laplace transform  $\mathbf{W}(p)/p$  we obtain impulse response function  $h(t)$ .

MathCAD help perform character direct and inverse Laplace transform.

**Example 49.** Determine voltage on capacitor and inductance current for diagram on Figure 41 knowing that initial conditions are zero.. Initial conditions:



$$E=30 \text{ V}; R=20 \text{ Ohm}; L=0,1 \text{ H}; C=100 \text{ mcF}.$$

**Solution.** Write Kirchhoff equation and form state space matrix and vector of output influence

$$\begin{cases} \left( C \frac{du_C}{dt} + i_L \right) R + u_C + RC \frac{du_C}{dt} = E \\ RC \frac{du_C}{dt} + u_C = i_L R + L \frac{di_L}{dt} \end{cases} \rightarrow$$

Given

$$(C \cdot u'_C + i_L) \cdot R + u_C + R \cdot C \cdot u'_C = E$$

$$R \cdot C \cdot u'_C + u_C = i_L \cdot R + L \cdot i'_L$$

$$A(i_L, u_C, \mathbf{E}) := \text{Find}(i'_L, u'_C) \rightarrow \begin{pmatrix} \frac{-1}{2} \cdot \frac{3 \cdot i_L \cdot R - u_C - E}{L} \\ \frac{-1}{2} \cdot \frac{i_L \cdot R + u_C - E}{R \cdot C} \end{pmatrix}$$

The impulse response function  $\mathbf{W}(p)$  in MathCAD can be easily obtained as shown below when state matrix and vector of output influence are known.

$$E := 10 \quad L := 0.1 \quad C := 100 \cdot 10^{-6} \quad R := 10$$

$$A := \begin{pmatrix} \frac{-3}{2} \cdot \frac{R}{L} & \frac{1}{2 \cdot L} \\ \frac{-1}{2 \cdot C} & \frac{-1}{2 \cdot R \cdot C} \end{pmatrix} \quad F := \begin{pmatrix} \frac{1}{2 \cdot L} \\ \frac{1}{2} \cdot \frac{1}{R \cdot C} \end{pmatrix} \cdot 1 \quad \text{eigenval}(A) = \begin{pmatrix} -250 \\ -400 \end{pmatrix}$$

$$W(p) := (\text{identity}(2) \cdot p - A)^{-1} \cdot F$$

$$W(p) \text{ float, 5} \rightarrow \begin{pmatrix} 5.0000 \cdot \frac{p + 500.}{p^2 + 650. \cdot p + .10000e6} + \frac{2500.0}{p^2 + 650.00 \cdot p + .10000e6} \\ \frac{-25000.}{p^2 + 650.00 \cdot p + .10000e6} + 500. \cdot \frac{p + 150.}{p^2 + 650. \cdot p + .10000e6} \end{pmatrix}$$

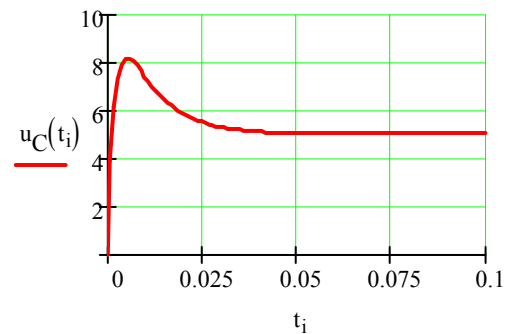
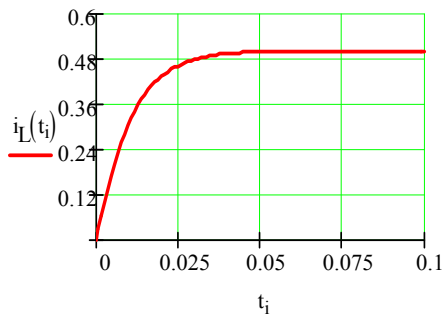
Now we can obtain analytical dependence for transient functions of inductance current  $h_L(t)$  and capacitor voltage  $h_C(t)$ , using inverse Laplace transform

$$h_L(t) := \frac{W(p)_0}{p} \left| \begin{array}{l} \text{invlaplace } p \\ \text{float, 4} \end{array} \right. \rightarrow .5000e-1 \cdot e^{(-400.) \cdot t} - .1000 \cdot e^{(-250.) \cdot t} + .5000e-1$$

$$h_C(t) := \frac{W(p)_1}{p} \left| \begin{array}{l} \text{invlaplace } p \\ \text{float, 4} \end{array} \right. \rightarrow (-2.500) \cdot e^{(-400.) \cdot t} + 2. \cdot e^{(-250.) \cdot t} + .5000$$

To obtain analytical dependence of inductance current  $i_L(t)$  and voltage on capacitor  $u_C(t)$  transient functions should be multiplied by emf  $E$ .

$$i_L(t) := h_L(t) \cdot E \quad u_C(t) := h_C(t) \cdot E \quad \underline{T} := 0.1 \quad \underline{N} := 100 \quad i := 0..N \quad t_i := \frac{T}{N} \cdot i$$



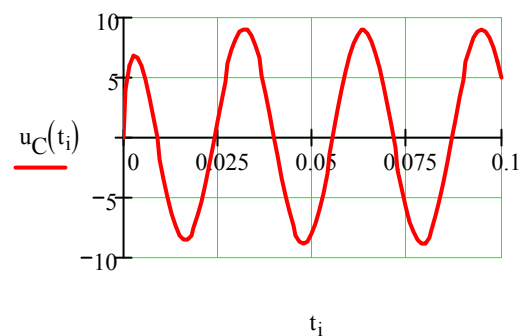
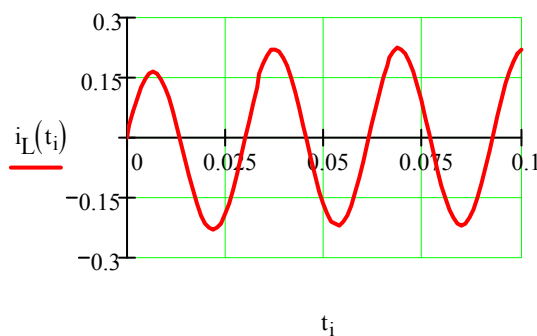
If input influence is time function  $e(t)$ , then to determine inductance current and voltage on capacitor one can apply Duhamel's integral:

$$i_L(t) = h_L(t)E(0) + \int_0^t \frac{de(\tau)}{d\tau} h_L(\tau - t) d\tau; \quad u_C(t) = h_C(t)e(0) + \int_0^t \frac{de(\tau)}{d\tau} h_C(\tau - t) d\tau.$$

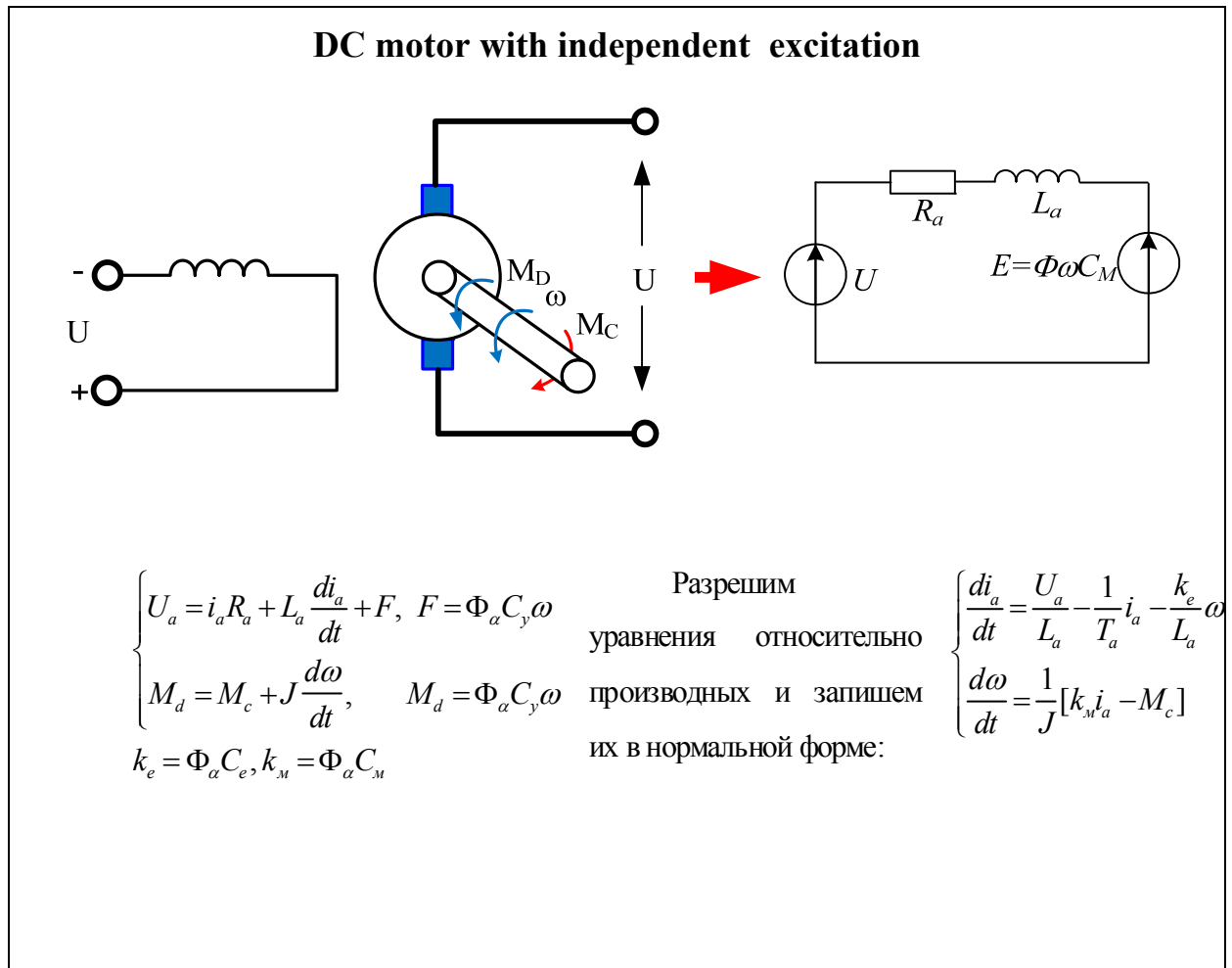
If, for instance, external influence in  $e(t) = 10\cos(200t)$  example, than in MathCAD the following should be written:

$$e(t) := 10 \cdot \cos(200 \cdot t) \quad \underline{i_L}(t) := \int_0^t h_L(t - \tau) \cdot \frac{d}{d\tau} e(\tau) d\tau + h_L(t) \cdot e(0)$$

$$\underline{u_C}(t) := \int_0^t h_C(t - \tau) \cdot \frac{d}{d\tau} e(\tau) d\tau + h_C(t) \cdot e(0)$$



The example below shows differential equation system solution of DC motor with independent excitation in **MathCAD** with state space method application in **Electronics Workbench** with structural simulation application.



Given

$$U_a = i_a \cdot R_a + L_a \cdot i_a' + \omega \cdot k_e$$

$$i_a \cdot k_M = M_c + J \cdot \omega'$$

$$A(i_a, \omega, M_c, U_a) := \text{Find}(i_a', \omega') \rightarrow \begin{bmatrix} \frac{-[(-U_a) + i_a \cdot R_a + \omega \cdot k_e]}{L_a} \\ \frac{i_a \cdot k_M - M_c}{J} \end{bmatrix}$$

$$A := \text{augment}(\mathbf{A}(1,0,0,0), \mathbf{A}(0,1,0,0)) \quad A \rightarrow \begin{pmatrix} \frac{-R_a}{L_a} & \frac{-k_e}{L_a} \\ \frac{k_M}{J} & 0 \end{pmatrix} \quad B := \mathbf{A}(0,0,M_c,0) + \mathbf{A}(0,0,0,U_a)$$

$$B \rightarrow \begin{pmatrix} \frac{U_a}{L_a} \\ -\frac{M_c}{J} \end{pmatrix}$$

### Given parameters

$$P_H := 0.17 \quad U_H := 110 \quad R_{OЯ} := 5.84 \quad L_a := 128 \cdot 10^{-3} \quad \eta := 47.5 \quad n := 750 \quad J := 0.004$$

$$I_H := \frac{P_H \cdot 10^5}{\eta \cdot U_H} \quad R_{доп} := 4.40 \quad R_a := R_{OЯ} + R_{доп} \quad R_a = 10.24 \quad I_H = 3.254 \quad \omega_H := \frac{\pi \cdot n}{30} \quad \omega_H = 78.54$$

$$k_M := \frac{10^3 \cdot P_H}{\omega_H \cdot I_H} \quad k_e := \frac{U_H - I_H \cdot R_a}{\omega_H} \quad k_M = 0.665 \quad k_e = 0.976 \quad T_a := \frac{L_a}{R_a} \quad T_a = 0.013 \quad T_M := \frac{J \cdot R_a}{k_M \cdot k_e}$$

$$T_M = 0.063$$

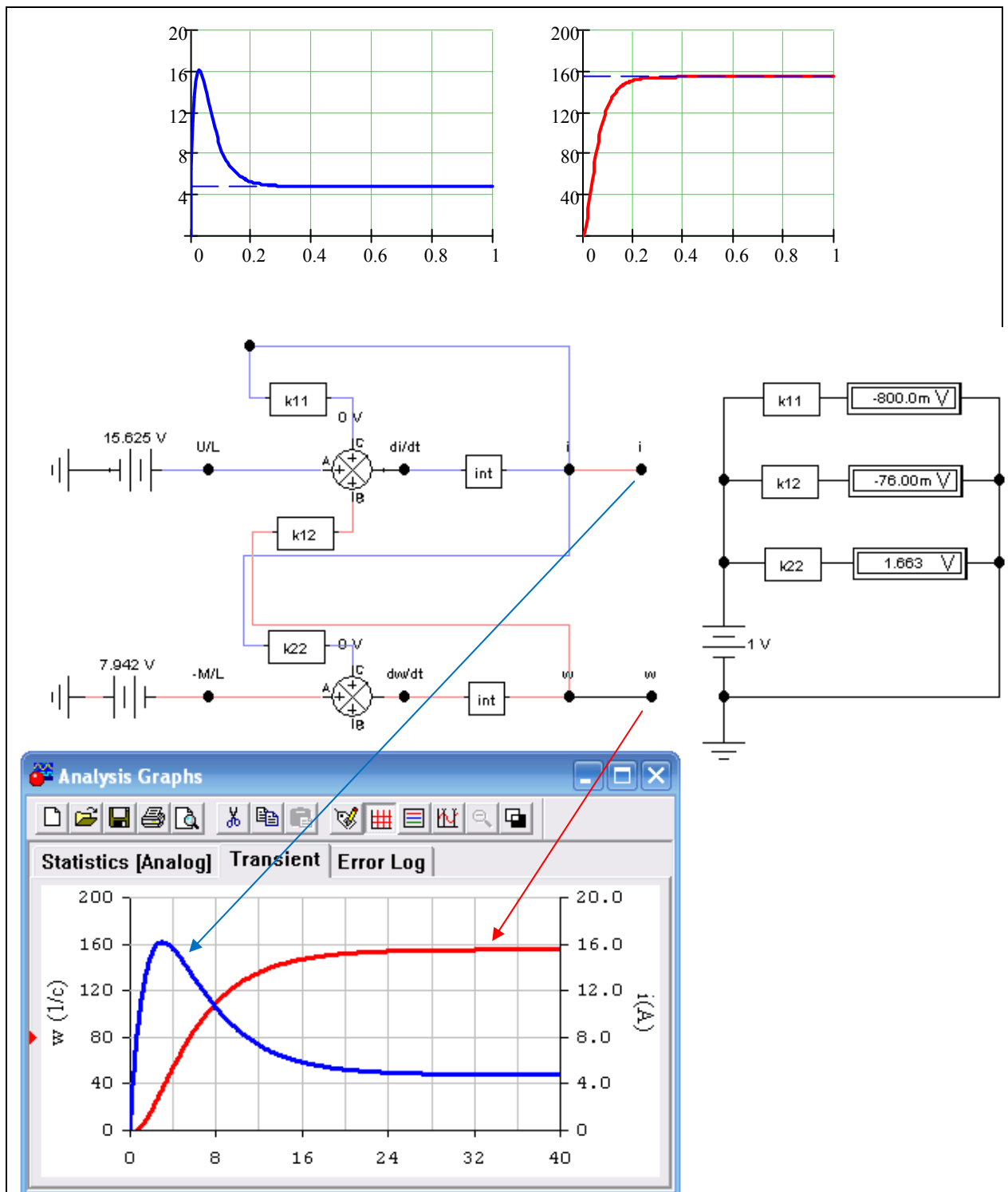
$$\frac{T_M}{T_a} = 5.045 \quad T := 1. \quad U := 200$$

$$M_c(t) := \begin{cases} I_H \cdot k_e & \text{if } 0 \leq t \leq T \cdot 0.4 \\ I_H \cdot k_e \cdot 2 & \text{if } T \cdot 0.4 \leq t \leq 0.8 \cdot T \\ I_H \cdot k_e \cdot 0.5 & \text{otherwise} \end{cases}$$

$$M_c(t) := I_H \cdot k_e$$

$$A := \begin{pmatrix} \frac{-R_a}{L_a} & \frac{-k_e}{L_a} \\ \frac{k_M}{J} & 0 \end{pmatrix} \quad B(t) := \begin{pmatrix} \frac{U}{L_a} \\ -\frac{M_c(t)}{J} \end{pmatrix} \quad \begin{pmatrix} I_{np} \\ \omega_{np} \end{pmatrix} := -A^{-1} \cdot B(T) \quad D(t,x) := A \cdot x + B(t) \quad N := 200$$

$$\begin{pmatrix} I_{np} \\ \omega_{np} \end{pmatrix} = \begin{pmatrix} 4.775 \\ 154.762 \end{pmatrix} \quad B(0) = \begin{pmatrix} 1562.5 \\ -794.16965 \end{pmatrix} \quad x := \text{rkfixed} \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0, T, N, D \right] \quad t := x^{\langle 0 \rangle} \quad i := 0..N$$



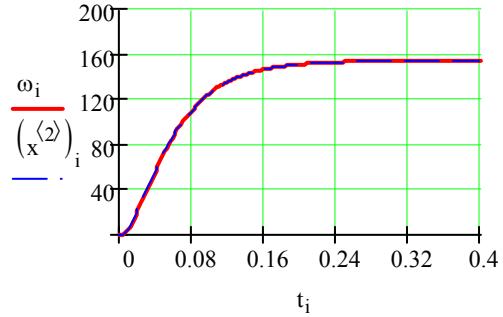
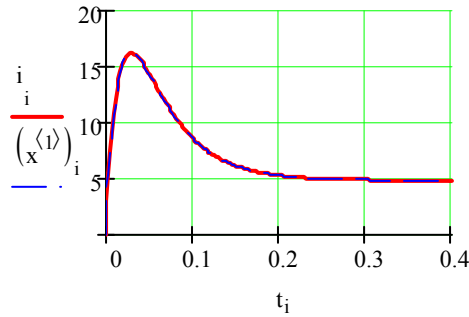
### *Substitution method of matrix equations into scalar equations*

$$\Lambda := \text{eigenvecs}(A) \quad \Lambda = \begin{pmatrix} -0.33 & 0.13 \\ 0.944 & -0.992 \end{pmatrix} \quad \lambda := \text{eigenvals}(A) \quad \lambda = \begin{pmatrix} -58.203 \\ -21.797 \end{pmatrix}$$

$$b := \Lambda^{-1} \cdot B(0) \quad b = \begin{pmatrix} -7.059 \times 10^3 \\ -5.919 \times 10^3 \end{pmatrix} \quad y_1(t) := \frac{b_0}{\lambda_0} \cdot (e^{\lambda_0 \cdot t} - 1) \quad y_2(t) := \frac{b_1}{\lambda_1} \cdot (e^{\lambda_1 \cdot t} - 1)$$

$$y_1(t_1) \text{ float}, 4 \rightarrow 162.0e^{(-21.80) \cdot t_1} - 162.0 \quad y_2(t_1) \text{ float}, 4 \rightarrow 416.7e^{(-58.20) \cdot t_1} - 416.7$$

$$\begin{pmatrix} i \\ i \\ \omega_i \end{pmatrix} := \Lambda \cdot \begin{pmatrix} y_1(t_i) \\ y_2(t_i) \end{pmatrix}$$



#### 1.7.4. Analytical method of transient processes calculation

Let differential equations system for state space vector  $\mathbf{x} = \{x_1, x_2, x_3 \dots x_n\}$  has the form of  $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{B}$ .

The following formula will be the solution of this system:

$$\mathbf{x}(t) = \mathbf{x}(0)e^{\mathbf{A}t} + \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{B}(\tau) d\tau$$

where  $\mathbf{x}(0) = \{x_1(0), x_2(0), x_3(0) \dots x_n(0)\}$  is initial conditions vector;  $e^{\mathbf{A}t}$  is matrix exponent.

Matrix exponent  $e^{\mathbf{A}t}$ , argument of which is square matrix  $\mathbf{A}t$  of  $n$  order, can be presented in the form of final row of  $n$  components  $e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 + \dots + \alpha_{n-1}(t)\mathbf{A}^{n-1}$ ,

where  $\alpha_k$  are row coefficients - time domain.

Coefficients of  $\{\alpha_0, \alpha_1, \alpha_2 \dots \alpha_{n-1}\}$  row can be found by solving,  $n$  system of algebraic equations written as state matrix own numbers **A**:

$$\begin{cases} \alpha_0 + \lambda_1 \alpha_1 + \lambda_1^2 \alpha_2 + \dots + \lambda_1^{n-1} \alpha_{n-1} = e^{\lambda_1 t} \\ \alpha_0 + \lambda_2 \alpha_1 + \lambda_2^2 \alpha_2 + \dots + \lambda_2^{n-1} \alpha_{n-1} = e^{\lambda_2 t} \\ \dots \\ \alpha_0 + \lambda_n \alpha_1 + \lambda_n^2 \alpha_2 + \dots + \lambda_n^{n-1} \alpha_{n-1} = e^{\lambda_n t} \end{cases},$$

or matrix form:

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \dots \\ \alpha_{n-1} \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \dots \\ e^{\lambda_n t} \end{pmatrix}.$$

Equation system solution according the given decomposition coefficients has the following form:

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \dots \\ \alpha_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \dots \\ e^{\lambda_n t} \end{pmatrix}.$$

After substituting the obtained coefficients into matrix exponent decomposition we obtain the required solution.

**Example 50.** Determine capacitance current and voltage on capacitance in circuit at Figure 42, if elements parameters are given:

$$E = 20 \text{ V}; R = 10 \Omega; C = 50 \text{ mcF}; L = 0,15 \text{ H}.$$

**Solution.** Write equation system according to Kirchhoff law and form state space matrix and vector of output influence:

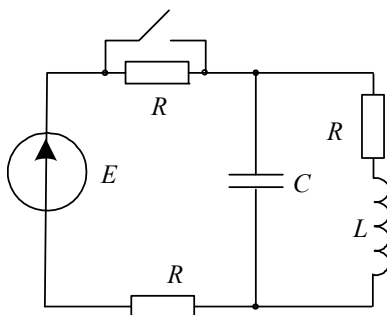


Рис. 42



$$L \frac{di_L}{dt} + i_L R = u_C;$$

$$L \frac{di_L}{dt} + i_L R + \left( i_L + C \frac{du_C}{dt} \right) R = E;$$

$$\mathbf{A} = \begin{pmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix}; \mathbf{F} = \begin{pmatrix} 0 \\ -\frac{E}{L} \end{pmatrix}.$$

All the following procedures are given in MathCAD.

Given

$$L \cdot i'_L + i_L \cdot R = u_C \quad L \cdot i'_L + i_L \cdot R + (i_L + C \cdot u'_C) \cdot R = E$$

$$A(i_L, u_C, \mathbf{E}) := \text{Find}(i'_L, u'_C) \rightarrow \begin{bmatrix} \frac{-(i_L \cdot R - u_C)}{L} \\ \frac{-(i_L \cdot R + u_C - E)}{R \cdot C} \end{bmatrix}$$

$$A := \text{augment}(\mathbf{A}(1, 0, 0), A(0, 1, 0)) \quad B := \mathbf{A}(0, 0, E)$$

$$A \rightarrow \begin{pmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{R \cdot C} \end{pmatrix} \quad B \rightarrow \begin{pmatrix} 0 \\ \frac{E}{R \cdot C} \end{pmatrix} \quad \underline{R} := 10 \quad E := 20 \quad \underline{C} := 50 \cdot 10^{-6} \quad \underline{L} := 0.15$$

$$\underline{A} := \begin{pmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{R \cdot C} \end{pmatrix} \quad A = \begin{pmatrix} -66.667 & 6.667 \\ -2 \times 10^4 & -2 \times 10^3 \end{pmatrix} \quad \underline{F} := \begin{pmatrix} 0 \\ \frac{E}{R \cdot C} \end{pmatrix} \quad F = \begin{pmatrix} 0 \\ 4 \times 10^4 \end{pmatrix}$$

Initial conditions 
$$\begin{pmatrix} i_{L0} \\ u_{C0} \end{pmatrix} := \begin{pmatrix} \frac{E}{3 \cdot R} \\ \frac{E}{3} \end{pmatrix} \quad \begin{pmatrix} i_{L0} \\ u_{C0} \end{pmatrix} = \begin{pmatrix} 0.667 \\ 6.667 \end{pmatrix}$$

matrix **A** eigenvalues

$$\lambda := \text{eigenvals}(\mathbf{A}) \quad \lambda = \begin{pmatrix} -138.285 \\ -1.928 \times 10^3 \end{pmatrix}$$

Matrix exponent decomposition coefficients

$$\alpha(t) := \begin{pmatrix} 1 & \lambda_0 \\ 1 & \lambda_1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} e^{\lambda_0 \cdot t} \\ e^{\lambda_1 \cdot t} \end{pmatrix}$$

$$\alpha(t)_0 \text{ float}, 5 \rightarrow 1.0773 \cdot e^{(-138.29) \cdot t} - .77250e-1 \cdot e^{(-1928.4) \cdot t}$$

$$\alpha(t)_1 \text{ float}, 5 \rightarrow .55863e-3 \cdot e^{(-138.29) \cdot t} - .55863e-3 \cdot e^{(-1928.4) \cdot t}$$

Matrix exponent

$$\mathbf{I} := \text{identity}(2) \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{Exp}(t) := \alpha(t)_0 \cdot \mathbf{I} + \alpha(t)_1 \cdot \mathbf{A}$$

$$\text{Exp}(t) \text{ float}, 3 \rightarrow \begin{bmatrix} 1.04 \cdot e^{(-138.) \cdot t} - .400e-1 \cdot e^{(-.193e4) \cdot t} & .372e-2 \cdot e^{(-138.) \cdot t} - .372e-2 \cdot e^{(-.193e4) \cdot t} \\ (-11.2) \cdot e^{(-138.) \cdot t} + 11.2 \cdot e^{(-.193e4) \cdot t} & (-.400e-1) \cdot e^{(-138.) \cdot t} + 1.04 \cdot e^{(-.193e4) \cdot t} \end{bmatrix}$$

