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On the strength calculation of the rotating parts

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Abstract

The existing solutions of differential equations of equilibrium of an infinitesimal element of the rotating parts of an isotropic elastic solid known as the Navier equilibrium equations are considered. Examples of the flat disk calculation by solving the differential equilibrium equations by the sweep method and the finite element method in the modern program "Autodesk Simulation Multiphysics" are represented; paradoxical changes of radial and hoop stresses are revealed. An original method of derivation formulas based only on the principle of d'Alembert to calculate radial and hoop stresses in parts that operate under centrifugal (inertial) forces is proposed. The solution for rotating disks of any profile that corrects unnatural classical solutions is obtained. Analysis of the obtained new formulas for calculating stresses shows that it is necessary to reject the concept of "equal-strength disk" because of the inability to provide the equality of the hoop and radial stress in all sections of the disk. A new method of the optimum strength disk profile calculation, which requires a restriction of outer radius disk, is suggested. In designing of optimum strength rotating parts is recommended to limit outer disk radius of $0{,}8\sqrt{[\sigma]/(\rho \cdot \omega^2)}$ where $[\sigma]$ − the allowable stress, ρ − density of the disk material; ω − angular velocity of disk rotation.

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1. Introduction

Various machines such as centrifuges and separators for division or filtration of suspensions and emulsions, gas and steam turbine engines, turbochargers, etc. incorporate in their construction the rotating parts of various shapes,

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which operate under inertial (centrifugal) force and surface loads. Problem of stress calculation in a rotating thin disk was considered for the first time in 1850 by Maxwell JC in the report "On the equilibrium of elastic solids" [1]. He used the works of Hooke R., Navier C.L.M.H., Poisson S.D., Lame G., Clapeyron B.P.E. and Stokes G.G. Stokes work was especially marked:

"…his equations are identical with those of this paper, which are deduced from the two following assumptions.

In an element of an elastic solid, acted on by three pressures at right angles to one another, as long as the compressions do not pass the limits of perfect elasticity –

1st. The sum of the pressures, in three rectangular axes, is proportional to the sum of the compressions in those axes.

2nd. The difference of the pressures in two axes at right angles to one another, is proportional to the difference of the compressions in those axes".

As it is known from history of the theory of elasticity[2], and to this day[3], differential equations of equilibrium of an elastic body in the analysis of operation of rotating details are used by huge number of scientists. These equations are also included in textbooks and reference books.

For example, the rotating circular disk that side surfaces are formed by the rotation of curve z = ±*f*(*r*) about the same axis is considered in[4] (Fig. 1). Using the equations of elementary volume equilibrium, generalized Hooke's law, and neglecting the axial disk displacement, the differential equilibrium equation of second-order from the amount of disk element displacement (u) at a distance r from the axis of rotation was obtained:

$$\frac{d^2u}{dr^2}+\left(\frac{1}{r}+\frac{1}{z}\cdot\frac{dz}{dr}\right)\frac{du}{dr}+\frac{1}{r}\left(\frac{\nu}{z}\cdot\frac{dz}{dr}-\frac{1}{r}\right)u+\frac{1-\nu^2}{E}\rho\cdot\omega^2\cdot r=0 \qquad (1)$$

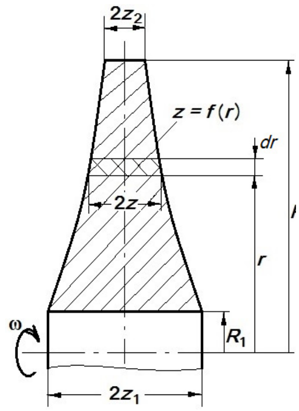

Fig. 1. Disk parameters

Analytical solution of this differential equilibrium equation can only be done for a flat disk (z = const). For a disk with a central hole and loads of p_1 and p_2, equation gives the formulas (2, 3) for calculating the hoop (σ_t) and radial (σ_r) stresses, where ρ − density of the disk material; ω - angular velocity of rotation of the disk; ν − Poisson's ratio; R_1, R_2 − hole radius and outer disk radius; r − radius of considered disk section.

