REPAIRABLE SYSTEMS

Maintenance defines the set of actions performed on the item to *retain it in* or to *restore it to* a specified state.

Thus, maintenance deals with

- *preventive maintenance*, carried out at predetermined intervals, e.g. to reduce wear-out failures,
- *corrective maintenance*, carried out at failure and intended to bring the item to a state in which it can perform the required function.

The goal of a preventive maintenance must also be to *detect & repair hidden failures & defects* (e.g. undetected failures in redundant elements).

Corrective maintenance, also known as repair, includes *detection, localization, correction, and checkout*.

To simplify the calculations, it is generally assumed that the item for which a maintenance action has been performed, is *as-good-as-new* after maintenance.

It is also assumed that each failed item in the multicomponent systems is repaired by individual repairman, i.e. all failed items are repaired simultaneously.

Item's time to repair is regarded as a continuous random variable.

The mean of the repair time is denoted by *MTTR (mean time to repair (restoration)).*

Just like failure probability in non-repairable systems, the probability of repair grows with time.

We will further assume that time to repair is an exponentially distributed random variable, which means that *repair rate* – a measure analogous to the hazard rate – is a constant number, μ.

Thus, $MTTR = \frac{1}{\mu}$.

Availability is a broad term, expressing the ratio of delivered to expected service.

It is often designated by *A* and used for the asymptotic & steady-state value of the *point availability*.

Point availability ($PA(t)$) is a characteristic of the item expressed by the probability that the item will perform its required function under given conditions at a stated instant of time t.

Availability calculation is often difficult, as *human aspects & logistic support* have to be considered. Ideal human aspects & logistic support are often assumed, yielding to the *intrinsic availability*.

Further assumptions for calculations are continuous operation and complete renewal of the repaired element.
In this case, the point availability of the one-item structure rapidly converges to an asymptotic & steady-state value, given by

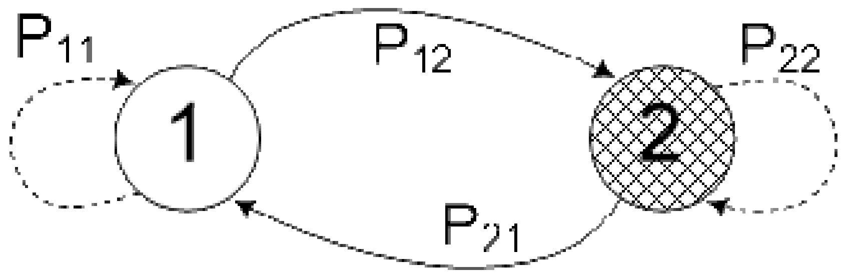
$$A = \frac{MTTF}{MTTF + MTTR}.$$

Ex.: Consider a single-item system, assuming that the following corrective maintenance strategy is adopted:

When the system fails, a repair action is initiated to bring it back to its initial functioning state. After the repair is completed, the system is assumed to be as-good-as-new.

We also assume that the system has constant hazard rate λ and constant repair rate μ.

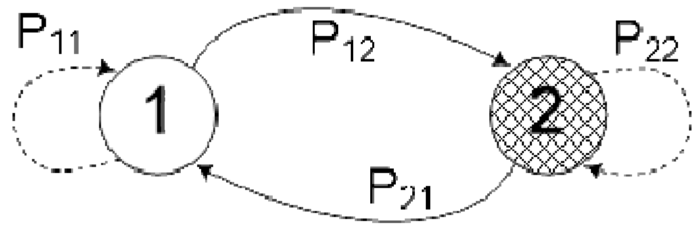
To evaluate the point availability of the system, a space-state method should be exploited. In order to do that, we need to compose a graph – a *state-transition diagram*.



In the diagram <u>circles represent</u> system's <u>states</u> and <u>arcs correspond to transitions</u> from one state to another.

Repairable Systems

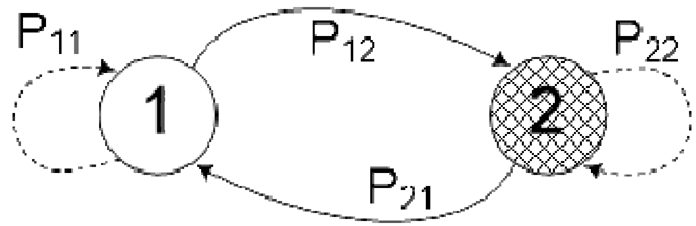
We denote an operable state as a state 1, and a failed state as 2.



At any single moment there is a probability that the system would transit from its current state to another one (P_{12} and P_{21}), and, also, that the system remains in its current state (P_{11} and P_{22}).

By definition, P_{12} is the instantaneous failure probability, and P_{21} – instantaneous repair probability.

Repairable Systems



Further calculations do not involve probabilities P_{ii}, so we won't draw loops in graphs anymore.

For any given state *i* the sum of P_{ij} adds up to 1.
State probabilities are obtained then by the following system of equations:

$$\begin{cases} P_1(t+\Delta t) = P_1(t)\cdot[1-P_{12}(\Delta t)] + P_2(t)\cdot P_{21}(\Delta t) \\ P_2(t+\Delta t) = P_2(t)\cdot[1-P_{21}(\Delta t)] + P_1(t)\cdot P_{12}(\Delta t) \end{cases}$$

In order to simplify the explanation, we will refer to the moment *t* as to "now", and to the moment *t+Δt* as to "next moment".

Then, the first equation is explained as follows:

$$P_1(t+\Delta t) = P_1(t)\cdot[1-P_{12}(\Delta t)] + P_2(t)\cdot P_{21}(\Delta t)$$

The probability that the system will find itself in the next moment in state 1 is equal to…

… the probability that it is already in state 1 and it won't transit to state 2 during time interval *Δt*…

… plus the probability that now the system is in state 2 and it will transit to state 1 during time interval *Δt*.

As Δt approaches to 0, $P_{ij}(\Delta t) \to \gamma_{ij}\Delta t$, where γ_{ij} is the transient rate.

Using the definition of derivative we obtain:

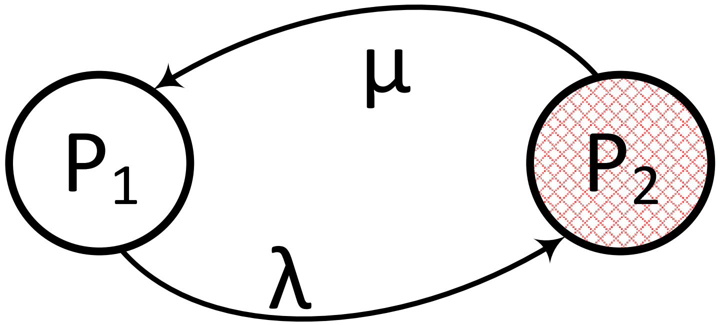
$$\begin{cases} P_1'(t) = -\lambda P_1(t) + \mu P_2(t) \\ P_2'(t) = \ \ \lambda P_1(t) - \mu P_2(t) \end{cases}$$

since $\gamma_{12} = \lambda$ and $\gamma_{21} = \mu$.