Analytical solution

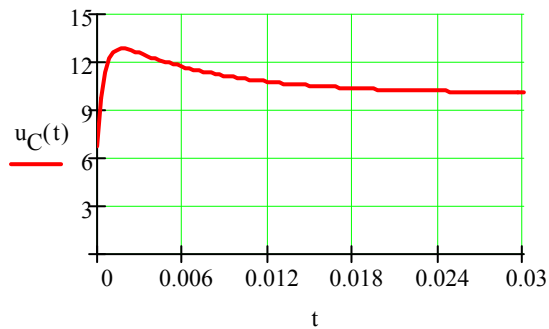
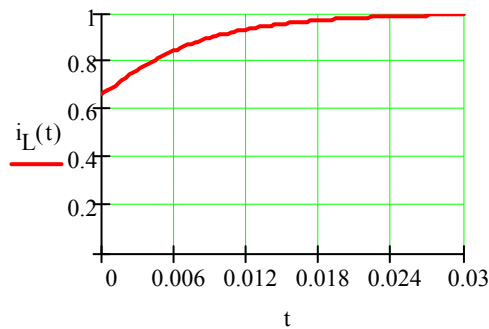
$$i_L(t) := \left[ \text{Exp}(t) \cdot \begin{pmatrix} i_{L0} \\ u_{C0} \end{pmatrix} \right]_0 + \int_0^t (\text{Exp}(t - \tau) \cdot \mathbf{F})_0 d\tau$$

$$u_C(t) := \left[ \text{Exp}(t) \cdot \begin{pmatrix} i_{L0} \\ u_{C0} \end{pmatrix} \right]_1 + \int_0^t (\text{Exp}(t - \tau) \cdot \mathbf{F})_1 d\tau$$

$$i_L(t) \text{ float,4} \rightarrow (-.3588) \cdot e^{(-138.3) \cdot t} + .2575e-1 \cdot e^{(-1928.) \cdot t} + 1.000$$

$$u_C(t) \text{ float,4} \rightarrow 3.855 \cdot e^{(-138.3) \cdot t} - 7.191 \cdot e^{(-1928.) \cdot t} + 10.00$$

$$T := 0.03 \quad t := 0, 0.01 \cdot T .. T$$

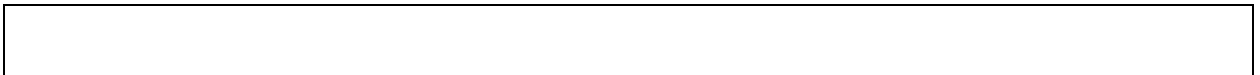


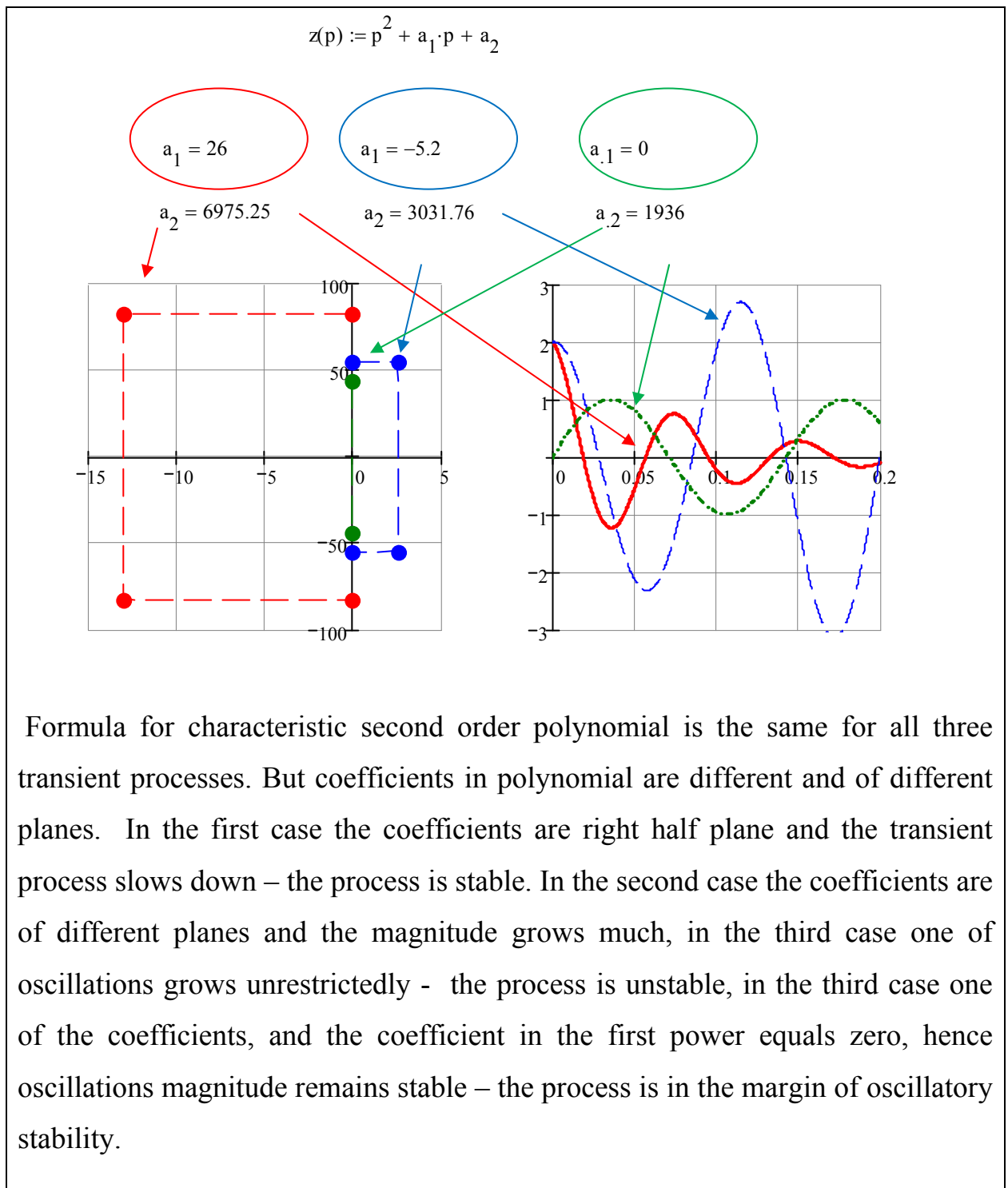
## 1.8. Automatic control linear system stability

When electric power system runs small disturbances always occur. These perturbations can occur because of power change, load change, short circuit or breaking of circuit. They make electrical or mechanical system unstable, it means that steady state mode breaks. In this case automatic system controls switch on, they perform according to the situation to balance the system. Such disturbances should not break down system stable performance. Never let disturbances increase. The system should be stable at small perturbations, i.e. it should keep static stability.

***Static stability is system ability to regain the initial mode after small disturbances or the mode close to initial mode (if disturbance remains).***

To state the system stability it is no need to calculate the roots of characteristic equation because indirect indicators which show roots real parts planes from which we state the system stability. These indirect indicators are called ***stability criteria***.





### 1.8.1. The Routh-Hurwitz stability criterion (algebraic stability criterion)

All the coefficients of the characteristic equation of stable system must have the same sign (the necessary condition of the Routh-Hurwitz stability criterion). If

one of the coefficients turns zero then the system is unstable. If all the coefficients are right half plane(RHP) then all the roots(if there are roots on the left half plane)will be left- half-plane(LHP). Complex roots can be right-half-plane.

If the free component coefficient  $a_n = 0$  turns zero it means there is a zero root , i.e. the system is in the range of dead-beat stability. If some intermediate coefficient turns zero it means some imaginary roots appear, i.e. the system is in the range of pendulum stability.

The necessary condition of stability criterion is the Routh –Hurwitz matrix which is formed as follows:

For characteristic polynomial

$$D(p) = a_0 p^n + a_1 p^{n-1} + a_2 p^{n-2} \dots + a_n$$

Form the matrix using the following pattern.

- Write diagonally all the coefficients starting from  $a_1$  to  $a_n$ .
- Complete the matrix from the bottom to the top in order of index increase
- In places where indices are outside the boundaries put zeros.

$$D(p) = a_0 p^4 + a_1 p^3 + a_2 p^2 + a_3 p + a_4 \rightarrow A = \begin{pmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{pmatrix}$$

If all the diagonal minors of matrix are right half plane then the system is stable – this condition is enough for system stability.

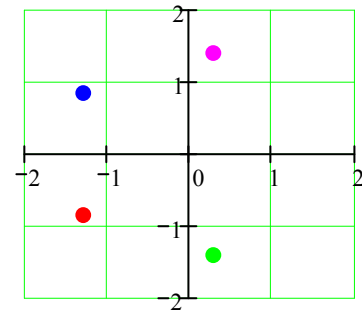
$$a := (1 \ 2 \ 3 \ 4 \ 5)^T$$

$$z(p) := a_0 \cdot p^4 + a_1 \cdot p^3 + a_2 \cdot p^2 + a_3 \cdot p + a_4$$

The necessary stability criterion is met(All the coefficients are right half plane)

Computing the coefficients

shows  $\lambda := z(p) \left| \begin{array}{l} \text{solve, p} \\ \text{float, 4} \end{array} \right. \rightarrow \begin{bmatrix} (-1.288) - .8579i \\ (-1.288) + .8579i \\ .2878 - 1.416i \\ .2878 + 1.416i \end{bmatrix}$



that the system is unstable

Form the Routh-Hurwitz matrix  $A := \begin{pmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{pmatrix} \quad |A| = -60$

$$a_1 = 2 \quad \Delta_2 := a_1 \cdot a_2 - a_0 \cdot a_3 \quad \Delta_2 = 2 \quad \Delta_3 := a_3 \cdot \Delta_2 - a_1 \cdot \begin{vmatrix} a_1 & 0 \\ a_0 & a_4 \end{vmatrix}$$

$\Delta_3 = -12 \quad \Delta_4 := \Delta_3 \cdot a_4 \quad \Delta_4 = -60$  The last minor is left half plane(LHP) so the system is unstable.

### Example 1. MathCAD

### Example 2.

$$a := (1 \ 2 \ 3 \ 4)^T \quad z(p) := a_0 \cdot p^3 + a_1 \cdot p^2 + a_2 \cdot p + a_3$$

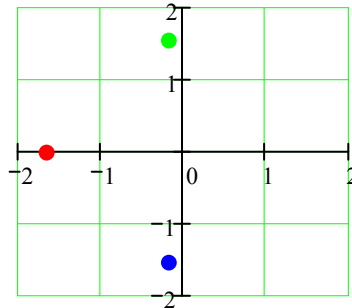
The necessary criterion is met.

$$A := \begin{pmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{pmatrix} \quad |A| = 8 \quad a_1 = 2 \quad \Delta_2 := a_1 \cdot a_2 - a_0 \cdot a_3 \quad \Delta_3 := a_3 \cdot \Delta_2 \quad \Delta_3 = 8$$

The necessary Routh-Hurwitz criterion is met thus the system is stable.

Computing the roots proves the system to be stable.

$$\lambda := z(p) \left| \begin{array}{l} \text{solve, } p \\ \text{float, 4} \end{array} \right. \rightarrow \begin{bmatrix} -1.651 \\ (-.1747) - 1.547i \\ (-.1747) + 1.547i \end{bmatrix}$$



### Example 3.

$$a := (1 \ 1 \ 4 \ 0)^T \quad z(p) := a_0 \cdot p^3 + a_1 \cdot p^2 + a_2 \cdot p + a_3$$

The necessary criterion is met (all the coefficients are right half plane)

The first diagonal minor is right half plane, the last but one equals zero

Thus polynomial is in the range of pendulum stability

$$A := \begin{pmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{pmatrix} \quad |A| = 0 \quad z(p) \left| \begin{array}{l} \text{solve, } p \\ \text{float, 4} \end{array} \right. \rightarrow \begin{bmatrix} -1. \\ 2 \cdot i \\ (-2) \cdot i \end{bmatrix}$$

$$a_1 = 1 \quad \Delta_2 := a_1 \cdot a_2 - a_0 \cdot a_3 \quad \Delta_2 = 0 \quad \Delta_3 := a_3 \cdot \Delta_2 \quad \Delta_3 = 0$$

### Example 4.

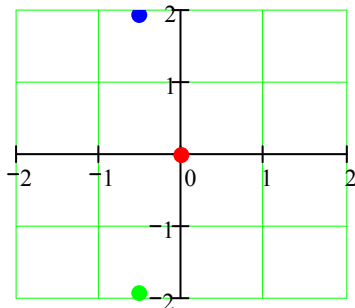
$$a := (1 \ 1 \ 4 \ 0)^T \quad z(p) := a_0 \cdot p^3 + a_1 \cdot p^2 + a_2 \cdot p + a_3$$

The necessary criterion is met (all the coefficients are right half plane)

The first diagonal minor is right half plane, the last minor equals zero. Thus the roots of polynomial is on aperiodic stability boundary.

$$A := \begin{pmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{pmatrix} \quad |A| = 0 \quad \lambda := z(p) \quad \left| \begin{array}{l} \text{solve, p} \\ \text{float, 4} \end{array} \right. \rightarrow \begin{bmatrix} 0 \\ (-.5000) + 1.937i \\ (-.5000) - 1.937i \end{bmatrix}$$

$$a_1 = 1 \quad \Delta_2 := a_1 \cdot a_2 - a_0 \cdot a_3 \quad \Delta_2 = 4 \quad \Delta_3 := a_3 \cdot \Delta_2 \quad \Delta_3 = 0$$



### Example 5.

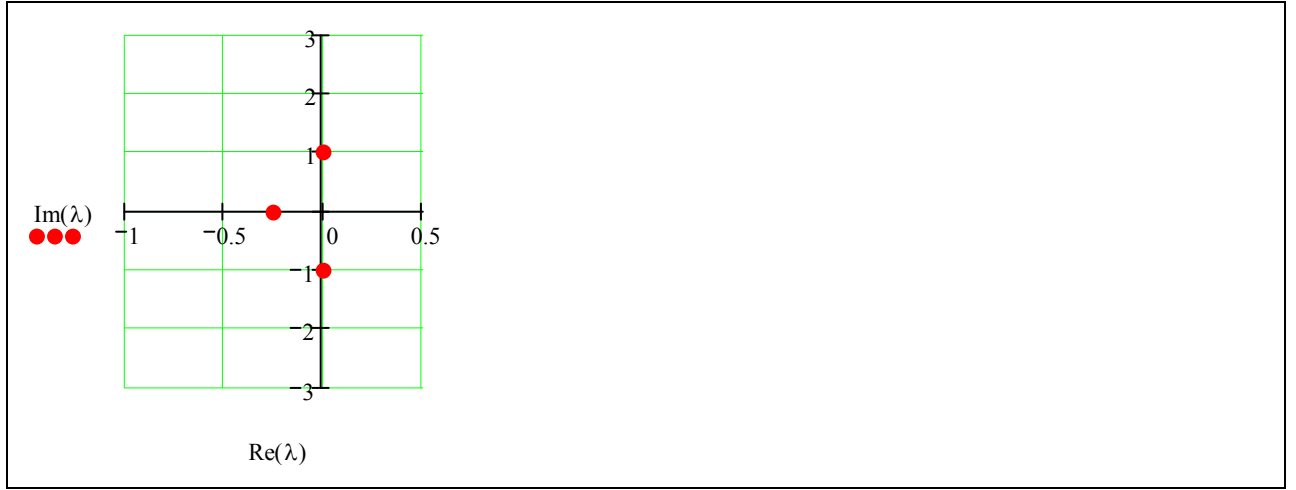
$$a := (4 \ 1 \ 4 \ 1)^T \quad Z(p) := a_0 \cdot p^3 + a_1 \cdot p^2 + a_2 \cdot p + a_3$$

The necessary criterion is met (all the coefficients are right half plane)

The first diagonal minor is right half plane. Thus the roots of polynomial is on the oscillatory stability boundary

$$\Delta_1 := \text{identity}(3) \quad \Delta_2 := \begin{pmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 4 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \lambda := Z(p) \quad \left| \begin{array}{l} \text{solve, p} \\ \text{float, 5} \end{array} \right. \rightarrow \begin{bmatrix} -.25000 \\ 1 \cdot i \\ (-1) \cdot i \end{bmatrix}$$





### 1.8.2. The Routh-Hurwitz algebraic criterion application example

We are given a characteristic polynomial for back pressure system control of turbine with two amplifying components [5]:

$$\tau_1 \tau_2 p^3 + (\tau_1 + \tau_2 + \delta_p \tau_1 \tau_2) p^2 + [(\tau_1 + \tau_2) \delta_p + 1] p + \delta_p + 1 = 0$$

$$a_0 p^3 + a_1 p^2 + a_2 p + a_3 = 0$$

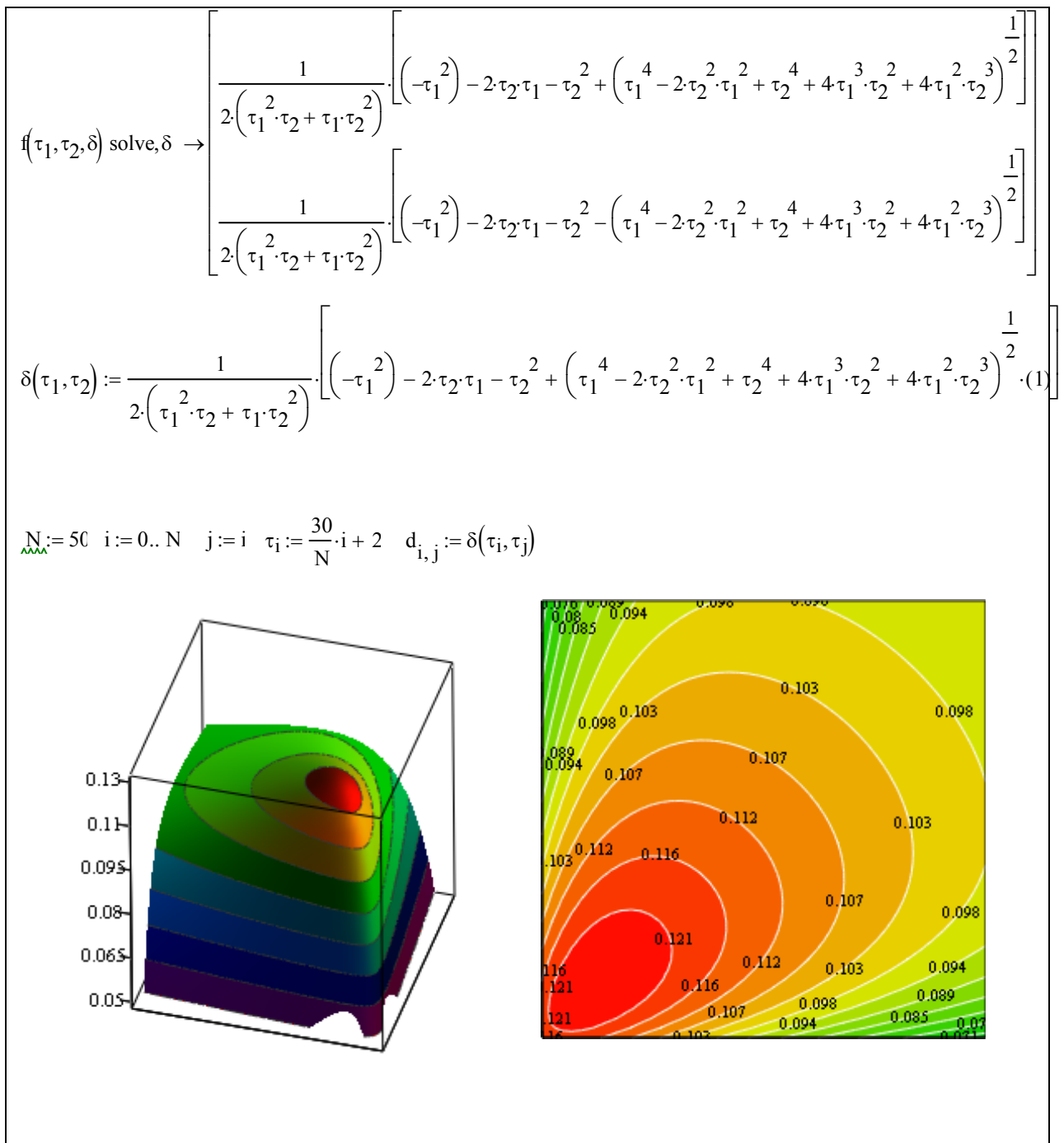
Here  $\tau_1 = T_1 / T_p$ ,  $\tau_2 = T_2 / T_p$  - relative time constants  $\delta_p$  - inequity degree of backpressure regulation

Using the Routh-Hurwitz criterion we obtain:

$$\Delta_2 = 0, \rightarrow \delta_p^2 \tau_1 \tau_2 (\tau_1 + \tau_2) + (\tau_1 + \tau_2)^2 \delta_p + (\tau_1 + \tau_2 - \tau_1 \tau_2) = 0$$

Let's plot/construct stability borders for turbine backpressure system with **MathCAD** help.

$$f(\tau_1, \tau_2, \delta) := \delta^2 \cdot \tau_1 \cdot \tau_2 \cdot (\tau_1 + \tau_2) + \delta \cdot (\tau_1 + \tau_2)^2 + \tau_1 + \tau_2 - \tau_2 \cdot \tau_1$$



Self regulation influences stability much. As  $\delta_p$  grows, stability area decreases and collapses at  $\delta_p = 0,125$ . При  $\delta_p \geq 0,125$  the system is stable at any time invariables of servomotor.

### 1.8.3. Mikhailov stability criterion

Let's analyze the characteristic polynomial of  $n$ power:

$$D(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots a_{n-1} \lambda + a_n .$$

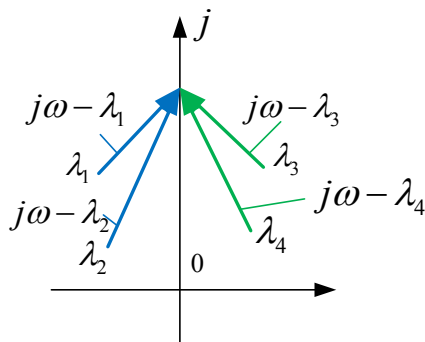
The polynomial can be depicted as :

$$D(\lambda) = a_0 (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n) .$$

Where amplitude of complex function has the form of

$$|D(\lambda)| = |a_0 (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)| ,$$

$$\arg(D(\lambda)) = \arg(\lambda - \lambda_1) + \arg(\lambda - \lambda_2) + \arg(\lambda - \lambda_3) + \dots \arg(\lambda - \lambda_n)$$



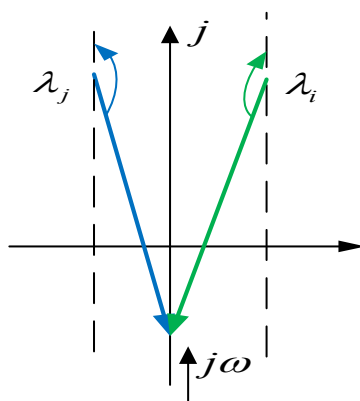
$n$ - number of roots,

$m$ -polynomial right half plane roots

Let's put  $p = j\omega$  in polynomial

$$\arg(D(j\omega)) = \arg(j\omega - \lambda_1) + \arg(j\omega - \lambda_2) + \dots \arg(j\omega - \lambda_n)$$

$$\arg(D(j\omega)) = \pi(n-m) - m\pi = (n-2m)\pi \quad -\infty < \omega < \infty$$



If all the roots are left half plane  $m = 0$ :

$$\arg(D(j\omega)) = \pi n \quad -\infty < \omega < \infty$$

At frequency change from zero to infinity obtain:

$$\arg(D(j\omega)) = \frac{\pi n}{2} \quad 0 < \omega < \infty$$

Thus Mikhailov criterion for stable system describes a curve (the Mikhailov hodograph) of characteristic polynomial which starts on the positive part of real semi-axis, generates an anti-clockwise motion through  $n$  quadrants.

Alternative definition of **Mikhailov criterion** For stable system it is required the zeros of real and imaginary parts of characteristic polynomial shift, and their gross number equals  $n$ .

**Пример.** System impulse response function has the form of

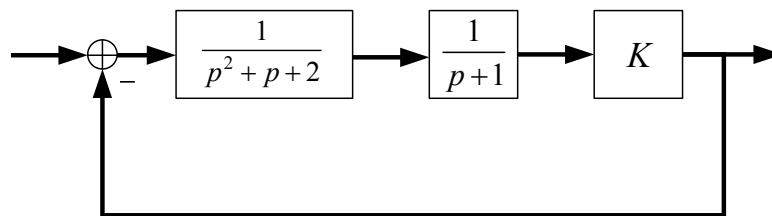


Fig.

Determine at which  $K$  values the system is stable, unstable or is in stability limit.

$$W_1(p) = \frac{1}{p^2 + p + 2}, \quad W_2(p) = \frac{1}{p + 1}, \quad W_3(p) = K$$

$$W(p) = \frac{K}{p^3 + 2p^2 + 3p + K + 2}$$

Solution is given in **MathCAD**

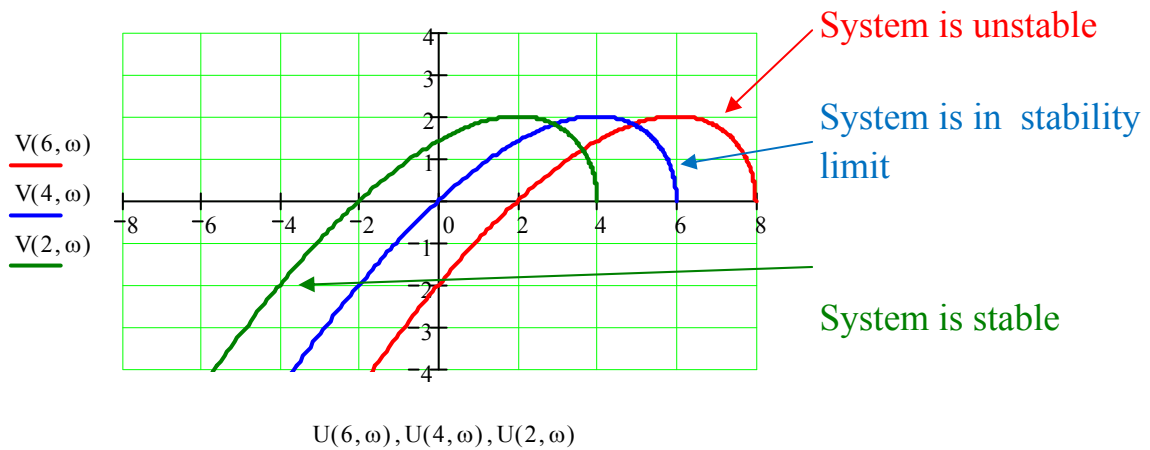
$$W_1(p) := \frac{1}{p^2 + p + 2} \quad W_2(p) := \frac{1}{p + 1} \quad W_3(p) := K \quad W(p) := \frac{W_1(p) \cdot W_2(p) \cdot W_3(p)}{1 + W_1(p) \cdot W_2(p) \cdot W_3(p)}$$

$$W(p, K) := \frac{K}{p^3 + 2 \cdot p^2 + 3 \cdot p + 2 + K} \quad D(K, p) := p^3 + 2 \cdot p^2 + 3 \cdot p + 2 + K$$

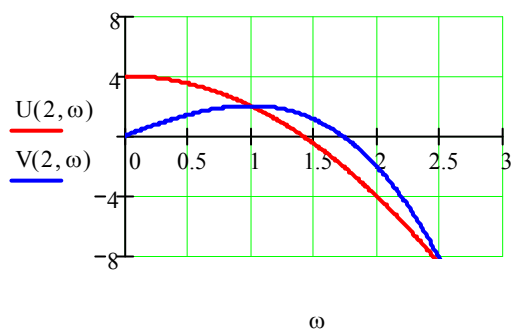
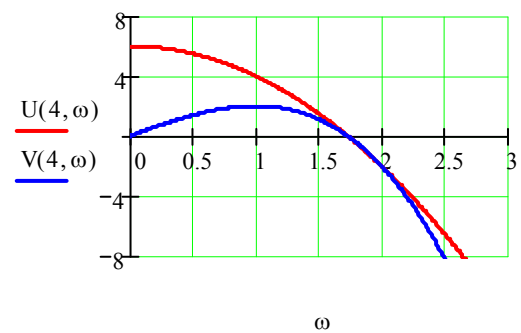
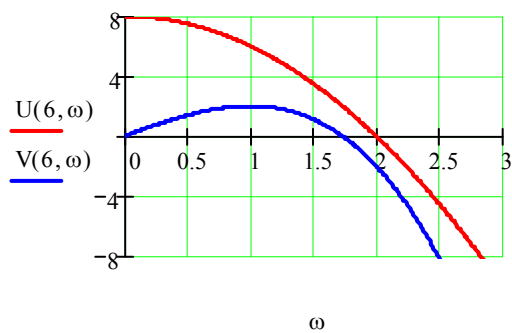
$$U(K, \omega) := \operatorname{Re}(D(K, i \cdot \omega)) \quad V(K, \omega) := \operatorname{Im}(D(K, i \cdot \omega))$$

$$U(K, \omega) \text{ complex} \rightarrow (-2) \cdot \omega^2 + 2 + K \quad V(K, \omega) \text{ complex} \rightarrow (-\omega^3) + 3 \cdot \omega$$

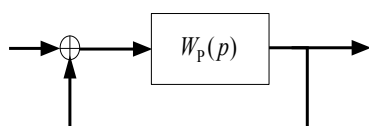
$$\omega := 0, 0.01..3$$



### Mikhailov stability criterion



### 1.8.4. Nyquist stability criterion



Nyquist criterion help determine closed loop system stability by open loop system stability

$$W_3(p) = \frac{W_p(p)}{1 + W_p(p)} = \frac{M(p) / D(p)}{1 + M(p) / D(p)} = \frac{M(p)}{D(p) + M(p)}$$

Characteristic polynomial of closed loop system has the form of

$$D(p) + M(p) = D(p) \left( 1 + \frac{M(p)}{D(p)} \right) = D(p) (1 + W(p))$$

Let's use Mikhailov's criterion

$$\Delta \arg_{0 \leq \omega < \infty} [D(j\omega) + M(j\omega)] = \Delta \arg_{0 \leq \omega < \infty} D(j\omega) + \Delta \arg_{0 \leq \omega < \infty} [1 + W_p(j\omega)] \quad (*)$$

If closed loop system is stable then the result should equal  $n \frac{\pi}{2}$ .

