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PHYSICS
Brief course

MECHANICS

Recommended for publishing as a study aid
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This book gives brief but thorough enough materials on "Kinematics", "Dynamics" and "Relativistic mechanics" of "General physics" course. It is prepared on department of theoretical and experimental physics of Tomsk polytechnic university. This textbook intended for the students of all non-humanitarian disciplines of technical universities.

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Contents

Chapter 1

Kinematics

Any motion of an **real object** from one space point to another is called **mechanical** motion. The purpose of classical mechanics is studying of mechanical motion. This purpose consists of two parts:

- mathematical description of all the possible mechanical motions of a object in study;

- explanation of motion causes in all the possible conditions that accompany the mechanical motion of the object.

The first problem is solved by branch of mechanics, called **kinematics**, the second – by branch of mechanics, called **dynamics**.

1.1   Physical reality and its modelling

The mathematical description of the motion of *real objects* of world around is *impossible* (at least with the help of the mathematics, that man has built to date). Modern mathematics can only describe a *mathematical model* of a real object. And of course, the simpler the model, the easier its description.

The simplest model of the object that can move is a **point particle**. As it is known *a point particle is usually referred to as a physical object*, the **own size** of that is small or negligible, or if its **geometrical properties and structure** *are irrelevant*.

Such a definition of a point particle allows one to create a sufficiently clear idea about this physical object, but it is *not mathematically constructive* (i.e. it does not contain elements of a mathematical model indicating how to describe the physical object in mathematical language).

We introduce a slightly different definition of a point particle. First of all, we emphasize that a **point particle** is a *mathematical abstraction* of a real object of Nature. And accordingly, by a **point particle** *will be called any point in space, which is attributed to* **all** *the physical parameters of the object.*

From such a definition of a point particle, it is clear that in terms of mathematics the description of a point particle location in space is equivalent to the description of any space point position. But to describe the position of a point in space, one must first define the **coordinate system** (abbreviated – CS).

Coordinate system is *a rule by which every point in space is associated with n numbers, called* **coordinates of a point**. *The minimum number* of coordinates needed to describe the position of a point in space, called the **dimension of the space**.

Our everyday experience shows us that *physical space* is **three-dimensional** (i.e., $n = 3$). The simplest example of coordinate system is **Cartesian CS**. It consists of three axes oriented perpendicularly to each other and the point where they meet is its **origin**. Coordinate axis are usually named as X, Y, and Z. On the figure, for example, a three dimensional Cartesian coordinate system is represented, with origin O and axis lines
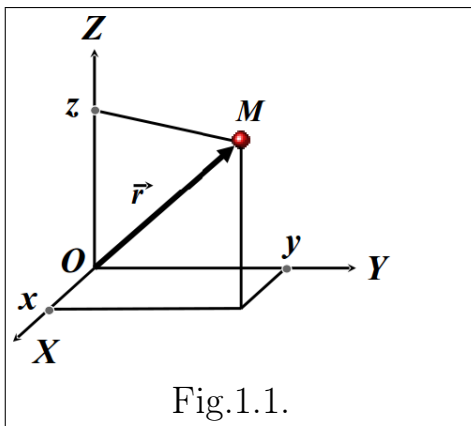
Fig.1.1.

X, Y and Z oriented as it is drawn by the arrows. Procedure for finding the coordinates (x, y, z) of a point particle M is qualitatively shown in Fig.1.1.

However, to describe not just the position of a point in space but its motion as well — we must be able to measure the change of the point position in space *over the time*. In physics, a procedure that allows to define coordinate system as well as the method of time measurement is called the procedure of specifying the **reference frame** (abbreviated - RF).

More precisely, the **reference frame** *is the set of* **basis** *and* **calibration**. Here:

- **basis** *is the set of physical laboratories (real or imagined), located at all points in space and equipped with instruments for measuring time*

and length intervals;

- **calibration** *is the rule that assigns 4 (four) numbers to each event in the physical space. Three (3) of this numbers specifies the coordinates of the physical event (i.e., they specify the CS), and the fourth number is the time of the physical event occurrence.*

According to this definition, the reference frame can be drawn in two ways:

- as a coordinate system one of which axes is the axe of time (it is possible, if the point moves on a plane, or along a straight line);

- as a coordinate system with the additional designation, indicating the possibility of measuring time (usually by letter K).

Consider the motion of a point relative to a RF K. Let in time interval from t_1 to t_2 a point particle has moved from the space point M_1 to some space point M_2 along L line (called, as it is well known, **trajectory** or the **path**). Let connect the origin of the CS (coincided with the selected RF) with the points M_1 and M_2 by segments directed to the points (OM_1 and OM_2 segments). Each of these segments is obviously a vector and is referred to as **radius vector** (position vector) of a point particle. So, we have:
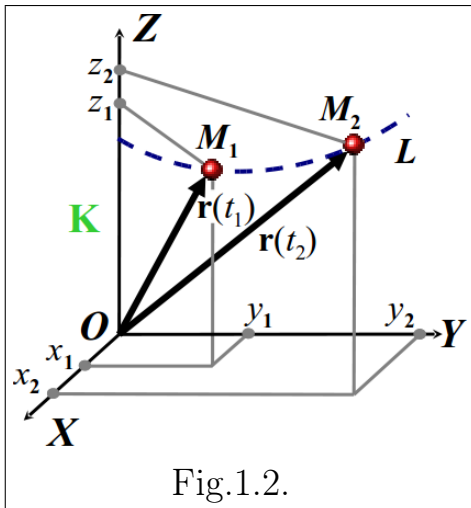
Fig.1.2.

$$OM_1 \ : \ \vec{r}_1 \ = \ (x_1, y_1, z_1), \qquad OM_2 \ : \ \vec{r}_2 \ = \ (x_2, y_2, z_2),$$

Therefore, specifying the radius vector of a point particle in some moment we specify the coordinates of the point in that moment and thus, we *completely define the position of the point relative to the selected RF.*

During the motion of a point particle along some curvilinear path the radius vector is changed in time. The change of the radius vector in the selected RF is described by equation that can be written in vector or coordinate form

$$\vec{r} = \vec{r}(t) \iff \begin{cases} x = x(t), \\ y = y(t), \\ z = z(t). \end{cases} \qquad (1.1.1)$$

These equations are called the **equations of motion** of a point particle. We recall, that equations (1.1.1) are the equations of a line in three dimensional space, written in parametric form. The time t plays the role of a parameter and the line is nothing but the trajectory of a point. Equations (1.1.1) can be written in the form of $f(x, y, z) = 0$ as well. To do that one needs to exclude the parameter t (time) from these equations and to find the relationship between the spatial coordinates of a moving point. The equations of motion, written in such a form, are referred to in physics as **phase trajectory equations**.
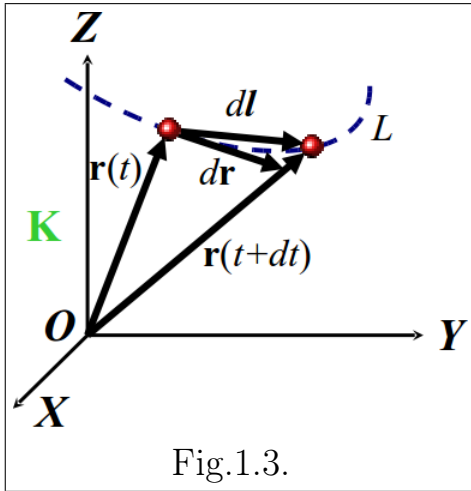
Consider an *elementary* (or, as mathematicians say, *infinitesimal*) displacement $\vec{dl}$ of a point particle along the trajectory L, which took place in an elementary time interval dt. Elementary interval $|\vec{dl}|$, which connects the initial and final position of the point particle in this case, should be a linear approximation corresponding to the curved segment $d\breve{l}$ of the trajectory L.

Then, according to the geometric meaning of the differential, segment $|\vec{dl}|$ must lie on a tangent line to the trajectory at the point corresponding to the initial position $\vec{r}(t)$. Therefore, the vector $d\vec{r}$, directed along the tangent to the trajectory of a particle, is called **an elementary displacement vector** and its length — elementary path dS:

$$|d\vec{r}| \ = \ dS \,. \tag{1.1.2}$$

The sum of elementary paths along the trajectory will give us, obviously, *the length of the trajectory*, that is the **distance** *travelled by the point particle*. By definition, the sum of infinitesimal quantities (i.e., elementary quantities) is an integral. So we have:

$$S \ = \ \int\limits_{L} |d\vec{r}| \,, \tag{1.1.3}$$

where S is the *distance* covered by a point particle, $\int\limits_{L}$ — notation for the *curvilinear integral* along the point trajectory L.

Fig.1.3.

Note that the definition (1.1.3) represents the distance as the sum of the modules (i.e., positive values) and hence the *distance covered by a point particle can* **only increase.**

Derivative of the radius vector with respect to time at any given point of the trajectory is called the **velocity vector** *of a point:*

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} \iff \begin{cases} v_x = dx(t)/dt, \\ v_y = dy(t)/dt, \\ v_z = dz(t)/dt. \end{cases} \tag{1.1.4}$$

According to the definition (1.1.4) and taking into account the meaning of the derivative, velocity vector $\vec{v}(t)$ measures the rate of the radius vector change in time during the motion of a point in space. Obviously, the velocity vector (as well as $d\vec{r}$) is directed at a tangent to the trajectory.

Magnitude of the velocity − speed − *is the module* (length) *of the velocity vector* and therefore by definition (1.1.4)

$$|\vec{v}(t)| = |d\vec{r}(t)/dt| \overset{dt>0}{=} |d\vec{r}(t)|/dt \underset{\text{according to } (1.1.2)}{=} dS/dt . \tag{1.1.5}$$

Thus the *speed (magnitude of the velocity)* is the time derivative of distance and characterizes the rate of change particle path length in time.

Further, a *derivative of the velocity vector with respect to time* at any given point of the trajectory is called the **acceleration vector** of a *point particle*

$$\vec{a} = \frac{d\vec{v}(t)}{dt} \iff \begin{cases} a_x = dv_x(t)/dt = d^2x(t)/dt^2, \\ a_y = dv_y(t)/dt = d^2y(t)/dt^2, \\ a_z = dv_z(t)/dt = d^2z(t)/dt^2. \end{cases} \tag{1.1.6}$$

in a similar way to velocity, acceleration measures the rate of change of velocity vector during the motion of a point in space.

1.2 Velocity for an arbitrary motion

We study the properties of the kinematic characteristics of motion – velocity and acceleration – at any given time moment.

To do this we need the notion of an **"instantaneous" coordinate system**: the "instantaneous" CS we will call so coordinate system, that set only to certain time moment.

As usual, we consider the motion of a point particle with respect to the reference frame K. Suppose that in an elementary time dt starting from initial moment t the point particle has moved from the space point M_1 to the space point M_2. We choose the direction of the axis OZ of the instantaneous Cartesian CS (coincided with the selected RF K) so that the points M_1 and M_2 lie in a plane parallel to the coordinate plane XY. We represent the radius vector of the point particle as the product of a unit vector $\vec{e}_r(t)$, which is directed along the radius vector (at any time), and of the module (length) of the radius vector:
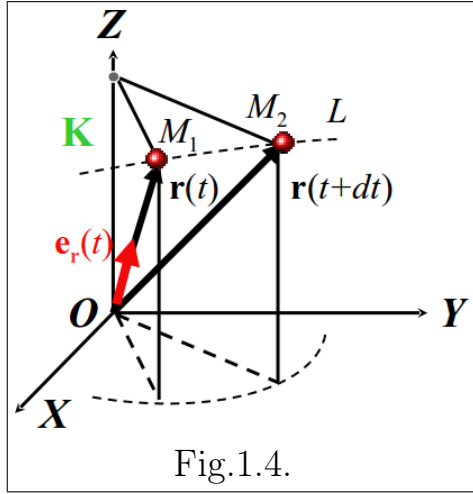
$$\vec{r}(t) \;=\; \vec{e}_r(t)|\vec{r}|. \tag{1.2.1}$$

Then the velocity vector takes the form

$$\vec{v}(t) \;=\; \frac{d\vec{r}(t)}{dt} \;=\; \vec{e}_r(t)\frac{|d\vec{r}(t)|}{dt} \;+\; |\vec{r}(t)|\frac{d\vec{e}_r(t)}{dt}. \tag{1.2.2}$$

Thus, the velocity vector of a point for any type of its motion can be represented as the sum of two components. One component is directed along the radius vector

$$\vec{v}_r(t) \;=\; \vec{e}_r(t)\frac{|d\vec{r}(t)|}{dt} \tag{1.2.3}$$

and is called **the linear motion velocity** of a point particle. The second component, is perpendicular to the radius vector:

$$\vec{v}_n(t) \;=\; |\vec{r}(t)|\frac{d\vec{e}_r(t)}{dt}. \tag{1.2.4}$$

The name of the component $\vec{v}_n$ we will introduced a little bit later. First we need to find out the meaning of time derivative of the unit vector $\vec{e}_r$.

To find this derivative we will complete the Fig.1.4: let's draw the unit vectors along the initial radius vector of the particle (i.e. in the direction of vector $\vec{OM_1} - \vec{e}_r(t)$) and along the radius vector for the time point $t + dt$ (i.e. in the direction of vector $\vec{OM_2} - \vec{e}_r(t + dt)$). The angle

between the OZ axis and the radius vector of a particle will be denoted by θ.
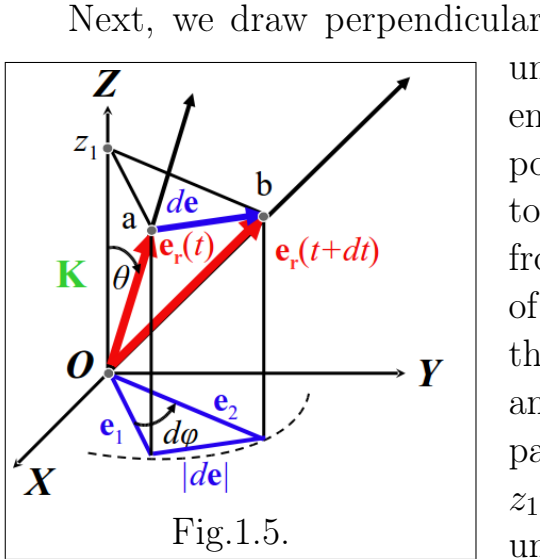
Next, we draw perpendiculars to the OZ axis from the end of the

Fig.1.5.

unit vectors. Due to the fact that the ends of the unit vectors (as well as the points M_1 and M_2) lie in the plane parallel to XY coordinate plane, perpendiculars from these points will fall into one point of the OZ axis — a point z_1. Accordingly, the perpendiculars form the segments z_1a and z_1b, as well as the triangle z_1ab are parallel to the XY plane. Then $e_1 = z_1a = |\vec{e}_r(t)| \sin\theta$ is the magnitude of the unit vector projection on the XY coordinate plane at the initial time point t, and $e_2 = z_1b = |\vec{e}_r(t + dt)| \sin\theta$ is the magnitude of the unit vector projection on the XY coordinate plane at the time point $t + dt$.

Let us denote the elementary angle between the initial and final positions of the point in the XY plane (i.e., between segments e_1 and e_2) by $d\varphi$. Then the length of the elementary segment $d|\vec{e}|$, connecting the ends of the segments e_1 and e_2, equals to

$$d|\vec{e}| \;=\; e_1 d\varphi \;=\; |\vec{e}_r| \sin\theta d\varphi. \tag{1.2.5}$$

We recall that the derivative of the rotation angle of a particle over time is called *angular velocity* of a particle

$$\omega \;=\; \frac{d\varphi}{dt}. \tag{1.2.6}$$

Consequently it follows from (1.2.5) that

$$\frac{d|\vec{e}_r|}{dt} \;=\; |\vec{e}_r|\omega \sin\theta. \tag{1.2.7}$$

By definition, the magnitude of the **vector product (cross product)** of any two vectors, such as $\vec{a}$ and $\vec{b}$, to take an example, is

$$|[\vec{a}, \vec{b}]| \;=\; |\vec{a}||\vec{b}| \sin(\widehat{\vec{a}, \vec{b}}). \tag{1.2.8}$$

Thus, assuming that the **angular velocity vector** *is directed* **along the axis of rotation** (i.e. along the Z axis), we obtain

$$\frac{d\vec{e_r}}{dt} \;=\; [\vec{\omega}, \vec{e_r}]. \tag{1.2.9}$$

According to the definition (1.2.4), the velocity component, perpendicular to the radius vector of a point, can be written in the next form

$$\vec{v}_n(t) \;=\; [\vec{\omega}, \vec{r}]. \tag{1.2.10}$$

From (1.2.10) it is clear that this velocity vector is a characteristic of rotational motion of a point (it shows the velocity of radius vector rotation around the axis of rotation – in our case around Z axis) and is called, respectively, **rotational velocity**.

So, we have found that

any motion of a particle can be decomposed into **two types of motion:** **linear (rectilinear)** – *along the radius vector* (with velocity $\vec{v}_r$) and **rotational** – *with respect to the reference frame origin* (with velocity $\vec{v}_n$)

Accordingly, the particle **velocity vector** *at any point of its trajectory* can be decomposed into **two components:**

 - velocity $\vec{v}_r$ of *linear motion* (eq. (1.2.3)

 - and velocity $\vec{v}_n$ of *rotational motion* (eq. (1.2.10).

So, the velocity vector for a general motion can be written in the form

$$\vec{v}(t) \;=\; \vec{e_r}(t)\frac{d|\vec{r}(t)|}{dt} \;+\; [\vec{\omega}(t), \vec{r}(t)]. \tag{1.2.11}$$

To be more precise, it should be said that the change in direction of the vector $\vec{\omega}(t)$ in time is equivalent to change in direction of the Z axis (as well as of X and Y axes) belonging to the "instantaneous" coordinate system.

1.3 Acceleration for an arbitrary motion

As in the previous paragraph, we study the motion of a point with respect to the reference frame K. Let us suppose, as before, that in an elementary time dt the point particle has moved from the space point M_1 to the space point M_2.

Velocity vector of a point particle $\vec{v}$ can be written in the form of a product of the unit vector $\vec{e}_\tau(t)$, which is directed along the velocity vector (for any certain time), and the module (length) of velocity vector:

$$\vec{v}(t) = \vec{e}_\tau(t)|\vec{v}|. \tag{1.3.1}$$

Then acceleration vector will have the form

$$\vec{a}(t) = \frac{d}{dt}\left(\vec{e}_\tau(t)|\vec{v}|\right) = \vec{e}_\tau(t)\frac{d|\vec{v}(t)|}{dt} + |\vec{v}(t)|\frac{d\vec{e}_\tau(t)}{dt}. \tag{1.3.2}$$

In the last term of equation (1.3.2) we change the variables:

$$\frac{d\vec{e}_\tau(t)}{dt} = \frac{d\vec{e}_\tau}{dS}\underbrace{\frac{dS}{dt}}_{} = |\vec{v}(t)|\frac{d\vec{e}_\tau}{dS} \tag{1.3.3}$$

$$\searrow \text{according to (1.1.5) it is } |\vec{v}|$$

and introduce new notation

$$\mathcal{R} = 1 \left/ \left|\frac{d\vec{e}_\tau}{dS}\right|\right. . \tag{1.3.4}$$

So, for the acceleration vector we obtain:

$$\vec{a}(t) = \vec{e}_\tau\frac{d|\vec{v}(t)|}{dt} + \frac{|\vec{v}|^2}{\mathcal{R}}\vec{e}_n, \tag{1.3.5}$$

where $\vec{e}_n$ is the unitary vector perpendicular to the velocity vector (i.e. perpendicular to the unitary vector $\vec{e}_\tau$). The first term in this equation is denoted by a_τ and is called **tangential** (i.e. tangent to the velocity vector) **acceleration**. Accordingly, the second term is denoted by a_n and is called **normal** (perpendicular to the velocity vector) **acceleration**.

Let us find out the meaning of the value $\mathcal{R}$, introduced in the equality (1.3.4). To do this, we consider the motion of a point particle along a

circular path with constant velocity (i.e. both angular velocity vector $\vec{\omega} = const$ and the radius vector $|\vec{r}| = const$). In this case, $\vec{v} = \vec{v}_n$ and we have for the acceleration

$$\vec{a}(t) \;=\; \frac{d\vec{v}_n}{dt} \;=\; \frac{d}{dt}\,[\vec{\omega},\vec{r}] \quad\overset{\omega\,=\,const}{\nearrow}\quad =\; \left[\vec{\omega}, \frac{d\vec{r}}{dt}\right] \;=\; \left[\vec{\omega}, [\vec{\omega},\vec{r}]\right]. \qquad (1.3.6)$$

$$\underset{\text{see }(1.2.10)}{\searrow}$$

The last expression in this equation contains a double vector (cross) product, which can be transformed according to the formula

$$\left[\vec{a}, [\vec{b}, \vec{c}]\right] = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b}). \qquad (1.3.7)$$

The relationship (1.3.7) is easy to remember by a mnemonic rule: **baccab**.

Consequently, $\vec{a}(t) = \vec{\omega}(\vec{\omega}, \vec{r}) - \vec{r}\omega^2$. But for the motion of a point along the circular path it's angular velocity vector $\vec{\omega}$ is perpendicular to the radius vector $\vec{r}$. This means that the scalar product of these vectors is zero $(\vec{\omega}, \vec{r}) = 0$. And we obtain for the acceleration

$$\vec{a}(t) \;=\; -\vec{r}\omega^2. \qquad (1.3.8)$$

The magnitude of acceleration vector equals to $|\vec{a}| = |\vec{r}|\omega^2$. Then, recalling that the magnitude of it's angular velocity vector $\omega = |\vec{v}|/R$, (here the letter R denotes the length of the radius vector $|\vec{r}|$, i.e. the radius of the circle) we arrive at the well-known expression

$$|\vec{a}(t)| \;=\; \frac{|\vec{v}|^2}{R} \;=\; |\vec{a}_c|. \qquad (1.3.9)$$

Hence, for the motion along a circular path with constant velocity, normal acceleration vector of a point particle is a *centrifugal acceleration* $|\vec{a}_c|$.

Comparing the formula for the normal acceleration (second term in (1.3.5)) with the last equation, we see that for the motion of a point along a circular path the value of $\mathcal{R}$ coincides with the radius R of the circle.

Obviously for an arbitrary motion of a point particle, value $\mathcal{R}$ equal to the radius of a **certain** (i.e., corresponding to a certain time) circle. In other words, the result obtained in equation (1.3.5), means that

> at any trajectory point,
> **point particle motion**
> can be regarded as
> **rotational motion along the certain circle arc**,
> *whose radius* equals to $\mathcal{R}$
> (with tangent $\vec{a}_\tau$ and normal $\vec{a}_n$ accelerations.)