We also can obtain such a system directly from the state-transition diagram

Repairable Systems

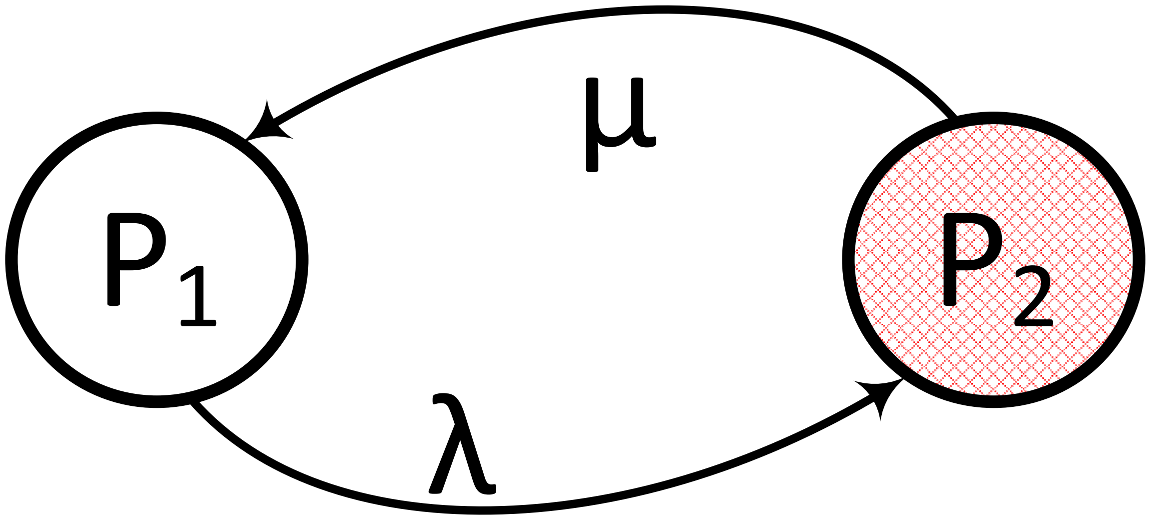
For each state in the diagram we have an equation with the derivative of the state probability on the left.



On the **right side** we have *as many terms as the number of arcs adjacent* to a current state.

Each arc is represented by its rate times the probability of the originate state.

The terms corresponding to incoming arcs go with "+" sign, outgoing arcs – with "-".



$$\begin{cases} P_1'(t) = -\lambda P_1(t) + \mu P_2(t) \\ P_2'(t) = \ \lambda P_1(t) - \mu P_2(t) \end{cases}$$

$$\begin{cases} P_1'(t) = -\lambda P_1(t) + \mu P_2(t) \\ P_2'(t) = \ \lambda P_1(t) - \mu P_2(t) \end{cases}$$

To solve this system we can use Laplace transform method, taking into account that

$$P(t) \overset{\mathcal{L}}{\Rightarrow} P(s)$$

$$P'(t) \overset{\mathcal{L}}{\Rightarrow} sP(s) - P(0^+)$$

Assuming that at time 0 the system is in state 1, we get:

$$\begin{cases} sP_1(s) = -\lambda P_1(s) + \mu P_2(s) + 1 \\ sP_2(s) = \lambda P_1(s) - \mu P_2(s) + 0 \end{cases}$$

$$\begin{cases} sP_1(s) = -\lambda P_1(s) + \mu P_2(s) + 1 \\ sP_2(s) = \lambda P_1(s) - \mu P_2(s) \end{cases}$$

From the second equation:

$$P_2(s) = \frac{\lambda}{s+\mu} P_1(s)$$

Substituting this result into the first equation, we get:

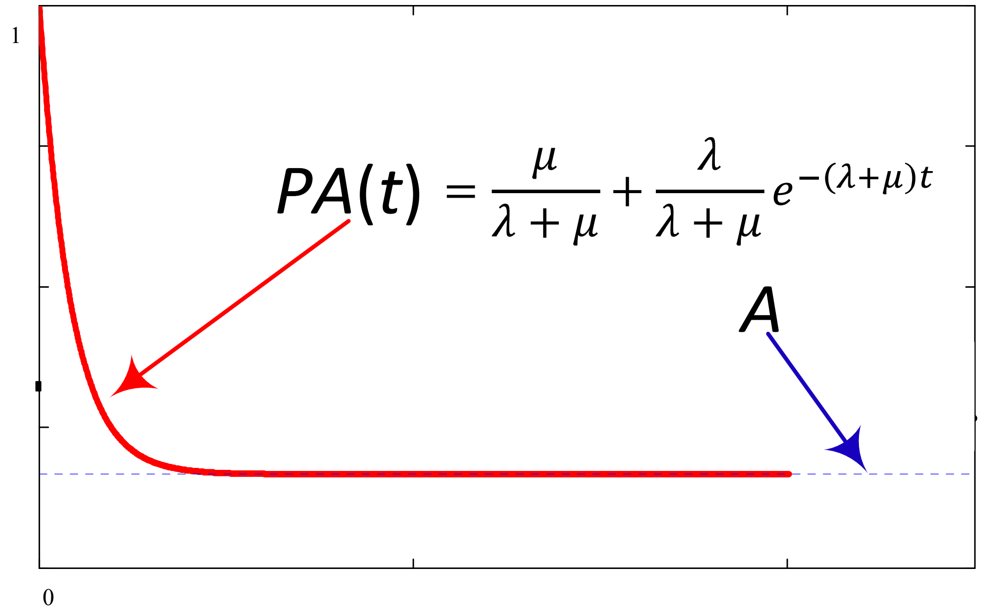
$$P_1(s) = \frac{s+\mu}{s(s+\lambda+\mu)} \qquad P_2(s) = \frac{\lambda}{s(s+\lambda+\mu)}$$

$$\begin{cases} sP_1(s) = -\lambda P_1(s) + \mu P_2(s) + 1 \\ sP_2(s) = \lambda P_1(s) - \mu P_2(s) \end{cases}$$

Inverse Laplace transform allows us to return into time domain:

$$P_1(t) = \mathcal{L}^{-1}\{P_1(s)\} = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

$P_1(t)$ is the probability that the system is in operable state, which, by definition, is the point availability $PA(t)$ of the repairable system.