According to Mikhailov criterion characteristic polynomial argument equals

$$\Delta \arg_{0 \leq \omega < \infty} D(j\omega) = \frac{\pi}{2} (\Lambda - \Pi) \text{ (system may be unstable too)} \quad (**)$$

Taking into consideration (\*) и (\*\*) we obtain the formula for variable argument  $1 + W(j\omega)$  formula:

$$\Delta \arg_{0 \leq \omega < \infty} [1 + W_p(j\omega)] = \frac{\pi}{2} n - \Delta \arg_{0 \leq \omega < \infty} D(j\omega) = \frac{\pi}{2} n - \frac{\pi}{2} (\Lambda - \Pi)$$

As  $\Lambda = n - \Pi - H$  then we finally obtain :

$$\Delta \arg_{0 \leq \omega < \infty} [1 + W_p(j\omega)] = \frac{\pi}{2} n - \frac{\pi}{2} (\Lambda - \Pi) = \frac{\pi}{2} n' - \frac{\pi}{2} (n' - \Pi - H - \Pi) = \frac{\pi}{2} (2\Pi - H)$$

*For stable system it is enough that frequency response locus of open loop system  $W_p(j\omega)$  at frequency change  $\omega$  from 0 up to  $\infty$  covers the point  $(-1, j)$  for angle*

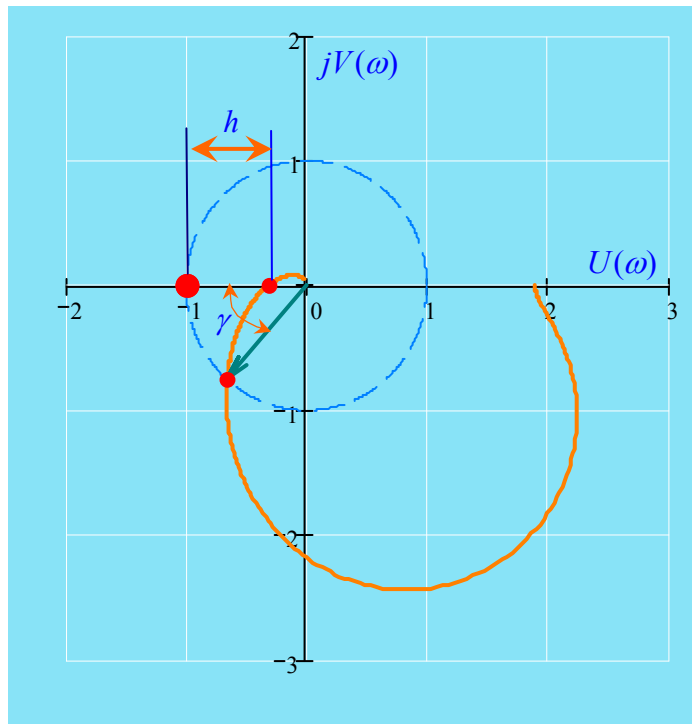
$$\frac{\pi}{2} (2\Pi - H)$$

$$\Delta \arg[1 + W_p(j\omega)] = \frac{\pi}{2}(2R - N) \quad \text{if the sum of number of right-2R and}$$

$$0 \leq \omega < \infty$$

neutral-N roots is equal zero then the angular is equal 0, i.e. the loop should not cover the point (-1, j).

Magnitude and phase criterion: System will be stable if locus of open loop system



$W_p(j\omega)$  does not cover the points with  $(-1, j0)$  coordinates

$h$  - minimum section of real axis, which characterizes the distance between a critical point and nearest point of root locus crossing with real axis, is called **modulus stability margins**.

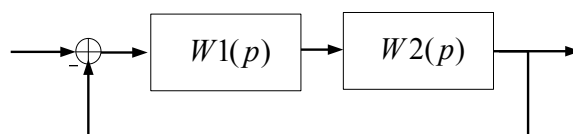
Minimal angle  $\gamma$  formed by the radius passing through the point of locus intersection with

unit radius circle (with the center at the beginning of coordinates) and negative part of real axis is called **phase stability margin**.

Nyquist – Mikhailov stability criteria let state control system stability with feedback by bode plot locus of closed loop. Criteria can be used in these cases when system differential equation (or separate components) is unknown but designer has the corresponding research characteristics.

Let's analyze some examples with the help of **MathCAD**

### 1. Example of stable open loop and stable closed loop systems



$$W1(p) := \frac{10}{p + 2} \quad W2(p) := \frac{1}{p^2 + 2 \cdot p + 4} \quad W_p(p) := W1(p) \cdot W2(p)$$

$$\text{denom}(W_p(p)) \left| \begin{array}{l} \text{solve}, p \\ \text{float}, 5 \end{array} \right. \rightarrow \begin{bmatrix} -2. \\ (-1.) + 1.7321i \\ (-1.) - 1.7321i \end{bmatrix} \quad \text{Open loop system is stable}$$

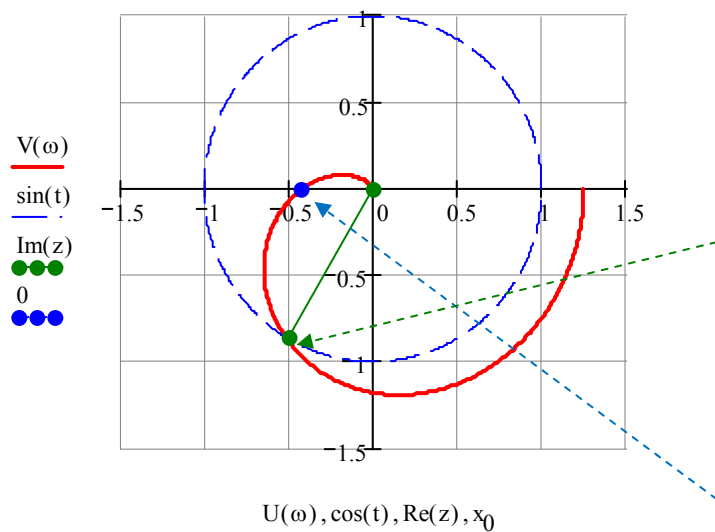
$$U(\omega) := \text{Re}(W_p(i\omega)) \quad \underline{V}(\omega) := \text{Im}(W_p(i\omega))$$

$$\underline{W}(p) := \frac{10}{p^3 + 4p^2 + 8p + 18} \quad \omega := 0..10^2 \quad t := 0..2\pi$$

With **Trace** function help(right click in plot field)determine phase and magnitude stability margins.

$$z := \begin{pmatrix} 0 \\ -0.5 - j \cdot 0.86 \end{pmatrix} \quad \text{Phase stability margin} \quad \phi := \frac{\pi + \arg(z_1)}{\text{deg}} \quad \phi = 59.826$$

$$x_0 := -0.43 \quad \text{Magnitude stability margin} \quad 1 + x_0 = 0.57$$





X

X-Value

-0.50395

Copy X

Y-Value

-0.86343

Copy Y

Y2-Value

Copy Y2

☐ Track data points

Close

X

X-Value

-0.43281

Copy X

Y-Value

0.0046296

Copy Y

Y2-Value

Copy Y2

☐ Track data points

Close

Determine analytically cutoff frequency – frequency at which locus crosses the unit circle

$$V(\omega) := \text{Im}(W_p(j\omega)) \quad \left| \begin{array}{l} \text{complex} \\ \text{simplify} \end{array} \right. \rightarrow 10\omega \cdot \frac{\omega^2 - 8}{(4 + \omega^2) \cdot (\omega^4 - 4\omega^2 + 16)}$$

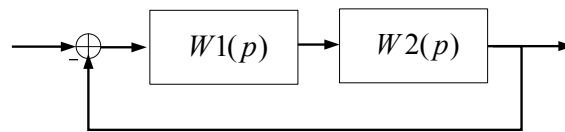
$$U(\omega) := \text{Re}(W_p(j\omega)) \quad \left| \begin{array}{l} \text{complex} \\ \text{simplify} \end{array} \right. \rightarrow (-40) \cdot \frac{\omega^2 - 2}{(4 + \omega^2) \cdot (\omega^4 - 4\omega^2 + 16)}$$

$$V(\omega) \rightarrow 10\omega \cdot \frac{\omega^2 - 8}{(4 + \omega^2) \cdot (\omega^4 - 4\omega^2 + 16)}$$

$$U(\omega) \rightarrow (-40) \cdot \frac{\omega^2 - 2}{(4 + \omega^2) \cdot (\omega^4 - 4\omega^2 + 16)}$$

$$\omega_C := \left( \begin{array}{l} V(\omega)^2 + U(\omega)^2 = 1 \\ \omega \geq 0 \end{array} \right) \quad \left| \begin{array}{l} \text{solve}, \omega \\ \text{float}, 4 \end{array} \right. \rightarrow \begin{bmatrix} 1.817 \\ (-.9085) + 1.574i \\ (-.9085) - 1.574i \end{bmatrix} \quad \omega_{C_0} = 1.817$$

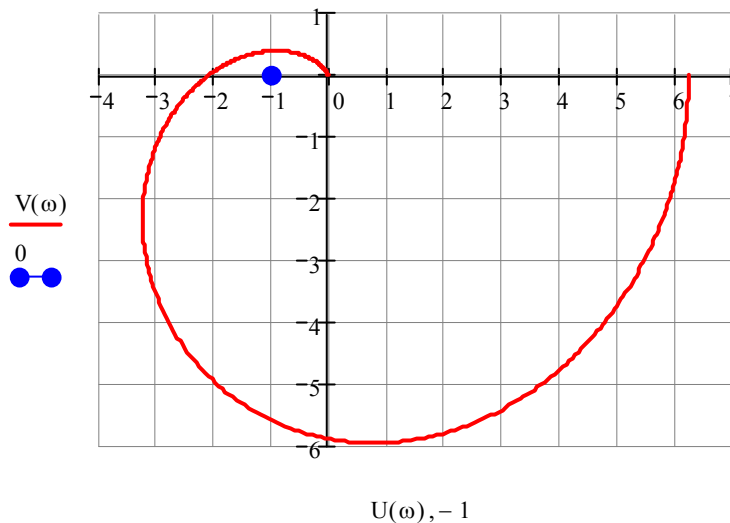
## 2. Example of stable closed loop system and unstable open loop system



$$W1(p) := \frac{10}{p+2} \quad W2(p) := \frac{5}{p^2 + 2 \cdot p + 4} \quad W_p(p) := W1(p) \cdot W2(p)$$

$$\text{denom}(W_p(p)) \left| \begin{array}{l} \text{solve, } p \\ \text{float, 4} \end{array} \right. \rightarrow \begin{bmatrix} -2. \\ (-1.) + 1.732i \\ (-1.) - 1.732i \end{bmatrix} \quad \underline{\underline{V(\omega)}} := \text{Im}(W_p(j \cdot \omega)) \quad U(\omega) := \text{Re}(W_p(j \cdot \omega))$$

$$\omega := 0, .01.. 10^2$$



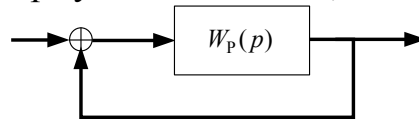
Locus covers the point  $(-1, j0)$ , hence closed loop system is unstable

$$\underline{\underline{W(p)}} := \frac{W1(p) \cdot W2(p)}{1 + W1(p) \cdot W2(p)} \text{ simplify } \rightarrow \frac{50}{p^3 + 4 \cdot p^2 + 8 \cdot p + 58}$$

Roots check proves the obtained result

$$\text{denom}(W(p)) \left| \begin{array}{l} \text{solve, p} \\ \text{float, 4} \end{array} \right. \rightarrow \begin{pmatrix} -4.831 \\ .417 - 3.441i \\ .417 + 3.441i \end{pmatrix}$$

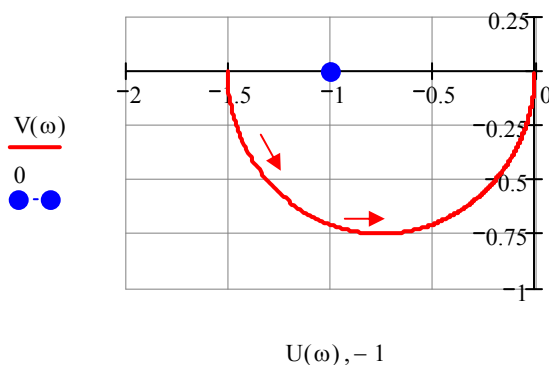
**3. Example** when open loop system is unstable, closed loop system is stable:



$$T := 1 \quad W_p(p) := \frac{1.5}{T \cdot p - 1} \quad \text{Open loop system is unstable}$$

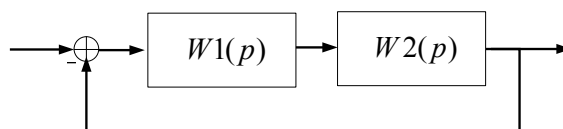
$$U(\omega) := \text{Re}(W_p(j \cdot \omega)) \left| \begin{array}{l} \text{complex} \\ \text{simplify} \rightarrow \frac{-1.500}{1. + \omega^2} \\ \text{float, 4} \end{array} \right. \quad V(\omega) := \text{Im}(W_p(j \cdot \omega)) \left| \begin{array}{l} \text{complex} \\ \text{simplify} \rightarrow (-1.500) \cdot \frac{\omega}{1. + \omega^2} \\ \text{float, 4} \end{array} \right.$$

$$\omega := 0, .01.. 10^2$$



Locus starts from  $U(0) = -1,5$  and moves to zero, hence closed loop system is stable.

**4. Example**, when open loop system has astaticism of the first order closed loop system is stable:



$$W1(p) := \frac{1}{p} \quad W2(p) := \frac{5}{p^2 + 2 \cdot p + 4} \quad W_p(p) := W1(p) \cdot W2(p) \quad W_p(p) \rightarrow \frac{5}{p \cdot (p^2 + 2 \cdot p + 4)}$$

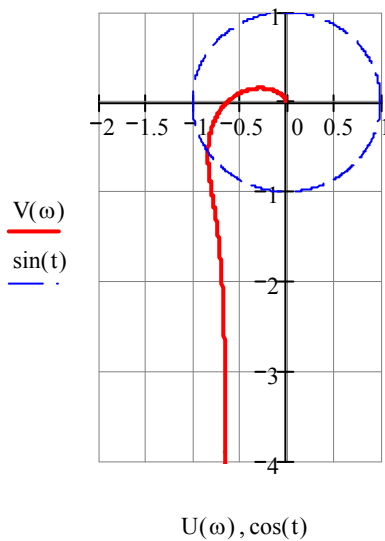
$$\text{denom}(W_p(p)) \left| \begin{array}{l} \text{solve}, p \\ \text{float}, 4 \end{array} \right. \rightarrow \begin{bmatrix} 0 \\ (-1.) + 1.732i \\ (-1.) - 1.732i \end{bmatrix}$$

Open loop system is in aperiodic stability margin

$$U(\omega) := \text{Re}(W_p(j\omega)) \left| \begin{array}{l} \text{complex} \\ \text{simplify} \rightarrow \\ \text{float}, 4 \end{array} \right. \frac{-10.}{\omega^4 - 4 \cdot \omega^2 + 16.}$$

$$V(\omega) := \text{Im}(W_p(j\omega)) \left| \begin{array}{l} \text{complex} \\ \text{simplify} \rightarrow 5 \cdot \frac{\omega^2 - 4.}{\omega \cdot (\omega^4 - 4 \cdot \omega^2 + 16.)} \\ \text{float}, 4 \end{array} \right.$$

$$\omega := 0..0.1 \cdot 10^2$$



Locus starts with negative imaginary axis and moves to zero around the point  $(-1, j0)$  below, hence closed loop system is stable.

5. Example when open loop system has astaticism of the first order a closed loop system is unstable:

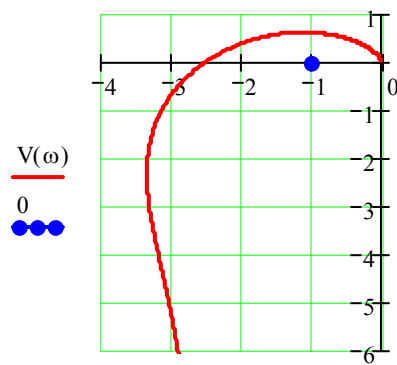
$$W1(p) := \frac{4}{p} \quad W2(p) := \frac{5}{p^2 + 2 \cdot p + 4} \quad W_p(p) := W1(p) \cdot W2(p)$$

$$\text{denom}(W_p(p)) \left| \begin{array}{l} \text{solve, } p \\ \text{float, } 5 \end{array} \right. \rightarrow \begin{bmatrix} 0 \\ (-1.) + 1.7321i \\ (-1.) - 1.7321i \end{bmatrix}$$

$$\underline{W}(p) := \frac{W_1(p) \cdot W_2(p)}{1 + W_1(p) \cdot W_2(p)} \quad \text{convert, parfrac, } p \rightarrow \frac{20}{p^3 + 2 \cdot p^2 + 4 \cdot p + 20}$$

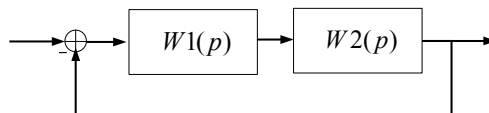
$$\text{denom}(W(p)) \left| \begin{array}{l} \text{solve, } p \\ \text{float, } 5 \end{array} \right. \rightarrow \begin{pmatrix} -2.9463 \\ .47313 - 2.5621i \\ .47313 + 2.5621i \end{pmatrix} \quad \underline{U}(\omega) := \text{Re}(W_p(i \cdot \omega)) \quad \underline{V}(\omega) := \text{Im}(W_p(i \cdot \omega))$$

$$\omega := 0, 0.01 \dots 10^2$$



$U(\omega), -1$

**Example.** When open loop system has astaticism of the first order closed loop system is at stability margin.

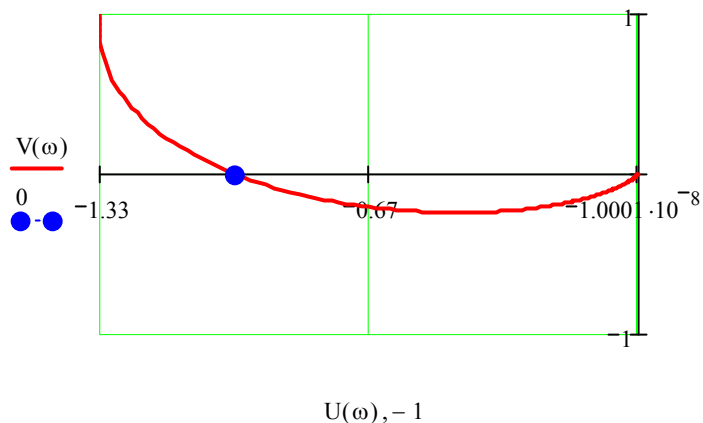


$$W_1(p) := \frac{1}{p} \quad W_2(p) := \frac{1}{p^2 + p + 1} \quad W_p(p) := W_1(p) \cdot W_2(p)$$

$$W_p(j\omega) \text{ complex} \rightarrow \frac{-1}{\left[(-\omega^2) + 1\right]^2 + \omega^2} - \frac{i}{\omega} \cdot \frac{(-\omega^2) + 1}{\left[(-\omega^2) + 1\right]^2 + \omega^2}$$

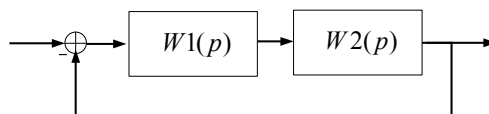
$$U(\omega) := \frac{-1}{(1 - \omega^2)^2 + \omega^2} \quad \underline{V(\omega)} := \frac{1}{\omega} \cdot \frac{1 - \omega^2}{(1 - \omega^2)^2 + \omega^2}$$

$$\omega := 0.1, 0.1 + 0.01 \cdot 10^2 \quad 1 + W_p(p) \left| \begin{array}{l} \text{solve}, p \\ \text{float}, 5 \end{array} \right. \rightarrow \begin{bmatrix} -1. \\ 1. \cdot i \\ (-1.) \cdot i \end{bmatrix}$$



System is at stability margin

**Example.**



$$\underline{W_1(p)} := \frac{2}{p + 1} \quad \underline{W_2(p)} := \frac{1}{p^2 + p + 1} \quad \underline{W_p(p)} := W_1(p) \cdot W_2(p)$$

$$W_p(i\omega) \left| \begin{array}{l} \text{complex} \\ \text{simplify} \end{array} \right. \rightarrow 2 \cdot \frac{(-2) \cdot \omega^2 + 1 + i \cdot \omega^3 - 2 \cdot i \cdot \omega}{(1 + \omega^2) \cdot (\omega^4 - \omega^2 + 1)}$$

$$U1(\omega) := (-2) \cdot \frac{2 \cdot \omega^2 - 1}{(1 + \omega^2) \cdot (\omega^4 - \omega^2 + 1)} \quad V1(\omega) := 2 \cdot \omega \cdot \frac{\omega^2 - 2}{(1 + \omega^2) \cdot (\omega^4 - \omega^2 + 1)}$$

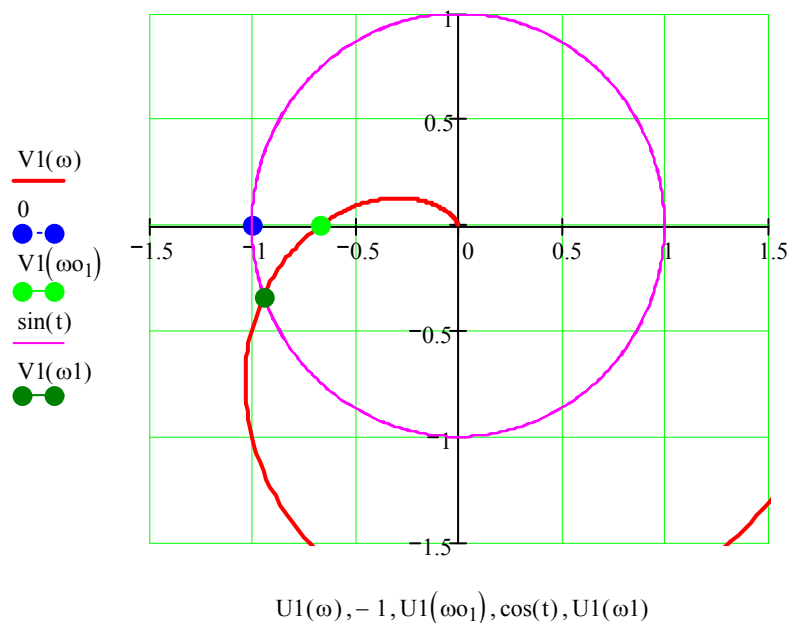
$$\omega_0 := \omega^2 - 2 \text{ solve, } \omega \rightarrow \begin{pmatrix} \frac{1}{2^2} \\ \frac{1}{-2^2} \end{pmatrix}$$

$$\omega_0 = \begin{pmatrix} 1.414 \\ -1.414 \end{pmatrix} \quad U1(\omega_0) = -0.667 \quad V1(\omega_0) = 0 \quad h := 1 + U1(\omega_0) \quad h = 0.333$$

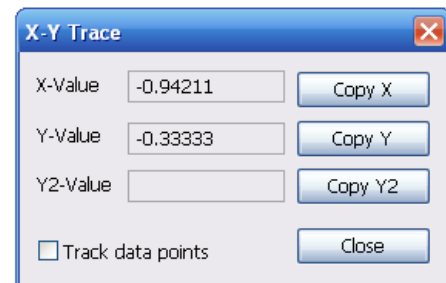
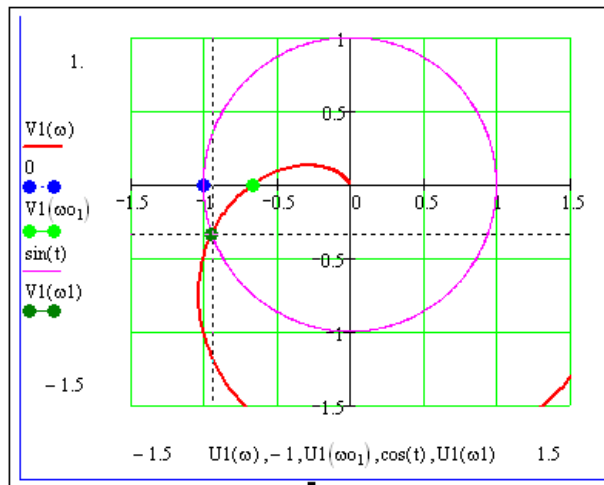
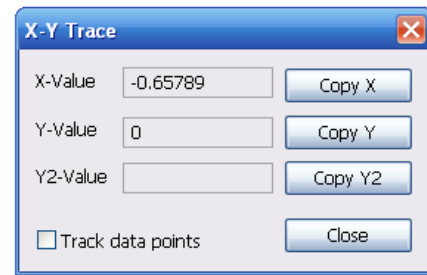
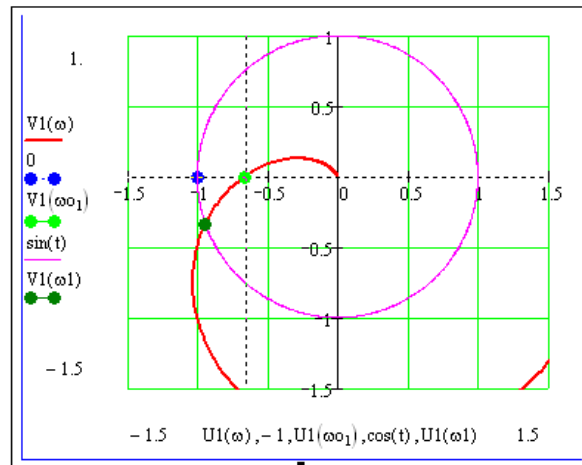
$$\omega_1 := 1.2 \quad \phi := \frac{\arg(W_p(\omega_1 \cdot i)) + \pi}{\text{deg}} \quad \phi = 19.669$$

$$\omega := 0.1, 0.1 + 0.01 \cdot 10^2$$

$$1 + W_p(p1) \left| \begin{array}{l} \text{solve, p1} \\ \text{float, 5} \end{array} \right. \rightarrow \begin{bmatrix} -1.8106 \\ (-.9474e-1) - 1.2837i \\ (-.9474e-1) + 1.2837i \end{bmatrix}$$



Determine modulus and phase stability margins(on diagram)



$$\frac{\arg(-0.94211 - i0.3333) + \pi}{\text{deg}} = 19.483$$

### 1.8.5. Logarithmic stability criterion

For the system which is stable being open-loop to be stable when being closed-loop it is enough at frequency  $\omega$ , for which  $L(\omega) > 0$ , difference between the number of positive and negative phase response crossing  $\phi(\omega)$  through the straight line  $\phi(\omega) = -\pi$  equal zero