$$\sigma_t = \frac{p_1 \cdot R_1^2 \cdot \left(r^2 + R_2^2\right) p_2 \cdot R_2^2 \left(r^2 + R_1^2\right)}{\left(R_2^2 - R_1^2\right) \cdot r^2} + \\ + \frac{\rho \cdot \omega^2 \cdot \left(R_2^2 - R_1^2\right) \left[(3+\mu) \cdot \left(R_1^2 \cdot R_2^2 + \left(R_1^2 + R_2^2\right) \cdot r^2\right) - (1+3\mu) \cdot r^4 \right]}{8\left(R_2^2 - R_1^2\right) \cdot r^2}, \tag{2}$$

$$\sigma_r = \frac{8 p_1 \cdot R_1^2 \cdot \left(R_2^2 - r^2\right) - \left(r^2 - R_1^2\right)\left(8 p_2 \cdot R_2^2 + (3+\mu) \cdot \rho \cdot \omega^2 \cdot \left(r^2 - R_2^2\right)\left(R_1^2 - R_2^2\right)\right)}{8 r^2 \cdot \left(R_1^2 - R_2^2\right)}. \tag{3}$$

Another original solution of this problem is given in the textbook[5]. It examines the differential equilibrium equation of disk element, which is obtained from the equation written in Lagrange form after the transformation of the volume integral into a surface integral by Gauss's formula. In cylindrical coordinates, this equation has the form:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_t}{r} + \rho \cdot \omega^2 \cdot r = 0. \tag{4}$$

To solve the equation a function F is introduced, which identically satisfies this equation. As a result, the analytical solution for the disk with a central hole and the loads of p_1 and p_2 gives the same formulae (2, 3).

Unnatural stress dependence on the current radius r, which is obtained as a result of using the received analytical formulas and numerical solution of differential equilibrium equations (1, 4) with the boundary conditions attract attention. For example, Fig. 2 shows the author-calculated dependencies for flat disk, when its inner cylindrical surface is loaded by the pressure (p_1 = 10 MPa), and an outer one has no load (p_2 = 0 MPa). The solid lines show the analytical dependencies received from solving the equation (1) by the sweep method at specified boundary conditions. Very similar results (points in Fig. 2) were obtained by disk calculation in modern program "Autodesk Simulation Multiphysics".

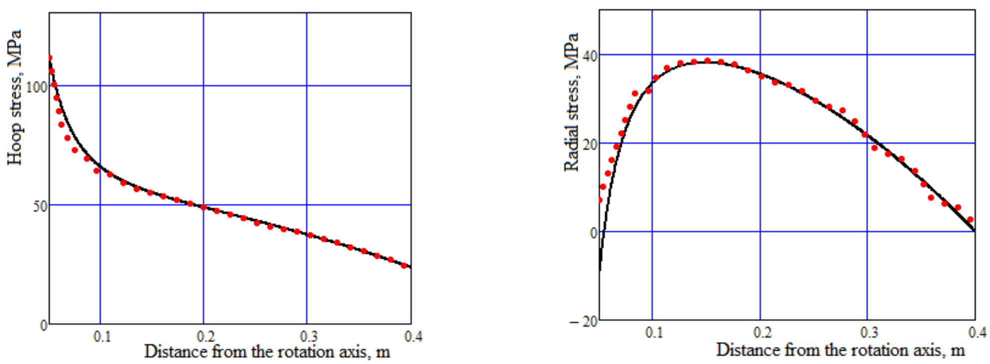

Fig. 2. The dependences of hoop and radial stresses on the current radius of the rotating flat disk with a central hole at R_1=0.05 m; R_2=0.4 m; p_1=10 MPa; p_2=0; ω=315 s^{-1}; μ=0.3; ρ=7800 kg/m^3 ; E=2·10^5 MPa

Unnatural character of the dependence of stress is as follows:

1) hoop stress reduction with increasing of current radius r at a quadratic radius dependence of centrifugal force that makes a main contribution to this stress, is contrary to mathematical logic;

2) it is difficult to give a physical interpretation marked extreme dependence of the radial stress: it grows up with increasing radius r and after reaching a certain radius value begins to decrease.