As $\frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$ rapidly tapers down to 0, $PA(t)$ approaches the value

$$A = \frac{\mu}{\lambda+\mu} = \frac{MTTF}{MTTF + MTTR}$$

Since usually $\mu \gg \lambda$, $PA(t)$ reaches its steady-state value very fast. So, instead of solving differential equations, we can obtain a steady-state solution a lot easier.

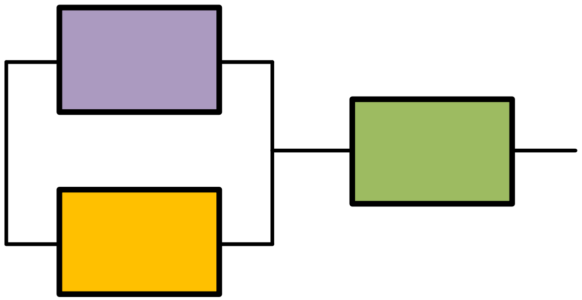
To achieve this, we *replace all derivatives with zeros* and supplement the system with an *additional equation*:

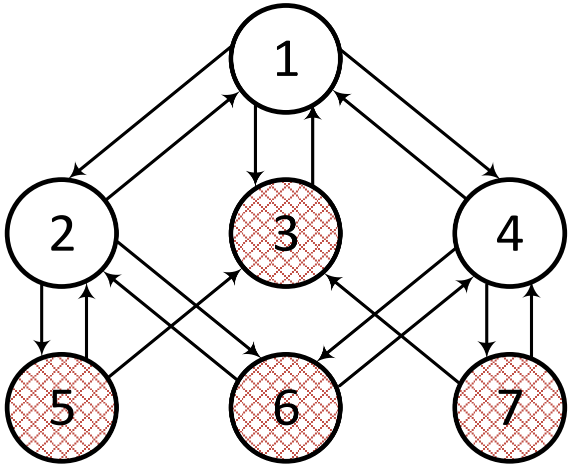
$$\begin{cases} 0 = -\lambda P_1 + \mu P_2 \\ 0 = \lambda P_1 - \mu P_2 \\ P_1 + P_2 = 1 \end{cases}$$

Solving this, we obtain $A = P_1$.

Repairable Systems

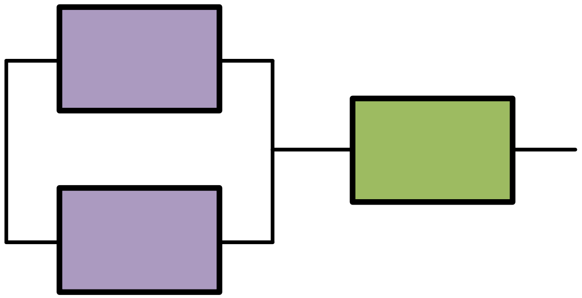
The major drawback of the state-space method is an avalanche-like increase of the number of states (and in turn, the number of equations) with increase of system complexity (number of components).

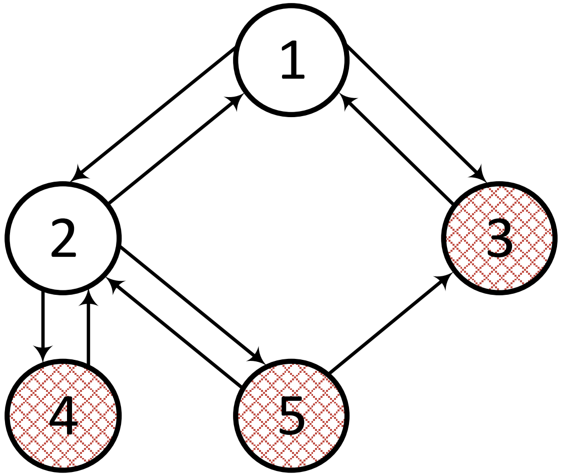




Sometimes the number of states *can be reduced*.

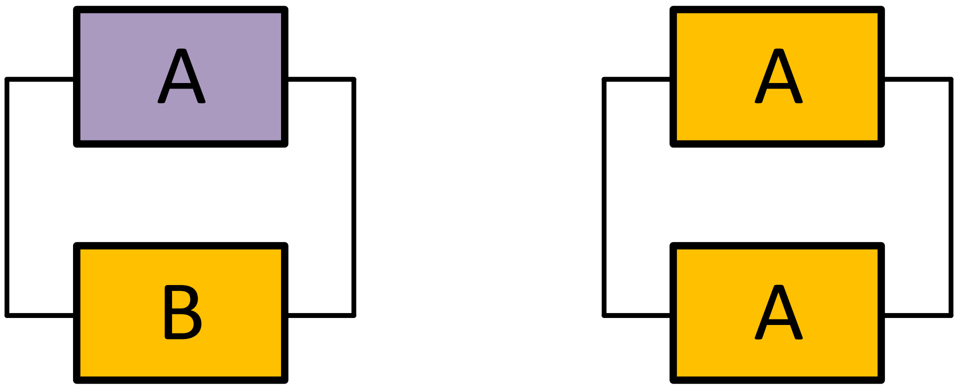
Usually, it is possible if components have the *same failure and/or repair rate* (which is common in parallel configurations).



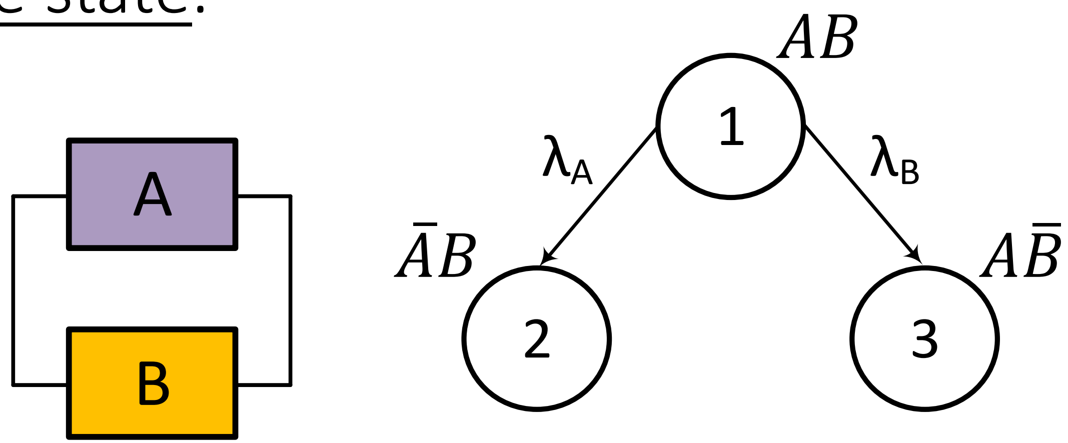


Ex.: Consider a system of 2 components in parallel (hot redundancy). Draw the state-transition diagram for 2 different cases:

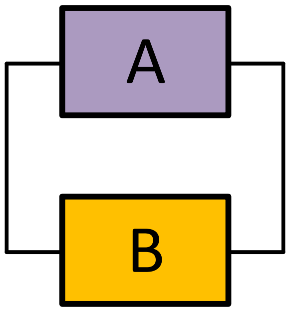
- both components have different failure and repair rates;
- both components are identical.