**Example 1.** Investigate closed loop system stability using logarithmic magnitude phase frequency response if transfer function of open-loop system has the form of:



$$\underline{W(p)} := \frac{960}{p^3 + 26 \cdot p^2 + 208 \cdot p + 480.}$$

$$\text{denom}(W(p)) \left| \begin{array}{l} \text{simplify} \\ \text{float}, 3 \end{array} \right. \rightarrow p^3 + 26 \cdot p^2 + 208 \cdot p + 480 \quad \text{denom}(W(p)) \left| \begin{array}{l} \text{solve}, p \\ \text{float}, 5 \end{array} \right. \rightarrow \begin{pmatrix} -10. \\ -4. \\ -12. \end{pmatrix}$$

**Solution.** Open-loop system is stable , all the poles are left half plane

$$\underline{W(p)} := \frac{960}{p^3 + 26 \cdot p^2 + 208 \cdot p + 480.} \quad \text{denom}(W(p)) \left| \begin{array}{l} \text{simplify} \\ \text{float}, 3 \end{array} \right. \rightarrow p^3 + 26 \cdot p^2 + 208 \cdot p + 480$$

$$\text{denom}(W(p)) \left| \begin{array}{l} \text{solve}, p \\ \text{float}, 5 \end{array} \right. \rightarrow \begin{pmatrix} -10. \\ -4. \\ -12. \end{pmatrix}$$

Determine the real and imaginary parts of magnitude frequency response

$$U(\omega) := \text{Re}(W(i \cdot \omega)) \left| \begin{array}{l} \text{complex} \\ \text{simplify} \end{array} \right. \rightarrow (-1920) \cdot \frac{13 \cdot \omega^2 - 240.}{260 \cdot \omega^4 + 18304 \cdot \omega^2 + 230400 + \omega^6}$$

$$\underline{V(\omega)} := \text{Im}(W(i \cdot \omega)) \left| \begin{array}{l} \text{complex} \\ \text{simplify} \end{array} \right. \rightarrow 960 \cdot \omega \cdot \frac{\omega^2 - 208.}{260 \cdot \omega^4 + 18304 \cdot \omega^2 + .23040\text{e}6 + \omega^6}$$

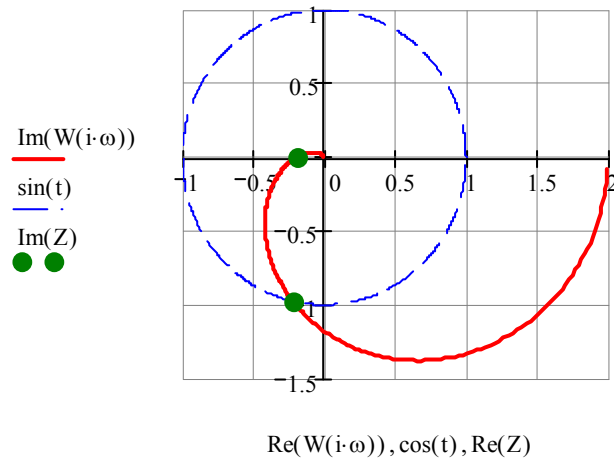
Determine the frequency at which the locus crosses the unit circle – crossover frequency

$$\omega_0 := \left( \begin{array}{l} |W(i \cdot \omega)| = 1 \\ \omega > 0 \end{array} \right) \left| \begin{array}{l} \text{solve}, \omega \\ \text{float}, 5 \end{array} \right. \rightarrow 5.1618$$

Determine the frequency at which locus crosses the real axis

$$\omega_1 := \left( \begin{array}{l} V(\omega_1) = 0 \\ \omega_1 > 0 \end{array} \right) \left| \begin{array}{l} \text{solve}, \omega_1 \\ \text{float}, 5 \end{array} \right. \rightarrow 14.422 \quad Z := \begin{pmatrix} W(\omega_0 \cdot j) \\ W(\omega_1 \cdot j) \end{pmatrix} \quad t := 0, .01.. 2 \cdot \pi$$

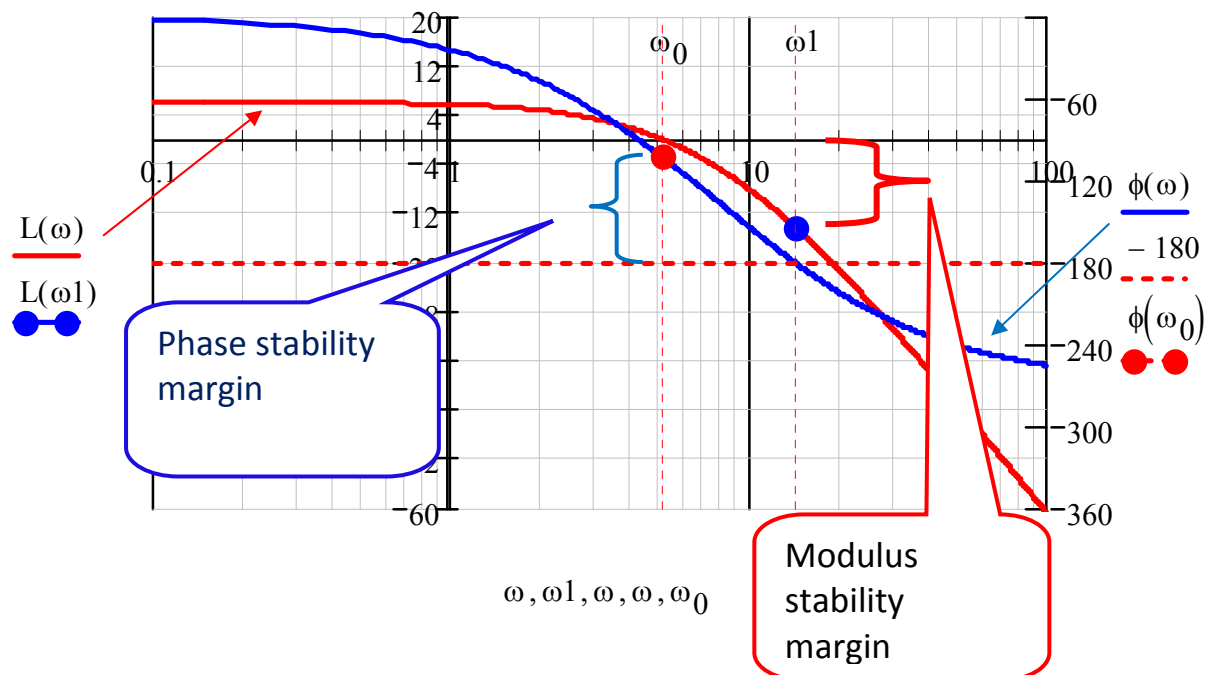
Plot locus



Plot logarithmic magnitude frequency response and logarithmic phase frequency response in one figure.

$$L(\omega) := 20 \cdot \log(|W(i \cdot \omega)|) \quad \omega := 0.1, 0.1 + 0.05 \cdot 10^2 \quad y := -360, -360 + 1 \cdot 0$$

$$\phi(\omega) := \text{if}(\arg(W(i \cdot \omega)) < 0, \arg(W(i \cdot \omega)), -\pi \cdot 2 + \arg(W(i \cdot \omega))) \cdot \text{deg}^{-1}$$



The graph shows that magnitude frequency response crosses  $L(\omega)=0$  earlier than phase frequency response crosses the axis  $\phi=-180^\circ$  thus the system is stable.

$$\phi(\omega_0) + 180 = 77.196 \text{ -phase stability margin}$$

$$\frac{L(\omega_1)}{1 - 10^{-20}} = 0.805 \text{ - modulus stability margin}$$

Compare to magnitude phase frequency response locus(see the graph above)

**Example 2.** Investigate the closed loop system stability according to logarithmic magnitude phase frequency response if the transfer function of open loop system has the form of:

$$W(p) := \frac{11 - 15 \cdot p}{(p^4 + 12 \cdot p^3 + 22 \cdot p^2 + 21 \cdot p + 10)} \quad \text{denom}(W(p)) \left| \begin{array}{l} \text{solve, } p \\ \text{float, 5} \end{array} \right. \rightarrow \begin{bmatrix} (-.50000) - .86603i \\ (-.50000) + .86603i \\ -1. \\ -10. \end{bmatrix}$$

**Solution .** The open-loop system is stable, all the poles are left half plane.

Determine the real and imaginary parts of magnitude phase frequency response.

$$U(\omega) := \text{Re}(W(i \cdot \omega)) \left| \begin{array}{l} \text{complex} \\ \text{simplify} \end{array} \right. \rightarrow \frac{(-557) \cdot \omega^2 + 110. + 191 \cdot \omega^4}{\omega^2 + 100 \cdot \omega^6 + 100. + \omega^8}$$

$$V(\omega) := \text{Im}(W(i \cdot \omega)) \left| \begin{array}{l} \text{complex} \\ \text{simplify} \end{array} \right. \rightarrow (-3.) \cdot \omega \cdot \frac{(-154) \cdot \omega^2 + 127. + 5 \cdot \omega^4}{\omega^2 + 100 \cdot \omega^6 + 100. + \omega^8}$$

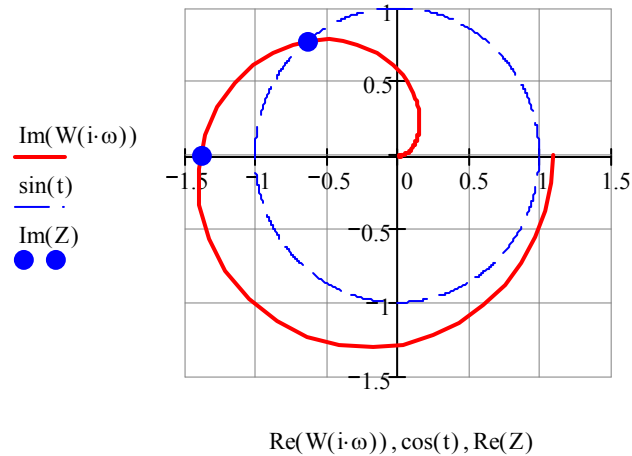
$$w(p) := \frac{W(p)}{1 + W(p)} \left| \begin{array}{l} \text{simplify} \end{array} \right. \rightarrow (-1.) \cdot \frac{(-11.) + 15 \cdot p}{p^4 + 12 \cdot p^3 + 22 \cdot p^2 + 6 \cdot p + 21.}$$

Determine the frequency at which the locus crosses the unit circle –crossover frequency

$$\omega_0 := \left( \begin{array}{l} |W(i \cdot \omega)| = 1 \\ \omega > 0 \end{array} \right) \left| \begin{array}{l} \text{solve, } \omega \\ \text{float, 5} \end{array} \right. \rightarrow 1.2370$$

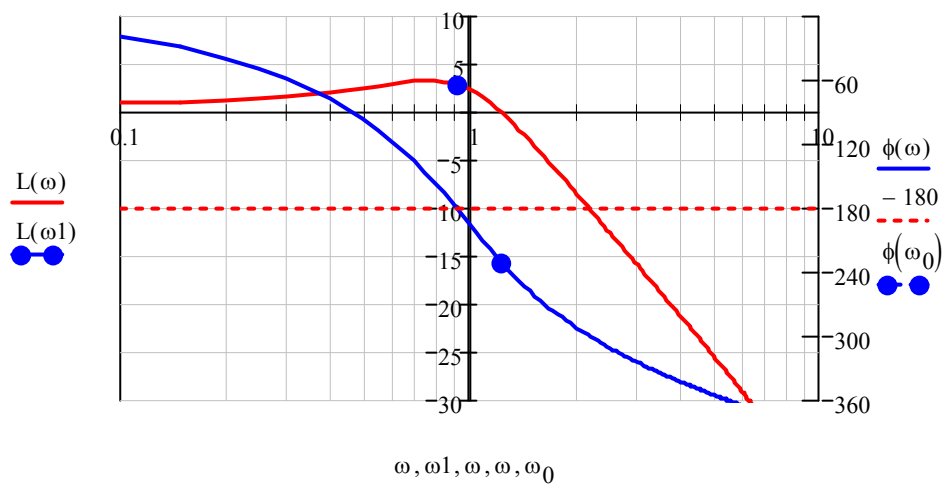
Determine the frequency at which the locus crosses the real axis

$$\omega_1 := \begin{pmatrix} V(\omega_1) = 0 \\ \omega_1 > 0 \\ U(\omega_1) < 0 \end{pmatrix} \left| \begin{array}{l} \text{solve, } \omega_1 \\ \text{float, 5} \end{array} \right. \rightarrow .9208 \quad Z := \begin{pmatrix} W(\omega_0 \cdot j) \\ W(\omega_1 \cdot j) \end{pmatrix} \quad t := 0, .01 \dots 2 \cdot \pi$$



Plot magnitude phase frequency response and phase frequency response in one figure

$$\begin{aligned} L(\omega) &:= 20 \cdot \log(|W(i \cdot \omega)|) \quad \omega := 0.1, 0.1 + 0.05 \dots 10 \quad y := -360, -360 + 1 \dots 0 \\ \phi(\omega) &:= \text{if}(\arg(W(i \cdot \omega)) < 0, \arg(W(i \cdot \omega)), -\pi \cdot 2 + \arg(W(i \cdot \omega))) \cdot \text{deg}^{-1} \\ \phi(\omega) &:= \text{if}(\phi(\omega) > -180 \wedge 5 \leq \omega, \phi(\omega) - 360, \phi(\omega)) \end{aligned}$$



The figure shows that magnitude phase frequency response crosses  $L(\omega) = 0$  later, than phase frequency response crosses the axis  $\phi = -180^\circ$  thus the system is unstable.

Compare magnitude phase frequency response to locus (see figure above)

### 1.8.6. Application of Niquist criterion and transient processes quality criteria

To evaluate the quality of the transient process the following quantity characteristics are used(Fig 14)

#### 1. Maximum overshoot

$$x_{\Pi m*} = \frac{x_{\Pi m}}{x_{\infty 2} - x_{\infty 1}} \quad \sigma = x_{\Pi m*} 100\% .$$

#### 2. Setting time.

The time during which the output variable differs from the steady value more than on five per cent.

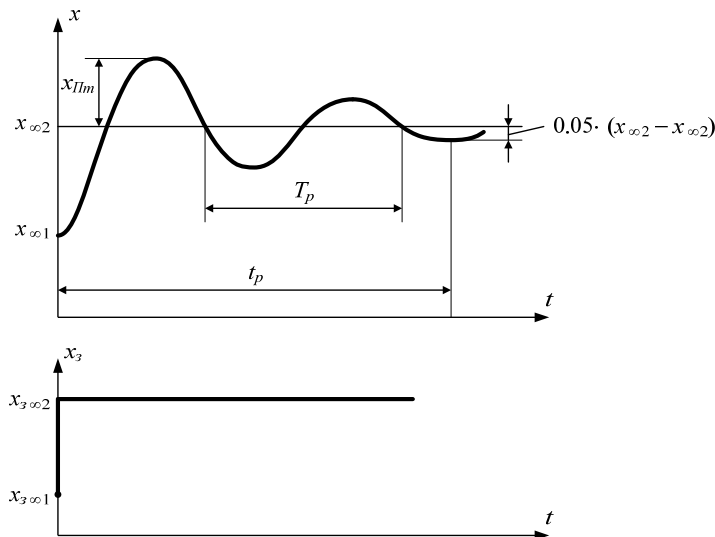


Fig 14.

#### 3. Oscillations

Full-wave oscillations number of output value change form regulation time.

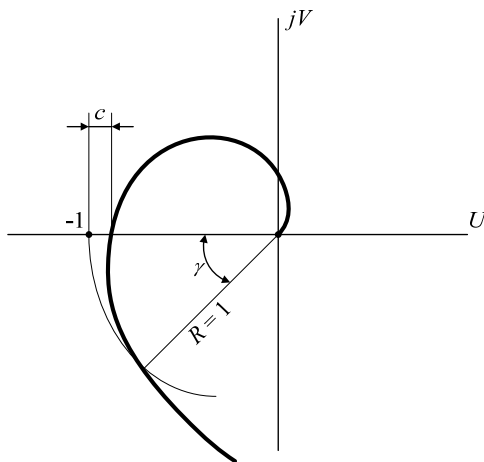
$$M = \frac{t_p}{T_p} .$$

Automatic control system which is in stability limit has bad transient process quality parameters. In this case the transient process represents undamped oscillations and that is why

$$x_{\Pi m*} = 1; \quad t_p \rightarrow \infty; \quad M \rightarrow \infty .$$

.It is evident the closer the system to stability margin the worse is transient process quality. Because of this fact we have indirect parameters of transient process quality using stability criteria.

Let us analyze such criteria based on Nyquist criterion. Figure 15 shows magnitude and phase characteristic of open loop system which is stable being closed.



**Figure 15**

As it is shown that the system can be in stability margin because of magnitude change or magnitude and phase characteristic vectors rotation to the point with coordinates  $-1; j0$ .

As indirect parameters of transient process quality we have the variables shown at Figure 15:

"c" –modulus stability margin

" $\gamma$ " – phase stability margin

### **1.8.7. Analysis of controller type and parameters on stability in frequency domain.**

Plotting open loop system magnitude and phase response on complex plane allows us examine the influence of controller type and parameters on stability margins.

By analyzing remember that object transfer function is known as well as its magnitude and phase characteristic.

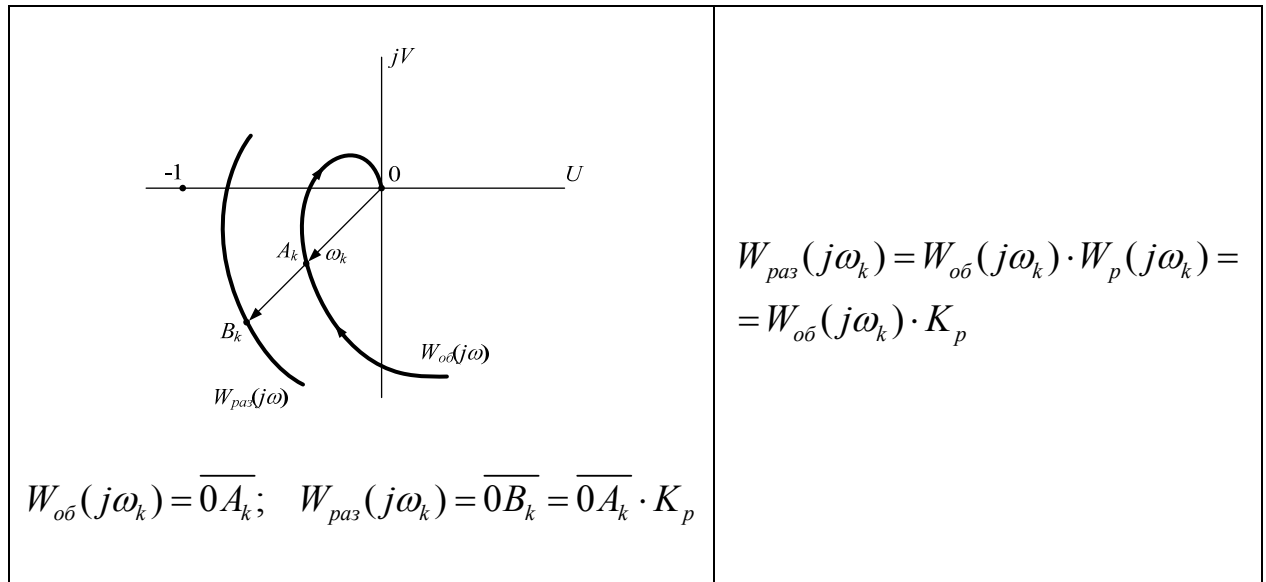
Cotroller type and its parameters influence can be detected when taking the usual the magnitude and phase locus vector being open at some  $\omega_k$  frequency in the following form

$$\begin{aligned}
W_{pa3}(j\omega_k) &= W_{o\bar{o}}(j\omega_k) \cdot W_p(j\omega_k) = \\
&= W_{o\bar{o}}(j\omega_k) \cdot \left[ K_p + K_d \cdot j\omega_k + \frac{K_u}{j\omega_k} \right] = \\
&= W_{o\bar{o}}(j\omega_k) \cdot K_p + W_{o\bar{o}}(j\omega_k) \cdot K_d \cdot \omega_k \cdot e^{j90^\circ} + W_{o\bar{o}}(j\omega_k) \cdot \frac{K_u e^{-j90^\circ}}{\omega_k}.
\end{aligned} \tag{51}$$

Formula (50) provides the rule of open loop system magnitude and phase characteristic visualizing.

Let us analyze special cases

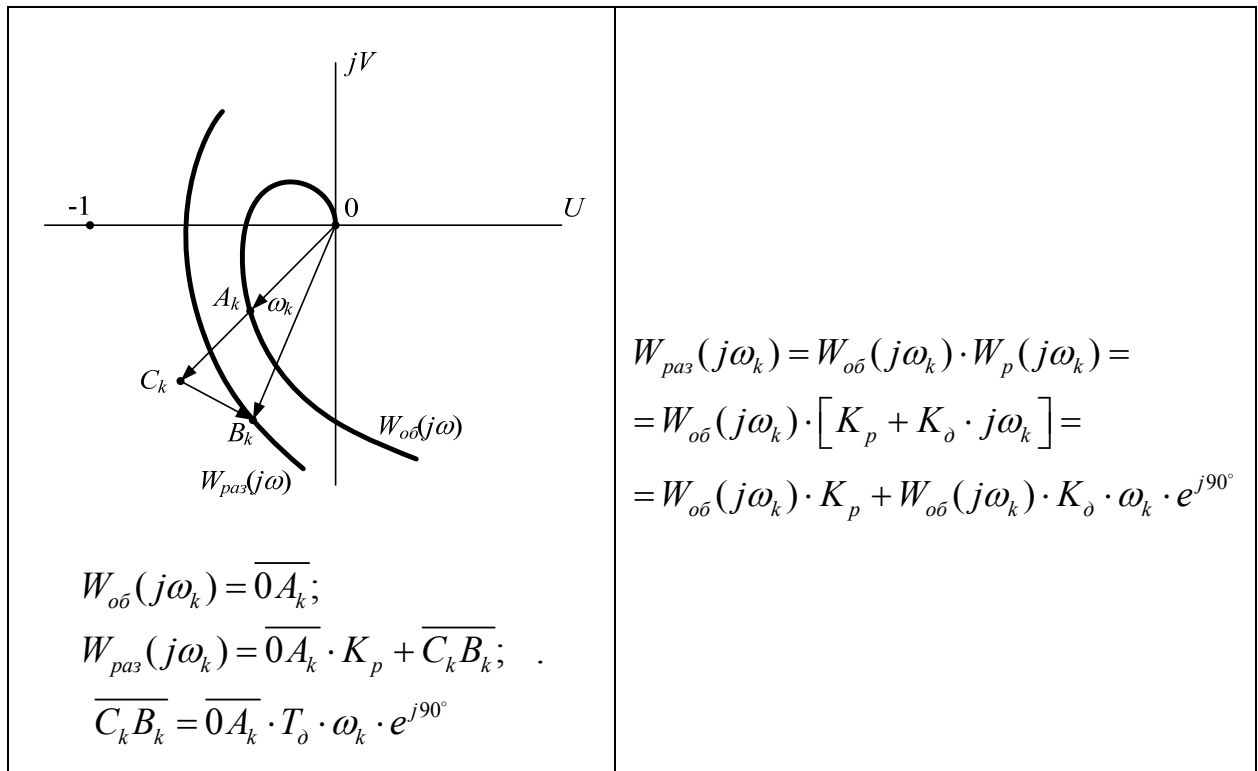
1 P controller (proportional controller) ( $K_p > 0$ ;  $K_d = 0$ ;  $K_u = 0$ ).



**Figure 16**

As shown above in system with P controller  $K_p$  gain ***causes static accuracy increasing***. At the same time as Figure 16 shows that  $K_p$  gain can cause stability margins decrease, at some big  $K_p$  ***value can break stability***.

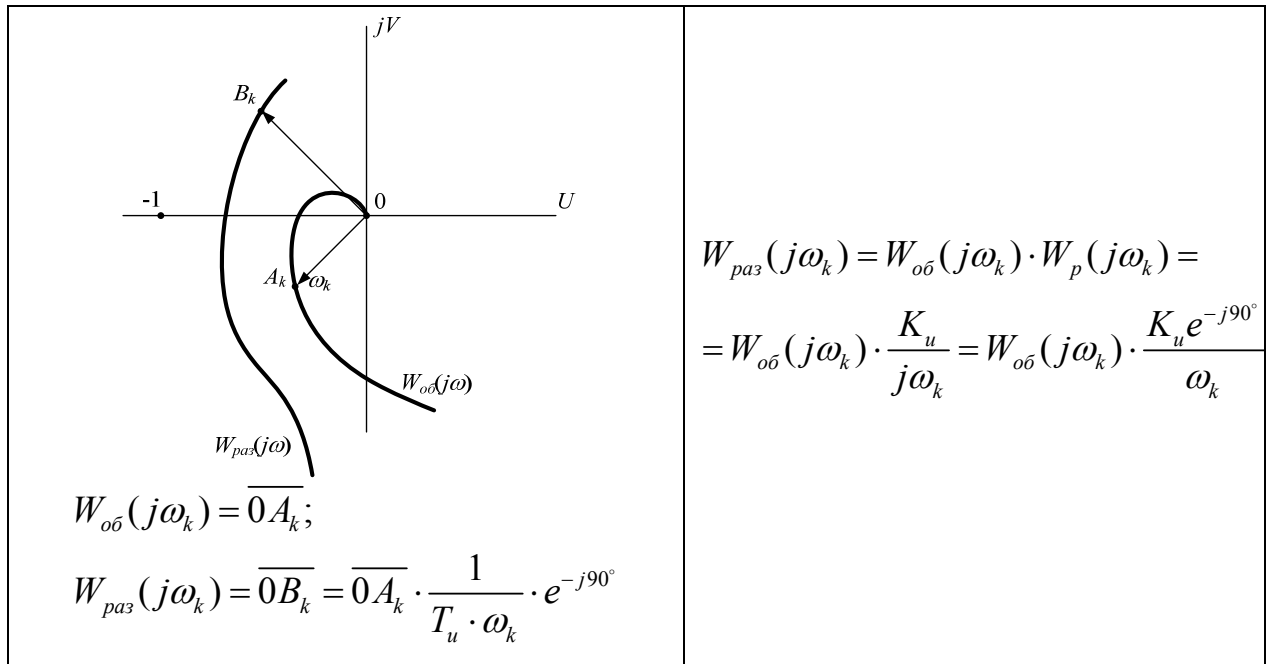
2. PD controller ( $K_p > 0$ ;  $K_d > 0$ ;  $K_u = 0$ ).



**Figure 17**

Derivative component *does not influence system properties* in steady state operating conditions, but Figure 17 *shows that it increases stability margins..*

**3. I controller** ( $K_p = 0$ ;  $K_\delta = 0$ ;  $K_u > 0$ ).

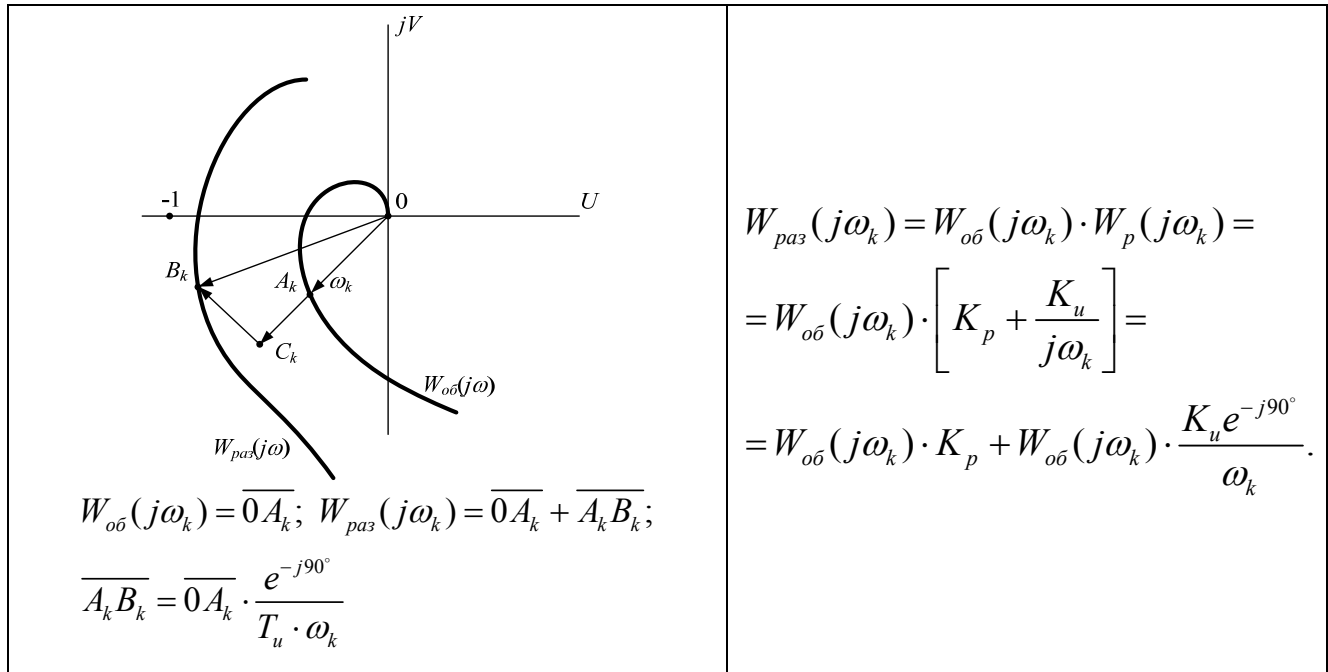


**Figure 18**



***In steady state operating conditions in system with I controller static errors don't occur*** . But integral controller makes stability margin decrease as all the vectors  $W_{pa3}(j\omega)$  rotate  $90^\circ$  to point side  $-1; j0$ .