Value $\mathcal{R}$ is called **the trajectory curvature radius** at a certain point.

The figure shows an example of decomposition of the total acceleration vector for a point particle (located at a preset space point M at a certain time) on its components: normal and tangential accelerations. For the same point M the certain circle with centre at the point O is constructed. The circle radius equals to the curvature trajectory radius $\mathcal{R}$ at the point M and numerically equals to the segment OM length.
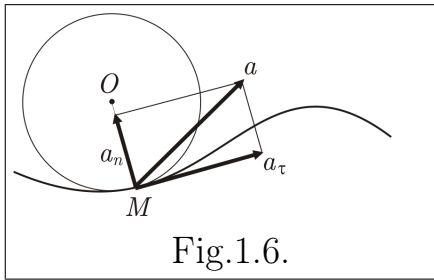
Fig.1.6.

1.4   Types of acceleration

In paragraph 1.2 it was found that any motion of a point particle can be decomposed into two types of motion: linear (along the radius vector) and rotational (with respect to the origin of RF). Accordingly, velocity vector can be represented as a vector sum of two vectors that characterize these motions (velocity vectors $\vec{v}_r$ and $\vec{v}_n$).

Obviously, such representation of the motion affects also by the acceleration. Let us find out what *types of acceleration* characterize the motion of a point particle taking into account the possibility of the motion decomposition on linear and rotational one.

According to the definition (1.1.6)

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt}.$$

If we recall here the expression for the velocity (1.2.11), then we obtain for the acceleration

$$\vec{a}(t) = \frac{d}{dt}\left(\vec{e}(t)\frac{d|\vec{r}(t)|}{dt} \ + \ [\vec{\omega}(t), \vec{r}(t)] \right) \qquad (1.4.1)$$

From this expression, using the product rule for derivatives, it's easy to find that (for the functions on time we use here the abbreviation f instead of $f(t)$ for brevity):

$$\vec{a}(t) = \vec{e}\,\frac{d^2|\vec{r}|}{dt^2} \; + \; \frac{d\vec{e}}{dt}\frac{d|\vec{r}|}{dt} \; + \; \left[\frac{d\vec{\omega}}{dt},\vec{r}\right] \; + \; \left[\vec{\omega},\frac{d\vec{r}}{dt}\right]. \qquad (1.4.2)$$

Derivative of the particle angular velocity over time is called *angular acceleration* of a particle

$$\vec{\varepsilon} \; = \; \frac{d\vec{\omega}}{dt}. \qquad (1.4.3)$$

The equation (1.2.9) gives us, as we had shown earlier, the derivative of the unit vector $\vec{e}$ on time t. Then

$$\vec{a} = \vec{e}\,\frac{d^2|\vec{r}|}{dt^2} \; + \; \frac{d|\vec{r}|}{dt}[\vec{\omega},\vec{e}] \; + \; [\vec{\varepsilon},\vec{r}] \; + \; \left[\vec{\omega},\underbrace{\left(\vec{e}\,\frac{d|\vec{r}|}{dt} + [\vec{\omega},\vec{r}]\right)}_{\text{velocity } \vec{v} = \dfrac{d\vec{r}}{dt}}\right] \qquad (1.4.4)$$

If we now open the brackets in the last vector product, we obtain

$$\vec{a} = \vec{e}\,\frac{d^2|\vec{r}|}{dt^2} \; + \; 2\left[\vec{\omega},\vec{e}\,\frac{d|\vec{r}|}{dt}\right] \; + \; [\vec{\varepsilon},\vec{r}] \; + \; \left[\vec{\omega},[\vec{\omega},\vec{r}]\right]. \qquad (1.4.5)$$

In paragraph 1.2, the value of $d|\vec{r}|/dt$ was named the velocity $\vec{v}_r(t)$ of a point linear motion. Accordingly, the value of $\vec{a}_r$

$$\vec{a}_r \; = \; \vec{e}\,\frac{d^2|\vec{r}|}{dt^2}, \qquad (1.4.6)$$

we will call **the linear motion acceleration** (along the radius vector) of a point particle

With the help of this notations, the equation for the acceleration of a point particle will have the form:

$$\vec{a} = \vec{a}_r \; + \; 2\,[\vec{\omega},\vec{v}_r] \; + \; [\vec{\varepsilon},\vec{r}] \; + \; \left[\vec{\omega},[\vec{\omega},\vec{r}]\right]. \qquad (1.4.7)$$

In this expression only the first term (i.e., $\vec{a}_r$) characterizes a *particle motion along the radius vector.* Hence the remaining three members characterize the *rotational motion* of a point particle *with respect to the reference frame origin.* Historically, these three components of particle acceleration are divided into two

$$\vec{a}_\xi = [\vec{\varepsilon}, \vec{r}\,] + \left[\vec{\omega}, [\vec{\omega}, \vec{r}\,]\,\right], \qquad (1.4.8)$$

$$\vec{a}_k = 2\,[\vec{\omega}, \vec{v}_r]. \qquad (1.4.9)$$

Component of the acceleration $\vec{a}_\xi$ is called **drag acceleration**. It *characterizes the change in the velocity* when the particle moves *along an arc of an instant circle.*

Component of the acceleration $\vec{a}_k$ is called **Coriolis acceleration** (Gaspard-Gustave de Coriolis was a French scientist, who studied the motion with such acceleration). This acceleration *characterizes the rate of change of velocity vector* for the motion of a point particle *along the radius* **of rotating** *instant circle.*

To better visualize the properties of these components of the total acceleration, we consider examples of a particle motion in which these components occur.

a) A particle moves *linearly.* Then

kinematic conditions of motion	kinematic characteristics of motion (Fig.1.8a)
$\vec{\omega} = 0$ $(\vec{\varepsilon} = d\vec{\omega}/dt = 0)$	$\vec{a} = \vec{a}_r = \vec{e}\,d^2|\vec{r}|/dt^2$ $\vec{v} = \vec{v}_r = \vec{e}\,d|\vec{r}|/dt$

a)   b)   c)
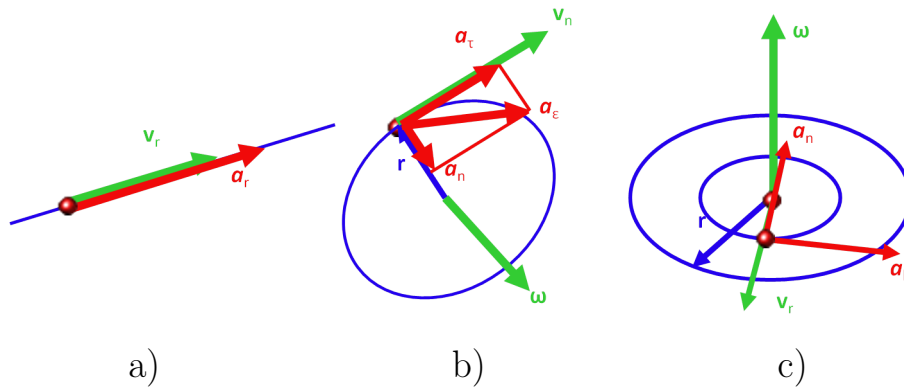
Fig.1.8.

b) A particle moves *along the arc of a circle.* Then

kinematic conditions of motion	kinematic characteristics of motion (Fig.1.8b)
$r = const$ $\vec{\omega} \perp \vec{r}$	$\vec{a} = \vec{a}_\xi = [\vec{\varepsilon}, \vec{r}] + [\vec{\omega}, [\vec{\omega}, \vec{r}]] =$ $= \vec{\omega}(\vec{\omega}, \vec{r}) - \vec{r}\omega^2 + [\vec{\varepsilon}, \vec{r}] =$ $= \vec{a}_n + \vec{a}_\tau$ $\vec{v} = \vec{v}_n = [\vec{\omega}, \vec{r}].$

Here, in obtaining an expression for the acceleration, we used the formula (1.3.7) to calculate the double vector product.

в) A particle moves *along the radius of a rotating circle* (disk). Then

kinematic conditions of motion	kinematic characteristics of motion (Fig.1.8c)
$\vec{\omega} = const$ $\vec{v}_r = const$ $\vec{\omega} \perp \vec{r}$	$\vec{a} = 2[\vec{\omega}, \vec{v}_r] - \vec{r}\omega^2 =$ $= \vec{a}_k + \vec{a}_n$ $\vec{v} = \vec{v}_r + [\vec{\omega}, \vec{r}] = \vec{v}_r + \vec{v}_n.$

It should be noted that it is **impossible** to build a motion in which the acceleration of a point would have *only the Coriolis acceleration component.*

1.5 Reconstructing motion equation (integral relations)

Knowing the equation of motion, i.e. the dependence of the radius vector on time $\vec{r} = \vec{r}(t)$, we can find the velocity $\vec{v}$ and acceleration $\vec{a}$ vectors of a point particle for any time point. This is so-called **direct problem**. The question arises: can we find the motion equation $\vec{r} = \vec{r}(t)$, knowing the time dependence of the velocity $\vec{v}(t)$ or acceleration $\vec{a}(t)$ vectors? The solution to this problem is called **the inverse problem**.

1.5.1 From the velocity vector

Suppose we are given the velocity vector of a point as a function of time

$$\vec{v} = \vec{v}(t) = \frac{d\vec{r}(t)}{dt} \tag{1.5.1}$$

and it is required to find the trajectory equation, that is, the dependence of $\vec{r} = \vec{r}(t)$, on the base of the knowledge the velocity vector. It is clear that from (1.5.1) we can find $d\vec{r}(t) = \vec{v}(t)dt$. Integration of this equation from the initial time point t_0 to any current time point t gives us

$$\vec{r}(t) - \vec{r}(t_0) = \int_{t_0}^{t} \vec{v}(t)dt , \tag{1.5.2}$$

or

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \vec{v}(t)dt , \tag{1.5.3}$$

where $\vec{r}(t_0)$ is the point particle radius vector at the initial time moment. Thus we see that

> for the **reconstructing motion equation**
> from the **velocity vector**
> one needs to know
> **initial position** of the point particle.

For example consider a particle moving so that its velocity vector remains constant in magnitude and direction (i.e., a point particle moves **uniformly and rectilinearly**):

$$\vec{v}(t) \; = \; const \; = \; \vec{v}_0 \tag{1.5.4}$$

Let us substitute this expression for the velocity in equation (1.5.3)

$$\vec{r}(t) \; = \; \vec{r}(t_0) + \int\limits_{t_0}^{t} \vec{v}_0 dt \; = \; \vec{r}(t_0) + \vec{v}_0 \int\limits_{t_0}^{t} dt \; = \; \vec{r}(t_0) + \vec{v}_0(t - t_0). \tag{1.5.5}$$

Obtained equation describes a particle in linear motion with a constant velocity and initial position defined by the radius vector $\vec{r}_0 = \vec{r}(t_0)$. If we assume that in the initial time moment $t_0 = 0$ the particle was at the reference frame origin, this equation takes its simplest form:

$$\vec{r}(t) \; = \; \vec{v}_0 t \,. \tag{1.5.6}$$

1.5.2   From the acceleration vector

Suppose we are given the acceleration vector of a point particle as a function of time

$$\vec{a} \; = \; \vec{a}(t) \; = \; \frac{d\vec{v}(t)}{dt} \,. \tag{1.5.7}$$

And it is required to find the trajectory equation, that is, the dependence $\vec{r} \; = \; \vec{r}(t)$, from this definition. It is clear that equation (1.5.7) reads $d\vec{v}(t) \; = \; \vec{a}(t)dt$. Integration of this equation from the initial time point t_0 to any current time point t gives us

$$\vec{v}(t) - \vec{v}(t_0) \; = \; \int\limits_{t_0}^{t} \vec{a}(t)dt \,, \tag{1.5.8}$$

or

$$\vec{v}(t) \; = \; \vec{v}(t_0) + \int\limits_{t_0}^{t} \vec{a}(t)dt \,, \tag{1.5.9}$$

where $\vec{v}(t_0)$ is the velocity vector at the initial time point.

Equation (1.5.9) determines the velocity vector as a function of time. Therefore now, to find the equation of the trajectory, we can use the result

of the preceding paragraph, where we had found trajectory equation for a given velocity. Substituting the right hand side of the last equation into the equation (1.5.3) we obtain:

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} \left[\vec{v}(t_0) + \int_{t_0}^{t} \vec{a}(t)dt \right] dt \qquad (1.5.10)$$

or

$$\vec{r}(t) = \vec{r}(t_0) + \vec{v}(t_0)\,(t - t_0) + \int_{t_0}^{t} \int_{t_0}^{t} \vec{a}(t)dt^2 \qquad (1.5.11)$$

where $\vec{r}(t_0)$ is the radius vector, and $\vec{v}(t_0)$ is the velocity vector at the initial time moment.

By this way we see, that

for **reconstructing motion equation**
from the **acceleration vector**
one needs to have **two parameters:**
initial position and **velocity vector**
of the point particle
at the **initial time point.**

When a particle moves so that its acceleration vector remains a constant (magnitude as well as direction are constant, so this is **uniformly accelerated** motion): $\vec{a}(t) = const = \vec{a}_0$, the corresponding equation (1.5.11) have the form:

$$\vec{r}(t) = \vec{r}(t_0) + \vec{v}(t_0)\,(t - t_0) + \frac{\vec{a}_0(t - t_0)^2}{2}. \qquad (1.5.12)$$

In this case, we have found the motion equation with constant acceleration $\vec{a}_0$, initial radius vector $\vec{r}_0 = \vec{r}(t_0)$ and initial velocity vector $\vec{v}_0 = \vec{v}(t_0)$. If we assume that at the initial time moment $t_0 = 0$ the particle was at the reference frame origin, this equation receive simplest form:

$$\vec{r}(t) = \vec{v}_0\,t + \frac{\vec{a}_0 t^2}{2}. \qquad (1.5.13)$$

1.6   Galilean transformations

Before this section, the mathematical description of a point particle mechanical motion was considered in a certain frame of reference

This means that if the necessary measurements have been made in a certain reference frame (for example, was measured the particle coordinates at the different time points), then these measurements allow to obtained any others kinematic parameters of this particle (velocity, acceleration, etc.).

However, the question arises: *how to obtain the description of a particle motion* in the reference frame K, *on the base of measurement results* in the reference frame K' *moving relatively to* RF K *with a certain velocity* $\vec{v}_0(t)$?

To answer this question, we consider the motion of a point particle with respect to two reference frames — K and K'. Radius vector connecting the origins of these reference frames (points O and O') we denote as $\vec{r}_o$. Suppose that in a specific time point t the particle is at a point M. Then the radius vector of the particle relative to the reference frame K is $\vec{r}(t)$ and relative to the reference frame K' is $\vec{r}\,'(t')$ (here t' is time measured in reference frame K' – in general case it is different from t).
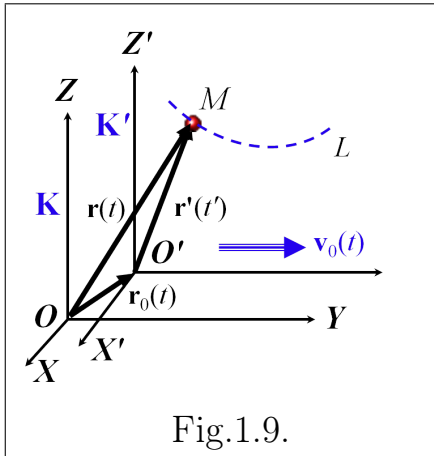
Fig.1.9.

These three vectors form a triangle and hence

$$\vec{r}(t) \;=\; \vec{r}_o(t) + \vec{r}\,'(t'). \qquad (1.6.1)$$

By definition, $\vec{v}\,'(t') \;=\; \dfrac{d\vec{r}\,'(t')}{dt'}$ is the velocity vector of a point particle relative to reference frame K'. Accordingly, $\vec{v}(t) \;=\; \dfrac{d\vec{r}(t)}{dt}$ is the velocity vector of the point particle relative to the reference frame K, and $\vec{v}_o(t) \;=\; \dfrac{d\vec{r}_o(t)}{dt}$ is the velocity vector of the reference frame K' itself with respect to K.

Taking the time derivative of expression (1.6.1) with respect to time t

defined in the reference frame K we obtain

$$\vec{v}(t) \; = \; \frac{d\vec{r}_o(t)}{dt} \; + \; \frac{d\vec{r}\,'(t')}{dt'}\frac{dt'}{dt} \; =$$

$$= \; \vec{v}_o(t) + \vec{v}\,'(t')\frac{dt'}{dt}. \qquad (1.6.2)$$

Here we've made the change of variables in the second term on the right hand side. Thus we see that **for an unambiguous determination of kinematic parameters describing the motion of a point particle relative to the reference frame K, by the measurements made in the reference frame K', one needs to know the connection of time moments t and t'.**

In classical mechanics the *problem* of time points relationship in various frame references is *solved* by use of *Galileo postulate:*

> **time points in different reference frames**
> **coincide with each other**
> **up to a constant,**
> **that is defined by the clock synchronization procedure,**

id est:

$$t \; = \; t' \; + \; const. \qquad (1.6.3)$$

It is usually assumed that clocks are synchronized so, that $t \; = \; t'$, i.e. $const = 0$. By using this method of synchronization the equation (1.6.1) takes the form

$$\vec{r} \; = \; \vec{r}_o \; + \; \vec{r}\,' \qquad (1.6.4)$$

(time points are not shown for the reason of their equality).This equation is called **Galileo transformation rule for the coordinates**, that were measured from an arbitrary **reference frames**.

Equation (1.6.2) does not depend on the method of time synchronization (i.e. on the choice of the constant in (1.6.3)):

$$\vec{v} \; = \; \vec{v}_o \; + \; \vec{v}\,' \qquad (1.6.5)$$

and is called **the law of velocities addition**.

To find the relationship between accelerations in different reference frames one needs (by definition) to take time derivative of equation (1.6.5). As the result we have:

$$\vec{a} \; = \; \vec{a}_o \; + \; \vec{a}\,' \, . \qquad (1.6.6)$$

Here $\vec{a}_o$ is **acceleration** of reference frame K' relative to K.

The set of all three equations (1.6.4 - 1.6.6) is called **Galileo transformation rules for the arbitrary reference frames**.

There is a *set of reference frames* among all the possible ones that occupies a special place: *they move relative to each other with constant velocity* (i.e., their relative acceleration is zero). Such reference frames are called **inertial reference frames (IRF)**.[1]

Let us find Galileo transformation rules for inertial reference frames. If the velocity vector of IRF K' ($\vec{v}_o = const$) is known, then the location of IRF K' origin could be found as equation of motion obtained on the base of known velocity (see, Section 1.5):

$$\vec{r}_o(t) \;=\; \vec{r}_o(t_o) \;+\; \vec{v}_o\,(t - t_o), \qquad (1.6.7)$$

where $\vec{r}_o(t_o)$ is the radius vector of IRF K' origin point (i.e. of the point O' - see, Fig.1.9) at the initial time moment t_o.

Let's take initial time point to be zero: $t_o = 0$ and origins of inertial reference frames K and K' at this moment coincide: $\vec{r}_o(t_o) = 0$. Then **Galileo transformation rules for inertial reference frames** will take the form

$$
\left\|
\begin{array}{l}
\vec{r}(t) \;=\; \vec{v}_o\,t \;+\; \vec{r}\,'(t), \\
\vec{v}(t) \;=\; \vec{v}_o \;+\; \vec{v}\,'(t), \\
\vec{a}(t) \;=\; \vec{a}\,'(t),
\end{array}
\right\| \qquad (1.6.8)
$$

The last equation in this transformation rules should be paid a special attention. It means that **acceleration vector of a point particle is the same in all of the inertial reference frames**.

This result is a quite clear evidence of exceptional properties of inertial reference frames. As the consequence, exactly these systems should be used to study of mechanical phenomena. Anywhere below, if it is not specified otherwise, such a choice of reference frame will be assumed.

---

[1]This frase **is not** a definition of inertial reference frames (it will be formulated in next chapter), but it gives a good logic and reasonable idea of such systems.

Chapter 2

Dynamics

Kinematics describes the motion of bodies not taking into account the question of why the body moves so (e.g. uniformly in a circle, or is uniformly accelerated in a straight line), and not otherwise.

Dynamics studies the motion of bodies in relation to those factors (interactions between bodies), which cause a particular pattern of motion. The basis of the so-called classical or Newtonian mechanics is the three laws of dynamics, formulated by Newton in the end of XVIth century.

In this chapter the dynamics of:

- a material point,

- system of material points,

- absolutely rigid body

is consecutively described. On this basis we set out the basic laws of absolutely rigid body dynamics, that is modelled by a system of material points (this allows us the use of all previous material). In further paragraphs concepts of work and energy are introduced with special attention paid to the concept of potential fields, potential energy and their properties. The laws of conservation for *full mechanical energy, linear and angular momentums* are derived. On the base of conservation laws some important types of mechanical interaction and motion are considered: absolutely rigid and non rigid collisions, the motion of a body of variable mass and Kepler's laws.

2.1   Dynamics of material point

The laws of physics, like of any other science — the laws that describe abstract models of real objects of Nature. Therefore, before formulating a law for the objects of interest in the selected RF, we need to build it's model that accurately reflects the properties of the object *in the selected RF*. It is clear that this type of description, will force us to re-build **all** the laws *for each new RF* – obviously an impossible task. Consequently, for the construction of the laws describing the *abstract* models of real objects we need to construct *such abstract* reference frames, that will conserve the form of physical laws (at least mechanical laws).

In other words, our laws should be valid in all (well-formed) abstract reference frames. In particular for problems of mechanics we can assume that there is a reference frame in which the acceleration of a point is entirely originated from interaction with other bodies. Then, a free particle that is not exposed to any other bodies, should move relative to a reference frame uniformly, or, as it is said, by inertia. Such *RFs* are called inertial reference frames, the qualitative concept of which has already been given in section 1.6.

The assertion that the *inertial reference frames exist* is the content of the first law of classical mechanics — the Galileo-Newton's law of inertia. In the next section we give a rigorous definition of the inertial reference frames and Newton's laws are formulated.

2.1.1   Laws of Newton

Newton's laws have resulted from the large number of theoretical generalization of experimental facts. The correctness of these laws, as well as any other, confirmed by the agreement of theoretical results of abstract models description with the observed behaviour of real objects.

Accordingly, if in some conditions, the behaviour of real objects begins to differ from predicted behaviour of their model, this means that the law is not satisfied. Therefore we need to either refine the existing law, or (if the predictions of the law essentially differ from the behaviour of real objects) – to build a new one. But, for the construction of a new law we need, of course, first accumulate a sufficiently large set of experimental facts to have the possibility of generalization.