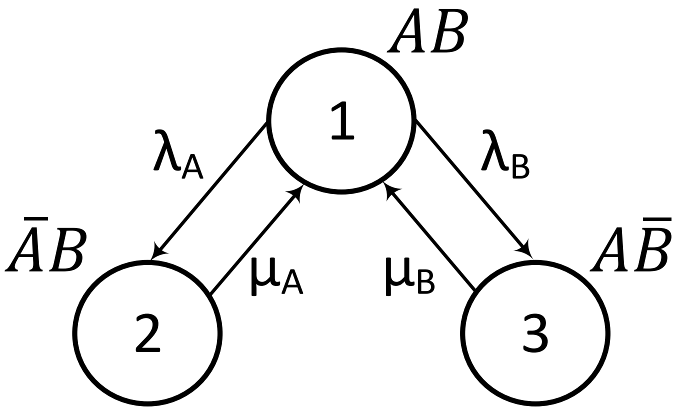


Starting from the initial state 1 in the first case, two events may happen: either component A or component B may fail (states 2 and 3). Those events are not the critical ones – the system is still in operable state.

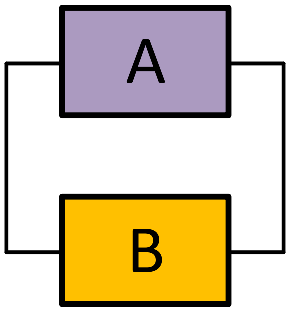


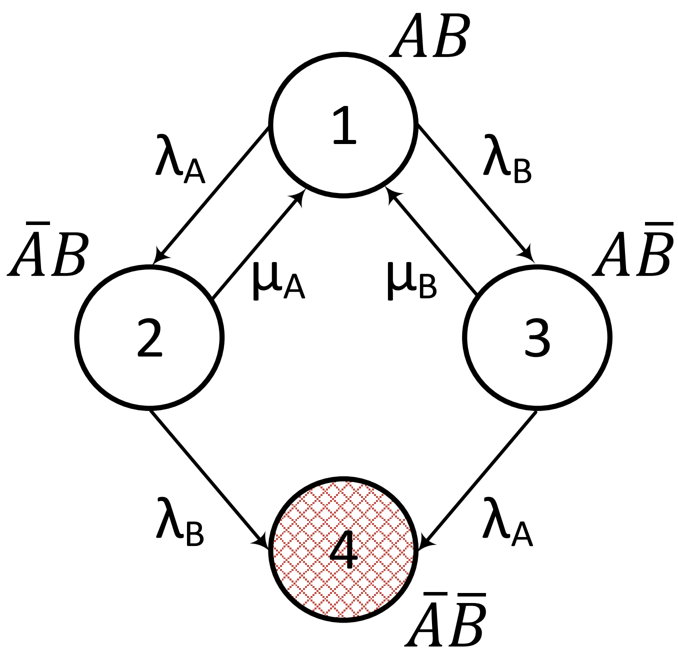
Failed components may be repaired, which return the system into the initial state.



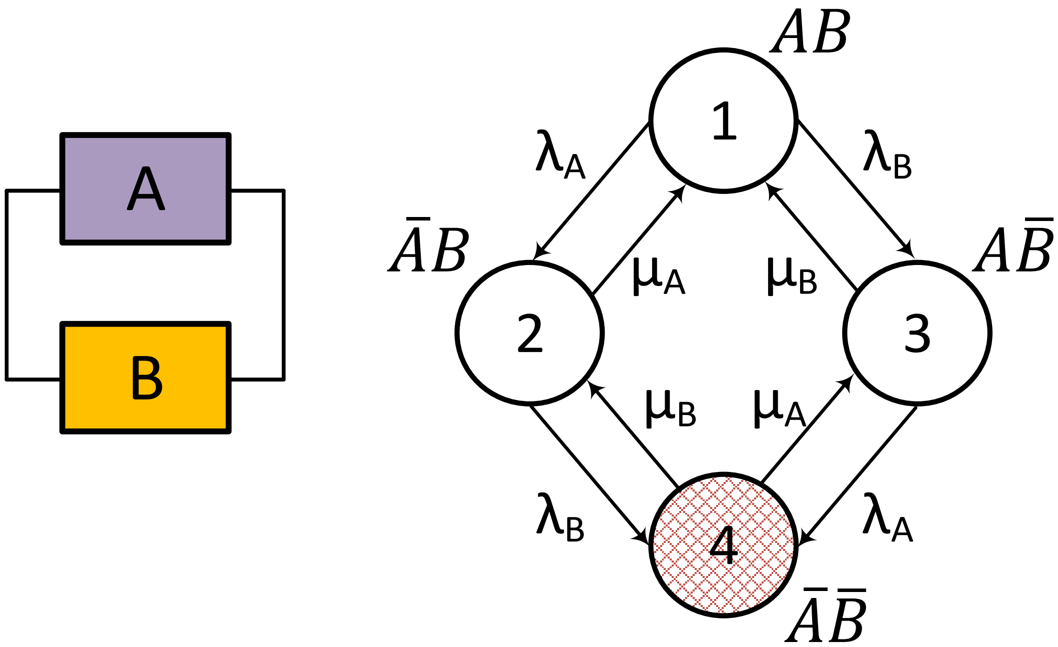


Components still operating in states 2 and 3 may fail as well, thus causing system failure (state 4).