4. PI Controller(proportional integral controller) ( $K_p > 0$ ;  $K_\delta = 0$ ;  $K_u > 0$ ).



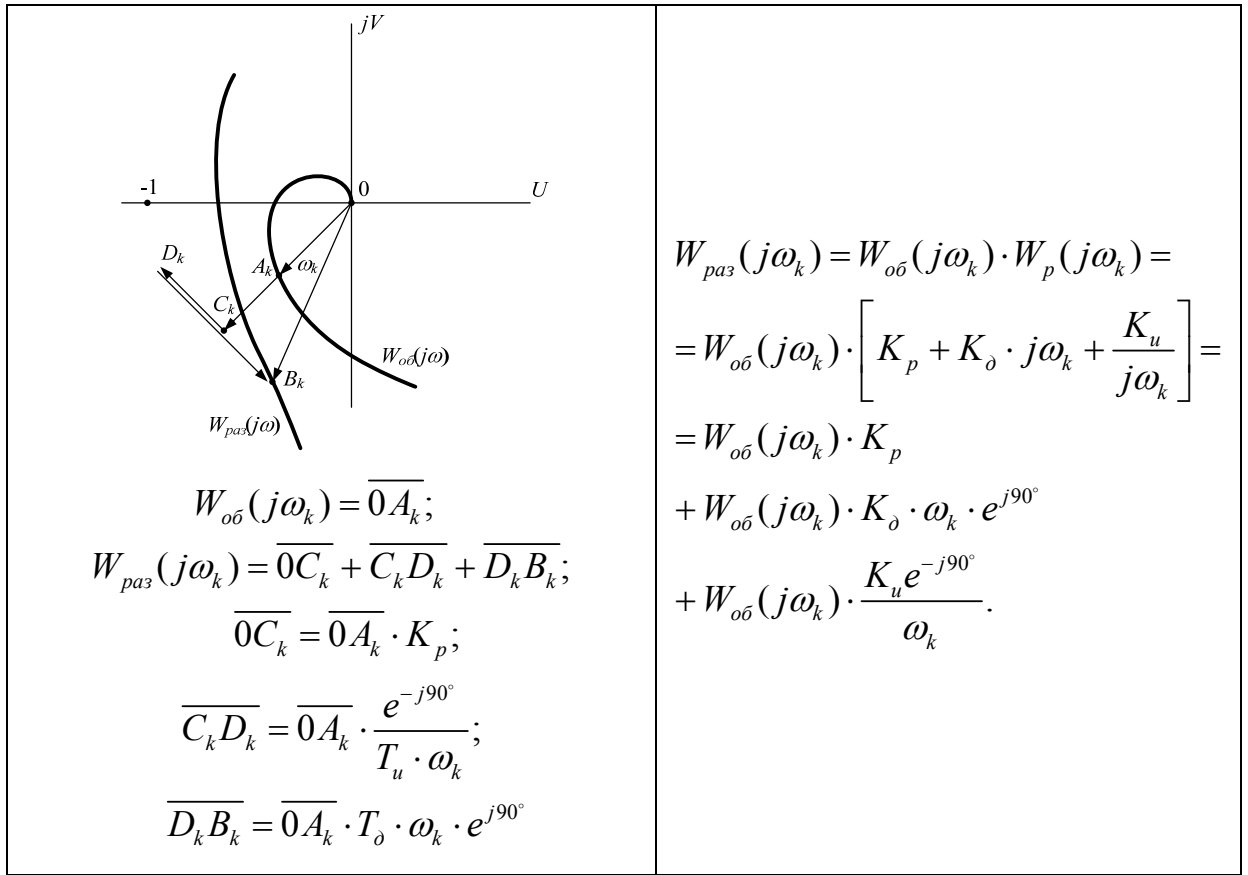
**Figure 19**

Integral component of regulation law **turns to complete static errors elimination but on equal terms decreases stability reserves (by using components  $B_k C_k$  ///)**

Correlation of controller type and its parameters influence both system property in steady state operating conditions and on stability margins makes it easy to determine that improving accuracy of the system and increasing stability margin are controversial

To solve this contradiction is to change structure of the system adding same subsidiary connections.

PID controller can be used to illustrate this example. Steady-state operating conditions of such a system are described above and it is seen that static errors occur in it. Figure 20 illustrates how three components influence magnitude and phase frequency response position in closed loop state.



**Figure 20**

From Figure 20 it's evident that stability reserves by integral component(vector  $C_k D_k$ ) can be eliminated by simultaneous use of differential component(vector  $D_k B_k$ )

**To remember.**

1. At amplification coefficient increase in proportional controller transient process time increase static error decreases, but overregulation increases. Uncontrolled coefficient's increase makes the system unstable.
2. Integral controller decreases the static error, but worsens system dynamic characteristics, i.e. increases transient process time increase, increases overregulation and regulation time.
3. Differential controller increases system stability, decreases overregulation and regulation time, i.e. improves system dynamic characteristics, doesn't influence the system in steady-state.

## 1.9. Stability areas determination

### 1.9.1. D-decomposition method

The D-decomposition method is used to investigate the influence of one or two parameters which are coefficients of system characteristic equation on CS stability. Let's analyze D –decomposition method by one parameter.

Write down the characteristic polynomial and change  $v$  varying coefficient

$$A(j\omega) = P(j\omega) + vQ(j\omega) \rightarrow v = -\frac{P(j\omega)}{Q(j\omega)} = U(\omega) + jV(\omega)$$

Let's plot the locus with axis  $U(\omega)$ ,  $jV(\omega)$ . The locus divides the complex plane into subareas with the certain range of varying parameter change. Selecting in subarea the arbitrary value of the investigated parameter  $v$  substitute its value into the characteristic polynomial and find polynomial roots. If **n**-left roots of **n** order characteristic polynomial are in area then this area is called **D(n)**- area. If n-m of left roots but m of right roots **m** then the area is called **D(n-m)** area.

Analyze **the example** using **MathCAD**

#### D-decomposition method

$$D(p, k) := p^3 + p^2 + p + k \quad k(\omega) := -\omega^2 + i \cdot (-\omega^3 + \omega)$$

$$U(\omega) := \omega^2 \quad \underline{V(\omega)} := \omega^3 - \omega$$

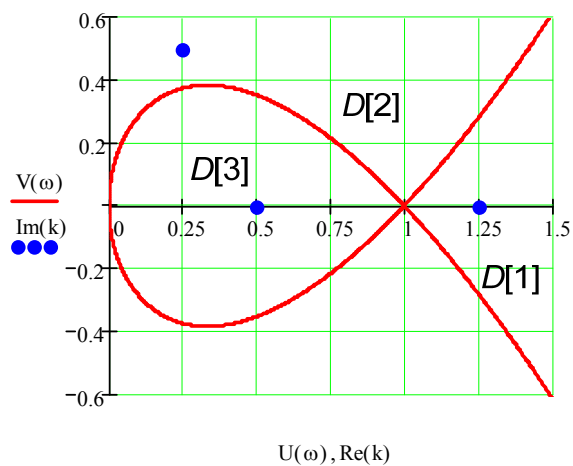
$$D(i\omega, k) \text{ complex} \rightarrow (-\omega^2) + k + i \cdot [(-\omega^3) + \omega]$$

$$\underline{k} := (0.5 \quad i \cdot 0.5 + 0.25 \quad 1.25)^T$$

$$D(p, k_0) \left| \begin{array}{l} \text{solve, p} \\ \text{float, 5} \end{array} \right. \rightarrow \begin{bmatrix} -0.64780 \\ (-0.17610) - 0.86072i \\ (-0.17610) + 0.86072i \end{bmatrix} \quad D(p, k_1) \left| \begin{array}{l} \text{solve, p} \\ \text{float, 5} \end{array} \right. \rightarrow \begin{bmatrix} (-0.76205) - 0.53915i \\ (-0.33671) + 1.0658i \\ 0.98763e-1 - 0.52661i \end{bmatrix}$$

$$D(p, k_2) \left| \begin{array}{l} \text{solve, p} \\ \text{float, 5} \end{array} \right. \rightarrow \begin{pmatrix} -1.1118 \\ .55901e-1- 1.0589i \\ .55901e-1+ 1.0589i \end{pmatrix}$$

$$\omega := -1.5, -1.5 + 0.001.. 1.5$$



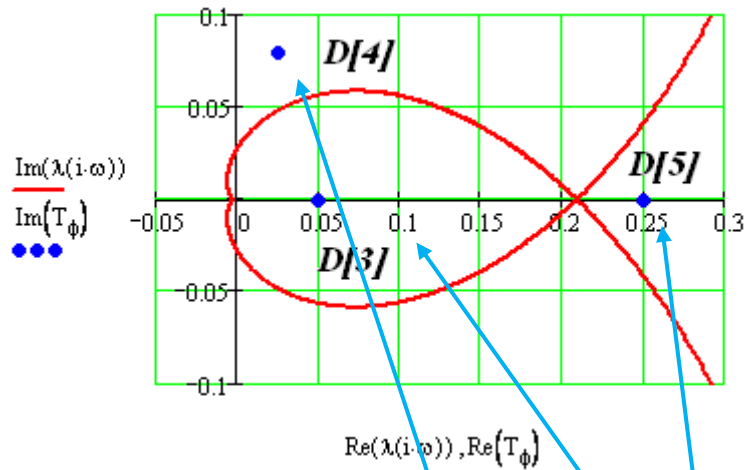
$$\text{ORIGIN} := 1 \quad T := (0.04 \ 0.08 \ 0.25)^T \quad T_p := 0.2 \quad T_\phi := 0.4$$

$$A(p, T_\phi) := \begin{pmatrix} T_\phi \cdot p & 0 & 0 & 0 & -1 \\ 1 & T_1 \cdot p + 1 & 0 & 0 & 0 \\ 0 & -1 & T_2 \cdot p + 1 & 1 & 0 \\ 0 & 0 & -1 & T_3 \cdot p & 0 \\ 0 & 0 & 0 & -1 & T_p \cdot p + 1 \end{pmatrix}$$

$$|A(p, T_\phi)| \left| \begin{array}{l} \text{collect, } T_\phi \\ \text{float, 3} \end{array} \right. \rightarrow (.160e-3p^5 + .880e-1p^3 + .49p^2 + p + .680e-2p^4) \cdot T_\phi + 1.$$

$$\lambda(p) := \frac{-1}{(.160e-3p^5 + .880e-1p^3 + .49p^2 + p + .680e-2p^4)}$$

$$\omega := -20, -20 + 0.01..20 \quad T_\phi := (0.05 \ 0.25 \ i \cdot 0.08 + 0.025)^T$$



$$T_{\phi_1} = 0.05$$

$$\left| A(p, T_{\phi_1}) \right| = 0 \quad \left| \begin{array}{l} \text{solve, p} \\ \text{float, 5} \end{array} \right. \rightarrow \begin{bmatrix} -25.614 \\ (-10.227) - 7.4396i \\ (-10.227) + 7.4396i \\ \boxed{1.7839 - 5.2278i} \\ \boxed{1.7839 + 5.2278i} \end{bmatrix}$$

$$T_{\phi_2} = 0.25$$

$$\left| A(p, T_{\phi_2}) \right| = 0 \quad \left| \begin{array}{l} \text{solve, p} \\ \text{float, 5} \end{array} \right. \rightarrow \begin{bmatrix} -25.134 \\ (-8.5027) - 4.9867i \\ (-8.5027) + 4.9867i \\ (-18014) - 3.1945i \\ (-18014) + 3.1945i \end{bmatrix}$$

$$T_{\phi_3} = 0.025 + 0.08i$$

$$\left| A(p, T_{\phi_3}) \right| = 0 \quad \left| \begin{array}{l} \text{solve, p} \\ \text{float, 5} \end{array} \right. \rightarrow \begin{bmatrix} (-25.146) + .37205i \\ (-11.589) - 4.6065i \\ (-7.4132) + 7.4857i \\ (-7.7487) - 6.0487i \\ \boxed{2.4230 + 2.7975i} \end{bmatrix}$$

## 1.9.2. Root locus analysis

Root locus is defined as characteristic polynomial roots trajectories at one parameter change of automatic system control from 0 to  $\infty$

Let characteristic polynomial has the form of:

$$A(p) = P(p) + \lambda Q(p) = 0$$

Where  $P(p), Q(p)$  are polynomials of  $n$  and  $m$  power,  $\lambda$  is the investigated parameter.

Let polynomial  $A(p)$  has the form of:

$$A(p) = \prod_{i=1}^n (p - z_i) + \lambda \prod_{i=1}^m (p - q_i) = 0$$

At  $\lambda = 0$  polynomial  $A(p)$  has  $n$  roots  $z_i$ , but at  $\lambda = \infty$  has only  $m$  roots  $q_i$ . Root locus will have  $n$  branches which start from  $n$  roots  $z_i$  and end in  $m$   $q_i$  roots. The rest  $n-m$  end in eternity. Asymptotes for branches ending in eternity come from one point – asymptotes point. Let's write down the formula of asymptote center determination.

$$x_0 = \frac{\sum_{i=1}^n z_i + \sum_{i=1}^m q_i}{n - m}$$

Angles between beams of the obtained star are determined by formula:

$$\phi = 2\pi / (n - m)$$

The nearest angle to real axis equals  $\phi / 2$ .

Root locus branches separation points from real axis are function extremum points.

$$\frac{P(p)}{Q(p)} = f(p) = -\lambda$$

Critical value of the required parameter at which the system is at the stability margin is determined at branches points of root locus crossing with imaginary axis. Critical value can be obtained by solving the equation:

$$A(j\omega) = P(j\omega) + \lambda Q(j\omega) = X(\omega, \lambda) + jY(\omega, \lambda) = 0$$

Solving system of equations

$$\begin{cases} X(\omega, \lambda) = 0 \\ Y(\omega, \lambda) = 0 \end{cases},$$

we obtain  $\omega_{kp}, \lambda_{kp}$ .

**Example:** Transfer function of open loop system is given as

$$W(p, k) = \frac{k}{p(p+1)(p+3)(p+4)}, \text{ determine at which } k \text{ values closed loop system}$$

will be stable.

Use **MathCAD** to solve the problem

### Root locus construction

$$W(p, k) := \frac{k}{p \cdot (p+1) \cdot (p+3) \cdot (p+4)} \quad W_c(p, k) := \frac{W(p, k)}{1 + W(p, k)}$$

$$W_c(p, k) \text{ simplify } \rightarrow \frac{k}{k + p^4 + 8 \cdot p^3 + 19 \cdot p^2 + 12 \cdot p}$$

Characteristic polynomial of a closed loop system

$$C(p, k) := k + p^4 + 8 \cdot p^3 + 19 \cdot p^2 + 12 \cdot p \quad P(p) := p^4 + 8 \cdot p^3 + 19 \cdot p^2 + 12 \cdot p \quad Q(p, k) := k$$

Root calculation at fixed value of system k -parameter

$$L(k) := \text{polyroots}((k \ 12 \ 19 \ 8 \ 1)^T)$$

$$\text{The beginning points of root locus branches } p := P(p) \text{ solve, } p \rightarrow \begin{pmatrix} 0 \\ -4 \\ -3 \\ -1 \end{pmatrix}$$

$$\text{Roots trajectories calculation in locus} \quad N := 500 \quad i := 0..N \quad x_i := \frac{100}{N} \cdot i + 0.1 \quad Lc_i := L(x_i)$$

Asymptotes center point calculation  $x_0 := \frac{\sum_{k=0}^3 p_k}{4} \quad x_0 = -2$


Locus branching points from real axis

$$\frac{d}{dp} C(p, k) \left| \begin{array}{l} \text{solve, } p \\ \text{float, } 5 \end{array} \right. \rightarrow \begin{pmatrix} -2. \\ -4188 \\ -3.5812 \end{pmatrix}$$

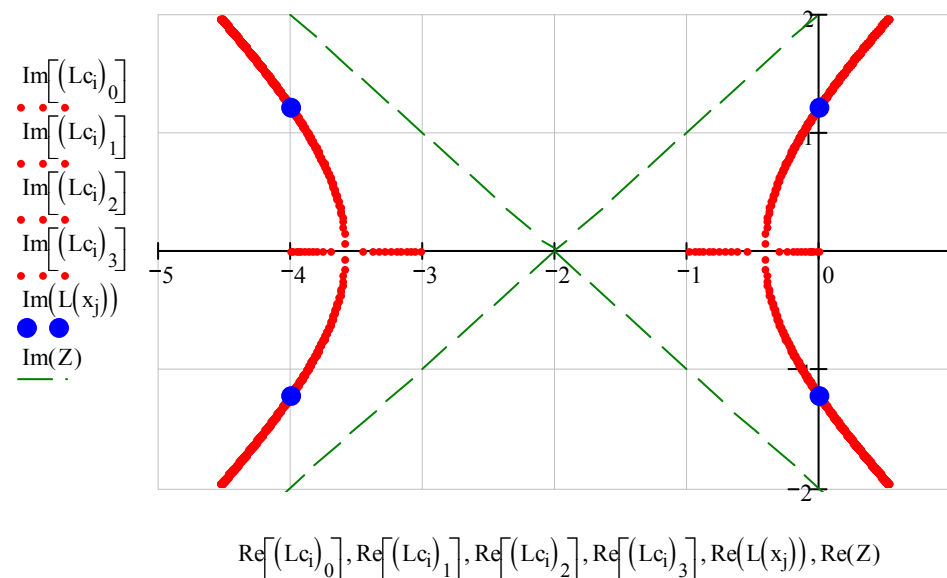
Auxiliary vector of plotting

$$Z := \left[ x_0 \quad x_0 + 10 \cdot e^{i \cdot \left(-\frac{\pi}{4}\right)} \quad 10 \cdot e^{i \cdot \left(\frac{\pi}{4}\right)} + x_0 \quad x_0 + 10 \cdot e^{i \cdot \left(\frac{\pi}{4} \cdot 3\right)} + x_0 \quad 10 \cdot e^{i \cdot \left(\frac{\pi}{4} \cdot 5\right)} + x_0 \quad x_0 \right]^T$$

Approximate critical value of the required parameter  $k=x_j$

$j :=$  

$$L(x_j) = \begin{pmatrix} -3.99141 + 1.21066i \\ -3.99141 - 1.21066i \\ -0.00859 - 1.21066i \\ -0.00859 + 1.21066i \end{pmatrix} \quad j = 127 \quad x_j = 25.5$$



Find critical parameters

$$C(j \cdot \omega, k) \text{ complex} \rightarrow k + \omega^4 - 19 \cdot \omega^2 + i \cdot [(-8) \cdot \omega^3 + 12 \cdot \omega]$$



$$\left[ \begin{array}{c} k + \omega^4 - 19\omega^2 \\ (-8)\omega^3 + 12\omega \end{array} \right] \left| \begin{array}{l} \text{solve,} \left( \begin{array}{c} \omega \\ k \end{array} \right) \\ \text{float, 5} \end{array} \right. \rightarrow \left( \begin{array}{cc} 0 & 0 \\ 1.2248 & 26.250 \\ -1.2248 & 26.250 \end{array} \right)$$

### 1.9.3. Poles placement method

One of the simplest methods of controller synthesis is ***poles placement*** of closed loop system transfer function which determine its dynamics namely operation speed and damping of oscillation (see chapter 6.8). The main idea that you should place these poles at given points of complex plane with the help of specially chosen controller. This problem can be solved with the help of linear equations system.

Let transfer function be polynomials relation

$$W_{oo}(p) = \frac{N(p)}{D(p)} = \frac{n_1 p + n_0}{p^2 + d_1 p + d_0}$$

Let's choose controller type

$$R(p) = \frac{N_R(p)}{D_R(p)} = \frac{a_1 p + a_0}{b_1 p + b_0}$$

where  $a_0, a_1, b_0, b_1$  are unknown coefficients to be determined

Characteristic polynomial of closed loop system is equal to

$$\begin{aligned} \Delta(p) &= N(p)N_R(p) + D(p)D_R(p) = (n_1 p + n_0)(a_1 p + a_0) + (p^2 + d_1 p + d_0)(b_1 p + b_0) \\ &= b_1 p^3 + (n_1 a_1 + d_1 b_1 + b_0) p^2 + (n_0 a_1 + n_1 a_0 + d_1 b_0 + d_0 b_1) p + n_0 a_0 + d_0 b_0 \end{aligned}$$

Suppose we want to choose the transfer function of controller to place polynomial roots  $\Delta(p)$  at given points to keep the identity:

$$\Delta(p) = p^3 + \delta_2 p^2 + \delta_1 p + \delta_0$$

where  $\delta_0, \delta_1, \delta_2$  are given numbers. Making the coefficient with the same power of  $p$  in the last two equations we obtain:

$$\begin{aligned}
p^3: & \quad b_1 = 1 \\
p^2: & \quad n_1 a_1 + d_1 b_1 + b_0 = \delta_2 \\
p: & \quad n_0 a_1 + n_1 a_0 + d_1 b_0 + d_0 b_1 = \delta_1 \\
p^0: & \quad n_0 a_0 + d_0 b_0 = \delta_0
\end{aligned}$$

or matrix form as follows :

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ n_1 & 0 & d_1 & 1 \\ n_0 & n_1 & d_0 & d_1 \\ 0 & n_0 & 0 & d_0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} 1 \\ \delta_2 \\ \delta_1 \\ \delta_0 \end{pmatrix}$$

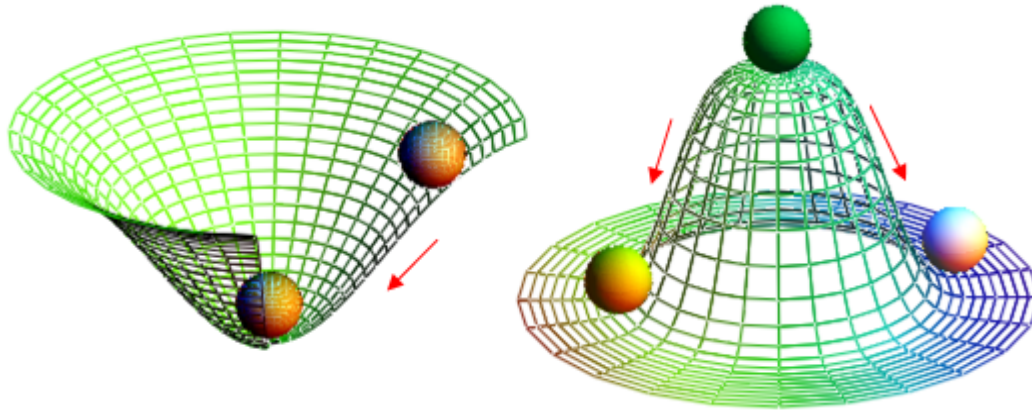
Equation solution can be written as:

$$\begin{pmatrix} a_0 \\ a_1 \\ b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ n_1 & 0 & d_1 & 1 \\ n_0 & n_1 & d_0 & d_1 \\ 0 & n_0 & 0 & d_0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \delta_2 \\ \delta_1 \\ \delta_0 \end{pmatrix}$$

Square matrix in this formula (it is called Sylvester matrix) needs to be reversible.

## CHAPTER 2. NONLINEAR DYNAMIC SYSTEMS

### 2.1. Phase space and dynamic systems phase portraits



**Figure 1**      Stable system

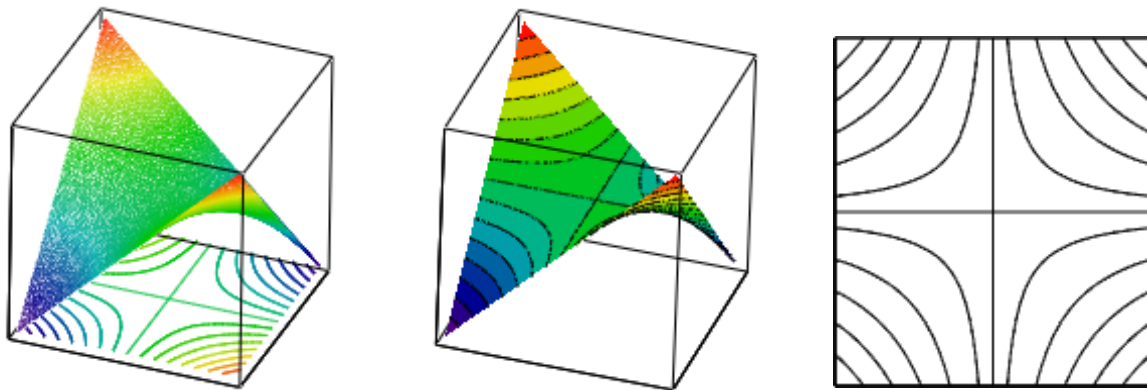
Unstable system

Any electromechanic system is a dynamic system. Elements forming the system can be nonlinear, hence differential equations describing dynamic systems are nonlinear.

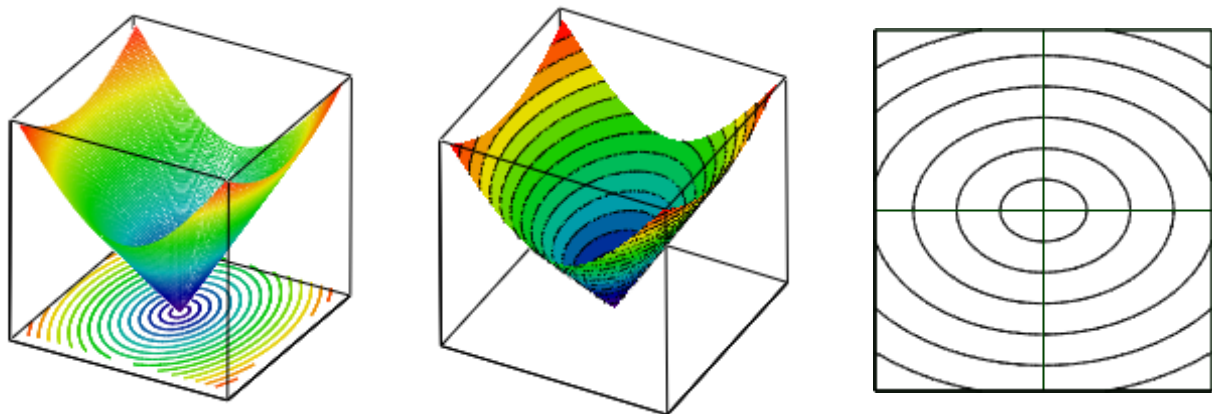
To investigate nonlinear systems and make the understanding of complex dynamic taking place in them *phase space* is used in which one can construct *phase portraits* (see Figure 2). Each dynamic system has its phase portrait.

On phase portrait there are *special points* – *equilibrium points*, which help foretell dynamic system behavior without solving differential equations. These equilibrium points can be stable and unstable. If dynamic system is in the range of stable equilibrium point, then small disturbances will not make the system unstable (see Figure 1). If equilibrium point is not stable, then disturbances persist and make the system "unstable" (see Figure 1).