S.P. Tymoshenko in the book "Course of elasticity theory"[6] notes that the solution of the differential equilibrium equations is determined by the relationship between stresses and strains that arise in an elastic solid under the action of external and internal forces. In all the papers cited above a linear relationship defined by the generalized Hooke's law is used. However, the practice of turbine rotor operation shows that the material works outside the proportionality band despite the fluidity and creep.

Dependencies that are significantly different from those shown in Fig. 2 may be found in modern publications. For example, in[7] the total strain is considered as a sum of the elastic and plastic strain. To solve the differential equilibrium equations of disk element the standard finite discretization approach is applied. The calculation results presented in this work indicate that the hoop stress in the disk at a constant temperature increases as the distance from the axis of rotation grows. Radial stress near the hole at first sharply decreases from the maximum positive value, reaches a minimum and then slowly increases to zero.

The problem of rotating part strength calculation is especially important in the design of turbojet engines, it is devoted to a series of papers, for example[8-10]. Therefore, the determination of the stresses in rotating parts without connection with the deformations and without introducing any additional hypotheses is of current importance.

2. Materials and Methods

The equilibrium of the element of a flat disk with a central hole selected by the two planes passing through the axis of rotation at an $d\varphi$ angle to each other is considered (Fig. 3*a*). The inner and outer cylindrical surfaces of the disk are loaded with p_1 and p_2 pressures. Force projection sum on the line that bisects the $d\varphi$ angle after $d\varphi$ reduction is as follows:

$$p_1 \cdot R_1 \cdot z_1 + \int_{R_1}^{R_2} \rho \cdot \omega^2 \cdot r^2 \cdot z \cdot dr + p_2 \cdot R_2 \cdot z_2 - \int_{R_1}^{R_2} \sigma_t \cdot z \cdot dr = 0\,, \tag{5}$$

where $z = f(r)$, $z_1 = f(R_1)$ and $z_2 = f(R_2)$ - disk thickness at a distance of r, R_1, and R_2 from the axis of rotation.

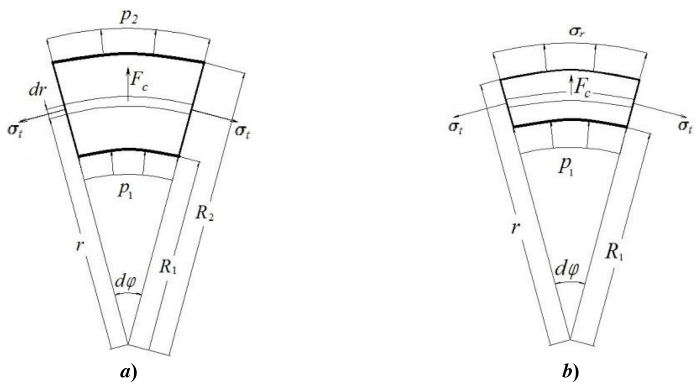

Fig. 3. On the formula derivation of hoop and radial stresses in a rotating disk with a hole

The products $p_i \cdot R_i \cdot z_i$ are represented in the form of definite integrals

$$p_i \cdot R_i \cdot z_i = \frac{p_i \cdot R_i \cdot z_i}{R_2 - R_1} \int_{R_1}^{R_2} dr$$

and substituted into the equation (5). Further, the simple definite integral property that allows getting free from the sum of integrals over the same limits is used: "If the function $f(x)$ is integrable in the interval [a; b] and $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f(x) \geq 0$". The result is a formula of hoop stress calculation:

$$\sigma_t = \frac{p_1 \cdot R_1 \cdot z_1 + p_2 \cdot R_2 \cdot z_2}{z \cdot (R_2 - R_1)} + \rho \cdot \omega^2 \cdot r^2. \quad (6)$$

The equilibrium equation of the part of disk or ring element, which is cut from the previous element of the cylindrical surface of r radius, is made in the same way Fig. 3b),

$$p_1 \cdot R_1 \cdot z_1 + \int_{R_1}^{r} \rho \cdot \omega^2 \cdot r^2 \cdot z \cdot dr + \sigma_r \cdot r \cdot z - \int_{R_1}^{r} \sigma_t \cdot z \cdot dr = 0. \quad (7)$$