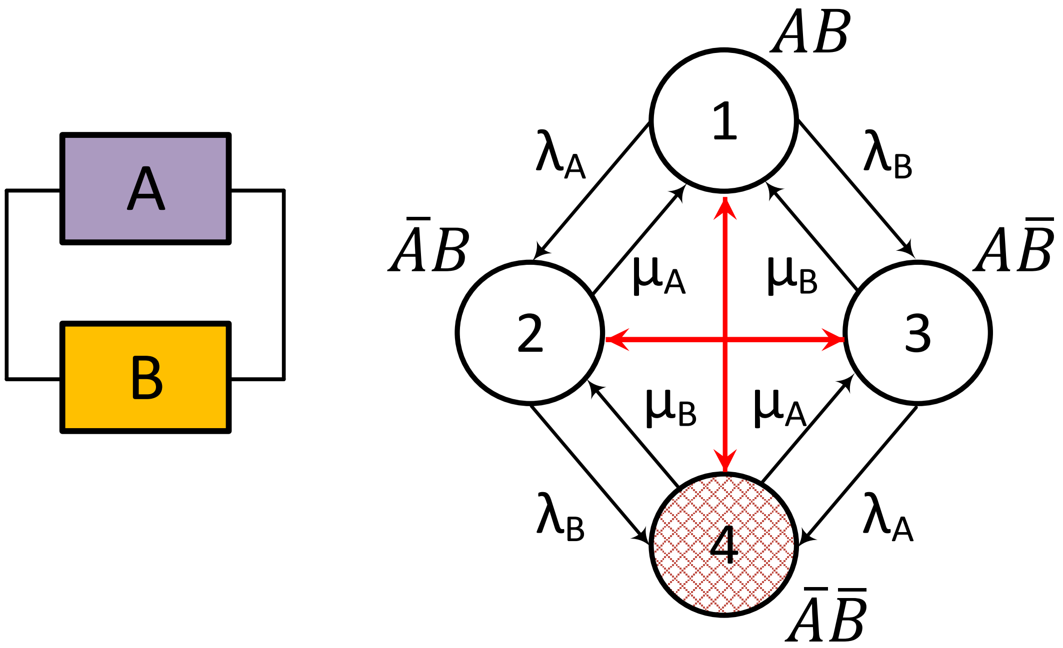




Finally, failed components from state 4 may be repaired, returning the system into states 2 or 3.

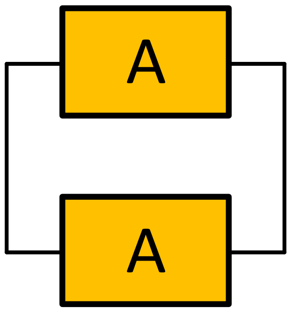


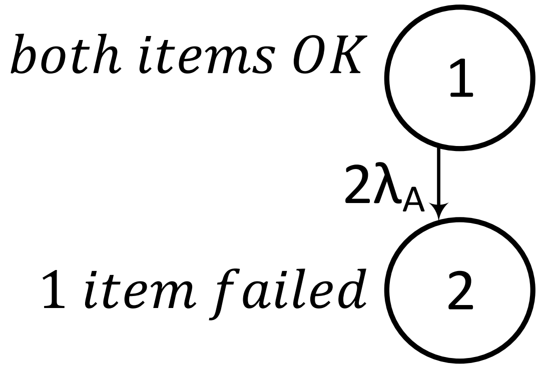
Note that each arc corresponds to a single failure/repair event – no two events can occur simultaneously!



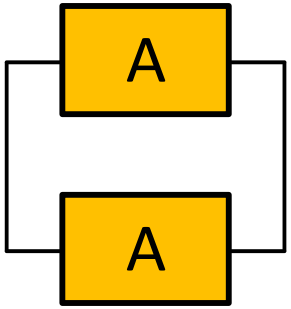
Since the components in the second case are statistically identical, it is irrelevant which item fails first – the consequences for the system are the same.

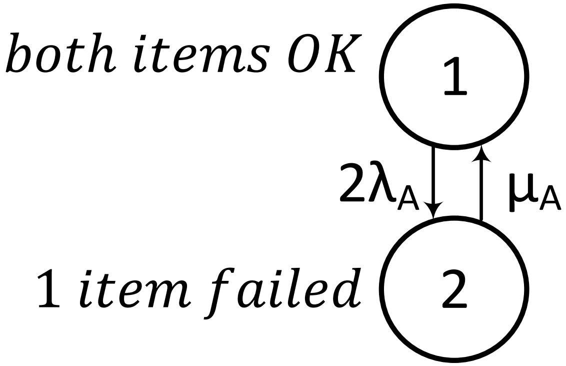
The transition rate is $2\lambda_A$ which indicates joint "effort" of both items to transfer the system into state 2.



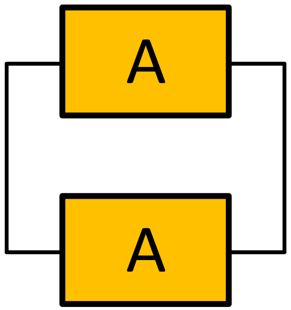


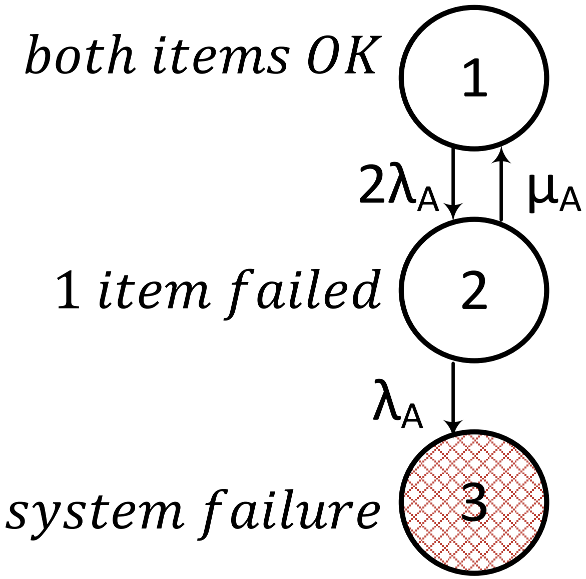
However, only one failed item is under repair in state 2, so the rate of reverse transition is still μ_A.



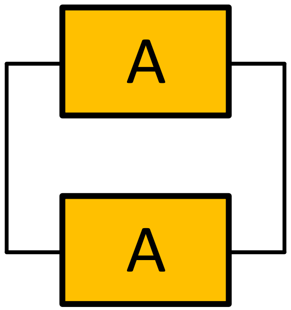


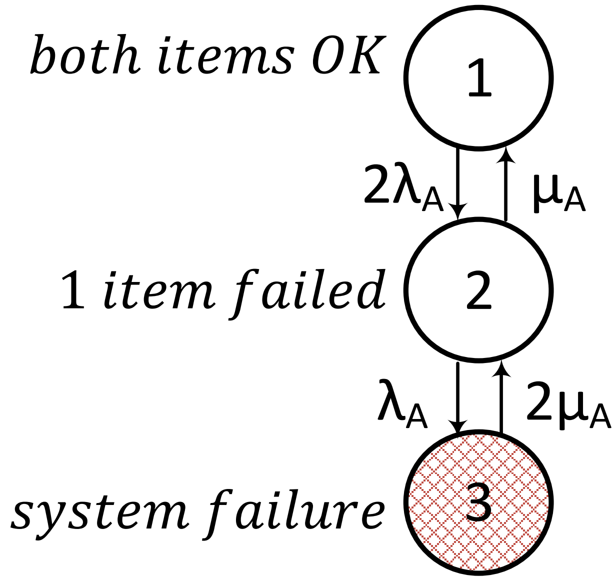
State 2 still has one operable component that may fail, resulting in system failure.