First Newton's law

Let us give a rigorous definition of inertial reference frames. Reference frames in which the law (equation) of motion is uniquely defined, if specified:

- initial conditions $\vec{r}_o = \vec{r}(t_o)$, $\vec{v}_o = \vec{v}(t_o)$,

- the function $\vec{F}(\vec{r}, t)$, describing the interaction with the environment

are called **inertial reference frames** (IRF).

Function of coordinates and time $\vec{F}(\vec{r}, t)$, describing the interaction of a point with the environment, is called **force**. The target of Newton laws — to find out the causes of motion and, as it will be shown later, this problem in each case is in determination of the function $\vec{F}(\vec{r}, t)$.

Having defined the concept of IRF, we can formulate Newton's first law:

> **Inertial reference frames exist,**
> as *a mathematical abstraction*
> of real reference frames.

The second Newton law

As mentioned above, the problem of Newton's laws - to find out the causes of movement. Let us compare the means of obtaining the equations of motion in the kinematics and dynamics.

From the viewpoint of kinematics, to recover the equations of motion for a given acceleration $\vec{a}(\vec{r}, t)$ one needs to know two parameters: the initial position of a material point and the speed of this point in the initial time moment.

From the point of view of dynamics equation of motion can be uniquely determined on the base of knowledge of the function $\vec{F}(\vec{r}, t)$ – force. In this case one needs to know two parameters: the initial position of a point and the speed of this point in the initial time moment.

Kinematics		Dynamics	
$\vec{r}_0, \vec{v}_0$	$\vec{r} = \vec{r}(t)$	$\vec{r}_0, \vec{v}_0$	$\vec{r} = \vec{r}(t)$
$\vec{a}(t)$		$\vec{F}(\vec{r}, t)$	

$$\Rightarrow \quad \boxed{\vec{a}(t) \Longleftrightarrow \vec{F}(\vec{r}, t)}$$

Thus the acceleration and force perform similar functions in the two sections of mechanics. As a consequence, we can formulate the second law of Newton:

> In *inertial reference frames*
> **interaction of an object with the environment (force)**
> **causes acceleration of the object.**

Historically, the particular mathematical form of the law connecting acceleration and force was chosen in the simplest form:

$$\vec{a}(t) = \frac{1}{m}\vec{F}(\vec{r}, t) \qquad (2.1.1)$$

where m is the coefficient of proportionality between acceleration and force. It characterizes the ability of the body to receive acceleration due to some force, i.e. is a measure of the body inertia and, accordingly, is called *inert* **mass**.

In essence, the equation (2.1.1) is the definition of the force concept.

Third Law of Newton

Newton's third law in general, is a universal law of interactions: *every action causes an equal in magnitude reaction.* The third law of Newton for physics:

> For **any physical interaction**
> the **action** *of one body on another*
> **causes** *equal in magnitude and opposite in direction*
> **reaction** of the *second one* on the first body.
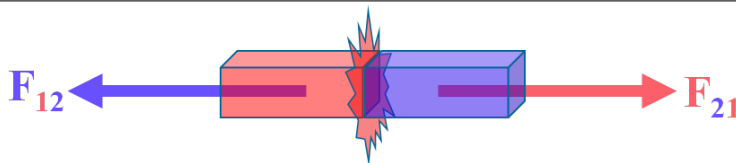
We emphasize that the forces related by Newton's third law are attached to *different* bodies and therefore can never start at one point.

This feature of the forces connected by Newton's third law (the impossibility to have the origin in one point), imposes certain restrictions on the form of the law itself. The fact of the matter is that

- *one vector equation may not contain vectors that originate in different points of the space.*

That is why *third* Newton's law **can not** be written in vector form. But we can write it in the next form:

$$|\vec{F}_{12}| = -|\vec{F}_{21}|, \qquad (2.1.2)$$

where the minus sign – the so-called "physical" minus which only shows that forces $|\vec{F}_{12}|$ and $|\vec{F}_{21}|$ are oppositely directed.

2.1.2 Types of forces in point mechanics

We emphasize that if one uses the model of material point to describe the motion of a physical body, all the forces applied to this body, must be appended precisely to this point (which, as we will see in further paragraphs, is the centre of mass). There are **four types of forces** in mechanics of a material point.

1. *Predefined forces (symbol - $\vec{F}$)*

Specifically defined force, the magnitude and direction of which are *defined a priory* (figure 2.2).

2. *Gravity force (symbol - $\vec{F}_g$)*

The origin of gravity force is the gravitational interaction of a body with the Earth. It is always directed to the centre of gravity of the Earth (or vertically down if it's possible to introduce the concept of vertical). The magnitude of the force: $F_g = mg$, where g – acceleration due to gravity (figure 2.2).

3. *Reaction forces*

This type of forces arise in the *direct contact* of the body in study with other bodies. Accordingly, the number of reaction forces equals to the number of bodies, with that the studied body is in direct mechanical contact. There are two types of reaction forces:

a) support reaction force or normal force (symbol – $\vec{N}$), perpendicular to the plane of contact (or to a tangent plane) out from the plane (figure 2.3);
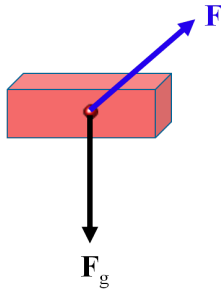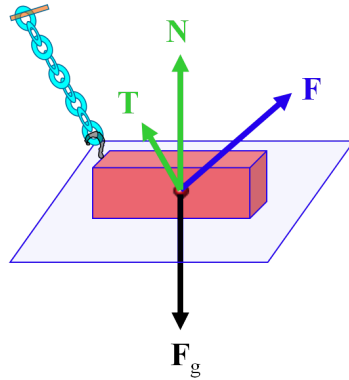
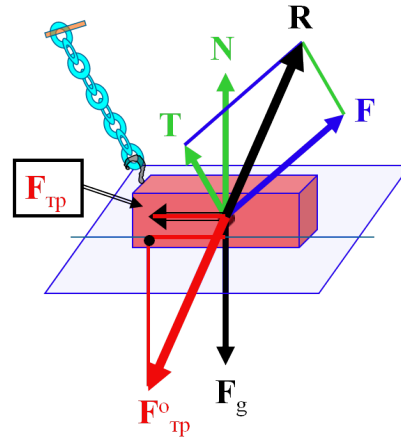Fig. 2.2.   Fig. 2.3.   Fig. 2.4

b) string reaction force (symbol – $\vec{T}$), is directed from the centre of mass along the string (on the figure 2.3 the string is drawn as a chain).

String reaction forces can be modelled using ideal strings which is unstretchable (and can be also massless and frictionless, if needed).

The magnitude of reaction forces can be determined **only** from the laws of Newton.

4. *Friction forces (symbol – $\vec{F}_{fr}$)*

Friction is a surface force that opposes relative motion (real or possible). There are two kinds of friction forces: **static friction** $(\vec{F}_{fr})_{st}$ and **kinetic friction** $(\vec{F}_{fr})_{kin}$.

Vector of the static friction force equals to *resulting* vector $\vec{R}$ of all other forces. But conventionally, by the force of static friction is called the projection vector $\vec{F}_{st}$ on the plane of motion (figure 2.4). Maximum value of static friction force (achieved when the body begins to move) is called the force of kinetic friction:

$$\left|(\vec{F}_{fr})_{kin}\right| = \left|\max\left((\vec{F}_{fr})_{st}\right)\right| = k\left|\vec{N}\right| \qquad (2.1.3)$$

where k – is the coefficient of static friction that is defined by the properties of the surfaces in contact.

The last equality in the ratio (2.1.3) is approximate and is satisfied for sufficiently small values of the coefficient of friction (usually for $k < 0.6$). For large values of friction force this relationship between magnitude of reaction force $|\vec{N}|$ and friction force is non-linear.

2.2   The system of interacting particles

2.2.1   Centre of mass

Definition: the set of material points bounded in space, arbitrarily moving in space is called *mechanical system.*

Let's consider such a system with respect to some inertial reference frame K (figure 2.5). For any mechanical system we can determine an imaginary point called **centre of mass**. This point has important properties required for the description of particles motion. The position of the **centre of mass** with respect to the given reference frame is characterized by **radius-vector** $\vec{r}_c$, defined as follows:
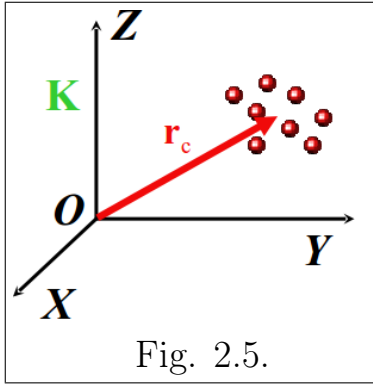
$$\vec{r}_c = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2 + ... + \vec{r}_n m_n}{m_1 + m_2 + ... + m_n}$$
$$= \frac{1}{M} \sum_{i=1}^{n} \vec{r}_i m_i, \qquad (2.2.3)$$

where i – is the index of a point, n – number of points, m_i – mass of ith point and M is the mass of the whole system. Consequently the speed of the centre of mass is, by definition:

Fig. 2.5.

$$\vec{v}_c = \frac{d\vec{r}_c}{dt} = \frac{1}{M} \sum_{i=1}^{n} m_i \frac{d\vec{r}_i}{dt} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{v}_i. \qquad (2.2.4)$$

In kinematics we have dealt with kinematic parameters of the particle motion (speed and acceleration). In dynamics, to describe the causes of motion, we introduce new parameters of motion – dynamic parameters. These parameters could be obtained by multiplying the kinematic parameters of a particle by its inertial mass. The value $\vec{p}_i = m_i \vec{v}_i$ is the *first dynamic parameter* of a particle and is called **impulse**. The value

$$\vec{P}_c = M\vec{v}_c = \sum_i m_i \vec{v}_i \qquad (2.2.5)$$

is called the **centre of mass impulse**. By this way we see, that impulse $\vec{P}_c$ is connected with velocity $\vec{v}_c$ by the same way as for material point with mass M (whole mass of the system).

In other words, the concept of the centre of mass is a *rigorous mathematical* procedure for mapping of an arbitrary mechanical system to a mathematically precise model – the centre of mass.

2.2.2   A theorem on the motion of the centre of mass

We now find the **acceleration of the centre of mass**. By definition, the acceleration vector is the first derivative of the velocity vector with respect to time:

$$\vec{a}_c = \frac{d\vec{v}_c}{dt} = \frac{1}{M}\sum_{i=1}^{n} m_i \frac{d\vec{v}_i}{dt} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{a}_i. \qquad (2.2.6)$$

The value $\vec{F}_i = m_i \vec{a}_i$ is the *second dynamic parameter* and, according to Newton's second law, is the force acting on the particle. Thus:

$$\vec{a}_c = \frac{1}{M}\sum_{i}^{n} \vec{F}_i. \qquad (2.2.7)$$

It should be noted that this formula makes sense only for *homogeneous* and *stationary* force fields, i.e.:

$$\vec{F}_i \neq \vec{F}_i(\vec{r}, t). \qquad (2.2.8)$$

The formula (2.2.7) is an analytic form of the **theorem on the motion of mass centre**: *for all the interactions of each particle of a mechanical system with the environment, the centre of mass of a mechanical system moves in such a way as if all the forces, acting on individual particles of the system, are attached to one point — the centre of mass.*

Consider now more in detail forces acting on the particles of a mechanical system. The forces acting on each point of the system could be divided into two types:

1. forces from all other particles in the system (internal forces);

2. the resultant of all external forces.

Thus, in general the force acting on a particle with the number i can be written as

$$\vec{F}_i = \sum_{k \neq i}^{n} \vec{F}_{ik} + \vec{F}_{i_{\text{ext}}} \qquad (2.2.9)$$

where $\vec{F}_i$ – the resultant of all forces acting on the i-th material point, $\vec{F}_{ik}$ – is the force from the particle with number k, applied to the i-th particle (inner force), and $\vec{F}_{i_{\text{ext}}}$ – the sum of all external forces acting on the i-th particle (figure 2.6).

We substitute this expression into a theorem of the centre of mass motion (2.2.7):

$$\vec{a}_c = \frac{1}{M} \left(\sum_{i,k \neq i}^{n} \vec{F}_{ik} + \sum_{i}^{n} \vec{F}_{i_{\text{ext}}} \right).$$

Let us consider the first sum in this equation more in detail. For the sake of simplicity, we assume that our mechanical system consists of three particles. Generalization to an arbitrary number of particles is not difficult.
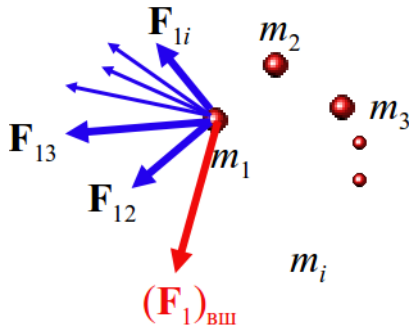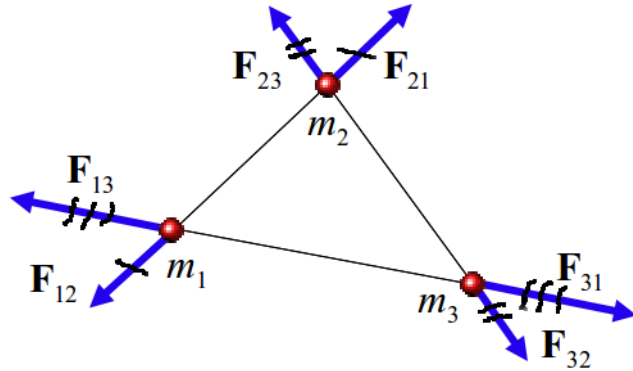

Fig. 2.6   Fig. 2.7

Let us draw all the forces acting **between** the points. To take an example, F_{12} and F_{13} are the forces acting on the first point due to the second and third points, respectively. Similarly, for the points numbered 2 and 3. Obviously, this is *interaction* forces, that is, they obey Newton's third law. In the figure pairs of forces related by Newton's third law are marked by one, two and three dashes respectively.

Sums of these pairs of forces is obviously zero. Consequently, according to Newton's third law in the summation of all the vectors of internal forces we will get zero and $\sum_{ik} \vec{F}_{ik} = 0$. Then the second **theorem of centre of mass motion** takes the form:

$$\vec{a}_c = \frac{1}{M} \sum_{i=1}^{n} \left(\vec{F}_i \right)_{\text{ext}}. \tag{2.2.10}$$

This form of the theorem implies that *if the system is in an external stationary and uniform field, no action within the system can not change the motion of the centre of mass of the system.*

2.2.3 Motion of a body with variable mass

Very often, in real situations, the body moves in such a way that the mass of the body is permanently changing during the movement (rocket, jet plane, platform loaded on the move, etc.). Let us deduce the equation of motion of such a body.

We model the studied body by system of material points, provided that the mass m of the system can be changed (particles can enter or leave the system).

Suppose that in an infinitesimal time dt mass of the system has changed by an infinitesimal amount dm (figure 2.8). If the velocity $\vec{u}$ with respect to the system (relative velocity) of an ejected particle of mass dm is not zero, then, the impulse of the system will be changed (velocity will increase by $d\vec{v}$).
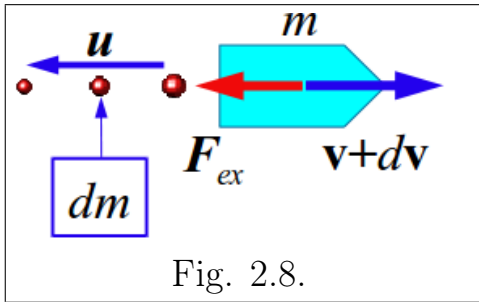
Fig. 2.8.

But, by the theorem of the motion of the centre of mass, impulse of the system can be changed also by an external force:

$$d\vec{P} = \vec{F}_{ext}dt. \tag{2.2.11}$$

Thus, the total change in impulse of the system in time dt is:

$$md\vec{v} = \vec{F}_{ext}dt + \vec{u}dm \tag{2.2.12}$$

and we receive *the equation of variable mass body dynamics*

$$\boxed{m(t)\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \vec{u}\frac{dm(t)}{dt}.} \tag{2.2.13}$$

It is called **Meshcherskij equation**. This equation was first obtained by Russian scientist V. Meshchersky in 1904.

Let us consider some particular examples.

Tsiolkovsky equation

We assume that there are no external forces ($\vec{F}_{ext} = 0$). Then the total change in the impulse of the system in time dt is:

$$m(t)d\vec{v} = \vec{u}dm(t). \tag{2.2.14}$$

If the velocity $\vec{u}$ of ejected particles with respect to the system (relative velocity) is constant, then for the velocity of the system $\vec{v}$ we get

$$\vec{v}(t) = \vec{v}_0 + \vec{u}\ln\frac{m(t)}{m_0} \tag{2.2.15}$$

reactive motion equation (**Tsiolkovsky equation**). In the expression (2.2.15) vector $\vec{v}_0$ and m_0 are initial (at time $t = 0$) velocity and mass of the system.

Normally, during the reactive motion, velocity vectors $\vec{v}$ (of a body) and $\vec{u}$ (of "fuel") are opposite, therefore

$$v(t) = v_0 + u\ln\frac{m_0}{m(t)}. \tag{2.2.16}$$

This equation was obtained by the Soviet (regardless the works of he V.M. Meshcherskij) scientist K. Tsiolkovsky in 1897.

Jet propulsion rocket

We assume that the rocket moves in the atmosphere – then as a model of external forces a resistance force proportional to the velocity vector can serve $\vec{F}_{ext} = -k\vec{v}$. In this case, the total change in linear momentum of the rocket during time dt equals to:

$$m(t)d\vec{v} = -k\vec{v}dt + \vec{u}dm. \tag{2.2.17}$$

If the velocity $\vec{u}$ of emitted particles relative to the rocket (the relative velocity) is constant in time and α is the fuel consumption per second (due to combustion), the law of the rocket mass change with time can be written as follows:

$$\frac{dm(t)}{dt} = -\alpha(m(t) - m_0), \tag{2.2.18}$$

where m_0 – rocket mass without fuel, $(m(t) - m_0)$ – fuel mass at time moment t.

In equation (2.2.17) we divide both sides of the equality on dt, as the result we get:

$$m(t)\frac{d\vec{v}}{dt} = -k\vec{v} + \vec{u}\,\frac{dm(t)}{dt}. \qquad (2.2.19)$$

To solve system of equations (2.2.18, 2.2.19) we choose next parameters: $m_0 = 10$ kg, $m_f = 100$ kg (initial mass of fuel), $k = 0.1$ kg/s^2, $\alpha = 0.2$s^{-1}, $u = 500$ m/s, $v_0 = 0$ m/s. Numerical solution of such differential system of equations for our choose of initial data is shown on the figure where the rocket velocity is presented as a function of time:
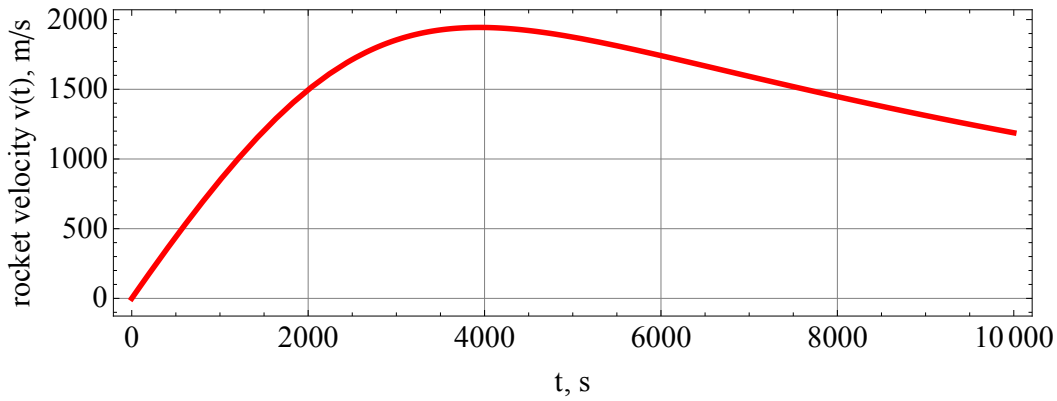

Fig. 2.9.

2.2.4   Conservation of linear momentum

Mechanical system is called **closed**, if the resultant of all the external forces acting on the system equals to zero:

$$\vec{F}_{\text{ext}} = \sum_{i=1}^{n} \left(\vec{F}_i\right)_{\text{ext}} = 0. \qquad (2.2.20)$$

Hence, for a closed system $\vec{a}_c = 0$. On the other hand, if the system is closed, then from the equation (2.2.6) it follows:

$$\sum_{i=1}^{n} m_i \vec{a}_i = \sum_{i=1}^{n} m_i \frac{d\vec{v}_i}{dt} = \frac{d}{dt}\sum_{i=1}^{n} m_i \vec{v}_i = 0. \qquad (2.2.21)$$

By definition (2.2.5) it can be written in the form

$$\frac{d\vec{P}_c}{dt} = 0 \qquad \Longleftrightarrow \qquad \vec{P}_c = const \qquad (2.2.22)$$

or $\vec{v}_c = const$. By this way we get the law that is called *impulse conservation law*:

> **the centre of mass impulse of a closed mechanical system is conserved.**

This means that *centre of mass of a closed mechanical system ether moves uniformly, or keeps its state of rest.*

From the last equality in expression (2.2.21) one can deduce *impulse conservation law* in more familiar form like the course of physics in senior schools:

$$\vec{P}_c = \sum_{i=1}^{n} m_i \vec{v}_i = \sum_{i=1}^{n} \vec{p}_i = const, \qquad (2.2.23)$$

id est, *vector sum of particles impulses is conserved in closed systems.*

2.3 Description of the rigid body motion

Following the standard scheme of any scientific description of real objects of Nature we have to build a rigid body model. In the description of motion of a rigid body we will study only the so called **absolutely rigid body** — an idealization of a solid body in which deformation is neglected. In other words, the distance between any two given points of a rigid body remains constant in time regardless of external forces exerted on it. This, of course, a qualitative definition.

Modelling the behaviour of an absolutely rigid body we will break it down (mentally) into as much as desired large number of sufficiently small parts – that is, we *model* the absolutely rigid body *by a mechanical system.*

Then, it is obvious that an arbitrary motion of an absolutely rigid body can be reduced to the sum of linear and rotational motions of all the points of the body.

In the following sections we give rigorous definitions of linear and rotational motions of a rigid body, and introduce new dynamic parameters necessary to describe the rotational motion of an absolutely rigid body.