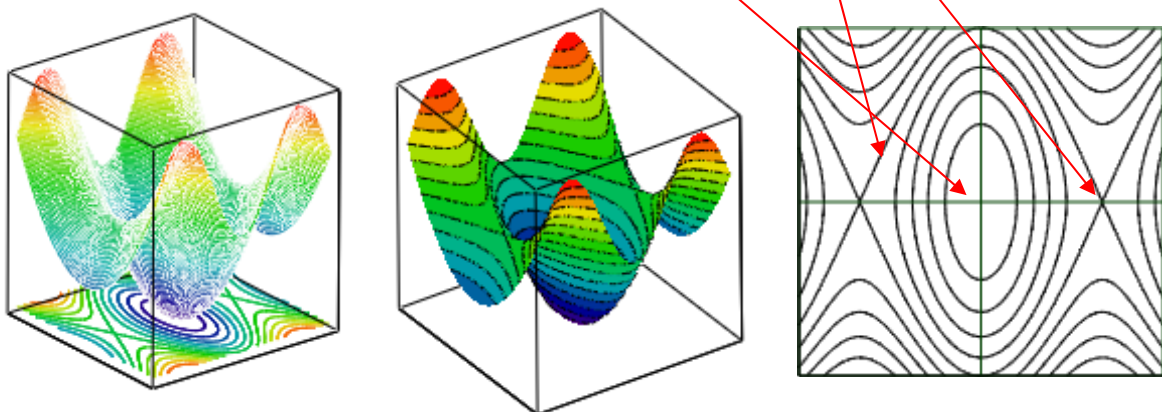
equilibrium point of "saddle" type



equilibrium point of "centre" type



equilibrium points of "centre" and "saddle" type,  
the line separating solutions—"separatrix"



**Figure 2.** Dynamic system phase portraits

Let's analyze phase space method relating to dynamic system of the second order.

$$\begin{cases} \frac{dx}{dt} = F_1(x, y) \\ \frac{dy}{dt} = F_2(x, y) \end{cases}, \quad (1)$$

where  $F_1(x, y)$  and  $F_2(x, y)$  are nonlinear functions of their own arguments.

To get phase trajectory time need to be eliminated. To do this divide the second equation by the first one

$$\frac{dy}{dx} = \frac{F_2(x, y)}{F_1(x, y)} \quad (2)$$

At equilibrium points time derivative turns zero, then we obtain the expression  $0/0$  which has no defined value and is called an indeterminate form:

$$\frac{dy}{dx} = \frac{F_2(x, y)}{F_1(x, y)} = \frac{0}{0} \quad (3)$$

The points at which there are uncertainties are called singular points. In singular points on phase plane solutions can bifurcate. Let us describe equilibrium state points search algorithm.

At equilibrium state the given variables  $x$  and  $y$  do not change, hence derivatives equal zero:

$$\begin{cases} 0 = F_1(x, y) \\ 0 = F_2(x, y) \end{cases} \quad (4)$$

By solving the given nonlinear equations system we find equilibrium state points  $x_0, y_0$ . After determining equilibrium state point coordinates one need to determine point type. To determine point type expand  $F_1(x, y)$  and  $F_2(x, y)$  functions in

equilibrium state points neighborhood  $x_0, y_0$ , and analyze the first three expansion terms:

$$\begin{cases} F_1(x, y) = F_1(x_0, y_0) + \frac{\partial F_1}{\partial x}(x - x_0) + \frac{\partial F_1}{\partial y}(y - y_0) + \dots \\ F_2(x, y) = F_2(x_0, y_0) + \frac{\partial F_2}{\partial x}(x - x_0) + \frac{\partial F_2}{\partial y}(y - y_0) + \dots \end{cases} \quad (5)$$

Put down coefficients at linear terms and obtain Jacobian matrix:

$$\mathbf{A}(x_0, y_0) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad (6)$$

Jacobian matrix elements  $a_{i,j}$  are constant variables.

Next step is to determine point character .

To do this one should find eigenvalues of Jacobian matrix  $\lambda$  by solving characteristic equation.

$$|\mathbf{I} - \lambda \mathbf{A}| = 0. \quad (7)$$

where  $\mathbf{I}$  is identity matrix.

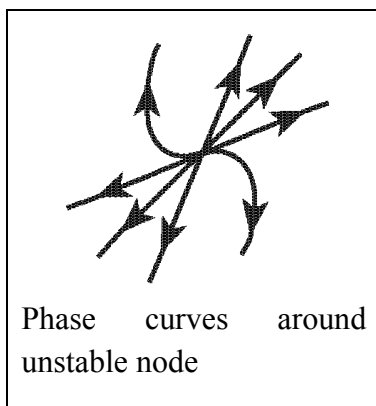
It is easy to determine eigenvalues by using **MathCAD** **eigenvalues(A)** function

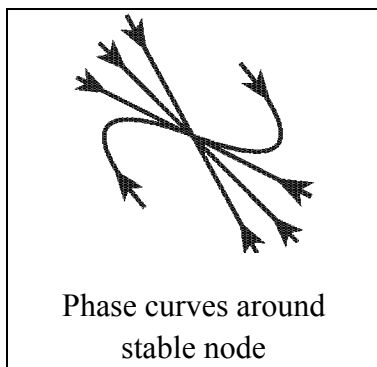
application, several variants are acceptable.

1. Both eigenvalues are real and positive:

$$\lambda_1 > 0, \lambda_2 > 0$$

Then, equilibrium state point with coordinates  $x_0, y_0$  is called an ***unstable node***

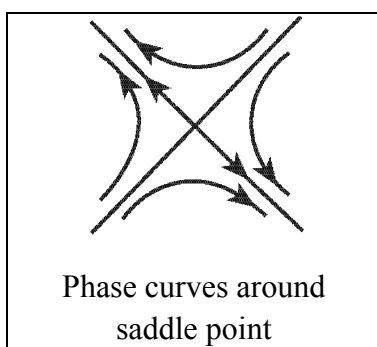




2. Both eigenvalues are real and negative:

$$\lambda_1 < 0, \lambda_2 < 0$$

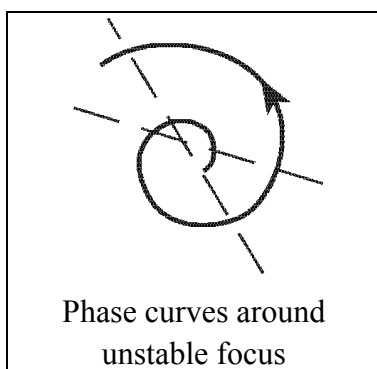
Then equilibrium state point with coordinates  $x_0, y_0$  is called a **stable node**.



3. Both eigenvalues are real and have different signs:

$$\lambda_1 < 0, \lambda_2 > 0 \quad \text{или} \quad \lambda_2 < 0, \lambda_1 > 0$$

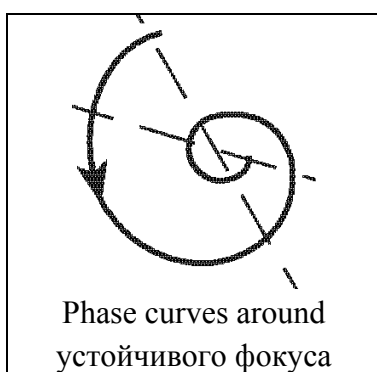
Then equilibrium state point  $x_0, y_0$  is called a **saddle or a saddle node**.



4. Eigenvalues are complex conjugated numbers which have positive real parts:

$$\lambda_1 = \beta + j\omega, \lambda_2 = \beta - j\omega, \quad \beta > 0$$

Then equilibrium state point  $x_0, y_0$  is called an **unstable focus**

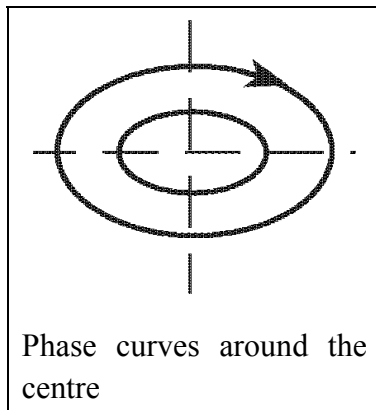


5. Eigenvalues are complex conjugated numbers which have negative real parts:

$$\lambda_1 = \beta + j\omega, \lambda_2 = \beta - j\omega, \quad \beta < 0$$

Then equilibrium state point  $x_0, y_0$  is called a **stable focus**

6. Eigenvalues are imaginary conjugated numbers:



$$\lambda_1 = j\omega, \lambda_2 = -j\omega$$

Then equilibrium state point with coordinates  $x_0, y_0$  is called a **centre**.

Let's analyze some examples of equilibrium state points determination and determine their type using **MathCAD**.

ORIGIN:= 1

**Example 1. Dynamic system is given**

$$\frac{d}{dt}x = x^2 + y^2 - 17$$

$$\frac{d}{dt}y = x \cdot y + 4$$

Find equilibrium points of dynamic system

$$x^2 + y^2 - 17 = 0$$

$$x \cdot y + 4 = 0$$

$$z := \begin{pmatrix} x^2 + y^2 - 17 \\ x \cdot y + 4 \end{pmatrix} \text{ solve, } \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -4 & 1 \\ -1 & 4 \\ 1 & -4 \\ 4 & -1 \end{pmatrix}$$

$$\text{xyz} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad y_0 := z \begin{pmatrix} 2 \end{pmatrix} \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\text{xyz}$  Expand nonlinear functions in the neighborhood of equilibrium state points.

$$x^2 + y^2 \text{ series, } x = -4, y = 1, 2 \rightarrow (-17) - 8 \cdot x + 2 \cdot y$$

$$a := \begin{pmatrix} -8 & 2 \\ 1 & -4 \end{pmatrix}$$



$\text{eigenvals}(a) = \begin{pmatrix} -8.449 \\ -3.551 \end{pmatrix}$  point with coordinates  $x=-4$  и  $y=1$  is a *stable node*

$x \cdot y$  series,  $x = -4, y = 1, 2 \rightarrow 4 - 4 \cdot y + x$

$$|a - \lambda \cdot I| \rightarrow 30 + 12 \cdot \lambda + \lambda^2$$

$x^2 + y^2$  series,  $x = -1, y = 4, 2 \rightarrow (-17) - 2 \cdot x + 8 \cdot y$

$$a := \begin{pmatrix} -2 & 8 \\ 4 & -1 \end{pmatrix}$$

$\text{eigenvals}(a) = \begin{pmatrix} -7.179 \\ 4.179 \end{pmatrix}$  point with coordinates  $x = -1$  и  $y = 4$  is called a *saddle node*.

$x \cdot y$  series,  $x = -1, y = 4, 2 \rightarrow 4 - y + 4 \cdot x$

$$|a - \lambda \cdot I| \rightarrow (-30) + 3 \cdot \lambda + \lambda^2$$

$x^2 + y^2$  series,  $x = 1, y = -4, 2 \rightarrow (-17) + 2 \cdot x - 8 \cdot y$

$$a := \begin{pmatrix} 2 & -8 \\ -4 & 1 \end{pmatrix}$$

$\text{eigenvals}(a) = \begin{pmatrix} 7.179 \\ -4.179 \end{pmatrix}$  point with coordinates  $x = 1$  и  $y = -4$  is called a *saddle*

$x \cdot y$  series,  $x = 1, y = -4, 2 \rightarrow 4 + y - 4 \cdot x$

$$|a - \lambda \cdot I| \rightarrow (-30) - 3 \cdot \lambda + \lambda^2$$

$x^2 + y^2$  series,  $x = 4, y = -1, 2 \rightarrow (-17) + 8 \cdot x - 2 \cdot y$

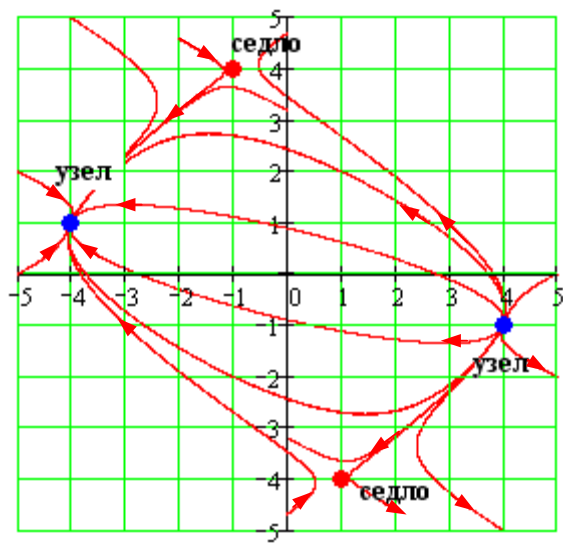
$x \cdot y$  series,  $x = 4, y = -1, 2 \rightarrow 4 + 4 \cdot y - x$

$$a := \begin{pmatrix} 8 & -2 \\ -1 & 4 \end{pmatrix}$$

$\text{eigenvals}(a) = \begin{pmatrix} 8.449 \\ 3.551 \end{pmatrix}$  point with coordinates  $x = 4$  и  $y = -1$  is called *an unstable node*.

$$|a - \lambda \cdot I| \rightarrow 30 - 12 \cdot \lambda + \lambda^2$$

## Phase portrait of dynamic system



### Example 2. Dynamic system is given

$$\frac{d}{dt}x = x^2 - y^2 - 5$$

$$\frac{d}{dt}y = x^2 + y^2 - 13$$

Find equilibrium points of dynamic system

$$x^2 - y^2 - 5 = 0$$

$$x^2 + y^2 - 13 = 0$$

$$z := \begin{pmatrix} x^2 - y^2 - 5 \\ x^2 + y^2 - 13 \end{pmatrix} \text{ solve, } \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 \\ -3 & 2 \\ 3 & -2 \\ -3 & -2 \end{pmatrix}$$

$$x_0 := z^{(1)} \quad y_0 := z^{(2)} \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Expand nonlinear functions in the neighbourhood of equilibrium state points

$$x^2 - y^2 \text{ series, } x = 3, y = 2, 2 \rightarrow (-5) + 6 \cdot x - 4 \cdot y$$

$$x^2 + y^2 \text{ series, } x = 3, y = 2, 2 \rightarrow (-13) + 6 \cdot x + 4 \cdot y$$

$$a := \begin{pmatrix} 6 & -4 \\ 6 & 4 \end{pmatrix} \quad |a - \lambda \cdot I| \rightarrow 48 - 10\lambda + \lambda^2$$

$\text{eigenvals}(a) = \begin{pmatrix} 5 + 4.796i \\ 5 - 4.796i \end{pmatrix}$  point with coordinates  $x=3$  и  $y=2$  is called an *unstable focus*

$$x^2 - y^2 \text{ series, } x = -3, y = 2, 2 \rightarrow (-5) - 6 \cdot x - 4 \cdot y$$

$$x^2 + y^2 \text{ series, } x = -3, y = 2, 2 \rightarrow (-13) - 6 \cdot x + 4 \cdot y$$

$$a := \begin{pmatrix} -6 & -4 \\ -6 & 4 \end{pmatrix} \quad |a - \lambda \cdot I| \rightarrow (-48) + 2\lambda + \lambda^2$$

$\text{eigenvals}(a) = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$  point with coordinates  $x=-3$  и  $y=2$  is called a *saddle*.

$$x^2 - y^2 \text{ series, } x = 3, y = -2, 2 \rightarrow (-5) + 6 \cdot x + 4 \cdot y$$

$$x^2 + y^2 \text{ series, } x = 3, y = -2, 2 \rightarrow (-13) + 6 \cdot x - 4 \cdot y$$

$$a := \begin{pmatrix} 6 & 4 \\ 6 & -4 \end{pmatrix} \quad |a - \lambda \cdot I| \rightarrow (-48) - 2\lambda + \lambda^2$$

$\text{eigenvals}(a) = \begin{pmatrix} 8 \\ -6 \end{pmatrix}$  point with coordinates  $x=3$  и  $y=-2$  is called a *saddle*.

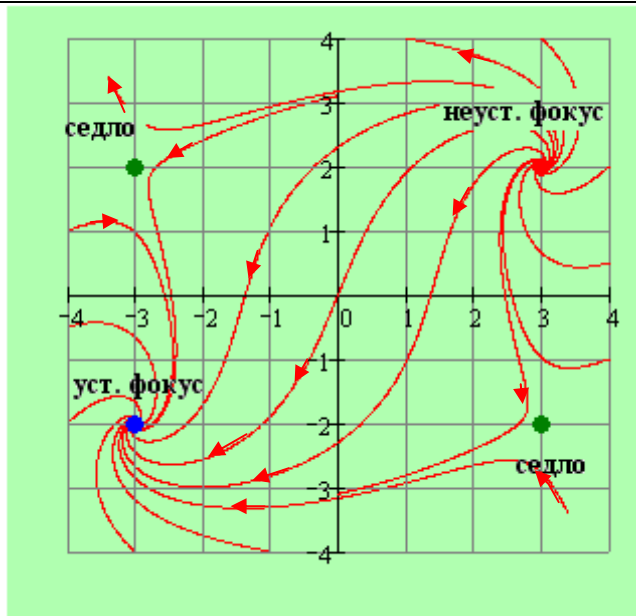
$$x^2 - y^2 \text{ series, } x = -3, y = -2, 2 \rightarrow (-5) - 6 \cdot x + 4 \cdot y$$

$$x^2 + y^2 \text{ series, } x = -3, y = -2, 2 \rightarrow (-13) - 6 \cdot x - 4 \cdot y$$

$$a := \begin{pmatrix} -6 & 4 \\ -6 & -4 \end{pmatrix} \quad |a - \lambda \cdot I| \rightarrow 48 + 10\lambda + \lambda^2$$

$\text{eigenvals}(a) = \begin{pmatrix} -5 + 4.796i \\ -5 - 4.796i \end{pmatrix}$  point with coordinates  $x=3$  и  $y=2$  is called a *stable focus*.

Phase portrait of dynamic system.



**Example 3. Determine equilibrium state point type of a linearized dynamic system.**

$$\frac{d}{dt}x = -x - 4y$$

$$\frac{d}{dt}y = 2x + 5y$$

$$\underline{\underline{A}} := \begin{pmatrix} -1 & -4 \\ 2 & 5 \end{pmatrix} \quad \lambda := \text{eigenvals}(A) \quad \lambda = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{Point type } x=0, y=0 \text{ } \textit{unstable node}$$

**Example 4. Determine equilibrium state point type of a linearized dynamic system.**

$$\frac{d}{dt}x = -x - y$$

$$\frac{d}{dt}y = x - 3y$$

$$A := \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix} \quad \underline{\underline{\lambda}} := \text{eigenvals}(A) \quad \lambda = \begin{pmatrix} -2 \\ -2 \end{pmatrix} \quad \text{Point type } x=0, y=0 \text{ } \textit{is a stable node.}$$

### 2.1.1. Phase portraits construction with the help of surfaces.

Let's analyze some more examples of surface construction which help determine decomposition structure of phase plane into trajectories. First additional calculations should be done. Put down Newton's equation of motion with unit mass on which force  $F(x)$  influences:

$$\frac{d^2x}{dt^2} = F(x) \quad (8)$$

Taking into account that force equals potential function gradient with negative sign the equation can be presented as:

$$F(x) = -\frac{dU(x)}{dx} \rightarrow \frac{d^2x}{dt^2} = -\frac{dU(x)}{dx} \quad (9)$$

This equation can be rearranged by multiplying each part by velocity  $v = dx / dt$

$$v \frac{dv}{dt} = -\frac{dU}{dx} \frac{dx}{dt} \rightarrow \frac{1}{2} \frac{dv^2}{dt} + \frac{dU}{dt} = 0 \rightarrow \frac{d}{dt} \left( \frac{v^2}{2} + U \right) = 0$$

From the obtained formula it follows that formula in brackets equals some constant  $E$

$$\frac{v^2}{2} + U(x) = E(x, v) \quad \text{где} \quad U(x) = -\int_0^x F(x) dx \quad (10)$$

Now let's investigate the dynamic system having the form of:

$$\begin{cases} \frac{dx}{dt} = 2y \\ \frac{dy}{dt} = 4x - 4x^3 \end{cases}$$

Denote velocity in the first equation by  $2y$ , in the second equation denote the right part by force influencing the particle with unit mass, then auxiliary potential function

$U(x)$  has the following form :

$$U(x) = -\int_0^x (4x - 4x^3) dx = -2x^2 + x^4$$

In this case constant energy surface expression can be presented as:

$$U(x) + 2y^2 = E \rightarrow (-2x^2 + x^4 + 2y^2) = E$$

Now we can plot the phase portrait of constant energy surface

To solve this equation we use **MathCAD**

$$\frac{d}{dt}x = 2y$$

$$\frac{d}{dt}y = 4x - 4x^3$$

$$F_1(x, y) := 2 \cdot y \quad F_2(x, y) := 4 \cdot x - 4 \cdot x^3$$

Find equilibrium state points

$$z := \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix} \left| \begin{array}{l} \text{solve,} \begin{pmatrix} x \\ y \end{pmatrix} \\ \text{float, } 5 \end{array} \right. \rightarrow \begin{pmatrix} 0 & 0 \\ 1. & 0 \\ -1. & 0 \end{pmatrix} \quad x := z^{(0)} \quad y := z^{(1)}$$

$$x = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Find Jacobian

$$A(x, y) := \begin{pmatrix} \frac{\partial}{\partial x} F_1(x, y) & \frac{\partial}{\partial y} F_1(x, y) \\ \frac{\partial}{\partial x} F_2(x, y) & \frac{\partial}{\partial y} F_2(x, y) \end{pmatrix} A(x, y) \rightarrow \begin{pmatrix} 0 & 2 \\ 4 - 12 \cdot x^2 & 0 \end{pmatrix}$$

Determine point type with coordinates 0, 0

$$A_1 := A(x_0, y_0) \quad A_1 = \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix} \quad \lambda := \text{eigenvals}(A_1)$$

$$\lambda = \begin{pmatrix} 2.828 \\ -2.828 \end{pmatrix} \quad \textbf{Saddle point}$$

Determine point type with coordinates 1, 0

$$A_2 := A(x_1, y_1) \quad A_2 = \begin{pmatrix} 0 & 2 \\ -8 & 0 \end{pmatrix} \quad \lambda := \text{eigenvals}(A_2)$$

$$\lambda = \begin{pmatrix} 4i \\ -4i \end{pmatrix} \quad \textbf{Centre}$$

Determine point type with coordinates -1, 0

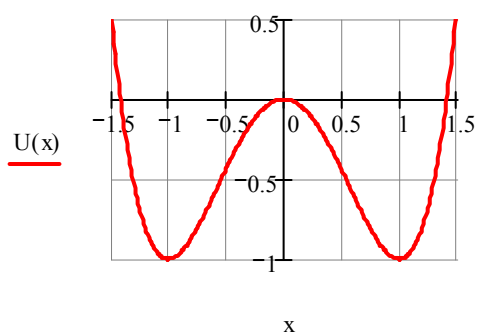
$$A_3 := A(x_2, y_2) \quad A_3 = \begin{pmatrix} 0 & 2 \\ -8 & 0 \end{pmatrix} \quad \lambda := \text{eigenvals}(A_3)$$

$$\lambda = \begin{pmatrix} 4i \\ -4i \end{pmatrix} \quad \textbf{Centre}$$

Auxiliary potential functions  $U(x)$  is written as

$$U(x) := - \int_0^x 4x - 4x^3 \, dx \rightarrow (-2) \cdot x^2 + x^4$$

$$x := -1.5, -1.5 + 0.01..1.5$$



Auxiliary energy function  $E(x, y)$  is written as

$$E(x, y) := y^2 \cdot 2 + U(x)$$

Plot phase portrait

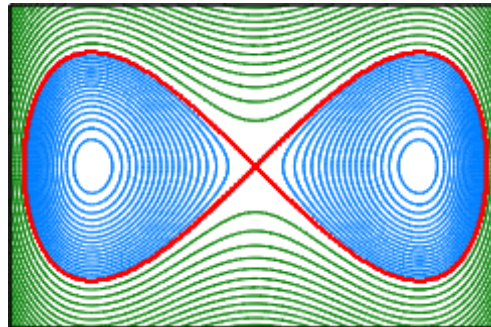
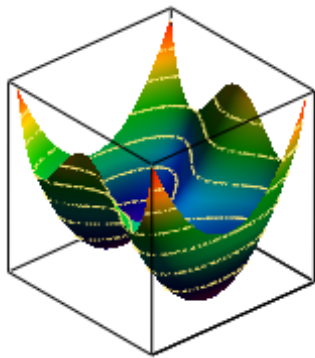
$$N := 251 \quad i := 0..N \quad j := i \quad x_1 := -1.5 + \frac{3}{N} \cdot i \quad y_j := -1 + \frac{2}{N} \cdot j$$

Surface  $\tilde{E}_{i,j} := E(x_i, y_j)$

Surface limited by separatrix  $E1_{i,j} := \text{if}(E_{i,j} < 0, E_{i,j}, 0)$

Surface behind separatrix  $E2_{i,j} := \text{if}(E_{i,j} > 0, E_{i,j}, 0)$

Separatrix  $\tilde{\delta} := 10^{-1}$   $E_{s,i,j} := \text{if}(E_{i,j} > 0, \delta, 0)$



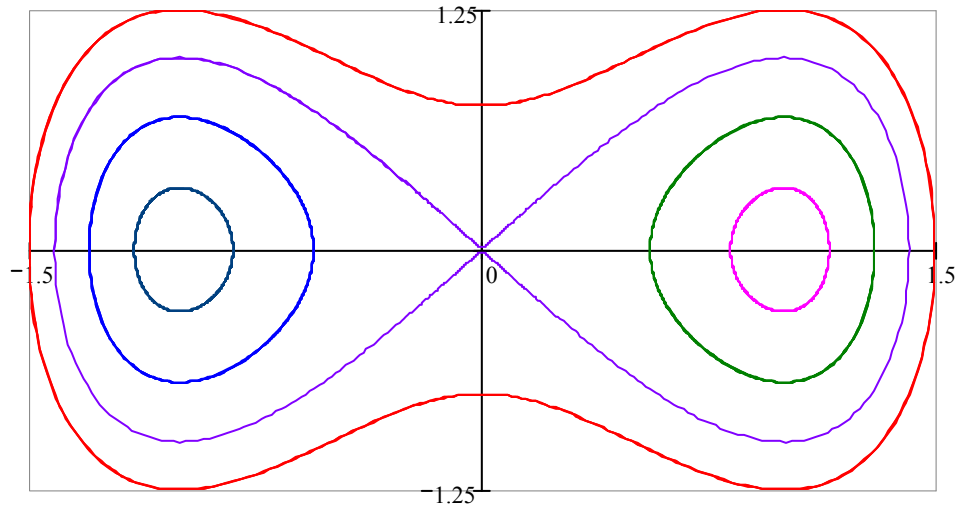
Direct differential equations solutions are given below.