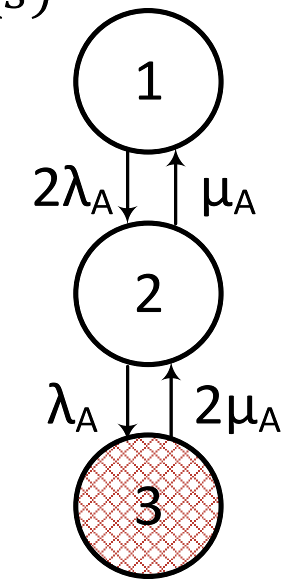
State 3 has two failed components, each being repaired individually by separate repairman. So, the rate of joint "effort" of returning back to state 2 is $2\mu_A$.





To find the point availability of the system we can perform the computation manually, which is rather tedious.

$$\begin{cases} sP_1(s) = -2\lambda_A P_1(s) + \mu_A P_2(s) + 1 \\ sP_2(s) = 2\lambda_A P_1(s) - (\lambda_A + \mu_A)P_2(s) + 2\mu_A P_3(s) \\ sP_3(s) = \lambda_A P_2(s) - 2\mu_A P_3(s) \end{cases}$$



Here, $PA(t) = P_1(t) + P_2(t) = 1 - P_3(t)$

Alternatively, we can resort to a mathematical software:

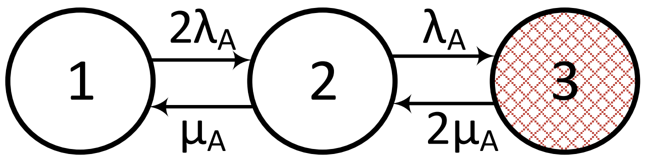
$$A(s) := \begin{pmatrix} s + 2\lambda & -\mu & 0 \\ -2\lambda & s + \lambda + \mu & -2\mu \\ 0 & -\lambda & s + 2\mu \end{pmatrix} \qquad B := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad P(s) := A(s)^{-1} \cdot B$$

$$P(s) \;\Big|\; \begin{matrix} \text{simplify} \\ \text{factor} \end{matrix} \rightarrow \begin{bmatrix} \dfrac{2\cdot\mu^2 + 3\cdot\mu\cdot s + s^2 + \lambda\cdot s}{s\cdot(2\cdot\mu + 2\cdot\lambda + s)\cdot(\mu + \lambda + s)} \\ \dfrac{2\cdot\lambda\cdot(2\cdot\mu + s)}{s\cdot(2\cdot\mu + 2\cdot\lambda + s)\cdot(\mu + \lambda + s)} \\ \dfrac{2\cdot\lambda^2}{s\cdot(2\cdot\mu + 2\cdot\lambda + s)\cdot(\mu + \lambda + s)} \end{bmatrix} \begin{matrix} P_1(s) \\ P_2(s) \\ P_3(s) \end{matrix}$$

However, if our goal is only to find the steady-state availability, we can achieve it easily with the following equations:

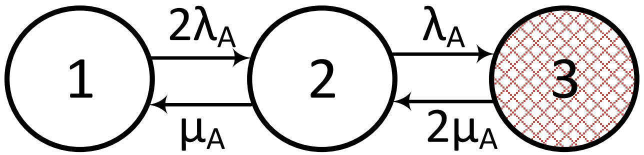
$$P_i = \frac{\lambda_{i-1,i}}{\mu_{i,i-1}} P_{i-1} \qquad P_i = \frac{\prod_{j=1}^{i-1} \lambda_{j,j+1}}{\prod_{j=1}^{i-1} \mu_{j+1,j}} P_1 \qquad \sum P_i = 1$$

$$P_2 = \frac{2\lambda_A}{\mu_A} P_1$$



$$P_3 = \frac{\lambda_A}{2\mu_A} P_2 = \frac{2\lambda_A \cdot \lambda_A}{2\mu_A \cdot \mu_A} P_1 = \frac{\lambda_A^2}{\mu_A^2} P_1$$

Please, note that this technique is applied only in cases, when state transition diagram is represented as a *birth-death process*.



$$P_1\left(1+\frac{2\lambda_A}{\mu_A}+\frac{\lambda_A^2}{\mu_A^2}\right)=1 \Rightarrow P_1=\frac{\mu_A^2}{(\lambda_A+\mu_A)^2}, P_2=\frac{2\lambda_A\cdot\mu_A}{(\lambda_A+\mu_A)^2}$$

$$A=P_1+P_2=\frac{\mu_A(\mu_A+2\lambda_A)}{(\lambda_A+\mu_A)^2}$$

Naturally, the state-space method can be applied for non-repairable systems as well.

However, this approach is appropriate only in the case of complex redundancy configuration, not covered by previously introduced formulae.

Moreover, the state-space method allows us to find the *MTTF* of repairable systems.

Let’s compare the following equations:

$$T = \int_{0}^{\infty} P(t)dt \qquad P(s) = \int_{0}^{\infty} P(t)e^{-st}dt$$

The left one is the formula for the *MTTF* of non-repairable system where reliability function is regarded as the probability of system being in operable state.

The right equation is the *Laplace transform*.

$$T = \int_0^{\infty} P(t)dt \qquad P(s) = \int_0^{\infty} P(t)e^{-st}dt$$

It's obvious that $P(s = 0) = T$.

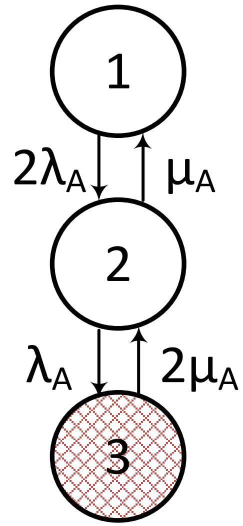
The *mean sojourn time* (*mean waiting time*) for a certain state in the state-transition diagram is evaluated as a Laplace transform of the probability of being in that state, assuming $s = 0$.

The algorithm for the MTTF calculation is as follows:

1. Obtain the system of equations in the form of Laplace transform.
2. Exclude from the system the equations which describe inoperable states.
3. Exclude from the remaining equations all terms which contain the probabilities of inoperable states.
4. Replace $P_i(s)$ with T_i.
5. Set $s = 0$.
6. Solve for T_i.
7. Find $\sum T_i$.