2.3.1 Rectilinear motion

Rectilinear motion of a rigid body is a motion of its *system* of material points, with a condition that *velocities* of rectilinear motion of these points with respect to any inertial reference frames are the *same*, while *angular velocities* with respect to the axis passing through the centre of mass *are zero* (figure 2.10).

From this definition it follows that the kinematic *conditions* of recti-linear motion are as follows:

Fig. 2.10.
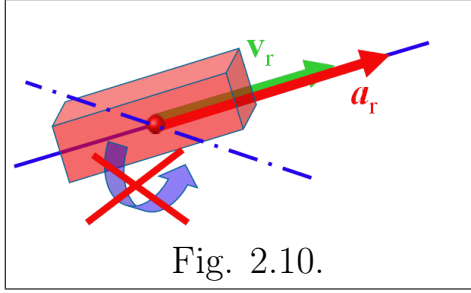
$$\vec{v}_i = \vec{e}_{r_i} \frac{d|\vec{r}_i|}{dt} = \vec{v}_r = const$$

$$\vec{\omega}_i = \vec{\omega} = 0, \tag{2.3.3}$$

where i – the index of a point, $\vec{v}_r$ – velocity of translational motion. Then *kinematic characteristics* of motion are:

$$\vec{v}_c = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{v}_i = \vec{v}_r; \qquad \vec{a}_r = \frac{d\vec{v}_r}{dt} = \frac{\vec{F}_{\text{ext}}}{M}, \tag{2.3.4}$$

where $\vec{v}_c$ – velocity of the centre of mass (2.2.4) and $\vec{a}_r$ – acceleration of translational motion of the body, $M = \sum_i m_i$ – the mass of the body.

Therefore, to describe the linear motion of an absolutely rigid body it is enough to describe the motion of a single point – the body centre of mass. All the other points move in the same way.

2.3.2   Rotational motion

Rotational motion of a rigid body is a motion of its *system* of material points, with a condition that rectilinear velocities of these points with respect to a given inertial reference frame are zero, and *angular velocities* with respect to a given axis are *identical*. From this definition it follows that kinematic conditions of rotational motion are as follows:

$$\vec{v}_{r_i} = 0; \qquad \vec{\omega}_i = \vec{\omega} = const, \tag{2.3.5}$$

where $\vec{v}_{r_i}$ – velocity of rectilinear motion of a point i (1.2.10) and $\vec{\omega}$ – angular velocity vector. Then *kinematic characteristics* of motion could be obtained by substitution of conditions (2.3.5) into equations (1.2.11) and (1.4.7):

$$\vec{v}_i = [\vec{\omega}, \vec{r}_i]; \qquad \vec{a}_i = [\vec{\varepsilon}, \vec{r}_i] - \vec{r}_i \, \omega^2. \tag{2.3.6}$$

Respectively, the second law of Newton for any point of an absolutely rigid body under the rotational motion becomes:

$$\vec{F}_i = m_i \vec{a}_i, \iff m_i \frac{d^2 \vec{r}_i}{dt^2} = [\vec{\varepsilon}, \vec{r}_i] - \vec{r}_i \, \omega^2. \tag{2.3.7}$$
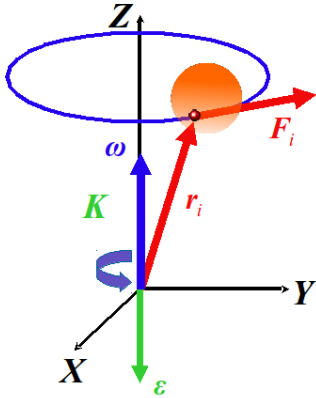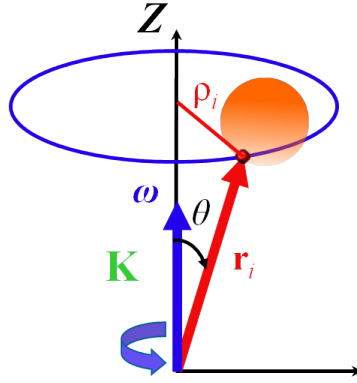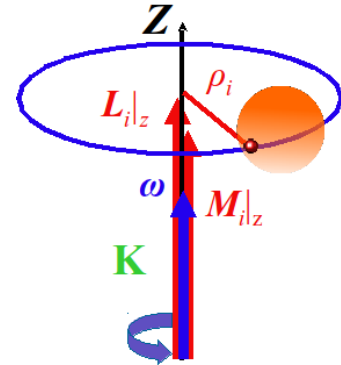
Fig. 2.11.   Fig. 2.12.   Fig. 2.13.

It is very difficult to find the law of motion $\vec{r}_i = \vec{r}_i(t)$ from this equation. Therefore there is a need to build some *new dynamics equations* describing exactly the rotational motion of a point.

To do that we introduce *new dynamic parameters*: the angular and force momentums.

Angular momentum *of a particle* with respect to some inertial reference frame is a vector $\vec{L}_i$ that equals to *vector product* of the material point i radius vector $\vec{r}_i$ (figure 2.11) with linear momentum $\vec{p}_i$:

$$\vec{L}_i = [\vec{r}_i, \vec{p}_i] = m_i[\vec{r}_i, \vec{v}_i]. \qquad (2.3.8)$$

Force momentum (torque) *of a material point* with respect to some inertial reference frame is a vector $\vec{M}_i$ that equals to *vector product* of the material point i radius vector $\vec{r}_i$ (figure 2.11) with force vector $\vec{F}_i$ applied to the point:

$$\vec{M}_i = [\vec{r}_i, \vec{F}_i] = m_i[\vec{r}_i, \vec{a}_i]. \qquad (2.3.9)$$

Projections of angular and force momentums

Let's find the projection of the expression (2.3.8, 2.3.9) on the axis of rotation Z. Using the kinematic conditions of rotational motion (2.3.5) we get:

$$[\vec{r}_i, \vec{v}_i] = [\vec{r}_i, [\vec{\omega}, \vec{r}_i]] = \vec{\omega}|\vec{r}_i|^2 - \vec{r}_i(\vec{\omega}, \vec{r}_i), \qquad (2.3.10)$$

where in the last equality we used the rule (1.3.7) BAC-CAB. As a result, we received a sum of vectors, the first one of which is directed along the axis of rotation Z, and the second along the radius vector of the point.

The projection of this expression on the axis of rotation Z is:

$$\left(\vec{\omega}|\vec{r_i}|^2 - \vec{r_i}(\vec{\omega}, \vec{r_i})\right)\Big|_z = \omega|\vec{r_i}|^2 - |\vec{r_i}|\cos\theta\,(\omega|\vec{r_i}|\cos\theta) =$$
$$\omega|\vec{r_i}|^2(1 - \cos^2\theta) = \omega\rho_i^2, \tag{2.3.11}$$

where ρ_i – distance from the axis to the point under consideration (figure 2.12).

Similarly for vector product $[\vec{r_i}, \vec{a_i}]$:

$$[\vec{r_i}, \vec{a_i}] = \left[\vec{r_i}, \left([\vec{\varepsilon}, \vec{r_i}] - \vec{r_i}\omega^2\right)\right] = \vec{\varepsilon}|\vec{r_i}|^2 - \vec{r_i}(\vec{\varepsilon}, \vec{r_i}) - [\vec{r_i}, \vec{r_i}]\,\omega^2. \tag{2.3.12}$$

Vector product of a vector on itself is zero: $[\vec{r_i}, \vec{r_i}] \equiv 0$. Therefore:

$$[\vec{r_i}, \vec{a_i}] = \vec{\varepsilon}|\vec{r_i}|^2 - \vec{r_i}(\vec{\varepsilon}, \vec{r_i}). \tag{2.3.13}$$

Obtained expression differs from (2.3.10) only by vector $\vec{\omega}$ instead of vector $\vec{\varepsilon}$. Both of the vectors are directed along the same axe – axe of rotation. Consequently, projection of vector product $[\vec{r_i}, \vec{a_i}]$ on the axe of rotation will be $\varepsilon\rho_i^2$.

As a result for projections of angular and force momentums on the axe of rotation (figure 2.13):

$$\vec{L_i}\Big|_z = m_i\rho_i^2\,\omega \tag{2.3.14}$$

and

$$\boxed{\vec{M_i}\Big|_z = m_i\rho_i^2\,\varepsilon = J_{iz}\,\varepsilon}, \tag{2.3.15}$$

where new notation is introduced:

$$J_{iz} = m_i\rho_i^2. \tag{2.3.16}$$

Quantity J_{iz} is called **moment of inertia** of a material point with respect to axe Z. Equation (2.3.15) is the law we looked for – **the law of rotational motion dynamics for a material point:** *force momentum projection* $\vec{M_i}\Big|_z$ *on rotational axe equals to inertial moment of the material point (measured with respect to rotational axe) multiplied by angular acceleration vector.*

Comment: angular acceleration vector $\vec{\varepsilon}$ and vector of angular velocity $\vec{\omega}$ are parallel to rotational axe, therefore absolute values $|\vec{\varepsilon}|$ and $|\vec{\omega}|$ are equal their own projections on rotational axe:

$$\varepsilon = \pm|\vec{\varepsilon}| = \vec{\varepsilon}|_z, \qquad \omega = \pm|\vec{\omega}| = \vec{\omega}|_z \tag{2.3.17}$$

But, it should be noted that, projection of a vector can have different signs as opposed to the length of the vector.

Equations (2.3.14, 2.3.15) are obtained for a single arbitrary point of a (rotating) rigid body. Summing up all over the body, we obtain

$$L_z = \sum_{i=1}^{n} \vec{L}_i\Big|_z = J_z\omega \qquad (2.3.18)$$

$$\boxed{M_z = \sum_{i=1}^{n} \vec{M}_i\Big|_z = J_z\varepsilon,} \qquad (2.3.19)$$

where $J_z = \sum_{i=1}^{n} J_{iz}$ – the moment of inertia of the rigid body.

Thus, the **moment of inertia of a rigid body** about an axis is a quantity that is a **measure of inertia** of the rigid body in the rotational motion around this axis and equals to the sum of products of masses (the body consists of) and of it's squared distances from the same axis.

The equation (2.3.19) is the **fundamental law of rotational motion dynamics for an absolutely rigid body**: *resulting force momentum M_z projected on the axe of rotation (acts on the whole of the body) equals to the product of the body inertial momentum (measured with respect to axe of rotation) and angular acceleration vector ε, projected as well on the axe of rotation.*

Comment (2.3.17) should be taken into account here as well as in the law of rotational motion dynamics for a material point.

It can be seen from Table 1 that formulas describing the rotational motion with respect to a fixed axis are similar to expressions for rectilinear motion. So, it is enough to replace the values $(m, \vec{r}, \vec{v}, \vec{a}, ...)$ by the corresponding angular values $(J, \vec{\varphi}, \vec{\omega}, \vec{\varepsilon}, ...)$ and we get all the laws and relationships for rotational motion.

Table 1: Relations between kinematic and dynamic parameters.

Rectilinear motion	Rotational motion	
$\vec{r}$	$\vec{\varphi}$	
$\vec{v} = d\vec{r}/dt$	$\vec{\omega} = d\vec{\varphi}/dt$	
$\vec{a} = d\vec{v}/dt$	$\vec{\varepsilon} = d\vec{\omega}/dt$	
m	J_z	
$\vec{p} = m\vec{v}$	$\vec{L} = [\vec{r}, \vec{p}],$	$L_z = J_z\omega$
$\vec{F} = m\vec{a}$	$\vec{M} = [\vec{r}, \vec{F}],$	$M_z = J_z\varepsilon$

To take an example, elementary work of force $\vec{F}$ on elementary displacement $d\vec{r}$ is by definition:

$$\delta A = \vec{F}d\vec{r}.$$

Then for elementary work of force momentum $\vec{M}$ on elementary angle displacement $d\vec{\varphi}$, we get, using the Table 1:

$$\delta A = \vec{M}d\vec{\varphi}.$$

Kinetic energy T for rectilinear motion is defined by

$$T = (mv^2)/2,$$

then for rotational motion it will have the form:

$$T = (J\omega^2)/2.$$

2.3.3   The Basic Law of a rigid body rotational dynamics in differential form

Let us consider the basic law of dynamics of a rigid body rotational motion:

$$M_z = J_z\varepsilon.$$

If we find the time derivative of angular momentum:

$$\frac{dL_z}{dt} = \frac{d}{dt}(J_z\omega) = J_z\varepsilon, \tag{2.3.20}$$

and compare this expression with (2.3.19) we will obtain the basic law of dynamics of a rigid body rotational motion in differential form:

$$\boxed{M_z = \frac{dL_z}{dt}}. \tag{2.3.21}$$

As before, the forces acting on each point of a rigid body could be divided into two kinds – internal and external forces:

$$M_z = \sum_{i=1}^{n} [\vec{r}_i, \vec{F}_i]\Big|_z = \sum_{i=1}^{n} \left[\vec{r}_i, \sum_{k \neq i}^{n} \left(\vec{F}_{ik}\right)_{in} + \left(\vec{F}_i\right)_{ext}\right]\Bigg|_z = \tag{2.3.22}$$

$$= M_{in_z} + M_{ext_z}.$$

By Newton's third law (as in (2.2.10))

$$M_{\text{in}_z} = \sum_{i=1}^{n} \left[\vec{r_i}, \sum_{k \neq i}^{n} \left(\vec{F}_{ik} \right)_{in} \right] \Bigg|_z = 0. \qquad (2.3.23)$$

As a consequence, the *basic law of dynamics of a rigid body rotational motion* (it is sometimes called *momentums equation*) takes the form:

$$M_{\text{out}_z} = \frac{dL_z}{dt}. \qquad (2.3.24)$$

The total force momentum projected on the axe of rotation equals the rate of change of angular momentum projected as well on the axe of rotation.

2.3.4 Law of conservation of angular momentum

If the total force momentum projection on the axe of rotation is zero $M_{\text{out}_z} = 0$, then from the equation (2.3.24) we can get that angular momentum projection on the axe of rotation L_z is conserved with time:

$$L_z = J_z \omega = const. \qquad (2.3.25)$$

It should be noted that, as before, obtained results are valid only for a *homogeneous and stationary* external field.

If the resultant moment of all external forces acting on a mechanical system in the state of rotational motion is zero, then this system is called **closed** with respect to the torques (moments of forces). More important role plays not the law of conservation of angular momentum but it's consequence:

> **No action within the closed system (with respect to moments of forces) in a homogeneous stationary field, can change the angular velocity of a system centre of mass.**

All mentioned above is valid for the model of on absolutely rigid body.

2.3.5   Moment of inertia of a rigid body

Consider a rigid body in rotational motion. Moment of inertia about any axis can be found by calculation or measured experimentally. The motion of a rigid body we describe by the model of mechanical system (a system of material points). An elementary piece of it has mass dm and elementary moment of inertia for this piece is

$$dJ_z = \rho^2 dm = r^2 \sin^2 \theta \, dm. \qquad (2.3.26)$$
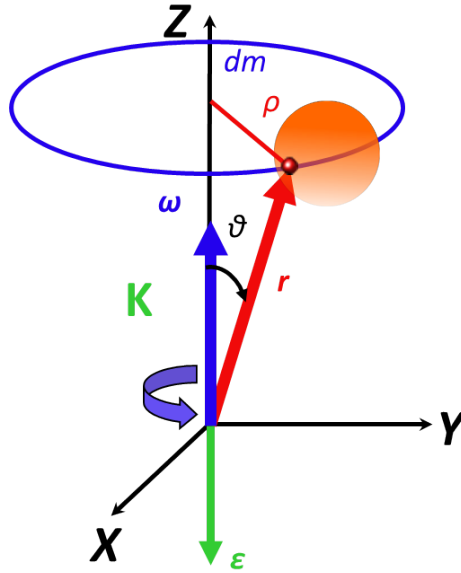
Then, the summation of these elementary moments overall the volume of the rigid body (i.e., integration), gives

$$J_z = \int_V r^2 \rho_0(\vec{r}) \sin^2 \theta \, dV, \qquad (2.3.27)$$

where $\rho_0(\vec{r}) = dm/dV$ – density of the body at the point with radius vector $\vec{r}$. Analytical calculation of such integrals is possible only in the simplest cases when the body's shape is geometrically simple (sphere, cylinder, etc.). For bodies of irregular shape such integrals can be found numerically.

Calculation of moments of inertia in many cases can be simplified using the ideas of similarity and symmetry, the theorem of Huygens-Steiner, as well as some other common relationship.

2.3.6 Huygens-Steiner theorem

The parallel axis theorem, also known as Huygens-Steiner theorem after Christiaan Huygens and Jakob Steiner, can be used to determine the moment of inertia J_z of a rigid body (of mass M) about any axis Z, given the body's moment of inertia J_0 about a parallel axis Z', passing through the object's centre of mass and taking place at a distance a from axis Z'.

As usual we modelize the body by a mechanical system of material points. For the solution of the problem we choose an arbitrary point of the body with mass m_i which is located at a distance ρ_i from the axis Z. Moment of inertia for this point with respect to axe Z equals to $J_{zi} = \rho_i^2 m_i$. Let us denote by $\vec{a}$ the vector perpendicular to both of the axes and directed towards the centre of mass. Then vectors $\vec{a}, \vec{\rho}_i$ and $\vec{\rho}_i'$ (figure 2.14.) form a triangle, so following the rule of vectors summation: $\vec{\rho}_i = \vec{a} + \vec{\rho}_i'$. Consequently:
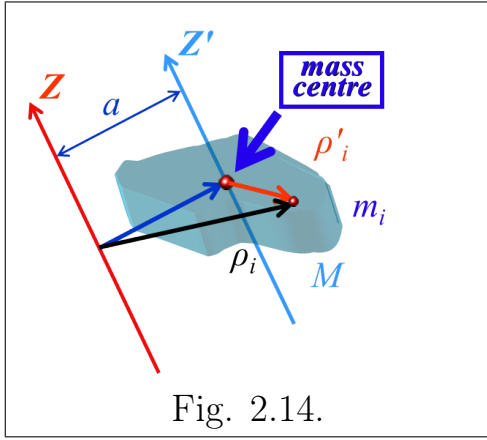
Fig. 2.14.

$$J_{zi} = \vec{\rho}_i^2 m_i = (\vec{\rho}_i' + \vec{a})^2 m_i. \qquad (2.3.28)$$

Expanding the brackets and summing over all the points of the body we obtain the moment of inertia of the whole body:

$$J_z = \sum_{i=1}^{n} \left(\vec{\rho}_i'^2 + 2\vec{a}\vec{\rho}_i' + \vec{a}^2 \right) m_i. \qquad (2.3.29)$$

After that we rewrite this expression by the next manner – we multiply and divide the second term on M – the mass of the whole body:

$$J_z = \sum_{i=1}^{n} \rho_i'^2 m_i + 2\vec{a}M \left(\frac{1}{M} \sum_{i=1}^{n} \vec{\rho}_i' m_i \right) + a^2 \sum_{i=1}^{n} m_i. \qquad (2.3.30)$$

The expression in brackets is by definition radius vector of the centre of mass (2.2.3), in our case coordinate system is chosen so that its origin coincides with the centre of mass, so its coordinate is zero and we have:

$$\boxed{J_z = J_0 + Ma^2} \qquad (2.3.31)$$

where $J_0 = \sum_{i=1}^{n} \rho_i'^2 m_i$ – is the inertia moment of the body in interest with respect to the axe Z' passing through the centre of mass.

So, *Huygens-Steiner theorem* states: a rigid body moment of inertia with respect to an arbitrary axe is equal to the sum of a moment of inertia of the same body with respect to the axe passing through the centre of mass, (both axes being parallel to each other) and the moment of inertia of the mass point (with the mass of the whole body) with respect to the initial axe.

As it seen from expression (2.3.31), moment of inertia of a rigid body with respect to an axe through the centre of mass is always less than the moment of inertia about any other axe parallel to the one passing through the centre of mass.

2.4   Work and energy

2.4.1   Work

Let us give, first, the definition of a physical field. We will say that a *physical field* is defined if there is a correspondence between every point in space and some certain value of a physical quantity.

Now we can consider the motion of a point particle in some **force field**, i.e. the motion due to some forces with known magnitude and direction for every point of trajectory L (figure 2.15).

Let in elementary time interval dt point particle makes elementary displacement $d\vec{r}$ due to force $\vec{F}$. The *process of displacement   due to*

Figure 2.15.

force is called the **work of the force**. By definition, elementary work of the force on an elementary displacement equals to scalar product of the force vector $\vec{F}$ and elementary displacement vector $d\vec{r}$:

$$\delta A = \vec{F} d\vec{r}. \qquad (2.4.3)$$
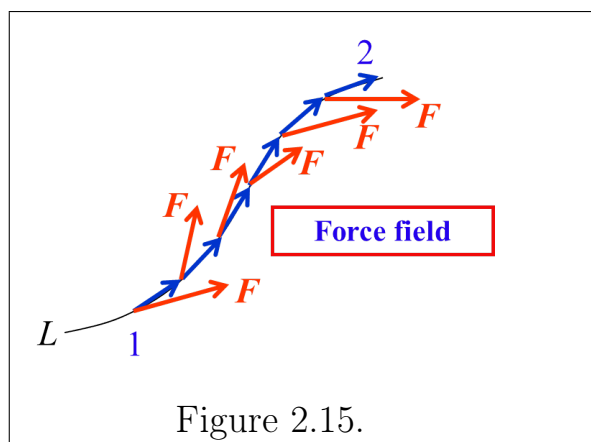
It should be pointed out here that for an elementary work we use notation δA (and not dA) to emphasize that in general case elementary work **do not equals** *the linear part of increment* (i.e. it is not a differential).

Obviously, for an arbitrary displacement, to take an example consider points 1 and 2 of the trajectory L, work of the force $\vec{F}$ for the total displacement is equal to the sum of elementary works along the trajectory: $A_{12} = \sum_1^2 \delta A$. The summation of infinitesimal quantities is, by definition, an integral.

Therefore, in general case, *the work on displacement of a point particle along an arbitrary trajectory L* equals to

$$\boxed{A = \int_L \vec{F} \, d\vec{r}.}$$
(2.4.4)

Here symbol «$\int_L$» is the integral along the trajectory L and is called curvilinear integral.

The results we have obtained (definitions (2.4.3, 2.4.4) are valid only for a point particle. But, developing the dynamics laws, we had found, that behaviour of physical objects of Nature could be described or by mechanical system model or by rigid body model. The number of models we can use is not limited by this two, we have named only the studied here models. It is possible to invent as many of it as we wish. Next reason is the theorem of centre of mass motion – to describe the behaviour of a mechanical system or of a rigid body it is enough to describe just the behaviour of one point – the centre of mass point. Consequently, obtained definitions are valid for such models as well, or, more precisely – for the description of centre of mass motion for corresponding objects. Therefore, talking further about a physical body we will mean it's model and all the forces will be attached to the only one point of the body – the centre of mass.