The calculation of the integrals by using the obtained formula (6) followed by simplification provides the formula for the radial stress calculating:

$$\sigma_r = \frac{p_1 \cdot R_1 \cdot z_1 \cdot (R_2 - r) + p_2 \cdot R_2 \cdot z_2 \cdot (R_1 - r)}{(R_1 - R_2) \cdot r \cdot z}. \quad (8)$$

Substituting the obtained formulas (6, 8) in a known differential equation of equilibrium (4) shows that they are its alternative solution. Comparison of formulas (6, 8) and formulas (3, 4) shows that the corresponding stresses differ significantly from each other. The use of well-known overlay method, the separate calculation of stresses from the surface and bulk loads, as it was done in solving this problem, for example, by N.I. Muskhelishvili[11], provides the formulas of hoop and radial stresses that are close to the formulas (6, 8) in nature and computed values. Muskhelishvili' formulas for pipe loaded only inner and outer pressures of p_1 and p_2 sum up with the stresses of centrifugal (inertial) load in a flat disk, derived from the formulas (6, 8) at z = const and $p_1 = p_2 = 0$. Then the formulas of total stresses of flat constant thickness disk will have the form:

$$\sigma_t = \frac{p_2 R_2^2}{R_2^2 - R_1^2}\left(1 + \frac{R_1^2}{R_2^2}\right) + \frac{p_1 R_1^2}{R_2^2 - R_1^2}\left(1 + \frac{R_2^2}{r^2}\right) + \rho \cdot \omega^2 \cdot r^2, \quad (9)$$

$$\sigma_r = \frac{p_2 R_2^2}{R_2^2 - R_1^2}\left(1 - \frac{R_1^2}{r^2}\right) + \frac{p_1 R_1^2}{R_2^2 - R_1^2}\left(1 - \frac{R_2^2}{r^2}\right). \quad (10)$$

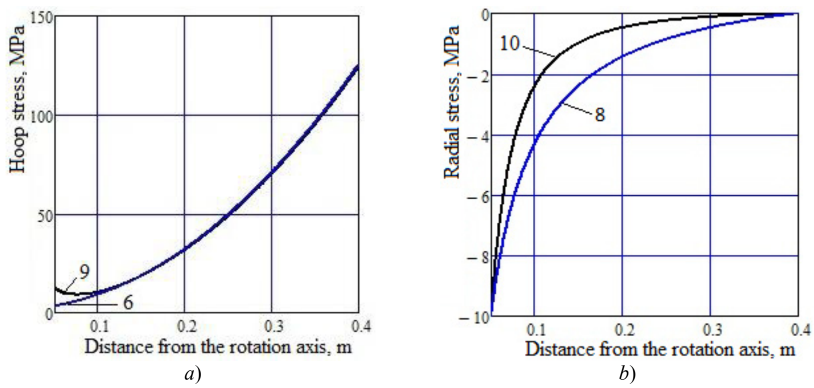

Fig. 4. The dependences of hoop and radial stresses on the current radius of the rotating flat disk with a central hole at the same parameters as in Fig. 2 calculated by the formulas 6, 9 (a) and 8, 10 (b)

Graphical comparison of the results obtained from the formulas (6, 8) and (9, 10) with the same parameters as in Fig. 2, is performed. As can be seen from the graphs shown in Fig. 4, hoop stresses, calculated by formulas (6, 9), and radial stresses calculated by formulas (8, 10), are slightly different from each other, but differ significantly from the graphs shown in Fig. 2, which are calculated by formulas if the elasticity theory.

Formulas for other profile disks which analytical solution is known to be very time-consuming[4], can be easily obtained by substituting of equation of lines $z = f(r)$, passing through two given points (R_1, z_1) and (R_2, z_2), in formulas (6, 8).