To provide an example, we consider the previous system - hot redundancy with two identical components:

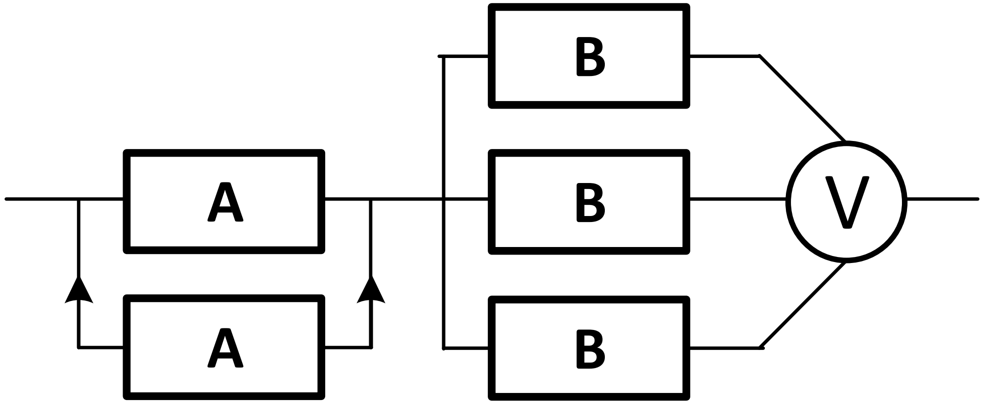
$$\begin{cases} sP_1(s) = -2\lambda_A P_1(s) + \mu_A P_2(s) + 1 \\ sP_2(s) = 2\lambda_A P_1(s) - (\lambda_A + \mu_A) P_2(s) + 2\mu_A P_3(s) \\ sP_3(s) = \lambda_A P_2(s) - 2\mu_A P_3(s) \end{cases}$$



$$\begin{cases} 0 = -2\lambda_A T_1 + \mu_A T_2 + 1 \\ 0 = 2\lambda_A T_1 - (\lambda_A + \mu_A) T_2 \end{cases} \qquad \begin{cases} 2\lambda_A T_1 - \mu_A T_2 = 1 \\ (\lambda_A + \mu_A) T_2 = 2\lambda_A T_1 \end{cases}$$

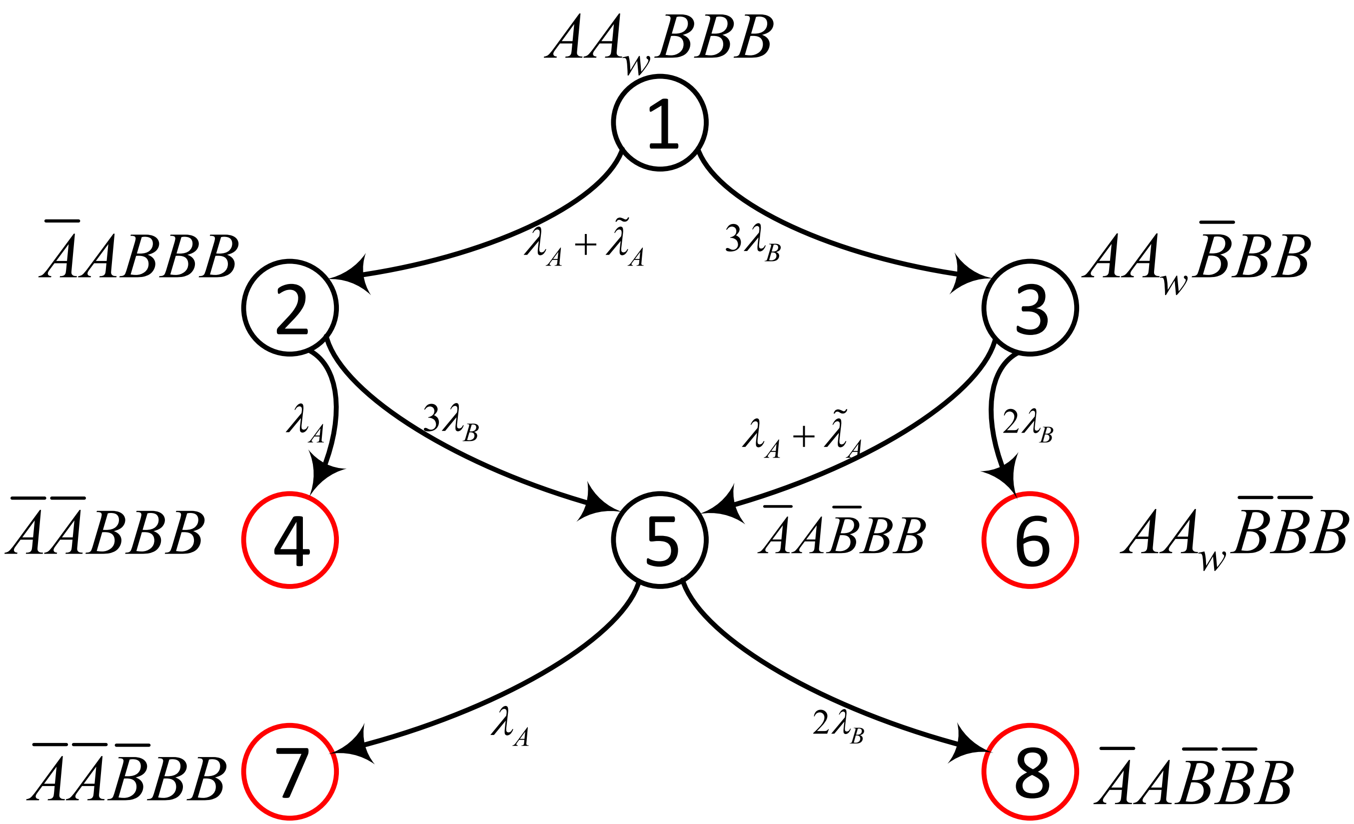
$$\begin{cases} T_2 = \dfrac{2\lambda_A}{\lambda_A + \mu_A} T_1 \\ 2\lambda_A T_1 - \mu_A T_2 = 1 \end{cases} \qquad \begin{cases} T_1 = \dfrac{\lambda_A + \mu_A}{2\lambda_A^2} \\ T_2 = \dfrac{1}{\lambda_A} \end{cases} \qquad T = T_1 + T_2 = \frac{3\lambda_A + \mu_A}{2\lambda_A^2}$$

Ex.: The system consists of 2 subsystems in series.
The first one has 2 identical components A with a warm standby redundancy.
The second one is a 2-out-of-3 system with identical components B.