2.4.2 Kinetic energy theorem

Let a point particle (or a centre of mass) made a displacement $d\vec{r}$ due to force $\vec{F}$, i.e. an elementary work δA have been done. Let us substitute the definition of second dynamical parameter – force into elementary work expression:

$$\delta A = \vec{F} d\vec{r} = m\frac{d\vec{v}}{dt} d\vec{r} = m\, \vec{v} d\vec{v}.$$
(2.4.5)

Obtained equation could be transformed with the use of next relation:

$$d(\vec{v}, \vec{v}) = (d\vec{v}, \vec{v}) + (\vec{v}, d\vec{v}) = 2(\vec{v}, d\vec{v}) \quad \Rightarrow \quad m(\vec{v}, d\vec{v}) = d\left(m\frac{(\vec{v}, \vec{v})}{2}\right)$$

Then elementary work formulae will have the form:

$$\delta A = d\left(\frac{mv^2}{2}\right). \qquad (2.4.6)$$

The quantity in brackets

$$T = \frac{mv^2}{2}$$

is called **kinetic energy of the body**.

As the result we have obtained the theorem on the kinetic energy:

– *for elementary displacements*:

Elementary work made on some body equals to differential of kinetic energy	$\delta A = dT$

– *for an arbitrary displacement*:

$$A_{12} = \int_1^2 dT = T_2 - T_1 \qquad \Rightarrow \qquad \boxed{A = \Delta T}$$

The work on displacement of a body between any two points in space equals to the difference of kinetic energies in final and initial points

2.4.3   Potential fields

Let us consider elementary displacement $d\vec{r}$ of a body due to external forces. Scalar product in work definition we expand through the known components of corresponding vectors:

$$\delta A = \vec{F}d\vec{r} = F_x dx + F_y dy + F_z dz. \qquad (2.4.7)$$

Suppose that components of resultant force vector acting on the body satisfy the next conditions in any space point:

$$\boxed{\begin{aligned} F_x &= -\frac{\partial}{\partial x}\Phi(x,y,z), \\ F_y &= -\frac{\partial}{\partial y}\Phi(x,y,z), \\ F_z &= -\frac{\partial}{\partial z}\Phi(x,y,z), \end{aligned}} \qquad (2.4.8)$$

where $\Phi(x, y, z)$ – is some scalar function. Id est, each component of force vector $\vec{F} = (F_x, F_y, F_z)$ could be expressed as a partial derivative of the scalar function $\Phi(x, y, z)$. Then, if we put equation (2.4.8) into (2.4.7), we obtain:

$$-\delta A = \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial y}dz. \qquad (2.4.9)$$

Right hand side of equation (2.4.9) is, by definition, differential of scalar function (x, y, z). Therefore, elementary work δA in the force field, satisfying condition (2.4.8), is equal to the full differential (with minus sign) of some scalar function $\Phi(x, y, z)$:

$$\delta A = -d\Phi(x, y, z). \qquad (2.4.10)$$

Function $\Phi(x, y, z)$ is called *potential function* of the force field.

Consequently, if a *force field, allows to introduce potential function*, id est it satisfies the relation (2.4.8), is called **potential force field**.

For an arbitrary displacement in a potential force field:

$$A_{12} = -\int_1^2 d\Phi = \Phi_1 - \Phi_2 \qquad \Rightarrow \qquad \boxed{A = -\Delta\Phi} \qquad (2.4.11)$$

so the *work on displacement of body in potential field do not depend on the form of trajectory and is defined only by initial and final positions of the body.*

According to the condition (2.4.11), if a body moves on a closed trajectory

$$A_{11} = -\int_1^1 d\Phi = \Phi_1 - \Phi_1 \equiv 0. \qquad (2.4.12)$$

Consequently the **work in a potential force field along any closed trajectory is zero**:

$$\oint \vec{F} \cdot d\vec{r} \equiv 0. \qquad (2.4.13)$$

Symbol «$\oint$» means that the integral should be taken along the closed curve.

Force field, satisfying *only* condition (2.4.13), is called **conservative**. As the result we obtain that *if a field is potential, then it is* **necessarily** *conservative.* The opposite, in general case, is incorrect: *conservative field* **is not necessarily** *potential.*

Obviously, expression (2.4.8) could be written in the next form

$$\vec{F} = -\left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right) \qquad (2.4.14)$$

or equivalently

$$\vec{F} = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \Phi. \qquad (2.4.15)$$

The brackets in last equation contain an expression that is formally a vector, but it's components are not numbers but some objects that "wants to take the derivative" from function to the right – it is called *operator-vector* and is denoted by symbol $\vec{\nabla}$:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \qquad (2.4.16)$$

In physics operator-vector $\vec{\nabla}$ is called operator "nabla". Operator "nabla" applied to a *scalar function* is called **gradient of the function**:

$$\vec{\nabla}\Phi = \operatorname{grad}\Phi. \qquad (2.4.17)$$

In geometry the gradient of a scalar field is a vector field that points in the direction of the greatest rate of increase of the scalar field, and whose magnitude is that rate of increase.
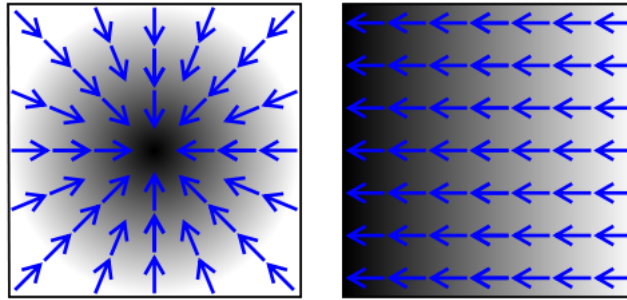
In the above two images, the scalar field is in black and white, black representing higher values, and its corresponding gradient is represented by blue arrows.

With the help of operator "nabla" equation (2.4.8) – **force field potentiality condition**, obtains the most laconic form:

$$\vec{F} = -\vec{\nabla}\Phi. \qquad (2.4.18)$$

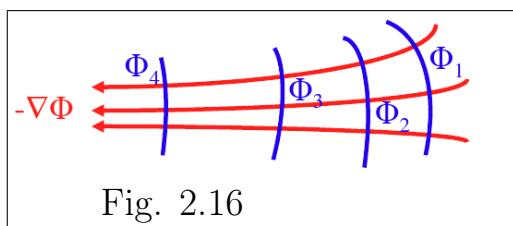To clarify the meaning of gradient we introduce the concept of *equipotential surface.* **Equipotential surface** in potential fields is the set of the field points in which the potential function has the same value. The equation of equipotential surface:

Fig. 2.16

$$\Phi(x, y, z) = const. \tag{2.4.19}$$

It is clear that for every value of *const* in this equation corresponds its own equipotential surface (Figure 2.16). In a potential field

$$\vec{F} = -\vec{\nabla}\Phi(x, y, z) \equiv -\operatorname{grad}\Phi, \tag{2.4.20}$$

force vector is directed towards the steepest decrease of the potential function ($\Phi_1 > \Phi_2 > \Phi_3$) and, therefore, *is always perpendicular to the equipotential surfaces.*

A line tangent to which in each point coincides with the direction of force vector at this point, is called **force line** (field line).

Consequently, in a potential field force lines and equipotential surfaces are always perpendicular to each other.

2.4.4 Potential energy

As we have found, work in a potential field is the difference between the values of the potential function at the initial and final points of the field:

$$A_{12} = \Phi_1 - \Phi_2. \tag{2.4.21}$$

The work is the physical quantity experimentally measurable. Accordingly, *potential function is measurable only up to an arbitrary constant.* This means that in a potential field any one of the potential surfaces can be taken as zero surface with function $\Phi_0 = 0$ (Figure 2.17). In this case, $A_{i0} = \Phi_i$, that is, the work of moving the body from a given point of the field to the point of zero potential numerically equals the value of the potential function at this point. The numerical value of the potential function at any point of the field (for a given zero surface) is called **potential energy** of the body at this point and is denoted by U_i.
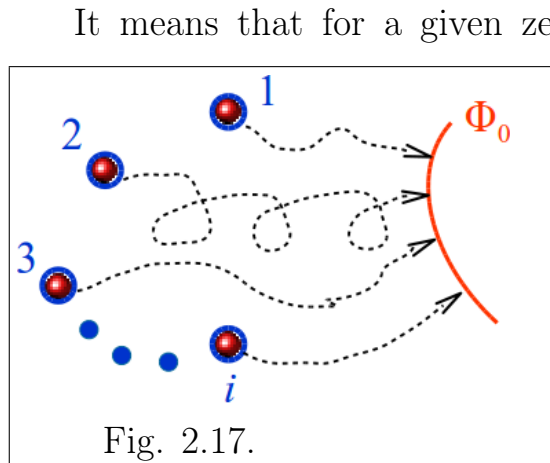
It means that for a given zero surface the values of potential energy and function are the same (numerically): $U_i = \Phi_i$. So, we have now recipe of potential energy calculation at a given point of space: *potential energy* **numerically equals the work,** that should be done, to move the body from a given point of the field at the point of zero potential. Taking into account that potential function is a way of describing the surrounding force fields influence of on the body (on the body centre of mass) we can formulate the next *qualitative* definition of the potential energy:

Fig. 2.17.

> **potential energy is the measure of the body interaction with the surrounding force field**
> *at a given point of the potential field.*

2.4.5   Energy conservation law

In general, conservation laws are just the reflection of certain symmetry properties of dynamics laws. That is why conservation laws permit to get the answers on some important questions easy enough without invoking the equations of motion. This circumstance makes the conservation laws a very effective instrument of researches.

Let us obtain the energy conservation law. For elementary displacements in a potential field the elementary work equals $\delta A_{\mathrm{p}} = -dU$ (equation (2.4.10)). Elementary work of all forces acting on a body (potential and non potential) by the kinetic energy theorem is $\delta A = \delta A_{\mathrm{p}} + \delta A_{\mathrm{np}} = dT$. Putting together both of this equations we obtain

$$d(T + U) = \delta A_{\mathrm{np}}. \qquad (2.4.22)$$

Quantity in brackets $E = T + U$ is called **total mechanical energy of a body**.

So, for elementary displacements

differential of total mechanical energy equals to elementary work of non potential forces:

$$\boxed{dE = \delta A_{\mathrm{np}}} \qquad (2.4.23)$$

For an arbitrary displacement we integrate equation (2.4.22) and obtain that:

the change of a body's total mechanical energy during any displacement equals to the work of non potential forces:

$$\boxed{\Delta E = A_{\mathrm{np}}.} \qquad (2.4.24)$$

From the last equation we see that if the work of non potential forces is zero ($A_{\mathrm{np}} = 0$) then $\Delta E = 0$, consequently total mechanical energy of the body is conserved ($E = const$). Therefore we obtain **the law of total mechanical energy conservation**:

> **total mechanical energy of a body is conserved**
> **in any states of the body,**
> **if the work of non potential fields in this states**
> **equals to zero.**

It should be noted here that it is the total mechanical energy **conserved**, while *kinetic* as well as *potential* energies in general case *change their value*. However, if law of total mechanical energy conservation is satisfied, then these changes are always so that the increment of one of them equals exactly the decrement of the second one: $\Delta T = -\Delta U$. But this does not mean that the kinetic energy is **converted** into potential energy (or vice versa). Only total mechanical energy can be converted into other forms of energy.

A special case of the total energy conservation law is the absence of non potential forces ($F_{\mathrm{np}} = 0$). Obviously, under this condition the work A_{np} is zero.

Consequence:

> *Total mechanical energy of a body*
> *moving in a potential field*
> *is conserved.*

In closed systems, i.e. in the absence of any kinds of forces (or, when their vector sum is zero), the total mechanical energy is conserved as well, but this case is of very limited practical value.

2.5   Collision of point particles

In this paragraph we will consider different kinds of point particles collisions **in a closed system**. As an instrument of investigation we will use the conservation laws of total energy and momentum.

Collision of point particles is a *mechanical* interaction through the
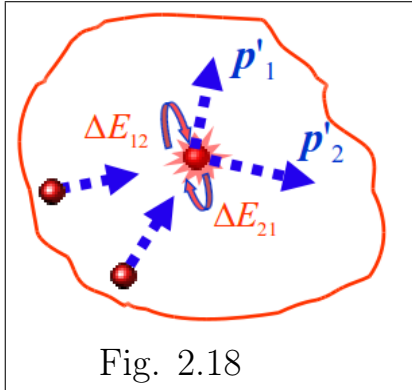

Fig. 2.18

direct contact in an infinitesimally small period of time* during that particles interchange their *energy and momentum* provided that the system of particles remains closed.

There are two types of collisions: perfectly-inelastic, such a collision that after the impact point particles move as a single unit and perfectly-elastic – after the impact point particles move with different velocity vectors provided that the laws of momentum and total mechanical energy conservation are satisfied.

Perfectly-elastic collision could be of two types: *central* and *non-central* collisions.

Non-central: after the collision particles move in different directions,

Figure. 2.19

that do not coincide with the direction of their relative motion before the interaction. This type of collision, for example, the collision of billiard balls (moving without torsion), that moved before the interaction on a straight line not coinciding with the one that pass through
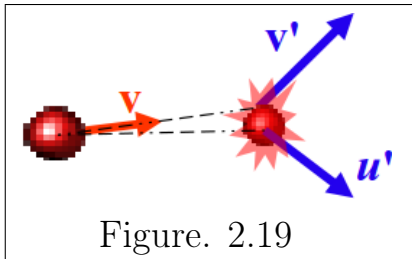
the centres of mass of the balls.

Central collision: is the one of point particles. This type of collision models, for example, the collision of billiard balls, that moved before the interaction on a straight line coinciding with the one that pass through the centres of mass of the balls.

Let us consider more in detail each one of this types of collisions.

2.5.1   Perfectly-inelastic collision

Hereafter, we will assume that a system consists of two particles or for an arbitrary system we will assume that in any time moment in a given point there is only one collision of two particles.

Following the definition of the perfectly-inelastic collision the momentum conservation law takes the form

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{u}. \tag{2.5.1}$$

From this equation we can find the system velocity vector after the interaction with respect to some inertial reference frame (figure 2.19):

$$\vec{u} = \frac{m_1\vec{v}_1}{m_1 + m_2} + \frac{m_2\vec{v}_2}{m_1 + m_2} = \frac{m_1 m_2}{m_1 + m_2}\left(\frac{\vec{v}_1}{m_2} + \frac{\vec{v}_2}{m_1}\right). \tag{2.5.2}$$

Let us consider this interaction in an inertial reference frame K_0, that moves with the system velocity $\vec{u}$. In this reference frame both of the particles have zero velocity after the interaction $u' = 0$. Correspondingly, the momentum conservation law (2.5.1) will have the form:

$$m_1\vec{v}_1' + m_2\vec{v}_2' = 0, \quad \Rightarrow \quad m_1\vec{v}_1' = -m_2\vec{v}_2'. \tag{2.5.3}$$

By this way we see that in inertial reference frame K_0 before the collision both of the particles move forward to each other with equal momentums. After the collision the new particle of mass $m_1 + m_2$ stands still. Therefore, kinetic energy of the system after interaction in reference frame K_0 is zero. Evidently, before the interaction the total kinetic energy of the system is not zero (the sum of two squared real numbers that are not zero could not be zero):

Fig. 2.19
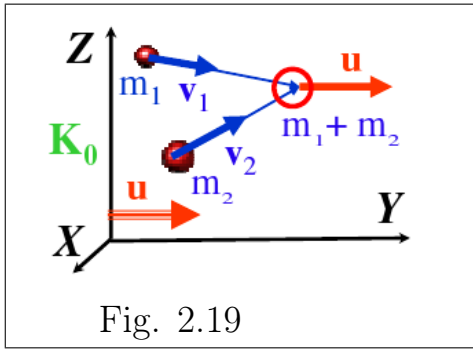
$$\frac{m_1(\vec{v}_1')^2}{2} + \frac{m_2(\vec{v}_2')^2}{2} \neq 0. \tag{2.5.4}$$

Consequently, **the law of total mechanical energy conservation for a perfectly-inelastic collision does not holds**.

Let us denote by Q the total mechanical energy of particles in inertial reference frame K_0 before the interaction and by μ – the so called reduced mass:

$$Q = \frac{m_1(\vec{v}_1')^2}{2} + \frac{m_2(\vec{v}_2')^2}{2}; \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{2.5.5}$$

Total mechanical energy Q can not "disappear" after the interaction. Therefore, the obtained result means that **for an perfectly-inelastic**

collision part of mechanical energy Q transforms into other type of energy – internal energy, id est the heat.

Now we find the quantity Q with respect to the initial (laboratory) inertial reference frame. To do that we express the energy Q in terms of particles momentums:

$$Q = \frac{(m_1 \vec{v}_1')^2}{2m_1} + \frac{(m_2 \vec{v}_2')^2}{2m_2} \qquad (2.5.6)$$

and we find the equation that binds together momentums $\vec{p}_1' = m_1 \vec{v}_1'$ and $\vec{p}_2' = m_2 \vec{v}_2'$ of the particles in reference frame K_0 with corresponding momentums in laboratory reference frame (with the use of Galileo transformation rules (1.6.8)):

$$m_1 \vec{v}_1' = m_1 (\vec{u} - \vec{v}_1) = m_1 \left(\frac{\vec{v}_1}{m_2} + \frac{\vec{v}_2}{m_1} \right) \frac{m_1 m_2}{m_1 + m_2} - m_1 \vec{v}_1. \qquad (2.5.7)$$

Here we have used the equation (2.5.2) for the velocity of the system after the perfectly-inelastic collision. Now we expand the brackets in the last equality in (2.5.7) and simplify it:

$$\begin{aligned}
& \frac{m_1}{m_2} \frac{m_1 m_2}{m_1 + m_2} \vec{v}_1 + \frac{m_1 m_2}{m_1 + m_2} \vec{v}_2 - m_1 \vec{v}_1 = \\
& m_1 \vec{v}_1 \left(\frac{m_1}{m_1 + m_2} - 1 \right) + \mu \vec{v}_2
\end{aligned} \qquad (2.5.8)$$

The expression in brackets on the right hand side after the simplification equals to the reduced mass μ. By this way, for the first particle momentum in inertial reference frame K_0 we have obtained:

$$m_1 \vec{v}_1' = \mu (\vec{v}_2 - \vec{v}_1). \qquad (2.5.9)$$

For the second particle momentum we obtain by analogy:

$$m_2 \vec{v}_2' = \mu (\vec{v}_1 - \vec{v}_2). \qquad (2.5.10)$$

As the result, the equation (2.5.6) takes the form:

$$Q = \frac{\mu^2 (\vec{v}_2 - \vec{v}_1)^2}{2m_1} + \frac{\mu^2 (\vec{v}_1 - \vec{v}_2)^2}{2m_2} \qquad (2.5.11)$$

and for the mechanical energy Q after some simple enough transformations (taking into account (2.5.5)) we get the expression in terms of quantities,

that characterise the motion of the system particles with respect to laboratory inertial reference frame:

$$Q = \frac{\mu(\vec{v}_2 - \vec{v}_1)^2}{2}. \tag{2.5.12}$$

As the result, **for an perfectly-inelastic collision part Q of the total mechanical energy of interacting particles is transformed (during the time of collision) into internal energy of the new particle**. The value of Q for a given pair of particles depends on their relative velocity only.

2.5.2 Perfectly-elastic collision, central

Let us consider closed system of particles with respect to two inertial reference frames – K (laboratory) and K_c. Reference frame K_c have the velocity $\vec{v}_c$ with respect to K (figure 2.20) where $\vec{v}_c$ is the velocity of the mass centre.

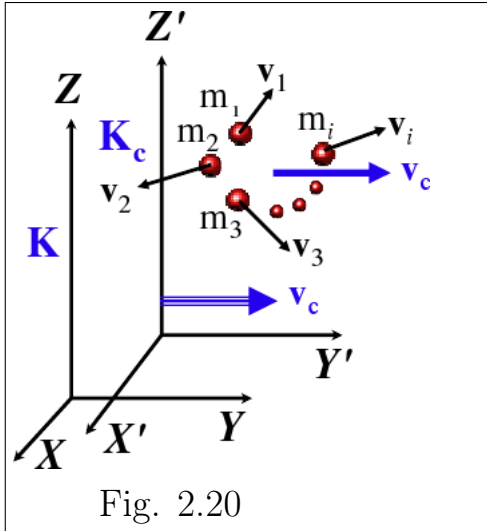We again assume that only two mass points, not more, can collide in any time moment in a given space point. Velocities of the particles before interaction we will denote by $\vec{v}_1, \vec{v}_2,...\vec{v}_i$ (in inertial reference frame K_c – the same symbols with primes – v_i') and velocities of the particles after the interaction we will denote by $\vec{u}_1, \vec{u}_2,..., \vec{u}_i$ (in inertial reference frame K_c – the same symbols with primes – u_i'). Let us consider the system in question from the reference frame K_c. The law of energy conservation in this system have the form:

Fig. 2.20

$$m_1\vec{v}_1'^2 + m_2\vec{v}_2'^2 = \\ = m_1\vec{u}_1'^2 + m_2\vec{u}_2'^2. \tag{2.5.15}$$

This equation multiplied by m_1 gives us:

$$m_1^2\vec{v}_1'^2 + m_1 m_2\vec{v}_2'^2 = \\ m_1^2\vec{u}_1'^2 + m_1 m_2\vec{u}_2'^2. \tag{2.5.16}$$

From the law of momentum conservation it is obvious that $(m_1 \vec{v}_1')^2 = (m_2 \vec{v}_2')^2$ and $(m_1 \vec{u}_1')^2 = (m_2 \vec{u}_2')^2$. Then we obtain:

$$m_2^2 \vec{v}_2'^2 + m_1 m_2 \vec{v}_2'^2 = m_2^2 \vec{u}_2'^2 + m_1 m_2 \vec{u}_2'^2 \qquad (2.5.17)$$

Now it's evident that

$$(m_2^2 + m_1 m_2) \vec{v}_2'^2 = (m_2^2 + m_1 m_2) \vec{u}_2'^2 \qquad (2.5.18)$$

Consequently, in inertial reference frame K_c $|\vec{v}_2'| = |\vec{u}_2'|$. For an arbitrary number of points $|\vec{v}_i'| = |\vec{u}_i'|$. The centre of mass velocity with respect to reference frame K_c is zero, so we obtain

$$\vec{v}_i' = -\vec{u}_i', \qquad (2.5.19)$$

id est, from the point of view of an observer in reference frame K_c the velocity vectors of particles will change their sign but not the module as the result of collision. To transform our equations into reference frame K we will use Galileo transformation rules:

$$\vec{u}_i = 2\vec{v}_c - \vec{v}_i. \qquad (2.5.20)$$

Particularly, for two mass points:

$$\vec{u}_1 = 2\frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} - \vec{v}_1; \qquad \vec{u}_2 = 2\frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} - \vec{v}_2. \qquad (2.5.21)$$

2.5.3   Perfectly-elastic collision, non central

To take an example of a non central perfectly-elastic collision we will consider two particles of different masses. We find the velocities of the particles after the interaction as well as the angle between this velocities. As a reference frame we choose such that moves with the same velocity as the particle of mass M, so this particle has zero velocity in this reference frame before a collision.