All curves on phase plane are closed curves hence the solutions will be periodic.

$$D(t, x) := \begin{bmatrix} 2 \cdot x_1 \\ 4 \cdot x_0 - 4 \cdot (x_0)^3 \end{bmatrix} \quad \tilde{N} := 10^3 \quad \tilde{T} := 10 \quad x(x_0, y_0) := \text{rkfixed} \left( \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, 0, T, N, D \right) \quad t := x(0, 0)^{\langle 0 \rangle}$$

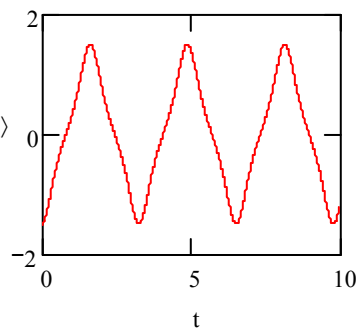


$x(-1.5, 0)^{\langle 2 \rangle}$   
 $x(-1.3, 0)^{\langle 2 \rangle}$   
 $x(1.3, 0)^{\langle 2 \rangle}$   
 $x(1.15, 0)^{\langle 2 \rangle}$   
 $x(-1.15, 0)^{\langle 2 \rangle}$   
 $x(-1.41425, 0)^{\langle 2 \rangle}$

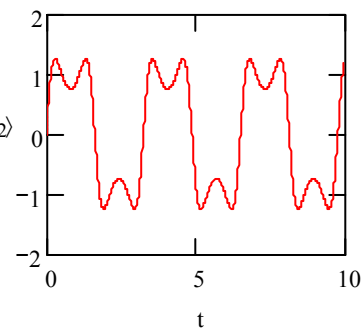


$x(-1.5, 0)^{\langle 1 \rangle}, x(-1.3, 0)^{\langle 1 \rangle}, x(1.3, 0)^{\langle 1 \rangle}, x(1.15, 0)^{\langle 1 \rangle}, x(-1.15, 0)^{\langle 1 \rangle}, x(-1.41425, 0)^{\langle 1 \rangle}$

$x(-1.5, 0)^{\langle 1 \rangle}$

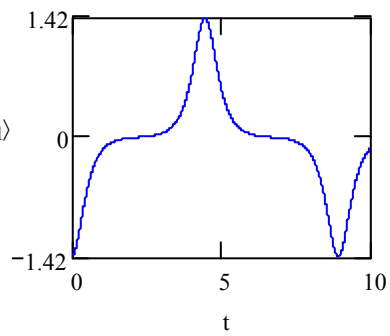


$x(-1.5, 0)^{\langle 2 \rangle}$

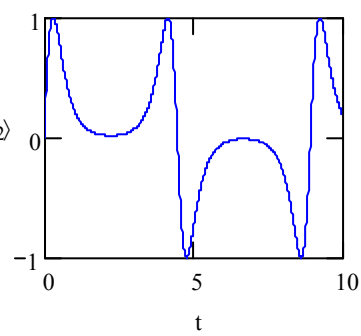


Solution behind the separatrix

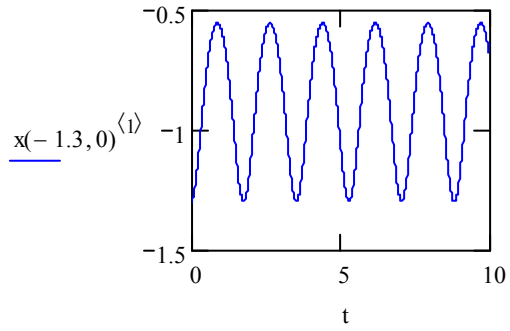
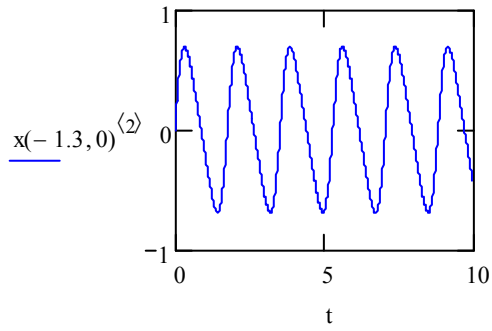
$x(-1.41425, 0)^{\langle 1 \rangle}$



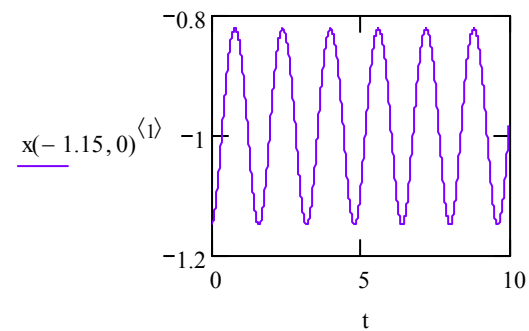
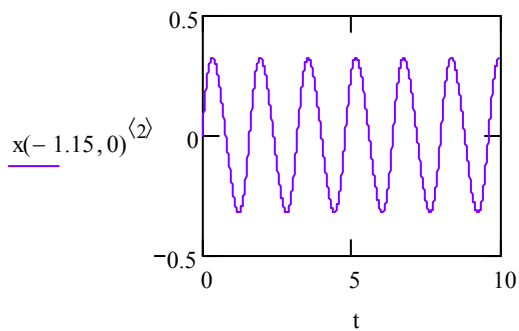
$x(-1.41425, 0)^{\langle 2 \rangle}$



Solution on the separatrix



Solution within the separatrix



Solution within the separatrix

The more is the field covered by phase curve the longer is oscillation period

Motion equation of generator rotor for small frequency change can be written as:

$$\tau \frac{d^2 \delta}{dt^2} = P_T - P_r \quad (11)$$

Rewrite the equation as the system of first order equations. Normalize the presented equations by  $\tau$  and taking into account that  $P_r = P_m \sin(\delta)$ :

$$\begin{aligned} \frac{d\omega}{dt} &= P_T - P_m \sin(\delta) \\ \frac{d\delta}{dt} &= \omega \end{aligned} \quad (12)$$

Find equilibrium state points of dynamic system and plot phase portrait using

**MathCAD**

$$P_T := 0.7 \quad P_m := 1$$

$$\left| \begin{array}{l} \frac{d}{dt} \delta = \omega \\ \frac{d}{dt} \omega = P_T - P_m \cdot \sin(\delta) \end{array} \right. \quad F_1(x, y) := y \quad F_2(x, y) := 0.7 - \sin(x)$$

Find equilibrium state points

$$z := \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix} \left| \begin{array}{l} \text{solve,} \begin{pmatrix} x \\ y \end{pmatrix} \\ \text{float, 5} \end{array} \right. \rightarrow (.77540 \ 0) \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} := z^T \quad x_1 := \pi - x_0 \quad y_1 := y_0 \quad x = \begin{pmatrix} 0.775 \\ 2.366 \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Find Jacobian

$$A(x, y) := \begin{pmatrix} \frac{\partial}{\partial x} F_1(x, y) & \frac{\partial}{\partial y} F_1(x, y) \\ \frac{\partial}{\partial x} F_2(x, y) & \frac{\partial}{\partial y} F_2(x, y) \end{pmatrix} \quad A(x, y) \rightarrow \begin{pmatrix} 0 & 1 \\ -\cos(x) & 0 \end{pmatrix}$$

Determine point type with coordinates 0.775 , 0

$$A_1 := A(x_0, y_0) \quad A_1 = \begin{pmatrix} 0 & 1 \\ -0.714 & 0 \end{pmatrix} \quad \lambda := \text{eigenvals}(A_1)$$

$$\lambda = \begin{pmatrix} 0.845i \\ -0.845i \end{pmatrix} \quad \textbf{Centre}$$

Determine point type with coordinates 2.366, 0

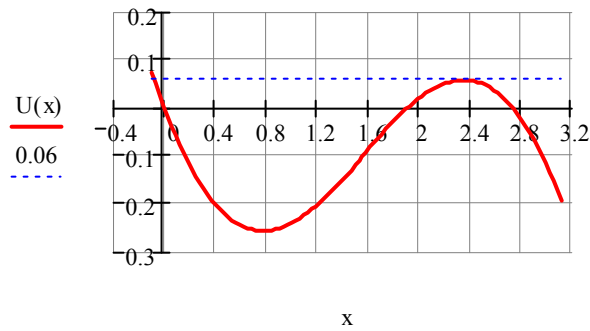
$$A_2 := A(x_1, y_1) \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0.714 & 0 \end{pmatrix} \quad \lambda := \text{eigenvals}(A_2)$$

$$\lambda = \begin{pmatrix} 0.845 \\ -0.845 \end{pmatrix} \quad \textbf{Saddle point}$$

Auxiliary potential function U(x) is written as :

$$U(x) := - \int_0^x (0.7 - \sin(x)) \, dx \text{ float, 5} \rightarrow (-.70000) \cdot x - 1 \cdot \cos(x) + 1.$$

$$x := -0.1, -0.1 + 0.01 \cdot \pi .. \pi$$



Energy auxiliary function  $E(x,y)$  is written as:

$$E(x,y) := y^2 + U(x)$$

Plot phase portrait

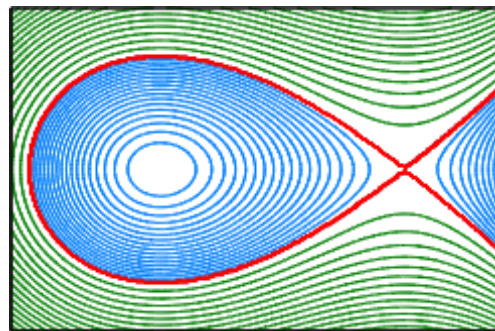
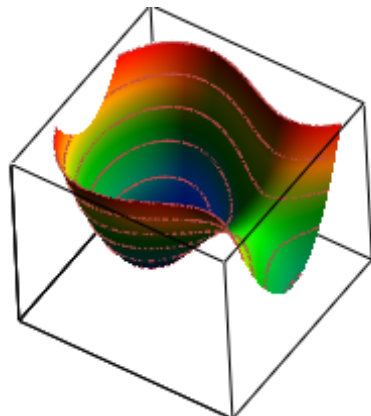
$$\underline{N} := 251 \quad i := 0..N \quad j := i \quad x_i := -0.2 + \frac{3.2}{N} \cdot i \quad y_j := -0.8 + \frac{1.6}{N} \cdot j$$

Surface  $\underline{E}_{i,j} := E(x_i, y_j)$

Surface within the separatrix  $E1_{i,j} := \text{if}(E_{i,j} < 0.058, E_{i,j}, 0.058)$

Surface behind the separatrix  $E2_{i,j} := \text{if}(E_{i,j} \geq 0.06, E_{i,j}, 0.06)$

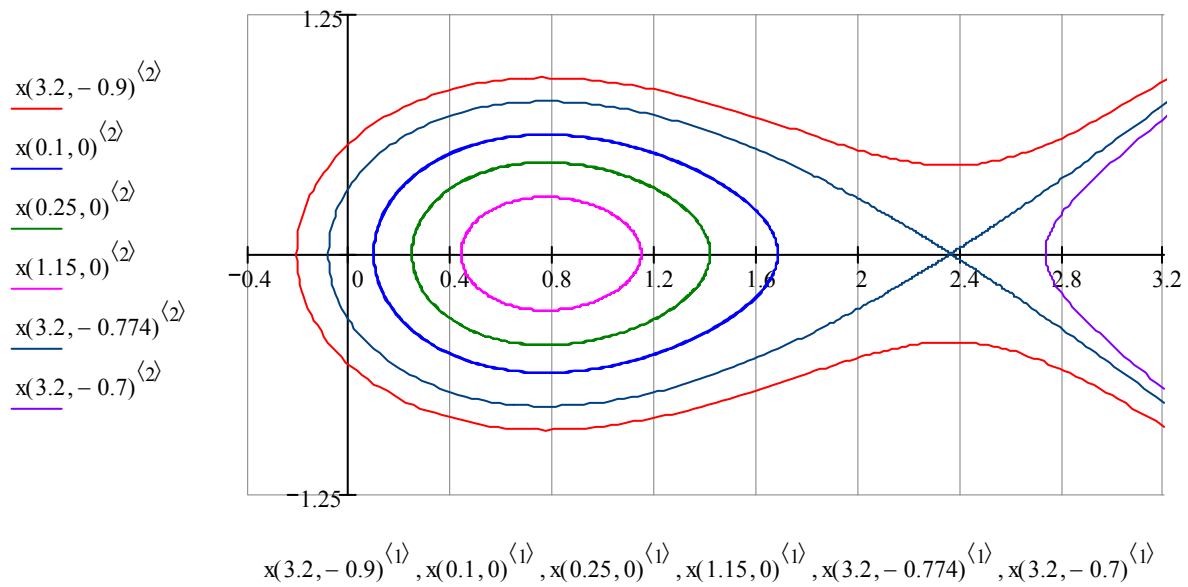
The separatrix  $\underline{\delta} := 10^{-3} \quad E_{s,i,j} := \text{if}(E_{i,j} \geq 0.058, \delta, 0)$



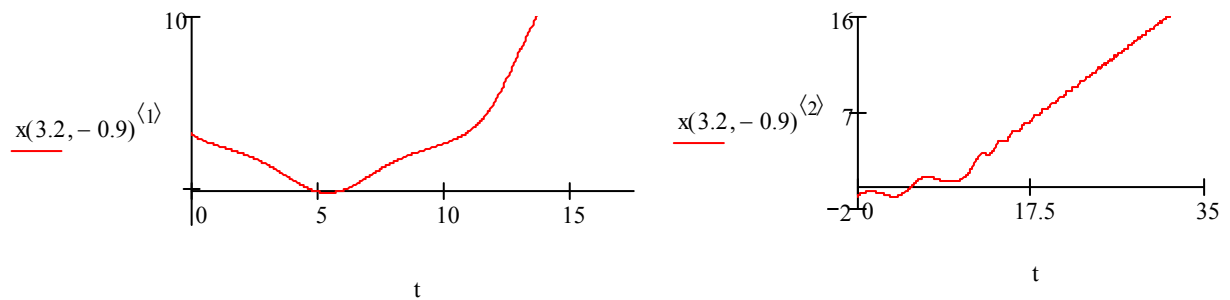
$$D(t,x) := \begin{pmatrix} x_1 \\ 0.7 - \sin(x_0) \end{pmatrix} \quad \underline{N} := 10^3 \quad \underline{T} := 35 \quad x(x_0, y_0) := \text{rkfixed} \left[ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, 0, T, N, D \right] \quad t := x(0,0) \langle 0 \rangle$$

Direct differential equations solutions are given below

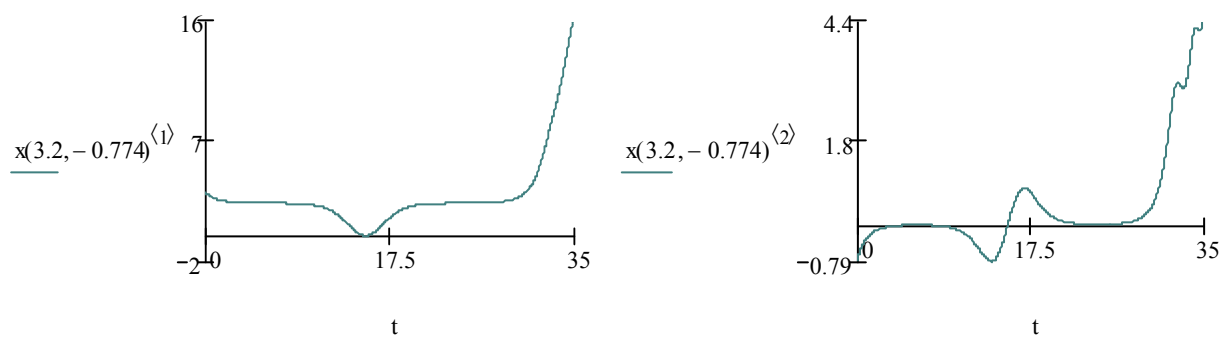
Closed curves on phase plane are periodic solutions, open curves are unstable solutions.



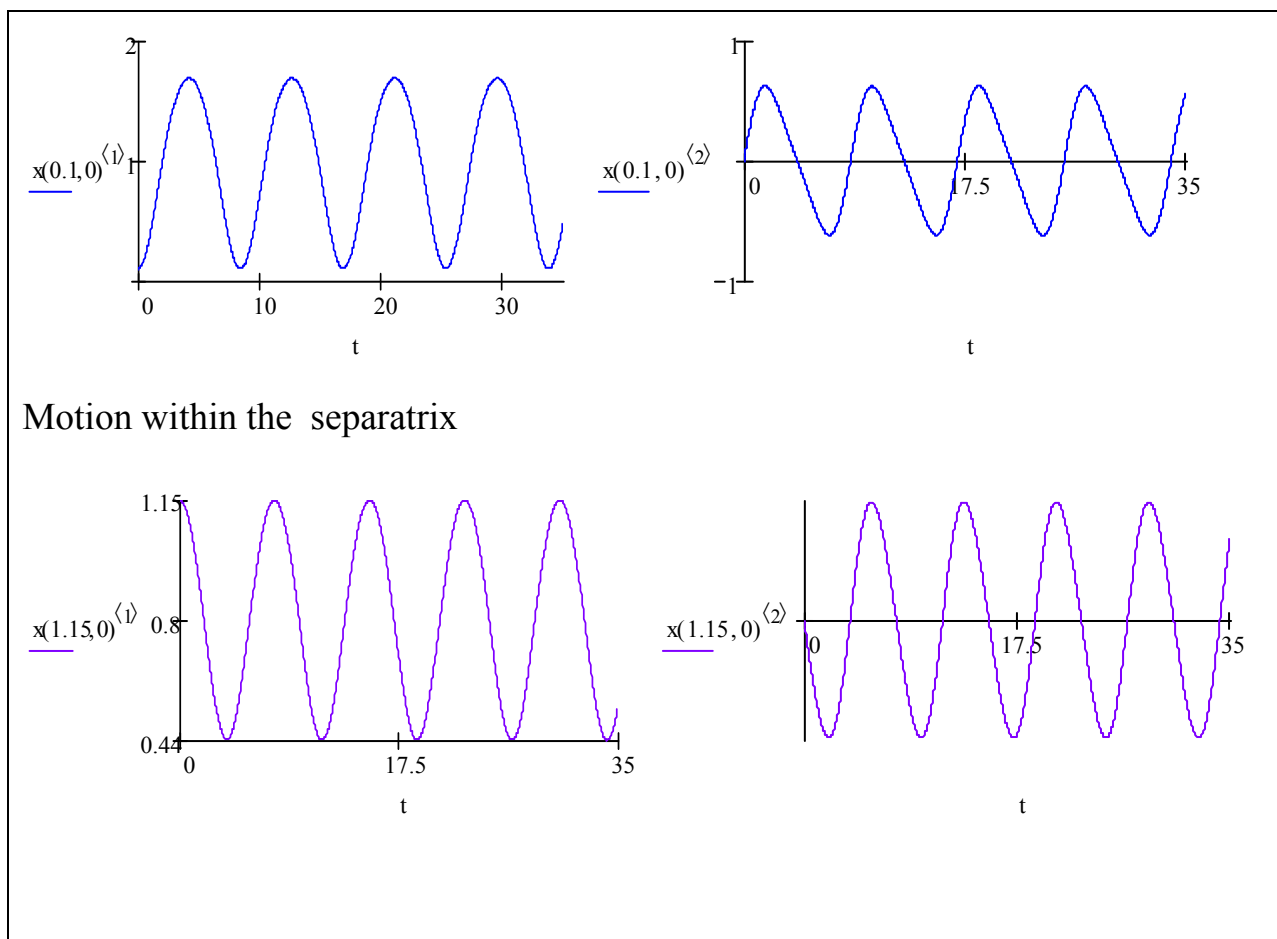
### Unstable motion behind the separatrix



### Motion on the separatrix



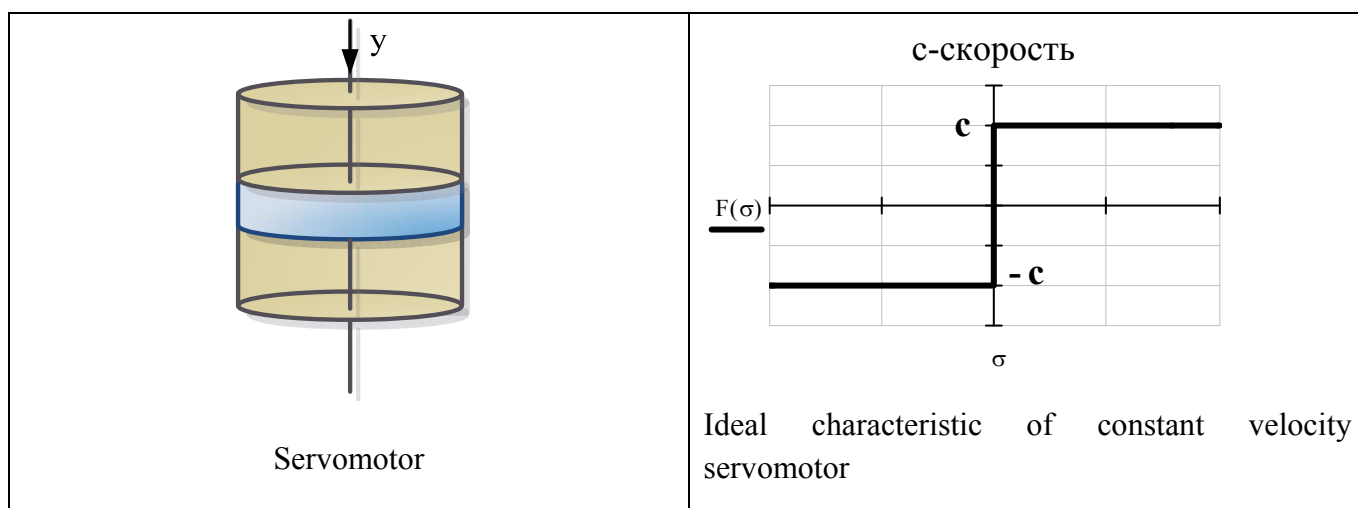
### Motion within the separatrix



## 2.2. Phase plane technique

### 2.2.1. Servomotor equation with ideal relay characteristic

Let us apply phase plane technique to detect important features of the process taking place in nonlinear system of turbine rotation frequency regulation by constant velocity servomotor described by equations [5]:



rotor

$$T_{\phi} \frac{d\phi}{dt} = \mu; \quad (13)$$

control element

$$\sigma = -\phi - \mu; \quad (14)$$

constant velocity servomotor

$$\frac{d\mu}{dt} = F(\sigma). \quad (15)$$

Where  $F(\sigma)$ :

$$F(\sigma) = c \cdot \text{sign}(\sigma). \quad (16)$$

As phase plane coordinates  $(x, y)$  choose

$$x = \phi, \quad y = \frac{d\phi}{dt}.$$

Dynamic system will be described by equations system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = \frac{F(\sigma)}{T_x} \end{cases} \quad \text{где } F(\sigma) = c \cdot \text{sign}(-x - yT_x), \quad \sigma = -x - yT_x \quad (17)$$

Find equation of servomotor switching line (vertical segment on plot)

$$\sigma = -x - yT_x = 0, \quad y = -x / T_x \quad (18)$$

Find phase portrait lines equation

In area  $\sigma < 0$  (negative values  $F$  in plot)

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{c}{T_x} \end{cases} \rightarrow \frac{dy}{dx} = -\frac{c}{T_x y} \rightarrow y^2 = -\frac{2cx}{T_x} + C_1 \quad (19)$$

In area  $\sigma > 0$  (positive values  $F$  in plot)

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = \frac{c}{T_x} \end{cases} \rightarrow \frac{dy}{dx} = \frac{c}{T_x y} \rightarrow y^2 = \frac{2cx}{T_x} + C_2 \quad (20)$$

Write the equation on vertical segment, equaling coefficients in equation of straight line and in phase plane

$$\frac{dy}{dx} = -\frac{c}{T_x y}, \quad y = -\frac{x}{T_x} \rightarrow -\frac{c}{T_x y} = -\frac{1}{T_x} \rightarrow y = c \quad (21)$$

Find differential equation on segment

$$\frac{dx}{dt} = y, \quad y = -\frac{x}{T_x} \rightarrow \frac{dx}{dt} = -\frac{x}{T_x} \rightarrow x(t) = C_3 e^{-t/T_x} \quad (22)$$

Sliding process

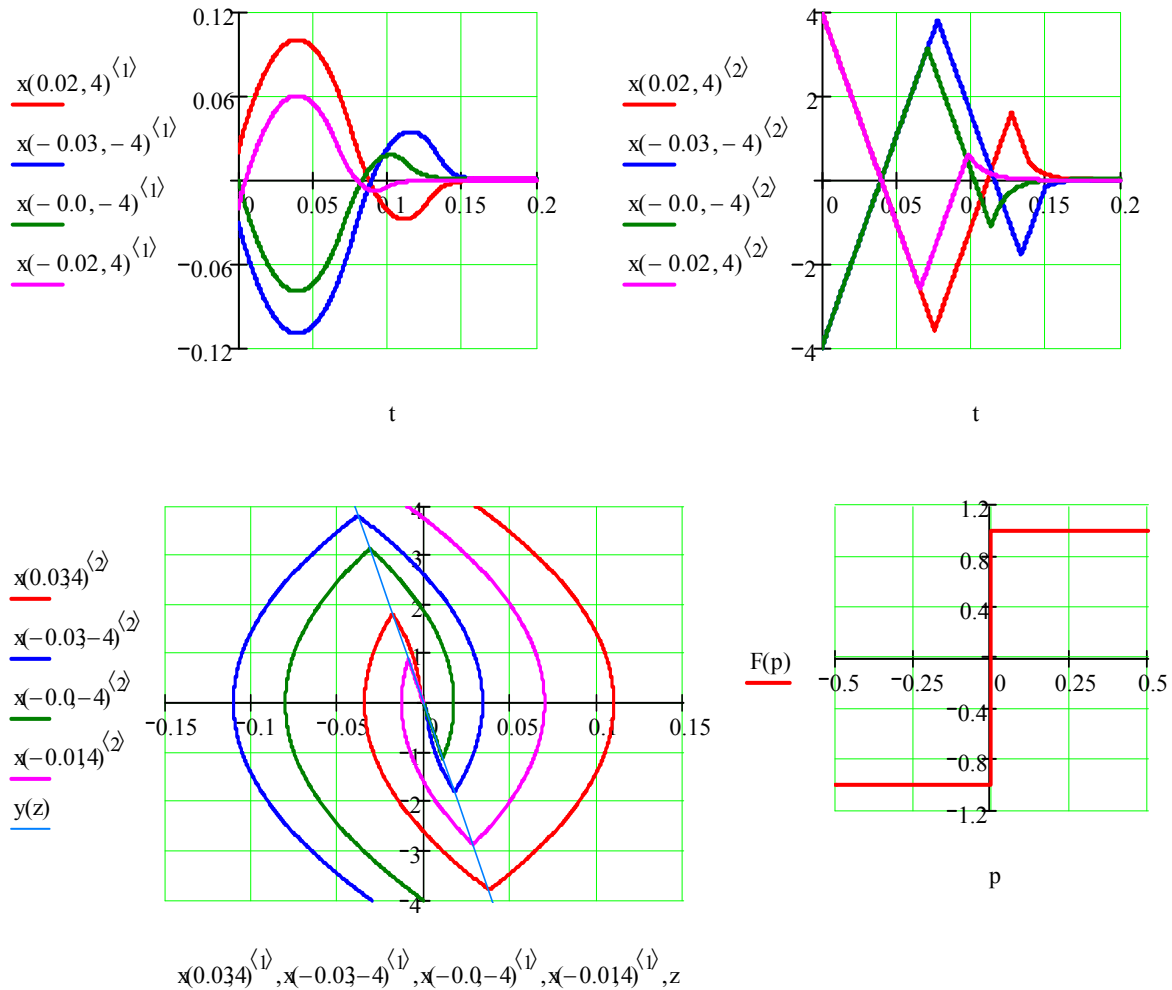
Let's analyze the solution with phase portrait plotting with use of **MathCAD**.