In addition, in the technical literature[12] there are the concepts of “equal strength disk” and “optimum strength disk”. In the equal strength disk the values of hoop and radial stresses are assumed equal in absolute value at all points. However, as follows from (6, 8), such a disk cannot exist. In case of equality of absolute stress values $|\sigma_t| = |\sigma_r|$ the following dependences of disk thickness on the current radius r of section under consideration.

$$f_1(r) = \frac{R_1 \cdot R_2 (p_1 \cdot z_1 + p_2 \cdot z_2)}{r^3 (R_1 - R_2) \rho \cdot \omega^2}; \; f_2(r) = \frac{p_1 \cdot R_1 (2r - R_2) z_1 + p_2 \cdot R_2 (2r - R_1) z_2}{r^3 (R_1 - R_2) \rho \cdot \omega^2} \; .$$

All the values included in the resulting expressions have a positive signs. Therefore, the values of functions $f_1(r)$ and $f_2(r)$ are negative, but it has no practical sense. In optimum strength disk the equality of allowable stress $[\sigma]$ to the maximum stress is taken to provide the minimum of disk weight at any section.

Hoop stress or the value of von Mises criterion computed for plane stress by the formula $\sigma_M = \sqrt{\sigma_t^2 + \sigma_r^2 - \sigma_t \sigma_r}$ can be the maximum stress. Implementation of these equations is possible only under certain ratio of thicknesses z_1, z_2 and loads p_1, p_2 for given values of radii R_1 and R_2. For example, the values of z_1 and p_1 are assigned, two expressions for value z_2 at the radii R_1 and R_2 is determined, they are taken to be equal to each other, the values of p_2 and z_2 in each case are calculated.

After that, all sizes and loads are known, the equality $\sigma_t = [\sigma]$ or $\sigma_M = [\sigma]$ allow finding an expression for disk thickness dependence on the arbitrary radius value $z = f(r)$. These calculations are quite cumbersome, therefore only an example of the graphs of functions obtained for both of the cases is given in Fig. 5, where p_{21} and p_{22} are the values of load p_2 calculated at the radius R_2=0.4 m, $\sigma_t = [\sigma]$ and $\sigma_M = [\sigma]$, respectively.

It should be noted that the disc thickness tends to infinity when the radius of the disc is approached to $R_{max} = \sqrt{[\sigma]/(\rho \cdot \omega^2)}$. It is obvious that it should be limited by a disk radius of 0,8R_{max} at the designing.

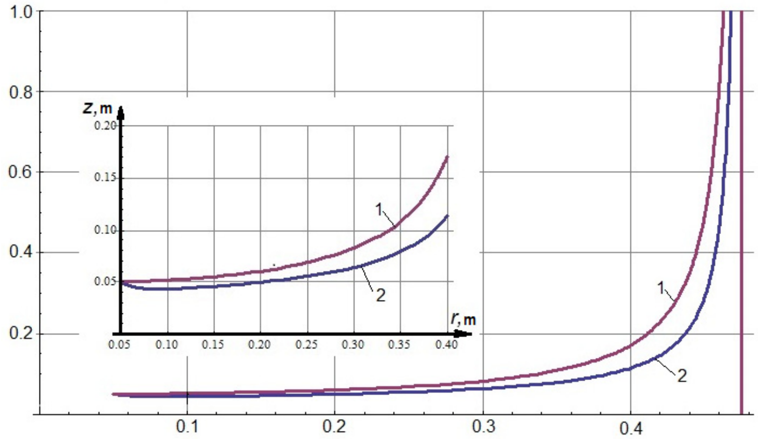

Fig. 5. The dependence of optimum disk thickness on the radius calculated on the maximum power (1) and by the criterion of von Mises (2) at ρ = 7800 kg/m3;" ω=320 s^{-1}; R1=0.05 m; R2=0.4 m; z1=0.05 m; [σ]=180 MPa; p1=10 MPa; p21=45.3 MPa; p_{22}=67.1 MPa

3. Results and Discussion

It has already been noted that the classical solution of the differential equations of equilibrium uses a linear dependence, which is determined by the generalized Hooke's law, despite the fact that the material of rotating parts can operate outside the limits of proportionality. In the above derivation method of integral equilibrium equation (5, 7), the principle of d'Alembert is only used and any assumptions are missing. Therefore, the formula (6, 8) received by this method for calculating radial and hoop stresses in rotating parts will give more accurate normal stress values. In addition, the validity of the formulas (6, 8) is confirmed by turbine collapse mode: sleeve remains intact and the whole periphery is destroyed. Classical hoop stress dependence excludes such destruction.