The laws of momentum and energy conservation in laboratory reference frame have the form:
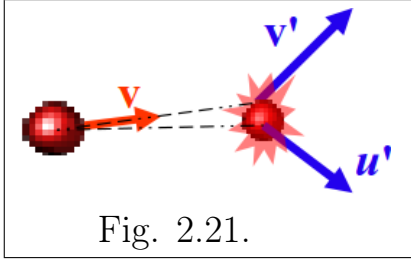

Fig. 2.21.

$$m\vec{v} = m\vec{v}\,' + M\vec{u}\,',$$
$$m\vec{v}\,^2 = m(\vec{v}\,')^2 + M(\vec{u}\,')^2. \qquad (2.5.26)$$

The second power of the first equation:

$$m^2\vec{v}\,^2 = m^2(\vec{v}\,')^2 + M^2(\vec{u}\,')^2 + 2mM(\vec{v}\,',\vec{u}\,'), \qquad (2.5.27)$$

and the second one we multiply by m:

$$m^2\vec{v}\,^2 = m^2(\vec{v}\,')^2 + mM(\vec{u}\,')^2. \qquad (2.5.28)$$

Now we subtract this equation from the previous one:

$$M^2(\vec{u}\,')^2 + 2mM(\vec{v}\,'\vec{u}\,') = mM(\vec{u}\,')^2. \qquad (2.5.29)$$

Expanding the scalar product here we regroup the terms of the sum:

$$2m\,v'u'\cos\alpha = (m - M)(\vec{u}\,')^2. \qquad (2.5.30)$$

Now we can find the angle between the velocities of the particles after the perfectly-elastic non central collision

$$\cos\alpha = \frac{u'}{2v'}\left(1 - \frac{M}{m}\right). \qquad (2.5.31)$$

The more interesting case is the one of the equal masses: in this case $\cos\alpha = 0$ and, consequently, the angle $\alpha{=}90°$, id est the particles of the same mass in a perfectly-elastic non central collision moves after the interaction in perpendicular directions.

2.6 Gravitational field

2.6.1 The universal law of gravitation

If the *acceleration of a test particle due to a force field does not depend on the mass of the particle, such force field will be called* **gravitational field**. Gravitational field supplies the gravitational interaction all the objects of the material world are involved in.

The universal law of gravitation is the law describing the gravitational interaction of *mass points* (figure 2.22):

$$\vec{F}_{12} = G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1) \qquad (2.6.1)$$

where $G = 6,673 \cdot 10^{-8} \text{cm}^3/\text{g·s}^2$ – gravitational constant, m_1 and m_2 – gravitational masses of particles. Due to the spherical symmetry of the right hand side of equation (2.6.1), it is valid not only for mass points but for spherically symmetrical bodies as well – particularly, for solar system planets. In this case radius vectors $\vec{r}_1$ and $\vec{r}_2$ in (2.6.1) are the radius vectors of the corresponding body centres.
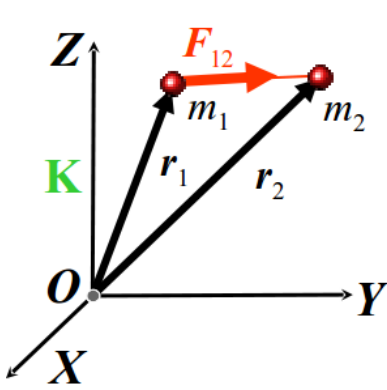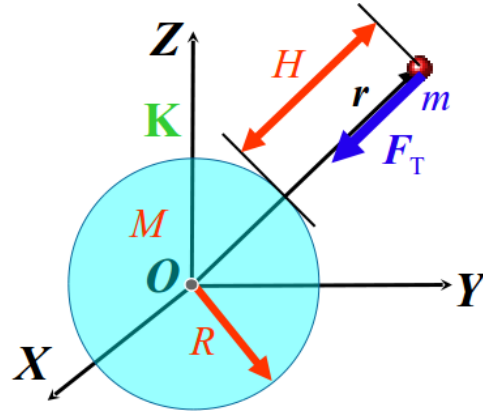

Fig. 2.22.

Fig. 2.23.

2.6.2   Equivalence principle

Let us consider the interaction of a mass point with the Earth. The force acting on the mass point from Earth is the gravitational force (figure 2.23)

$$\vec{F}_{\text{gr}} = -G\frac{Mm}{|\vec{r}|^3}\vec{r} \quad \Rightarrow \quad F_{\text{gr}} = G\frac{Mm}{(R+H)^2} \qquad (2.6.2)$$

where $R = 6,371 \cdot 10^8$ cm – radius of the Earth, H – the height above the ground where the point mass is situated, $M = 5,977 \cdot 10^{27}$ g – Earth's mass.

Following the second Newton law the point mass m_{i} gains acceleration a_g due to gravity:

$$a_g = \frac{F_{\text{gr}}}{m_{\text{i}}} = G\frac{M}{(R+H)^2}\frac{m}{m_{\text{i}}} = g\frac{m}{m_{\text{i}}} \qquad (2.6.3)$$

where m_i – inertial mass of the mass point and g is the acceleration of a freely falling object. Therefore, from the second Newton law we see that acceleration a_g due to gravity is not the same as acceleration of a freely falling object

$$a_g \neq g. \tag{2.6.4}$$

However, experiments on measuring gravitational m and inertial m_i masses for different bodies shows that **numerically** they are equal with a very high precision — relative error is not greater than 10^{-13}. On this basis the **principal of equivalence** was formulated. This principal states that

> **gravitational and inertial masses of a body numerically are always equal.**

Thereby:

$$a_g = G \frac{M}{(R+H)^2} \equiv g. \tag{2.6.5}$$

Thus, the value of acceleration due to gravity a_g and the value of a freely falling body acceleration g **are the same** if the principle of equivalence is satisfied.

2.6.3 Potentiality of gravitational field

Consider the gravitational interaction of two point bodies of mass m_1 and m_2. In order to make sure that the gravitational field of a mass point m_1 is potential, we need to ensure that there exists a scalar function $\Phi_{12}(\vec{r})$ so that its gradient (grad) equals to the strength of gravitational interaction of the mass point with any other point body of mass m_2. To do that we need to calculate the result of the operator $\vec{\nabla}$ action on the function $\Phi_{12}(\vec{r})$.

The solution of the problem for $\Phi_{\vec{12}}(r)$ function we will look for in the next form:

$$\Phi_{12}(\vec{r}) = G \frac{m_1 m_2}{|\vec{r}|}. \tag{2.6.6}$$

Operator nabla $\vec{\nabla}$ is by definition:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \tag{2.6.7}$$

so we need just to calculate the differentials of function (2.6.6) with respect to coordinates. As we see from (2.6.6) differentials of Φ are the differentials of inverse absolute value of radius-vector $|\vec{r}| = (x^2 + y^2 + z^2)^{1/2}$. The task is not so hard but, in view of further use of operator nabla we will do it otherwise, namely, with the help of operator nabla $\vec{\nabla}$ properties:

$$\left(\vec{a}\,\vec{\nabla}\right)\Phi = \vec{a}\left(\vec{\nabla}\Phi\right), \qquad \vec{\nabla}|\vec{r}|^n = n|\vec{r}|^{n-2}\vec{r}, \qquad \vec{\nabla}\left(\vec{a}\,\vec{r}\right) = a. \quad (2.6.8)$$

Then the second formula from (2.6.8) gives us:

$$\vec{\nabla}\Phi_{12}(\vec{r}) = Gm_1 m_2 \vec{\nabla}|\vec{r}|^{(-1)} = -G\frac{m_1 m_2}{|\vec{r}|^3}\vec{r}. \qquad (2.6.9)$$

Now comparing obtained expression with the universal law of gravitation (2.6.1), we see that

$$F_{12} = -\vec{\nabla}\Phi_{12}(\vec{r}). \qquad (2.6.10)$$

By this way we have proven that the force of gravitational interaction of two point masses m_1 and m_2 is defined by the gradient of a scalar function $\Phi_{12}(\vec{r})$. This function is called the potential function of gravitational interaction. Consequently, **gravitational field is potential.**

2.7   Motion in a central field

Central field is a potential field having the potential function depending on absolute value of radius-vector only, id est it is defined by the distance of a point to the *origin of the force field.*

Therefore, potential function of a central force field should have the form:

$$\Phi_c(\vec{r}) = U(|\vec{r}|). \qquad (2.7.1)$$

Correspondingly the force acting on a test particle in the central field:

$$\vec{F}_c = -\nabla U(|\vec{r}|) = -\frac{\partial U(|\vec{r}|)}{\partial \vec{r}} = -\frac{dU(|\vec{r}|)}{d|\vec{r}|}\frac{\vec{r}}{|\vec{r}|} \qquad (2.7.2)$$

is always opposite to the direction of the radius vector $\vec{r}$, id est it is directed towards the origin of the force field. With the use of angular momentum definition we can show that in the central field it is zero:

$$\vec{M}_c = [\vec{r}, \vec{F}_c] = -\frac{dU(|\vec{r}|)}{d|\vec{r}|}\frac{1}{|\vec{r}|}[\vec{r}, \vec{r}] \equiv 0. \qquad (2.7.3)$$

The consequence of this fact is the conservation of angular momentum in central force field. Let us prove it. The law of rotational motion dynamics in differential form:

$$\vec{M}_c = \frac{d\vec{L}_c}{dt} \equiv 0 \qquad (2.7.4)$$

– gravitational force momentum acting on a test particle is zero, so the derivative of corresponding angular momentum with respect to time is zero too. Therefore it is conserved:

$$\vec{L}_c = m[\vec{r}, \vec{v}_c] = const. \qquad (2.7.5)$$

On the base of this information it can be shown that a trajectory of any test particle in central force field is planar. Let us find the scalar product of angular momentum $\vec{L}_c$ on radius vector of the test particle:

$$\vec{L}_c \vec{r} = m\vec{r}\,[\vec{r}, \vec{v}] = m\vec{v}\,[\vec{r}, \vec{r}] \equiv 0 \quad \Rightarrow \quad \vec{L}_c \perp \vec{r}$$

(here in last equality we have made cyclic permutation of vectors in mixed product $\vec{r}\,[\vec{r}, \vec{v}]$). The zero scalar product $\vec{L}_c \vec{r}$ means that vectors $\vec{L}_c$ and $\vec{r}$ are always mutually perpendicular for any type of a test particle motion in a central force field. Therefore, the radius vector $\vec{r}$ remains all the time in the same plane perpendicular to the vector $\vec{L}$, that is, the trajectory of a particle in a central force field lies entirely in one plane.

The scalar product $\vec{L}_c \vec{r}$ can be written as:

$$L_x x + L_y y + L_z z = 0. \qquad (2.7.6)$$

The last expression is the equation of a plane passing through the origin, and indeed in that the particle moves.

Let us study more in detail the motion of a test particle in a central force field. We choose an inertial reference frame with the origin at the centre of force field O. From the expression for the total mechanical energy of the particle it is easy to find the connection between the $d|\vec{r}|$ and dt.

It was shown in paragraph 1.2 that any motion of a particle can be decomposed into two types of motion: linear (rectilinear) – along the radius vector (with velocity $\vec{v}_r$) and rotational – with respect to the reference frame origin (with velocity $\vec{v}_n$). Then we can write total mechanical energy of the particle in a central force field as the sum of it's potential energy $U(|\vec{r}|)$ and kinetic energy: kinetic energy of rectilinear motion along the

radius vector and kinetic energy of rotational motion with respect to the force field origin:

$$E = \left(\frac{mv_r^2}{2} + \frac{J\omega^2}{2}\right) + U(|\vec{r}|). \tag{2.7.7}$$

Using the definitions (1.2.3) and (2.3.18) we rewrite the last expression:

$$E = \frac{m}{2}\left(\frac{d|\vec{r}|}{dt}\right)^2 + \frac{L^2}{2J} + U(|\vec{r}|) = \frac{m}{2}\left(\frac{d|\vec{r}|}{dt}\right)^2 + \frac{L^2}{2m|\vec{r}|^2} + U(|\vec{r}|) \tag{2.7.8}$$

and express from the resulting one the differential from absolute value of radius vector with respect to time:

$$\left(\frac{d|\vec{r}|}{dt}\right)^2 = \frac{2}{m}(E - U(|\vec{r}|)) - \frac{L^2}{m^2|\vec{r}|^2}. \tag{2.7.9}$$

Then

$$dt = d|\vec{r}|\left(\frac{2}{m}(E - U(|\vec{r}|)) - \frac{L^2}{m^2|\vec{r}|^2}\right)^{-1/2}. \tag{2.7.10}$$

Integrating this differential equation, we can find the time interval during which a particle moves from one field point to any other one.

By definition of angular momentum projection on the axe of rotation $L = J\omega = mr^2 d\phi/dt$, therefore:

$$d\phi = \frac{L}{mr^2}dt \tag{2.7.11}$$

If we substitute here the expression for dt from (2.7.10) we will obtain:

$$d\phi = \frac{L}{mr^2}d|\vec{r}|\left(\frac{2}{m}(E - U(|\vec{r}|)) - \frac{L^2}{m^2|\vec{r}|^2}\right)^{-1/2}. \tag{2.7.12}$$

This is a differential equation of the particle trajectory in the polar coordinate system. Integrating this equation we can find the explicit form of the trajectory, that is, the dependence $\phi = \phi(\vec{r})$.

2.7.1 Kepler problem

Kepler problem is the problem of describing the motion of a particle in a central force field with corresponding potential function of the form

$$U(|\vec{r}|) = \frac{\alpha}{|\vec{r}|} \tag{2.7.13}$$

For the Kepler problem the total mechanical energy of a particle in the points where the velocity $\vec{v}_r = 0$ is zero (turning points) equals to:

$$E = \frac{L^2}{2m|\vec{r}|^2} - \frac{\alpha}{|\vec{r}|}. \tag{2.7.14}$$

This expression leads to quadratic equation on $|\vec{r}|$:

$$|\vec{r}|^2 + 2a|\vec{r}| - b^2 = 0, \tag{2.7.15}$$

where we denoted:

$$a = \frac{\alpha}{2E}, \qquad b = \frac{L}{\sqrt{2mE}} \tag{2.7.16}$$

Roots of this equation are:

$$|\vec{r}|_{1,2} = -a\left(1 \pm \sqrt{1 + (b/a)^2}\right) \tag{2.7.17}$$

Let us consider two cases: parameter $\alpha < 0$ – negative (repulsive centre) and $\alpha > 0$ – positive (attractive centre).

In the first case ($\alpha < 0$), if the total mechanical energy $\mathbf{E > 0}$ then parameters $a < 0$, $b > 0$ and $b \in Re$. *Only one real root* is possible:

$$|\vec{r}|_1 = |a|\left(1 + \sqrt{1 + (b/a)^2}\right)$$

If the total mechanical energy is negative: $\mathbf{E < 0}$ then parameters $a > 0$ and $b \in Im$. There are not any positive roots.

So, for an repulsive centre energy could be *only* positive $\mathbf{E > 0}$ and **the motion is infinite**.

In the second case ($\alpha > 0$) of attractive centre if the total mechanical energy $\mathbf{E > 0}$ positive, then parameters $a > 0$, $b > 0$ and $b \in Re$. *Only one real root* is possible:

$$|\vec{r}|_1 = a\left(-1 + \sqrt{1 + (b/a)^2}\right)$$

and **the motion is infinite**.

If the total mechanical energy is negative: $\mathbf{E < 0}$ then parameters $a < 0$ and $b \in Im$. There are *two roots*:

$$|\vec{r}|_{1,2} = -a\left(1 \pm \sqrt{1 - (b/a)^2}\right)$$

and the motion **is finite**.

Let us consider the Kepler problem more in detail for an attractive force centre.

With notations (2.7.16) equation (2.7.12) takes the form:

$$d\phi = \frac{b}{r^2}dr\left(1 + \frac{2a}{r} - \frac{b^2}{r^2}\right)^{-1/2}. \tag{2.7.18}$$

Integrating this equation we obtain:

$$\cos\phi = \left(\frac{b^2}{ra} - 1\right)\left(1 + \frac{b^2}{a^2}\right)^{-1/2} \qquad (2.7.19)$$

and after some transformations:

$$1 + e\cos\phi = p \qquad (2.7.20)$$

where we denoted $e = \sqrt{1 + (b/a)^2}$ – *the eccentricity* and $p = b^2/ra$ – *orbit parameter*. Expression (2.7.20) is the equation of conic section (with the focus at the origin of coordinate system) in polar coordinates. On the figure all the possible types of conic sections are presented.
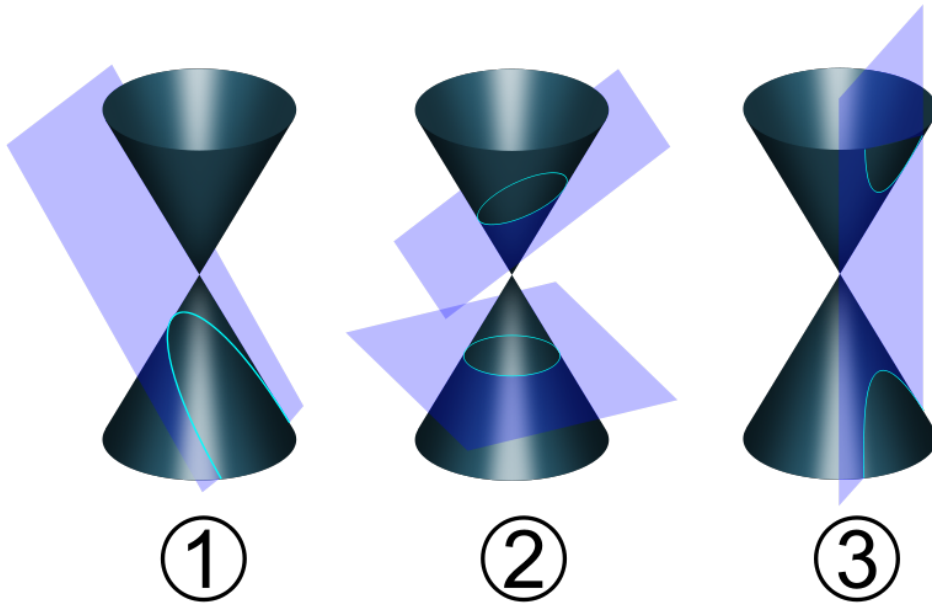
Thus, the motion of a particle in a central field with an attractive centre could have only three possible types of trajectories (figure):

1. parabola ($E = 0$, $e = 1$);

2. ellipse ($E < 0$, $e < 1$).

3. hyperbola ($E > 0$, $e > 1$);

For an ellipse the quantities a and b are the major and minor axis.

2.7.2 Kepler laws

The results of the Kepler problem solution for a particle moving in a central force field with the attractive centre are called *Kepler's laws.* For example the laws of planetary motion.

First Kepler law

By definition, Kepler's laws corresponds to the Kepler problem with parameters $\alpha > 0$ and $E < 0$. Therefore:

> **the only one possible closed trajectory**
> **of a point particle in a central force field**
> **with attractive force centre**
> **could be an ellipse**
> (with the force centre in the focus).

Second Kepler law

Consider a particle moving in a central field along a closed path. As we have already shown, for the motion in a central field the angular momentum of a particle is conserved $\vec{L} = const$ (2.7.5). From the other point of view

$$L = J\omega = mr^2\omega = mr^2\frac{d\varphi}{dt} = m\frac{r(rd\varphi)}{dt}, \qquad (2.7.21)$$

here we have used the definition of angular momentum for a material point (2.3.16) and the definition of angular velocity (1.2.6). Area of an ellipse sector could be found by the next formula: $dS = r^2d\varphi/2$, then:

$$L = 2m\frac{dS}{dt} = const. \qquad (2.7.22)$$

On the figure 2.24 a closed trajectory of a point particle in a central force field is shown. The force centre is in one of the focuses. The shaded area – the area dS of the elementary sector formed by two infinitely close radius vectors $\vec{r}(t)$ and $\vec{r}(t + dt)$ and by an element of the path arc.
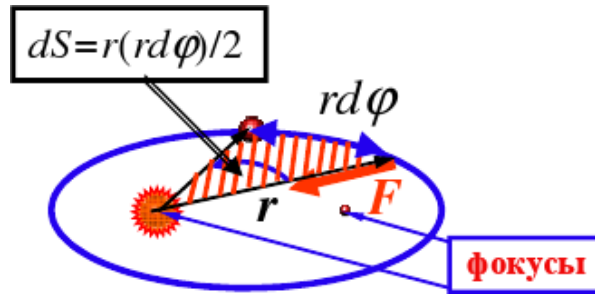
Fig. 2.24.