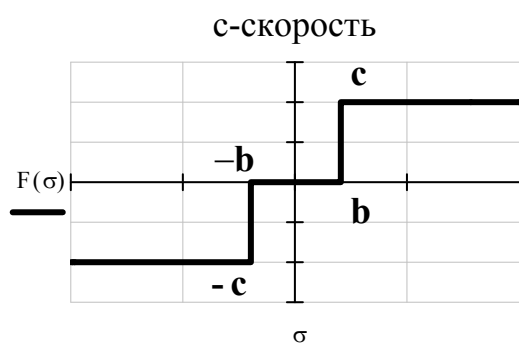
$$F(x) := \text{sign}(x) \cdot 1 \quad T1 := 0.01 \quad N := 10^3$$

$$D(t, x) := \begin{pmatrix} x_1 \\ \frac{F(-x_0 - T1 \cdot x_1)}{T1} \end{pmatrix} \quad T := 0.2 \quad x(x_0, x_1) := \text{rkfixed} \left[ \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, 0, T, N, D \right]$$

$$y(z) := -z \cdot \frac{1}{T1} \quad z := -0.05, -0.05 + 0.01 \dots 0.1 \quad p := -0.5, -0.5 + 0.001 \dots 0.5 \quad t := x(0, 0)^{(0)}$$



### 2.2.2. Servomotor equation with actual relay characteristic.



Two vertical segments correspond to servomotor switching.

$$\sigma = -x - yT_x = -b \rightarrow x = -yT_x + b$$

$$\sigma = -x - yT_x = b \rightarrow x = -yT_x - b$$

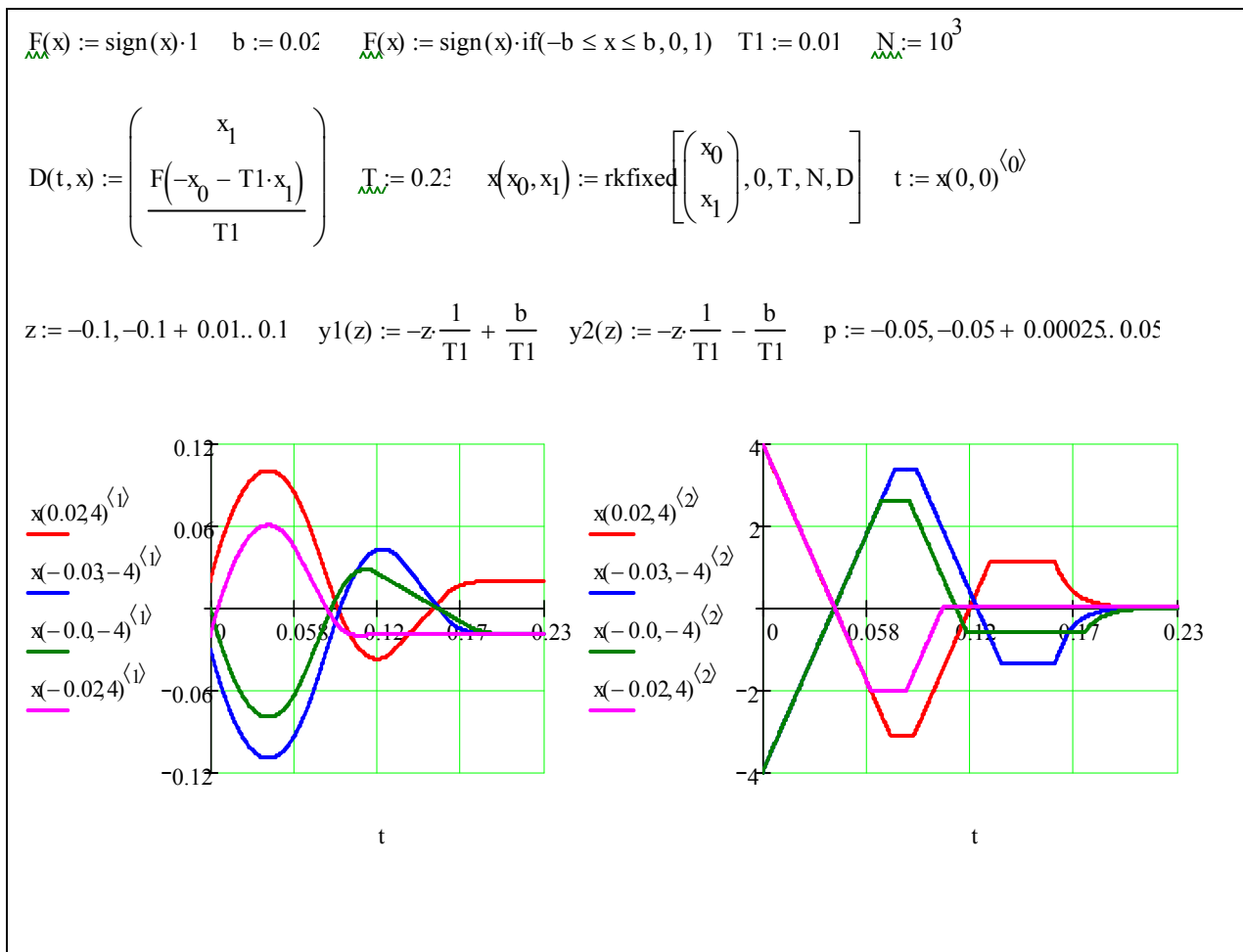
Plot its slopes on phase plane (see **MathCAD** document below)

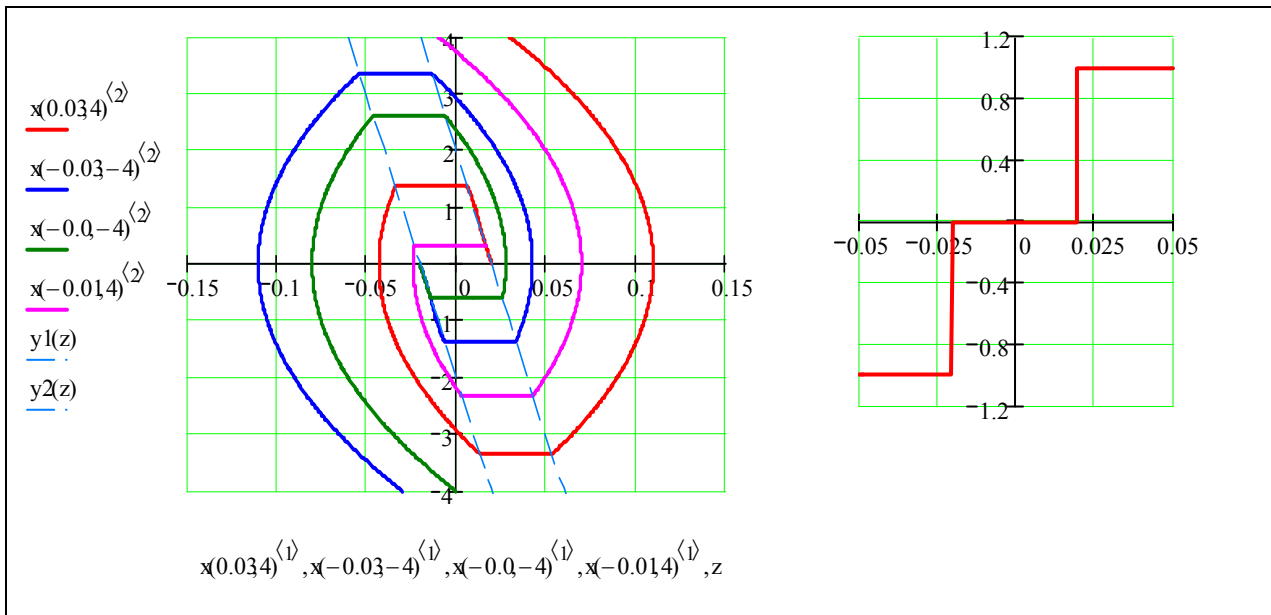
In difference from the above ideal example in this case one more area appears

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = 0 \end{cases} \rightarrow \frac{dy}{dx} = 0 \rightarrow y = \text{const}$$

Straight lines are parallel to axis  $x$ . Now equilibrium state is not one point but the segment.

$$y=0, \quad -b < x < b$$





### 2.3. Lyapunov straight method Прямой метод Ляпунова

Method is based on scalar functions applications having special properties on dynamic system solutions and being called Lyapunov functions. Lyapunov functions help estimate system stability and quality and construct control algorithms providing the required quality process properties.

For system described by the system of differential equations

$$\begin{cases} \frac{dx}{dt} = f_1(x, y) \\ \frac{dy}{dt} = f_2(x, y) \end{cases} \quad (23)$$

where functions  $f_1, f_2$  are arbitrary and possess any kind of nonlinear nature but always satisfies the condition  $f_1 = f_2 = 0$ , при  $x_1 = x_2 = 0$ , as in steady state all variables deviations and their derivatives equal zero. A certain function of system all phase coordinates (23)  $V(x, y)$  can be entered where  $x, y$  are variables deviations from some steady values. The function can be presented in two-dimensional space. Then in each point of phase space  $V$  will possess the definite value and at the beginning of coordinates will equal zero.

The function  $V$  is said to be **definite sign function** in some area if in any point inside the area function  $V$  has definite sign and turns to zero at the beginning of coordinates.

Let us analyze the example of definite sign positive function of the second order system  $n = 2$

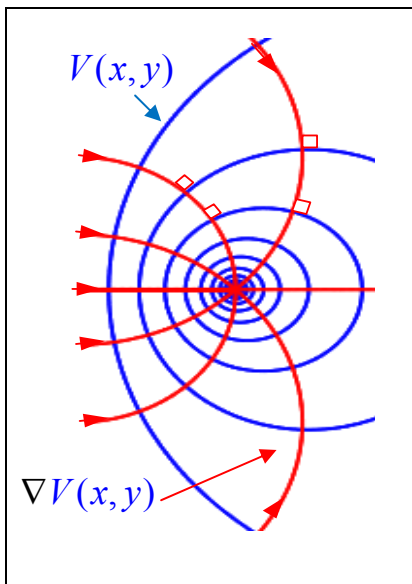
$$V(x, y) = x^2 + y^2 \quad (24)$$

it is clear that  $V > 0$ , и  $V = 0$ , only at  $x = y = 0$ .

Function  $V$  is called constant sign function, if it is of the same sign but it turns to zero not only at the beginning of the coordinates but in other points of the given area.

Function  $V$  is called variable sign function, if it in the given area around coordinates beginning can have different signs.

Arbitrary function  $V = V(x, y)$  which turns to zero only at  $x = y = 0$ , and where  $x, y$  are deviations in which the equation of system motion is written is called **Laypunov function**. Let's determine time derivative of function  $V$ . Recollect what gradient operator  $\nabla$  stands for:



$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} \rightarrow \nabla V(x, y) = \mathbf{grad} V(x, y)$$

– direction of fast change of function. Now time derivative of function  $f(x, y)$  can be written down, considering  $x, y$  depend on  $t$ :

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial y} \frac{dy}{dt} = \cancel{\frac{\partial V}{\partial t}} + \frac{\partial V}{\partial x} f_1(x, y) + \frac{\partial V}{\partial y} f_2(x, y)$$

$$\frac{dV}{dt} = (\nabla V, \mathbf{v}), \mathbf{v} = (f_1, f_2)$$

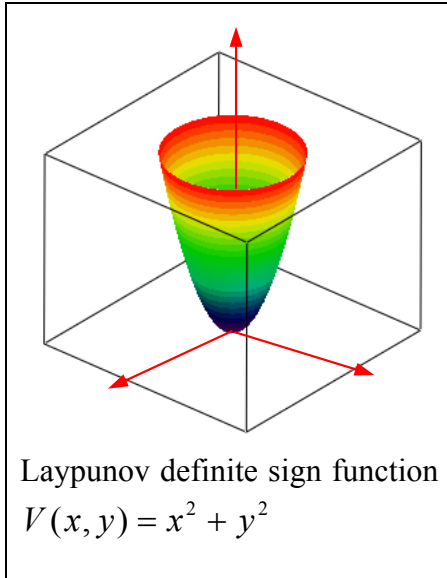
**Laypunov theorem about nonlinear systems stability:** if for equations of the system (23) one can choose the stable Lyapunov function  $V(x, y)$ , such that its time derivative function  $dV / dt$  was also definite sign function (or constant sign function), but have the sign opposite to sign  $V$ , then the given system is stable.

**Example:** Let nonlinear system be described by equations

$$\begin{cases} \frac{dx}{dt} = -x - x^3 \\ \frac{dy}{dt} = -y^3 \end{cases}$$

Let us choose Lyapunov definite sign function of type:

$$V(x, y) = x^2 + y^2$$



Find time derivative of Laypunov function

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial y} \frac{dy}{dt} = \frac{\partial V}{\partial x} f_1(x, y) + \frac{\partial V}{\partial y} f_2(x, y)$$

$$\frac{dV}{dt} = 2x(-x - x^3) + 2y(-y^3) = -2(x^2 + x^4 + y^4)$$

Function  $dV / dt$  is definite sign function but it is opposite to function sign  $V(x, y)$ , hence the system is stable.

## 2.4. Harmonic linearization method

One of the main methods of higher order nonlinear systems at present time is approximative method of harmonic linearization. Let's analyze the example when the object with linear transfer function is controlled by the block with nonlinear characteristic

$$y = F(x). \quad (25)$$

Suppose harmonic signal enters nonlinear component input

$$x = A \sin(\omega t) \quad (26)$$

Output signal will be periodic hence it can be decomposed into

$$y = F(A \sin(\omega t)) = q(A) \sin(\psi) + q'(A) \cos(\psi) + \text{high harmonics}$$

Fourier series

$$\begin{aligned}
y = F(A \sin(\omega t)) &= \frac{1}{2\pi} \int_0^{2\pi} F(A \sin(\psi)) d\psi \\
&+ \left[ \frac{1}{\pi} \int_0^{2\pi} F(A \sin(\psi)) \sin(\psi) d\psi \right] \sin(\psi) + \left[ \frac{1}{\pi} \int_0^{2\pi} F(A \sin(\psi)) \cos(\psi) d\psi \right] \cos(\psi) + \\
&+ \text{high harmonics}
\end{aligned}$$

There will be no constant component for odd symmetry of nonlinear characteristic that is why we have

$$\frac{1}{2\pi} \int_0^{2\pi} F(A \sin(\psi)) d\psi = 0 \quad (27)$$

Then output signal can be written down

$$y = q(A)x + q'(A) \frac{\dot{x}}{\omega} + \text{high harmonics} \quad (28)$$

Linear part of closed CS вследствие инерционности является фильтром низких частот, т.е. высокие гармоники проходят ее со значительно большим ослаблением, чем первая:

$$\begin{aligned}
y &= q(A)a + q'(A) \frac{\dot{x}}{\omega} + \text{high harmonics} \\
&= \left[ q(A) + q'(A) \frac{p}{\omega} \right] x, \quad p \rightarrow \frac{d}{dt}
\end{aligned} \quad (29)$$

Such a representation is called harmonic linearization of non-linearity and variables are determined by formulae

$$q(A) = \frac{1}{\pi A} \int_0^{2\pi} F(A \sin(\psi)) \sin(\psi) d\psi, \quad q'(A) = \frac{1}{\pi A} \int_0^{2\pi} F(A \sin(\psi)) \cos(\psi) d\psi, \quad (30)$$

are called harmonic linearization coefficients.

Transfer function of linear component has the form of

$$y = \left[ q(A) + q'(A) \frac{p}{\omega} \right] x, \rightarrow W(A, p) = q(A) + q'(A) \frac{p}{\omega}, \quad p = j\omega \quad (31)$$

$$W(A, j\omega) = q(A) + q'(A) \frac{j\omega}{\omega} = q(A) + q'(A)j$$

Transfer function does not depend on frequency. It increases the magnitude of input signal and is called complex amplifying coefficient.

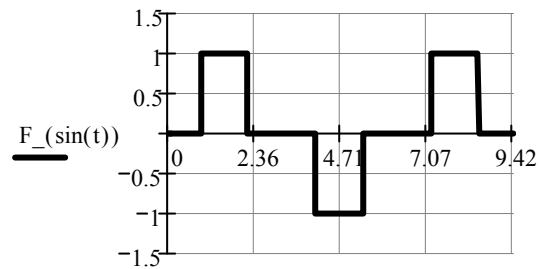
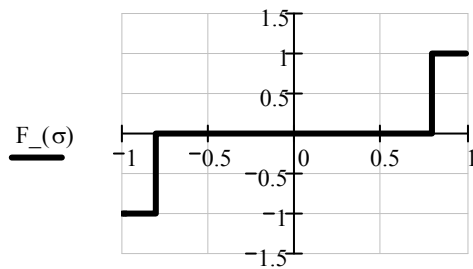
Thus nonlinear element can be substituted by linear element. Its frequency response depends on amplitude of input signal.

**Example.** Find amplification complex coefficient of given nonlinear component. We use **MathCAD**

$F(\sigma) := \text{if}(\sigma < 0, -1, 1) \quad b := 0.8$

$F_-(\sigma) := \text{if}(-b \leq \sigma \leq b, 0, F(\sigma))$  nonlinear component characteristic

$\sigma := -1, -1 + 10^{-4} \dots 1$



Nonlinear element characteristic and output signal

Find amplification coefficient **a** for different magnitudes, **b=0** as function is even

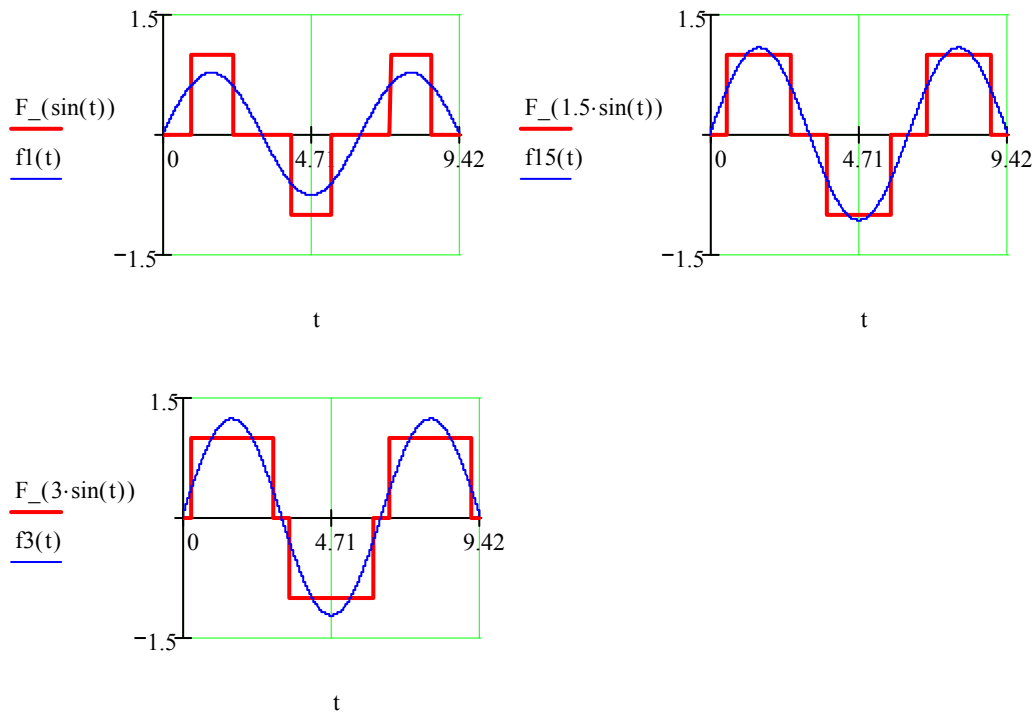
$$a(A) := \frac{1}{\pi \cdot A} \cdot \int_0^{2 \cdot \pi} F(A \cdot \sin(t)) \cdot \sin(t) \, dt$$

Find amplification coefficients and equivalent sinewaves at different magnitudes.

$$a(1) = 0.764 \quad f1(t) := a(1) \cdot \sin(t)$$

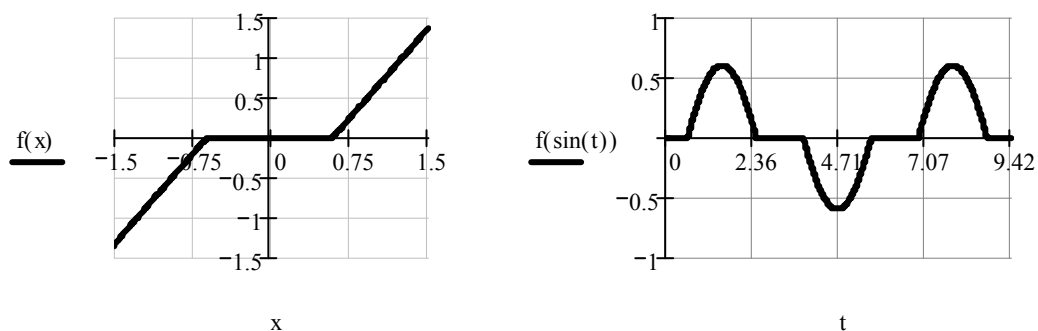
$$a(1.5) = 0.718 \quad f15(t) := a(1.5) \cdot 1.5 \cdot \sin(t)$$

$$a(3) = 0.409 \quad f_3(t) := a(3) \cdot 3 \cdot \sin(t)$$



$$A_- := 1.5 \quad f(x) := x \cdot A_- - B \quad x := -1.5, -1.5 + 0.01.. 1.5 \quad f(x) := \text{if}(x > 0, f(x), -f(-x))$$

$$f(x) := \text{if}(-bb < x < bb, 0, f(x))$$



Nonlinear element characteristic and output signal

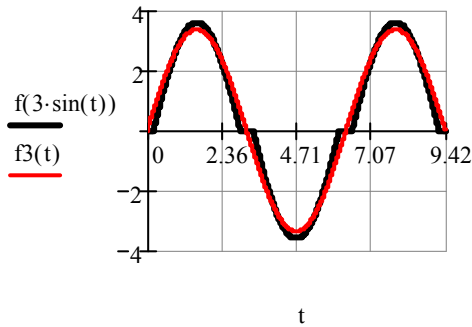
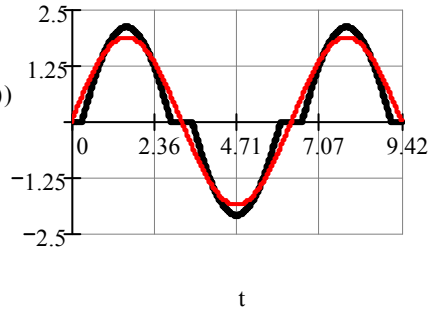
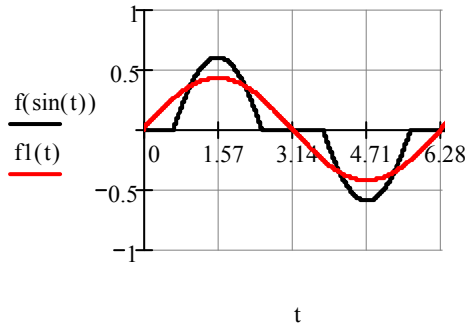
Find amplification coefficient **a** for different magnitudes, **b**=0 as function is even.



$$a(A) := \frac{1}{\pi \cdot A} \cdot \int_0^{2 \cdot \pi} F(A \cdot \sin(t)) \cdot \sin(t) dt$$

Find amplification coefficients and equivalent sinewaves at different magnitudes

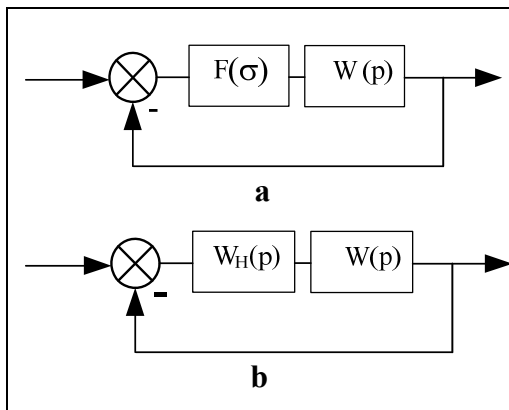
$$a(1) = 0.427 \quad f_1(t) := 0.427 \cdot \sin(t) \quad a(2) = 0.936 \quad f_2(t) := 0.936 \cdot 2 \cdot \sin(t) \quad a(3) = 1.121 \quad f_3(t) := 1.121 \cdot 3 \cdot \sin(t)$$



## 2.5. Self-exciting oscillations analysis algorithm.

With the help of nonlinear element harmonic linearization the closed loop system(Fig.a) equates to the system with linear equivalent element(Fig

b).Investigation of system with nonlinear element results in linear system investigation.



Find characteristic equation

$$W(p)W_H(A) = -1, \quad W(p) = -1 / W_H(A),$$

$$W(j\omega) = U(\omega) + jV(\omega), \quad W_H(A) = q(A) + jq'(A)$$

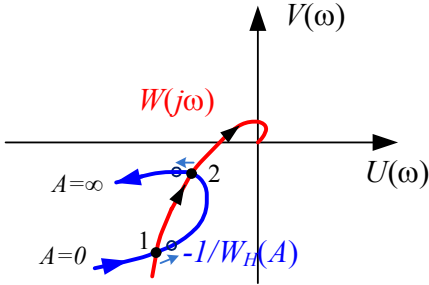
From the last equation it follows that:

$$U(\omega)q(A) - q'(A)V(\omega) = 1$$

$$U(\omega)q'(A) + q(A)U(\omega) = 0$$

Where  $q(A)$ ,  $q'(A)$  are harmonic linearization coefficients. Equation solution provides us equilibrium state points  $\omega_{II}$ ,  $A_{II}$ .

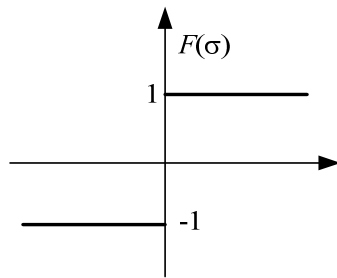
At solving the problem **Goldfarb** diagram is easy to use, its algorithm is given below.

	<b>Goldfarb diagram</b>
	<ol style="list-style-type: none"> <li>1. Plot the locus <math>W(j\omega)</math></li> <li>2. Plot the locus <math>-1/W_H(A)</math></li> <li>3. Find <math>\omega_{II}</math>, <math>A_{II}</math> solving the equation : <math display="block">U(\omega)q(A) - q'(A)V(\omega) = 1</math> <math display="block">U(\omega)q'(A) + q(A)U(\omega) = 0</math> </li> <li>4. If at motion along locus <math>-1/W_H(A)</math> в сторону увеличения <math>A</math> the point is covered by locus <math>W(j\omega)</math>, then magnitude <math>A</math> will correspond unstable self-excited oscillations (point 1). If the point is not covered by locus <math>W(j\omega)</math>, then at magnitude <math>A</math> stable self-excited oscillations occur (point 2)</li> </ol>

Analyze the example with **MathCAD** use.

Investigate the system on self-exciting oscillations presence

$\underline{W}(p) := \frac{3}{p^3 + 3 \cdot p^2 + p}$	$U(\omega) := \text{Re}(W(i \cdot \omega)) \quad \underline{V}(\omega) := \text{Im}(W(i \cdot \omega))$
$\underline{U}(\omega) := U(\omega) \left  \begin{array}{l} \text{complex} \\ \text{simplify} \rightarrow \\ \text{float, 5} \end{array} \right. \frac{-9}{7 \cdot \omega^2 + \omega^4 + 1}$	$\underline{V}(\omega) := V(\omega) \left  \begin{array}{l} \text{complex} \\ \text{simplify} \rightarrow \\ \text{float, 5} \end{array} \right. \frac{3}{\omega} \cdot \frac{\omega^2 - 1}{7 \cdot \omega^2 + \omega^4 + 1}$



$$q(a) := \frac{4}{\pi \cdot a}$$

Harmonic linearization coefficient  
of nonlinear component

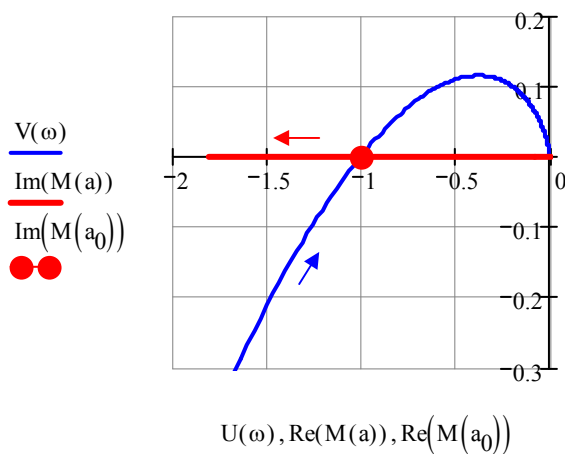
Find crossover point, frequency and magnitude

$$W_H(a) := q(a)$$

$$w := \left( \begin{array}{l} U(\omega) \cdot q(a) = -1 \\ V(\omega) \cdot q(a) = 0 \\ \omega > 0 \end{array} \right) \left| \begin{array}{l} \text{solve,} \left( \begin{array}{l} \omega \\ a \end{array} \right) \\ \text{float, 5} \end{array} \right. \rightarrow (1. \ 1.2732)$$

$$(\omega_0, a_0) := w \quad \omega_0 = 1 \quad a_0 = 1.273$$

$$\omega := 0, .01.. 10 \quad M(a) := \frac{-1}{W_H(a)} \quad U(1) = -1 \quad a := 0, 0.1.. 2.3$$



The obtained values of  
parameters  $\omega_0 = 1$  and  $a_0 = 1.273$   
correspond to stable self-excited  
oscillations because by point  
moving across the locus in the  
direction of increasing  
magnitude –  $a$ , locus does not  
cover the point

## **Literature**

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3. E.A.Nikulin. Backgrounds of Automatic Control Theory. StPb.:BHV- St. Petersburg 2007. 640p.