The reason of the paradoxical dependences of the radial and hoop stresses seems to be the transformation of volume forces into the surface forces in the derivation of Navier equations using Gauss's formula. This assumption is confirmed by the derivation of the formulas (9, 10) made by the overlay method when the bulk and surface strength are separately considered. The solutions obtained do not differ significantly from the formulas (6, 8). Both decisions are made without the use of laws and hypotheses related to the movements. Therefore, they give reason to search for another approach to obtain the alternative solutions of differential Navier-Stokes equations in other theories.

4. Conclusions

1. An original way to derive the formulas for calculating the radial and hoop stresses in parts that operate under the influence of centrifugal (inertial) forces was proposed. It relies only on the principle of d'Alembert.
2. New approach of V.M. Belyaev to derive the radial and hoop stresses in rotating parts corrects the paradoxical nature of obtained dependencies and allows increasing the reliability of engineering methods of strength calculation such dangerous to operate rotating parts as turbines of gas and steam engines, rotors of centrifuges and separators.
3. Analysis of the causes of paradoxical nature of classical solutions of differential Navier equations in the elasticity theory in the presence of internal forces gives the reason of the change in approach to solve differential Navier-Stokes equations in other theories and make adjustments in engineering methods of strength calculation of rotating machine parts that are dangerous to operate.

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References

1 The Scientific Papers of James Clerk Maxwell, Edited by W. D. NIVEN, M.A., F.R.S Two Volumes Bound As One, Dover Publications, INC., New York., 1965. 607 p.
2 Timoshenko S.P., The history of the science of strength of materials with a summary of the history of the theory of elasticity and theory of structures. M.: *Gos. izdatelstvo tekhniko-teoreticheskoj literatury*, 1957. 536 p. (in Russain)
3 Hu, Shaowei Study on the elastic–plastic interface and large deformation of axisymmetric disks under rotating status. *J. Applied Clay Science*. 2013; **79:** 41–48.
4 Kantorovich Z.B., Basis of calculation of chemical machinery and apparatus. *M.: Mashinostroenie*, 1960. 744 p. (in Russain)
5 Rabotnov Yu.N., *Mechanics of Solids*. M.: Nauka. Gl. red. fiz.-mat. lit. 1988; 712 p. (in Russain)
6 Timoshenko S.P., *Course of the theory of elasticity*. Kiev: Naukova dumka.1972; 508 p. (in Russain)
7 Sharma J.N., Sharma D., Kumar S. Analysis of Stresses and Strains in a Rotating Homogeneous Thermoelastic Circular Disk by using Finite Element Method. *International Journal of Computer Applications*. 2011; **35**:13.
8 Weiss H.J. and Prager W. The bursting speed of a rotating plastic disc. Graduate division of applied mathematics. Brown University, Providence, R.I., September, 1953.
9 Mazière M., Besson J., Foresta S., Tanguy B., Chalons H., Vogelb F. Overspeed burst of elastoviscoplastic rotating disk – Part I: Analytical and numerical stability analysess. *European Journal of Mechanics A/Solids*. 2009; **28:** 36–44.
10 Mazière M., Besson J., Foresta S., Tanguy B., Chalons H., Vogelb F. Overspeed burst of elastoviscoplastic rotating disks: Part II – Burst of a superalloy turbine disk. *European Journal of Mechanics A/Solids*. 2009; **28:** 428–432.
11 Muskhelishvili N.I. *Some basic problems of mathematical theory of elasticity*. M.: Nauka, 1966; 708p. (in Russain)
12 Demyanushko I. V., Birger I. A., *Stress calculation of rotating disks*. M.: Mashinostroenie, 1978; 247p. (in Russain)