Thereby from (2.7.21) we obtain:

$$\frac{dS}{dt} = const. \tag{2.7.23}$$

The differential dS/dt is called areal velocity. Expression (2.7.23) is called the second Kepler's law – *for the motion in a central field in an elliptical orbit, the areal velocity remains constant.* Or:

> **Radius vector of a particle**
> *that moves along an elliptical orbit*
> *in a central force field*
> **sweeps out equal areas**
> **during equal intervals of time.**

Third Kepler's law

From the second Kepler law:

$$2m\frac{dS}{dt} = L = const. \tag{2.7.24}$$

Then, for the period of revolution of a point particle moving in an elliptical orbit in the central field we obtain:

$$T = \frac{2mS}{L}. \tag{2.7.25}$$

Given that the area of the ellipse $S = \pi ab$ from the expression (2.7.25) with the use of notation (2.7.16) we can write

$$2m\pi ab = TL = Tb\sqrt{2mE} = Tb\sqrt{2m\frac{\alpha}{2a}}. \tag{2.7.26}$$

Where, after simplification we find the period T:

$$T = 2\pi a^{3/2}\sqrt{\frac{m}{\alpha}}. \tag{2.7.27}$$

Raising this expression in the second power we find the the ratio of periods for two trajectories (with different a – semi-major axes of the ellipses) and obtain the third law (see Fig. 2.25) of Kepler:

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}. \tag{2.7.28}$$

> **The square of the orbital period of particles**
> *in gravitational field* **is directly proportional**
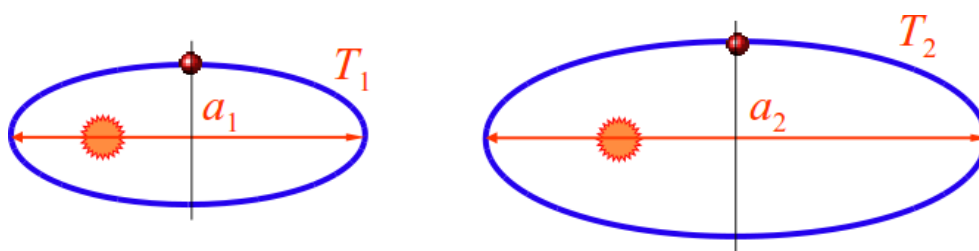> **to the cube of the semi-major axis of its orbit.**

Fig. 2.25

2.8 Non-inertial reference frames

The study of the dynamics we had started with Newton's laws and found that Newton's second law (2.1.1) is valid **only** in inertial reference frames. However, all the *real* reference frames could taken as inertial only approximately. In particular – a frame of reference associated with any point on Earth's surface (except the poles) is non-inertial. The cause is in daily rotation of the Earth. Accordingly, the lower the velocity $v_r = \omega R$ (where ω is the angular velocity of the rotation and v_r is the rotational velocity of the point on the surface of the Earth), the more accurate will be the model of inertial reference frame for this point.

Naturally, the question arises: how to write a law of motion for a particle in a non-inertial reference frame K', moving with respect to an inertial reference frame K with a known acceleration a_0?

Consider the motion of a material point of mass m along some trajectory L from two reference frames – inertial K and non-inertial K'. The magnitude and direction of the non-inertial reference frame K' acceleration $\vec{a}_0$ we chose to be equal to the acceleration $\vec{a}_0$ of the particle (with respect to an inertial reference frame K).
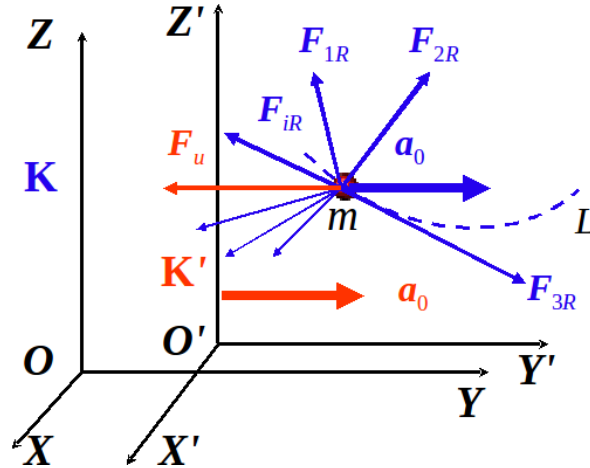
Fig. 2.26

A particle moves in the field of real forces F_{iR}, acting on the particle. Then the law of dynamics in the inertial reference frame K will be written as follows:

$$\sum_{i=1}^{n} \vec{F}_{iR} = \vec{a}_0, \tag{2.8.1}$$

In a non-inertial reference frame K' particle acceleration is zero and the law of dynamics must be of the form

$$\sum_{i=1}^{n+1} \vec{F}_i = 0. \tag{2.8.2}$$

The number of particles in mechanical system as well as the number of external real forces equals to n. The upper boundary $n + 1$ in the sum means that in non-inertial reference frame one extra force *is needed* so that it will equilibrate all the real forces. Let us invent new notation:

$$\vec{F}_u = -m\vec{a}_0. \tag{2.8.3}$$

Quantity $\vec{F}_u$ is an additive (**fictive**) force, that arise in non-inertial reference frames and is called "force" of *inertia*. Thus the law of material point dynamics in non-inertial reference frame K' takes the form:

$$\sum_{i=1}^{n} \vec{F}_i + \vec{F}_u = 0. \tag{2.8.4}$$

We emphasize that the inertia force $\vec{F}_u$ does not describe any physical interaction. In this sense, the force of inertia has no physical meaning – it's just the quantity that has the dimension of power.

2.8.1 Gravitational force and weight of a body

Let us consider a small enough body (figure 2.27), attached by a rope on some (not very big) height H. The Earth rotates (daily rotation) – all the bodies on the surface of the Earth take part in this rotational motion. Gravity force acts on the body due to gravitational interaction with the Earth. In inertial reference frame K with the origin in the Earth's centre the law of dynamics for the particle in question have the form

$$\vec{F}_{\text{gr}} + \vec{T} = m\vec{a}_{\text{cf}}, \tag{2.8.5}$$

where $\vec{T}$ – reaction force of the rope, $\vec{a}_{\text{cf}}$ – centrifugal acceleration.
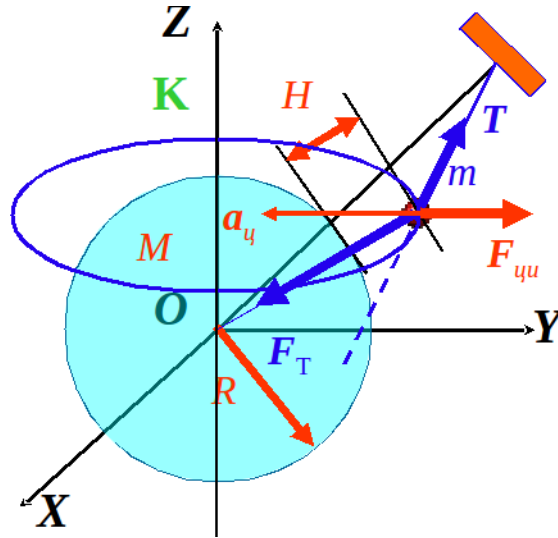

Fig. 2.27

The surface of the Earth is non-inertial reference frame that rotates with acceleration $\vec{a}_{\text{cf}}$. Therefore the law of dynamics in non-inertial reference frame bound to the surface of the Earth have the form:

$$\vec{F}_{\text{gr}} + \vec{T} + \vec{F}_{\text{cf}} = 0, \tag{2.8.6}$$

where $\vec{F}_{\text{cf}} = -m\vec{a}_{\text{cf}}$ – centrifugal force of inertia.

> **The weight of the body is the force, acting on a horizontal support or vertical suspension.**

Hence the weight P of a body of mass m

$$|\vec{P}| = |-\vec{N}|. \tag{2.8.7}$$

Writing this formula, we have taken into account that the weight of a body $\vec{P}$ and the reaction force of the thread $\vec{T}$ are the forces bound by Newton's third law, and therefore can not be reduced to a single point in space. As a (mathematical) consequence, these forces can not be written in one vector equation (see Section 2.1.1, Newton's Third Law).

Then, having $\vec{a}_{\text{cf}} = \vec{\rho}\omega^2$, where ρ is the radius of the circle along that the particle under the consideration moves together with the Earth, we obtain:

$$|\vec{P}| = |m\vec{g} + m\vec{\rho}\omega^2|. \tag{2.8.8}$$

Let us introduce notation:

$$\vec{g}_R = \vec{g} + \vec{\rho}\omega^2. \tag{2.8.9}$$

As the result the weight of a body of mass m

$$P = mg_R, \tag{2.8.10}$$

where g_R – acceleration due to gravity at the latitude at which the particle is located.

Chapter 3

Relativistic mechanics

3.1 Kinematics of special theory of relativity (STR)

3.1.1 Introduction

The special theory of relativity (SRT) is a mechanic, taking into account the peculiarities of time and space for objects moving at very high velocities relative to the observer - relativistic mechanics.

Historically, the basis of SRT are the Lorentz transformation rules. The first results related to the Lorentz transformation rules were obtained by himself (Hendrik Antoon Lorentz) in 1885 and by Woldemar Voigt in 1887. In 1904 came the work of Lorentz where transformation rules were firstly formulated as an independent scientific result. Then, in June 1905, a French mathematician Henri Poincaré has given a new form to transformation rules proposed by Lorenz (it was Poincaré who gave them the name of Lorentz), and established their group[1] nature. Due to these transformations the speed of light is constant and Maxwell's equations are invariant and thus satisfied the principle of relativity (the relativity principle of Poincaré): "the laws of physics must be the same for a stationary observer as well as for the observer involved in a uniform motion, so that we do not have and can not have any means of knowing whether or not we are in the state of such a motion".

At the end of September 1905 came the work of Albert Einstein, in which the same principle of relativity was formulated and Lorentz transformation rules were obtained.

The main difference of Einstein's work was the fact that the new prin-

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[1]The Lorentz group is the group of Lorentz transformations of Minkowski space, preserving the origin. The group of all Lorentz transformations, including translation is called the Poincaré group.

ciple of relativity and the constancy of the speed of light in all inertial reference frames (IRF) were **initial postulates** whilst the Lorentz transformation rules only a *consequence* of these postulates. In the works of Poincaré all results were obtained as *a consequence of ether properties* (ether – special physical substance in which all the observed physical objects are moving).

Thus, Einstein first proposed the idea that the unusual (in terms of Newtonian mechanics) form of the Lorentz transformation rules is the property of space itself (and not the consequence the unobserved ether properties).

3.1.2   Relativity principle

In classical mechanics, relativity principle was formulated by **Galileo Galilei** and can be reduced to the following statement:

> all the laws of mechanics do not depend on the choice of the inertial reference frame (IRF), that is they *are invariant* with respect to the choice of IRF.

This means that by no means (mechanical experiment) within any inertial reference frame it is impossible to determine whether the system is moving or at rest.

Einstein and Poincaré had generalized the classical Galilean relativity principle to include all the Natural phenomena (and therefore corresponding laws). Einstein-Poincaré principle of relativity can be formulated as follows:

> all the laws of Nature do not depend on the choice of inertial reference frame (IRF), that is they *are invariant* with respect to the choice of IRF.

This means that *by no means* within any inertial reference frame *it is impossible* to determine whether the system is moving or at rest.

Hence the principle of relativity (both classical and relativistic) claims the equality of all inertial reference frames.

3.1.3   Basic principals of SRT

The basic postulate of classical (ie Galilean) principle of relativity, which provides the equality of inertial reference frames in classical me-

chanics, is the postulate of Galileo, which states that in all of IRF time behaves the same way (ie, the time is absolute).

What are the changes introduced by Einstein and Poincaré in the classical views of inertial reference frames?

- The most important difference of Einstein-Poincaré principle of relativity to the classical one is that they refused the absoluteness of time with respect to the choice of IRF.

- As a consequence, the length of a segment in different IRF should also be different, i.e. it depends on the choice of IRF.

So, according to the ideas of the relativistic principle of relativity

$$\boxed{dt \neq inv} \; \boxed{dl \neq inv}$$

where dt and dl are elementary time and length intervals accordingly, inv – invariant (with respect to the choice of IRF). *Thus, time and length intervals in STR are relative values.*

A question arises – what is invariant (with respect to the choice of IRF) in relativistic principle of relativity?

This question is answered by **basic postulate** of SRT (Einstein's postulate), which states that

$$\boxed{\begin{array}{c} \text{the speed of light } c \text{ does not depend on the choice of IRF,} \\ \text{ie in all of the IRF is the same} \end{array}}$$

The second (or additional) postulate is the postulate of *space and time isotropy and homogeneity.* By logical reasoning one can deduce from the speed of light invariance the invariance of the value $s^2 = -c^2\Delta t^2 + \Delta l^2$, called the *interval.*

To eliminate the need to carry out a fairly complex (and most importantly – not very convincing) logical reasoning, we proceed differently – we introduce new invariant quantity instead of non-invariant with respect to the choice of IRF units dt and dl: $ds^2 = -c^2dt^2 + dl^2$. This quantity is called *elementary interval* and we **postulate** it invariance with respect to the choice of inertial reference frame (**additional** postulate):

$$\boxed{ds^2 \;=\; -c^2dt^2 + dl^2 \;=\; inv}$$

Here c – is the speed of light.

Thus, two postulates correspond to the principle of relativity. They can shortly be written as:

$$\boxed{c \;=\; inv} \;\boxed{ds^2 \;=\; inv}$$

3.1.4   Clock synchronization in SRT

To describe the motion of a point it is necessary to be able to measure the change in the position of the studied mass point in space over time. In physics (and particularly in SRT) the procedure, allowing to define both the coordinate system and method of time measurement is called setting of reference frame.

By definition (1.1) the reference frame is the set of basis and calibration. Basis is the set of physical laboratories (real or imagined), located at all points in space and equipped with instruments for measuring time and length intervals. In order to be able to measure the time it is first necessary to agree the procedure of clock synchronisation.

The synchronization procedure is based on the basic postulate of special relativity ($c = inv$):

1. we choose the base clock and *set* on them time t_0;

2. on all the ather clocks we *set* time t_{0i} (preliminary having measured distances l_i to each clock):

$$t_{0i} = l_i/c$$

3. at time moment t_0 **we start** base clocks and *simultaneously* send the signal with *the speed of light* to **all** of the other clocks. The *signals* reaching the clocks *start* them.
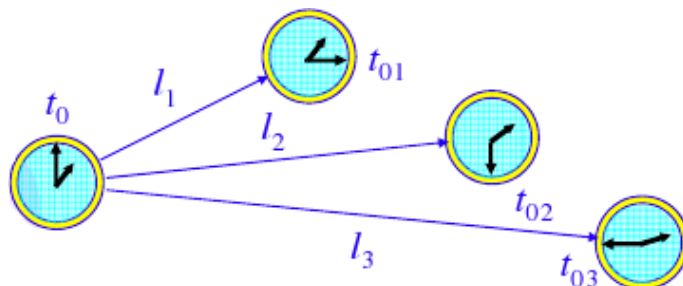

Fig. 3.1

3.1.5 Lorentz transformation

Time intervals in different IRF

In the previous section we have established that, under the principle of relativity, relativistic time intervals depend on the choice of the inertial reference frame and, therefore, are different in different IRF. Let us find out the connection between time periods in inertial reference frames moving relative to each other at a speed of $V = const$.
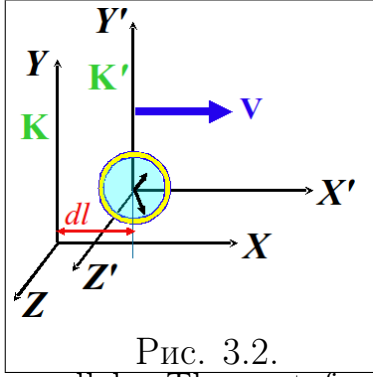
To simplify the calculations, consider inertial reference frames, moving so that their axes are parallel. The rest frame (laboratory one) will be denoted by K, and moving frame – by K' (see Figure 3.2). Assume that at the initial time moment the origins of IRF K and K' coincide.

Рис. 3.2.

Let us look for the clocks in the origin of K' IRF. In elementary time interval dt from the point of view of an observer in the rest IRF K, the clocks moved on elementary distance dl. However, according to the relativistic principle of relativity, the observer in an inertial reference frame K', will measure the duration of the same process dt' – different time interval. His clocks in IRF K' did not moved. The above can be summarized in the following form:

$$K : \ dt \ \longrightarrow \ dl \qquad K' : \ dt' \ \longrightarrow \ 0.$$

According to the second (additive) postulate of SRT, $ds^2 = ds'^{\,2}$. That is

$$-c^2 dt^2 + dl^2 \ = \ -c^2 dt'^{\,2} + \underbrace{dl'^{\,2}}_{\equiv\,0} . \tag{3.1.1}$$

And we easily find that

$$dt' \ = \ dt \sqrt{1 - \left(\frac{dl}{dt}\right)^2 / c^2}. \tag{3.1.2}$$

As the clocks move together with IRF K', then dl/dt equals to V – the velocity of IRF K', ie $dl/dt \ = \ V$. Hence, elementary time intervals dt and dt' in different inertial reference frames are bound by next formula:

$$dt' \ = \ dt \sqrt{1 - V^2/c^2}, \tag{3.1.3}$$

where V is the relative velocity of inertial reference frames. Integrating the last expression we will obtain the relation or to be more precise the rule of recalculation of time intervals measured from different IRF:

$$\Delta t' = \Delta t \sqrt{1 - V^2/c^2}. \qquad (3.1.4)$$

Time in the rest frame is called own time and is denoted by letter τ (we have $\tau = t'$). Time in all the other inertial reference frames is called world time and is denoted by t. Then equation (3.1.3) will have the form

$$\boxed{d\tau = \frac{dt}{\gamma},} \qquad (3.1.5)$$

where γ is the so called relativistic factor (Lorentz factor):

$$\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2}. \qquad (3.1.6)$$

Lorentz factor varies from 1 (for $v \ll c$ – non-relativistic limit) to infinity (at velocities close to the speed of light).

Obviously, **time intervals measured by own clocks are always minimal**.

In relativistic mechanics, due to technical impossibility (for now) experimentally verify the theoretical results, abstract problem method called a "thought experiment" is widely used. If the solution of such an abstract problem ("thought experiment") leads to an obvious contradiction, the problem is called a *paradox*.

One of the most well-known paradoxes associated with the formulas (3.1.4, 3.1.5) that allows to recalculate periods of time from one inertial reference frame to another is the so called *twin paradox*.

It should be noted that formula (3.1.5) is symmetric with respect to each inertial reference frame K and K'. In other words, if in terms of K-system clocks in K'-system are "slower", on the contrary, from the point of view of K'-system, clocks in K-system are "slower" (in the same rate). It is a necessary consequence of the special relativity postulates, and can not be attributed to any change in the properties of clocks due to their movement.

In other words, formula (3.1.5) is simply **the rule of recalculation** of time intervals measuring results from one inertial reference frame to another.

Length of a segment in SRT

In previous paragraph 3.1.3 we have established that according to the principle of relativity in SRT $dl \neq inv$ i.e. the length of a segment is different in different inertial reference frames. Let us find out the relation between measurement results of the same segment in different inertial reference frames moving with respect to each other with velocity $V = const$.

To simplify calculations we choose inertial reference frames so that

their coordinate axes are parallel and at the initial time moment both origins of K and K' systems coincide. Here K is the rest reference frame (laboratory) and K' – moving reference frame (figure 3.3).
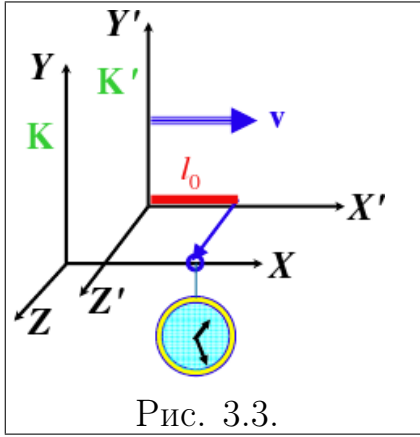
Consider a rod fixed in reference frame K' so that one end is in the origin and the other points out along the Ox axe. We will lead measurements from the point of view of an observer in K' reference frame, but the time will be measured with the help of clocks in K reference frame (figure 3.3).

Рис. 3.3.

It always important to clearly specify the scheme of time measurement in special theory of relativity so that it is in accordance with time synchronisation procedure. To do that let us mentally place two sources of electro-magnetic waves at two ends of the rod. This sources are able to emit the waves strictly along the Z axe (figure 3.3). By this way we can measure the time Δt needed for the rod to pass along the clocks and calculate the length l of the rod in reference frame K:

$$l = V\Delta t. \tag{3.1.7}$$

Analogously we can calculate the length of the rod in IRF K' by fixing the time moments t'_1 and t'_2 when both of the rod's ends pass in front of the clocks in inertial reference frame K'. Then the length of the rod l_0 in the rest frame equals to:

$$l_0 = V\Delta t'. \tag{3.1.8}$$

The ratio of rod's length measured from different IRF K and K':

$$\frac{l}{l_0} = \frac{\Delta t_0}{\Delta t'}. \tag{3.1.9}$$

The time in reference frame where the clocks are at rest is the own time: $\Delta t_0 \equiv \Delta \tau$. Then clearly:

$$\boxed{l_0 = \gamma l.} \tag{3.1.10}$$

Summarising we can say that, **the length of the rod in the rest frame (own length) is always maximum.**

As well as in the previous paragraph, equation (3.1.10) is just the **rule of recalculation**: how can we calculate the rod's length if the results of this length measurements from other – moving IRF are provided.

Lorentz transformations

Consider two inertial reference frames K and K'. Let K'-frame be moving with respect to K-frame with velocity $\vec{V}$. Let us choose coordinate axes as it is shown on the figure 3.4: axes X and X' coincide and directed parallel to the vector $\vec{V}$, axes Z and Z' both are parallel. Then we fix similar clocks and synchronise them separately in K-frame and in K'-frame. Finally, we start the measurements at time moment when the origins O and O' coincide (at $t = t' = 0$).
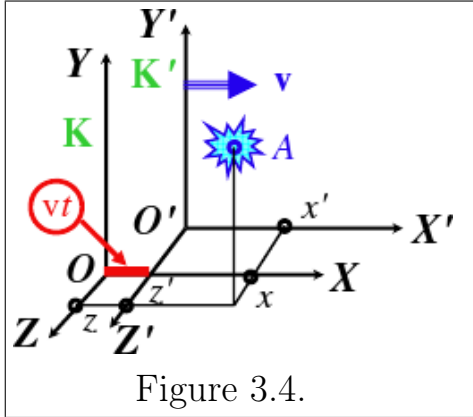

Figure 3.4.

An event A considered **from the IRF** K' has coordinate along the X' axe equal to the distance from point x' on this axe to the origin O'; we denote this distance by $x' = l_0$ (length of the segment measured in rest frame). In reference frame K the same distance equals to $x - Vt = l$ (length of the same segment measured from moving reference frame):

$$\left.\begin{array}{rlll} K': & x' = l_{O'x'} & = & l_0 \\ K: & x - Vt = l_{O'x} & = & l \end{array}\right\} \quad \Leftarrow \quad l = \frac{l_0}{\gamma}. \tag{3.1.11}$$

With the use of relation of lengths of segments l_0 measured in own reference frame and l we obtain:

$$x - Vt = x'/\gamma. \tag{3.1.12}$$

Expressing x' we get: $x' = \gamma(x - Vt)$.

The event A considered **from the IRF** K has coordinate along the X axe equal to the distance l_{Ox} from point x on this axe to the origin

O; now this length is the own one and should be denoted by l_0 while the same distance in IRF K' equals to $x' + Vt'$:

$$K: \qquad\quad x = l_{Ox} = l_0 \atop K': \; x' + Vt' = l_{Ox'} = l \Bigg\} \quad \Leftarrow \quad x' = \gamma(x - Vt). \quad (3.1.13)$$

Using the relation (3.1.10) between length of the segment l_0 measured in own IRF and l together with expression for x':

$$\gamma(x - Vt) + Vt' = \frac{x}{\gamma}. \tag{3.1.14}$$

Then obviously:

$$Vt' = \frac{x}{\gamma} - x\gamma + Vt\gamma = \gamma\left(Vt - x + \frac{x}{\gamma^2}\right). \tag{3.1.15}$$

Multiplier in brackets could be simplified by substitution of Lorentz-factor γ in the full form:

$$Vt' = \gamma\left(Vt - x\frac{V^2}{c^2}\right). \tag{3.1.16}$$

So we have obtained expression that relates time moment t in rest reference frame with t' time moment in moving reference frame with velocity V:

$$t' = \gamma\left(t - \frac{Vx}{c^2}\right). \tag{3.1.17}$$

As the result we have get Lorentz transformations for coordinates and time:

$$\boxed{\begin{aligned} &y' = y, \quad z' = z, \\ &x' = \gamma(x - Vt), \\ &t' = \gamma\left(t - \frac{Vx}{c^2}\right). \end{aligned}} \tag{3.1.18}$$

3.1.6   Addition law for velocities in SRT

Consider arbitrarily moving mass point from two inertial reference frames K and K'. And, as before we choose the moving system K' so that the axis X' is parallel to the axis X. Let the velocity of the particle in inertial reference frame K is known. Let us find what is its speed in the reference frame K'. By definition:

Fig. 3.5.
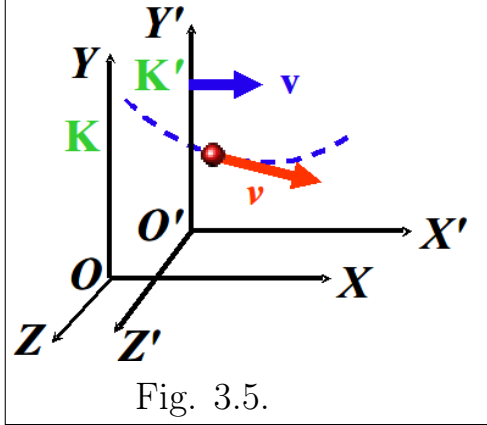
$$v'_x = \frac{dx'}{dt'}, \quad v'_y = \frac{dy'}{dt'}, \quad v'_z = \frac{dz'}{dt'}.$$

Therefore we need to find relations $dx' \to dx, dy' \to dy, dz' \to dz$ and $dt' \to dt$. For this we take the differential from the left and right sides of the Lorentz transformation (3.1.18) and obtain:

$$dx' = (dx - V\,dt)\gamma, \quad dt' = (dt - \frac{V}{c^2}dx)\gamma, \tag{3.1.19}$$
$$dy' = dy, \quad dz' = dz.$$

Substituting these expressions in the definition of velocity, we obtain

$$v'_x = \frac{dx'}{dt'} = \frac{(dx - V\,dt)\gamma}{(dt - V/c^2 dx)\gamma}. \tag{3.1.20}$$

Divide the numerator and denominator by dt:

$$v'_x = \frac{\left(\frac{dx}{dt} - V\right)}{\left(1 - \frac{V}{c^2}\frac{dx}{dt}\right)} \tag{3.1.21}$$

and again, using the definition of velocity we arrive at the final expression for the mass point velocity component along the direction of motion of the inertial reference system K':

$$v'_x = \frac{v_x - V}{1 - v_x V/c^2}. \tag{3.1.22}$$

For y component of the velocity in the moving reference frame we have $v'_y = dy'/dt'$. Substituting the relation (3.1.19) into this definition we

obtain the velocity addition law for the component v'_y perpendicular to the direction of motion of the reference frame K':

$$v'_y = \frac{dy}{(dt - V/c^2 dx)\gamma}.$$

(3.1.23)

Divide the numerator and denominator by dt:

$$v'_y = \frac{\frac{dy}{dt}}{\left(1 - \frac{V}{c^2}\frac{dx}{dt}\right)\gamma} = \frac{v_y}{(1 - \frac{v_x V}{c^2})\gamma}.$$

(3.1.24)

Similarly, for the last component of the velocity vector corresponding expression can be obtained, and, finally, we get the **velocity addition law**:

$$\boxed{v'_x = \frac{v_x - V}{1 - \frac{v_x V}{c^2}}, \quad v'_y = \frac{v_y}{(1 - \frac{v_x V}{c^2})\gamma}, \quad v'_z = \frac{v_z}{(1 - \frac{v_x V}{c^2})\gamma}.}$$

(3.1.25)

The addition law is also called a *composition law for velocities*.

3.1.7 Transformation of acceleration vector components

Consider arbitrarily moving mass point from two inertial reference frames K and K'. We choose the moving system K' so that the axis X' is parallel to the axis X. Let the acceleration vector of the particle in inertial reference frame K is known. Let us find what is its acceleration in the reference frame K'.
By definition:

$$a'_x = \frac{dv'_x}{dt'}, \quad a'_y = \frac{dv'_y}{dt'}, \quad a'_z = \frac{dv'_z}{dt'}.$$

Therefore we need to find relations $dv'_x \to dv_x, dv'_y \to dv_y, dv'_z \to dv_z$ and $dt' \to dt$. The relation for time we have already obtained in equation (3.1.19). Let us find the relation for velocity components differentials. For x component we have:

$$dv'_x = d\frac{v_x - V}{\left(1 - \frac{V v_x}{c^2}\right)} = \frac{dv_x}{\left(1 - \frac{V v_x}{c^2}\right)} + \frac{(v_x - V)V dv_x}{c^2 \left(1 - \frac{V v_x}{c^2}\right)^2}.$$

(3.1.26)

After combining similar terms and simplifications the differential of the velocity component dv_x will have the form:

$$dv'_x = dv_x \frac{1 - \frac{V^2}{c^2}}{(1 - \frac{V v_x}{c^2})^2} = \frac{dv_x}{\gamma^2 \left(1 - \frac{V v_x}{c^2}\right)^2}.$$

(3.1.27)

Now transform the expression (3.1.19):

$$dt' = \gamma \left(dt - \frac{V\,dx}{c^2} \right) = \gamma dt \left(1 - \frac{V}{c^2}\frac{dx}{dt} \right), \qquad (3.1.28)$$

and substitute it together with equation (3.1.27) into definition of acceleration:

$$a'_x = \frac{dv'_x}{dt'} = \frac{dv_x}{dt\,\gamma^3 \left(1 - \frac{V v_x}{c^2} \right)^3}, \qquad (3.1.29)$$

here we have took into account that dx/dt is by definition v_x. As the result we obtain:

$$\boxed{a'_x = \frac{a_x}{\gamma^3 \left(1 - \frac{V v_x}{c^2} \right)^3}} \qquad (3.1.30)$$

Similarly, for y component of the velocity vector (for the latter z component it is likely to be the same up to the change of indices $y \to z$) differential will be:

$$dv'_y = d\frac{v_y}{\gamma(1 - \frac{v_x V}{c^2})} = \frac{dv_y}{\gamma \left(1 - \frac{V v_x}{c^2} \right)} + \frac{V v_y dv_x}{c^2 \gamma \left(1 - \frac{V v_x}{c^2} \right)^2}. \qquad (3.1.31)$$

After some simplifications:

$$dv'_y = \frac{dv_y + \frac{V}{c^2}(v_y dv_x - v_x dv_y)}{\gamma \left(1 - \frac{V v_x}{c^2} \right)^2}. \qquad (3.1.32)$$

Then, by definition of acceleration $a'_y = dv'_y/dt'$ with the use of the resulting differential of velocity component dv'_y and of expression for dt' (3.1.28):

$$a'_y = \frac{dv_y + \frac{V}{c^2}(v_y dv_x - v_x dv_y)}{dt\,\gamma^2 \left(1 - \frac{V v_x}{c^2} \right)^3}. \qquad (3.1.33)$$

Dividing the numerator and denominator by dt (by analogy we can write the expression for the component z of the acceleration):

$$\boxed{\begin{aligned} a'_y &= \frac{a_y + \frac{V}{c^2}(v_y a_x - v_x a_y)}{\gamma^2 \left(1 - \frac{V v_x}{c^2} \right)^3}, \\ a'_z &= \frac{a_z + \frac{V}{c^2}(v_z a_x - v_x a_z)}{\gamma^2 \left(1 - \frac{V v_x}{c^2} \right)^3}. \end{aligned}} \qquad (3.1.34)$$

Thus, from the expressions (3.1.30) and (3.1.34) it is clear that the transition to a moving coordinate system K' the components of the acceleration $\vec{a}'$ transform not only in terms of the original vector $\vec{a}$ components, but (unlike the Galilean transformations (1.6.6)) depend on the velocity of the particle as well.

3.1.8 Relativity of simultaneity

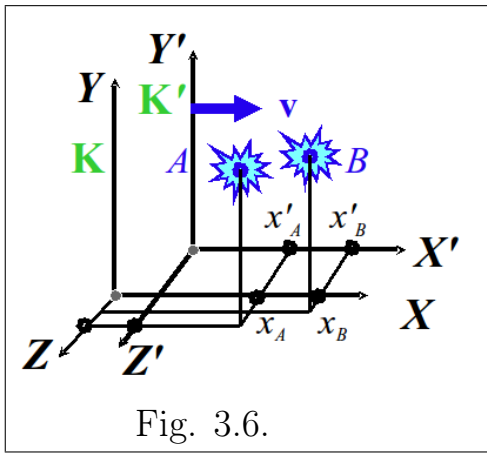Let us consider two events A and B from two inertial reference frames K and K', such that axis X' is parallel to axis X during the relative motion of frames. We find the time interval $\Delta t'$ between this events in reference frame K' provided it is known in inertial reference frame K. Coordinates of the event A in IRF K are: (x_A, y_A, z_A, t_A). Coordinates of the event B in IRF K: (x_B, y_B, z_B, t_B). Then with the use of Lorentz transformation (3.1.18) we obtain:

Fig. 3.6.

$$\Delta t' = t'_B - t'_A = \gamma(t_B - x_B V/c^2) - \gamma(t_A - x_A V/c^2) \qquad (3.1.35)$$

and regrouping the terms:

$$\Delta t' = \gamma(\Delta t - \Delta x V/c^2) \qquad (3.1.36)$$

where it is denoted $\Delta t = t_B - t_A$ и $\Delta x = x_B - x_A$.

Let the events A and B be simultaneous from the point of view of an observer in inertial reference frame K, id est $\Delta t = 0$. Then, for any observer from the inertial reference frame K':

$$\Delta t' = -\Delta x \frac{\gamma V}{c^2}. \qquad (3.1.37)$$

Consequently: **events (taking place at different points), simultaneous in one inertial reference frame, are not simultaneous in all other IRF.**

Note that at the same time *causality principle* is not violated.

3.2 Dynamics of the special theory of relativity

3.2.1 The basic dynamic characteristics

The concept of force in SRT

In classical mechanics, Newton's law determines the force $\vec{F} = m\vec{a} = d\vec{p}/dt$, where $\vec{p}$ – particle momentum. In SRT force definition remains unchanged:

$$\vec{F} = \frac{d\vec{p}}{dt}. \qquad (3.2.1)$$

But the definition of momentum is improved by taking into account the dependence of time intervals on the choice of inertial reference frame – derivative in the definition of momentum is taken with respect to own time:

$$\vec{p} = m\frac{d\vec{r}}{d\tau} = m\gamma\frac{d\vec{r}}{dt} = m\gamma\vec{v} \qquad (3.2.2)$$

where $d\tau = dt/\gamma$.

Mass and energy in SRT

In classical mechanics it is easy to obtain the expression for kinetic energy differential with the use of its definition (2.4.6):

$$dT = d\left(\frac{mv^2}{2}\right) = \vec{v}\,d\vec{p}. \qquad (3.2.3)$$

In special theory of relativity, this definition remains unchanged (with momentum, of course, defined as in (3.2.2)):

$$dT = \vec{v}\,d\vec{p} = \vec{v}\,d(\gamma m\vec{v}). \qquad (3.2.4)$$

After some simple transformations the last formula takes the form

$$dT = d(\tilde{m}c^2), \qquad (3.2.5)$$

where $\tilde{m}$ is the notation for the so-called **relativistic mass**:

$$\tilde{m} = \gamma m. \qquad (3.2.6)$$

We present here these transformations. By definition, the derivative of the product

$$dT = \vec{v}\,d(\tilde{m}\vec{v}) = v^2 d\tilde{m} + \tilde{m}\vec{v}d\vec{v}. \qquad (3.2.7)$$

Now we divide the definition of relativistic mass $\tilde{m}$ (3.2.6) on γ and multiply on c^2:

$$\tilde{m}c^2/\gamma = mc^2. \tag{3.2.8}$$

Then we express Lorentz-factor in exact form: $\gamma = 1/\sqrt{1 - v^2/c^2}$ and find the second power of the resulting equation:

$$\tilde{m}^2 c^4 \left(1 - v^2/c^2\right) = m^2 c^4. \tag{3.2.9}$$

Expanding and dividing on c^2 we obtain:

$$\tilde{m}^2 c^2 - \tilde{m}^2 v^2 = m^2 c^2. \tag{3.2.10}$$

Let us find now differential of the last equation:

$$2 c^2 \tilde{m} \, d\tilde{m} - 2 v^2 \tilde{m} \, d\tilde{m} - 2 \tilde{m}^2 v \, dv = 0 \tag{3.2.11}$$

and after the simplifications:

$$c^2 \, d\tilde{m} - (v^2 \, d\tilde{m} + \tilde{m} v \, dv) = 0. \tag{3.2.12}$$

If we compare now equation (3.2.12) with the expression for the differential of kinetic energy dT from (3.2.7) we will obtain: $dT = c^2 d\tilde{m}$, thus we have come to equation (3.2.5) that describe kinetic energy of a free particle. Integrating it we obtain:

$$T = \int_0^v c^2 d\tilde{m} = \tilde{m}c^2 - mc^2 = mc^2(\gamma - 1). \tag{3.2.13}$$

We find now conditions under which the expression (3.2.13) transforms into the expression for the classical kinetic energy. Expanding the parameter γ in a Taylor series with respect to small parameter v^2/c^2 up to the third term:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + O(v^6/c^6), \tag{3.2.14}$$

we see that

$$T = \frac{mv^2}{2} + m\frac{3}{8}\frac{v^4}{c^2} + O(v^6/c^6), \tag{3.2.15}$$

that is, the relativistic formula for the kinetic energy (3.2.13) becomes classical one, if $(v/c)^2 \ll 1$ and the term proportional to $(v/c)^4$ can be neglected.

The quantity

$$E = \tilde{m}c^2 = T + mc^2 \tag{3.2.16}$$

is called **total mechanical energy** of a free particle. For a particle in rest ($T = 0$) total energy is $E = mc^2$ and this quantity is called **rest energy** E_0:

$$E_0 = mc^2. \tag{3.2.17}$$

Total mechanical energy could be expressed in terms of rest energy

$$E = \gamma E_0. \tag{3.2.18}$$

The relationship between the total mechanical energy of the particle and its momentum

With the notation for relativistic mass (3.2.6), the definition of momentum (3.2.2) takes the form:

$$\vec{p} = \tilde{m}\vec{v}. \tag{3.2.19}$$

We emphasize that **relativistic mass** *is the third concept of mass* (in addition to the inertial and gravitational masses), and by the physical meaning relativistic mass is just **energy** *(measured in kilograms)*.

From equations (3.2.16) and (3.2.19) it is possible to obtain relationship between momentum and total mechanical energy of a free particle. We find the second power of this equations: $E^2 = \gamma^2 m^2 c^4$, $p^2 = \gamma^2 m^2 v^2$ and minus them:

$$E^2/c^2 - p^2 = \gamma^2 m^2 c^2 (1 - v^2/c^2)$$

or

$$E^2 = c^2(m^2 c^2 + p^2). \tag{3.2.20}$$

This expression plays very important role in special theory of relativity. From this formula, in particular it follows that the rest mass m in SRT is a non-additive value – the sum of the masses of individual particles in a system is not equal to the mass of the system. Let us prove it.

Rest mass of a system of particles

Definitions (3.2.2)-(3.2.19) do not, obviously, violate additivity properties of energy and momentum. We will use it for the calculation of these

characteristics for a system of two particles. System in consideration energy and momentum are, respectively:

$$E = E_1 + E_2; \qquad \vec{p} = \vec{p}_1 + \vec{p}_2. \tag{3.2.21}$$

We square both of them:

$$E^2 = E_2^2 + 2\,E_1\,E_2 + E_1^2; \qquad \vec{p}^{\,2} = \vec{p}_1^{\,2} + 2\vec{p}_1\vec{p}_2 + \vec{p}_2^{\,2} \tag{3.2.22}$$

and use the definitions of momentum $\vec{p} = \gamma m \vec{v}$ and energy $E = \gamma m c^2$:

$$
\begin{aligned}
E^2 &= \left(\gamma_1^2 m_1^2 + \gamma_2^2 m_2^2 + 2\gamma_1\gamma_2 m_1 m_2\right) c^4; \\
\vec{p}^{\,2} &= \gamma_1^2 m_1^2 v_1^2 + \gamma_2^2 m_2^2 v_2^2 + 2\gamma_1\gamma_2 m_1 m_2 \vec{v}_1\vec{v}_2
\end{aligned}
\tag{3.2.23}
$$

where it is denoted:

$$\gamma_1 = \left(1 - \frac{v_1^2}{c^2}\right)^{-1/2}, \qquad \gamma_2 = \left(1 - \frac{v_2^2}{c^2}\right)^{-1/2}. \tag{3.2.24}$$

Now let us substitute expression (3.2.23) into equation (3.2.20): $E^2/c^2 - p^2 = m^2 c^2$. After some simplifications we obtain:

$$
\begin{aligned}
m_1^2\gamma_1^2 &\left(1 - \frac{v_1^2}{c^2}\right) c^2 + m_2^2\gamma_2^2 \left(1 - \frac{v_2^2}{c^2}\right) c^2 \\
&+ 2m_1 m_2 \gamma_1\gamma_2 \left(1 - \frac{\vec{v}_1\vec{v}_2}{c^2}\right) c^2 = m^2 c^2.
\end{aligned}
\tag{3.2.25}
$$

From where it follows:

$$m^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma_1\gamma_2 \left(1 - \frac{\vec{v}_1\vec{v}_2}{c^2}\right). \tag{3.2.26}$$

It is not so hard to verify that coefficient $\gamma_1\gamma_2 \left(1 - \frac{\vec{v}_1\vec{v}_2}{c^2}\right) \geq 1$, id est

$$\boxed{m \geq (m_1 + m_2).} \tag{3.2.27}$$

Thus, the mass of the system of particles equals to the sum of masses of the individual particles only in the case where all of the particles are at rest or move at the same speed rectilinearly in one direction. Neither of this conditions is realistic for a real system.

   This result is one of the most important practical conclusions of special relativity theory.

To prove that $\gamma_1 \gamma_2 \left(1 - \frac{\vec{v}_1 \vec{v}_2}{c^2}\right) \geq 1$ we'll use the exact form of γ_i:

$$\left(1 - \frac{\vec{v}_1 \vec{v}_2}{c^2}\right) \geq \frac{1}{\gamma_1 \gamma_2} = \sqrt{\left(1 - \frac{v_1^2}{c^2}\right)\left(1 - \frac{v_2^2}{c^2}\right)} \qquad (3.2.28)$$

Let introduce another notation: $\vec{\beta} = \vec{v}/c$, then

$$1 - \vec{\beta}_1 \vec{\beta}_2 \geq \sqrt{(1 - \beta_1^2)(1 - \beta_2^2)}.$$

square the resulting expression:

$$1 + \beta_1^2 \beta_2^2 - 2\vec{\beta}_1 \vec{\beta}_2 \geq (1 - \beta_1^2 - \beta_2^2 + \beta_1^2 \beta_2^2).$$

Obtained inequality:

$$\beta_1^2 + \beta_2^2 - 2\vec{\beta}_1 \vec{\beta}_2 = (\vec{\beta}_2 - \vec{\beta}_1)^2 \geq 0$$

is definitely truth.

3.2.2   Equations of motion in SRT

According to the definitions (3.2.1) and (3.2.2), equation of motion for a particle have the form:

$$\vec{F} = \frac{d(\gamma m \vec{v})}{dt} = m \frac{d(\gamma \vec{v})}{dt} = m\gamma \frac{d\vec{v}}{dt} + m\vec{v} \frac{d\gamma}{dt}. \qquad (3.2.29)$$

It is easy to take the derivatives and write the equations of motion through the classical dynamic variables. The derivative in the first term is the acceleration vector $\vec{a}$ (by definition). Let us take the derivative of the Lorentz factor of γ.

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = -\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left(\frac{-2v}{c^2}\frac{dv}{dt}\right) = \gamma^3 \frac{v a_\tau}{c^2},$$

where a_τ is the absolute value of tangential acceleration (see equation 1.3.5). Now we express velocity vector $\vec{v}$ in the second term 3.2.29 in terms of it absolute value v and unity vector $\vec{e}_\tau$ that tangential to the trajectory: $\vec{v} = v\vec{e}_\tau$. As the result we obtain (with the use of relativistic mass $\tilde{m} = \gamma m$ notation):

$$\vec{F} = \tilde{m}\vec{a} + \tilde{m}\vec{e}_\tau a_\tau \gamma^2 \frac{v^2}{c^2} = \tilde{m}\vec{a} + \tilde{m}\vec{a}_\tau \left(\frac{\gamma v}{c}\right)^2 \qquad (3.2.30)$$

Consequently, **force**, in general, **is not** the cause of acceleration, as in classical mechanics – ie force is not linear function of acceleration.

We consider a special cases of a body motion.

- Particle moves *linearly.* In this case $\vec{a} = \vec{a}_\tau$ and we obtain:

$$\vec{F} = \tilde{m}\vec{a}_\tau \left(1 + \frac{v^2}{c^2}\gamma^2 \right) = \tilde{m}\gamma^2 \vec{a}_\tau$$

- Particle moves *on a circle.* In this case $\vec{a} = \vec{a}_n$, $a_\tau = 0$ and we get:

$$\vec{F} = \tilde{m}\vec{a}_n$$

Thus, the **magnitude of the force** measured for the same force interaction *by different observers* – **is different**. And the magnitude of the forces measured by *each observer*, depends not only on the acceleration of the interacting objects (as in classical mechanics), but also on the speed of the interacting objects relative to the inertial reference frame of the observer.

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