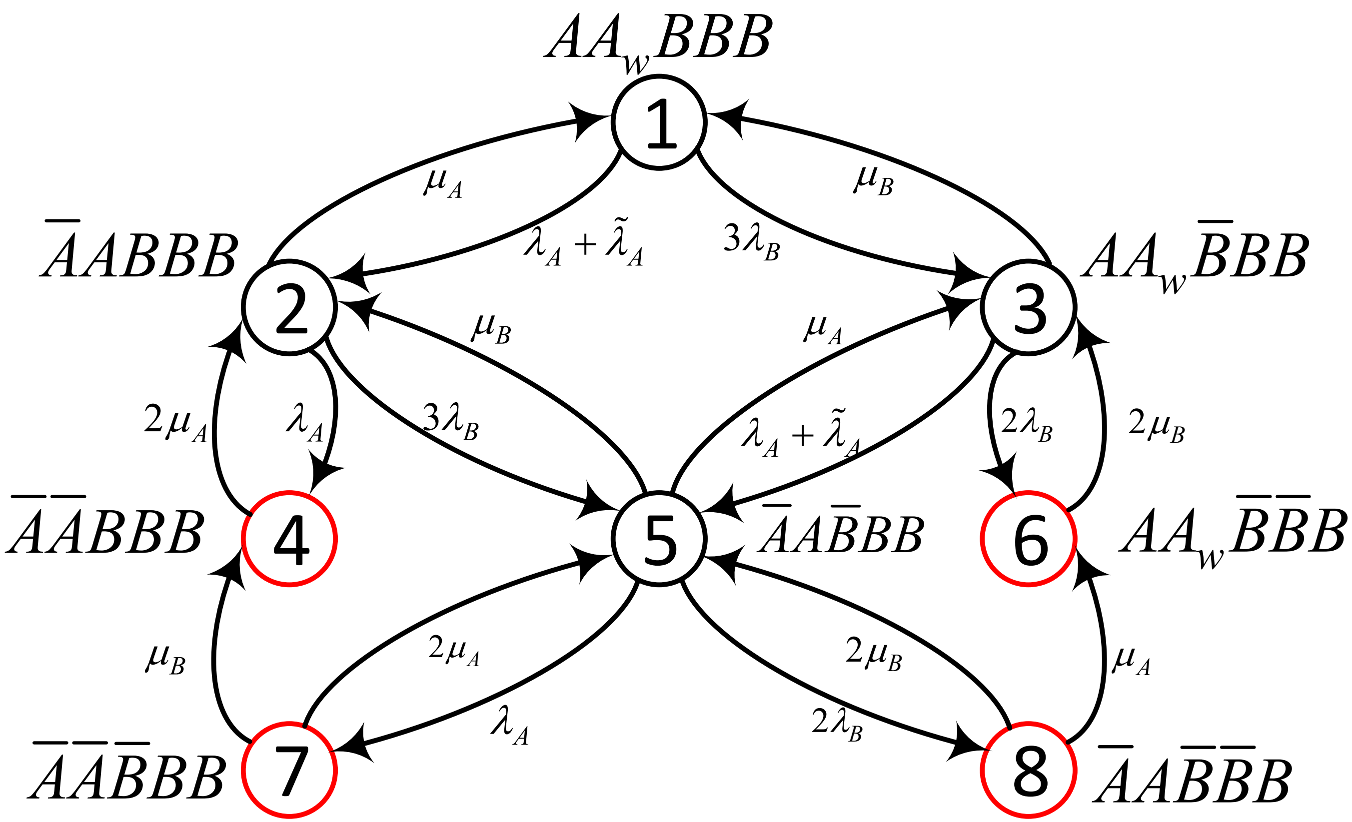


Draw a state-transition diagram for the system.

At first, let's draw transitions representing failures:



Now, let's complete the graph with transitions representing recoveries:



Next, let's write down the system of differential equations that describes the obtained graph:

$$p'_1(t) = -(\lambda_A + \tilde{\lambda}_A + 3\lambda_B)p_1(t) + \mu_A p_2(t) + \mu_B p_3(t)$$

$$p'_2(t) = (\lambda_A + \tilde{\lambda}_A)p_1(t) - (\lambda_A + 3\lambda_B + \mu_A)p_2(t) + 2\mu_A p_4(t) + \mu_B p_5(t)$$

$$p'_3(t) = 3\lambda_B p_1(t) - (\lambda_A + \tilde{\lambda}_A + 2\lambda_B + \mu_B)p_3(t) + \mu_A p_5(t) + 2\mu_B p_6(t)$$

$$p'_4(t) = \lambda_A p_2(t) - 2\mu_A p_4(t) + \mu_B p_7(t)$$

$$p'_5(t) = 3\lambda_B p_2(t) + (\lambda_A + \tilde{\lambda}_A)p_3(t) - (\lambda_A + 2\lambda_B + \mu_A + \mu_B)p_5(t) + 2\mu_A p_7(t) + 2\mu_B p_8(t)$$

$$p'_6(t) = 2\lambda_B p_3(t) - 2\mu_B p_6(t) + \mu_A p_8(t)$$

$$p'_7(t) = \lambda_A p_5(t) - (2\mu_A + \mu_B)p_7(t)$$

$$p'_8(t) = 2\lambda_B p_5(t) - (\mu_A + 2\mu_B)p_8(t)$$

The Laplace transform of the obtained system is as follows:

$$sP_1(s) = -\left(\lambda_A + \tilde{\lambda}_A + 3\lambda_B\right)P_1(s) + \mu_A P_2(s) + \mu_B P_3(s) + \mathbf{1}$$

$$sP_2(s) = \left(\lambda_A + \tilde{\lambda}_A\right)P_1(s) - (\lambda_A + 3\lambda_B + \mu_A)P_2(s) + 2\mu_A P_4(s) + \mu_B P_5(s) + \mathbf{0}$$

$$sP_3(s) = 3\lambda_B P_1(s) - \left(\lambda_A + \tilde{\lambda}_A + 2\lambda_B + \mu_B\right)P_3(s) + \mu_A P_5(s) + 2\mu_B P_6(s) + \mathbf{0}$$

$$sP_4(s) = \lambda_A P_2(s) - 2\mu_A P_4(s) + \mu_B P_7(s) + \mathbf{0}$$

$$sP_5(s) = 3\lambda_B P_2(s) + \left(\lambda_A + \tilde{\lambda}_A\right)P_3(s) - (\lambda_A + 2\lambda_B + \mu_A + \mu_B)P_5(s) + 2\mu_A P_7(s) + 2\mu_B P_8(s) + \mathbf{0}$$

$$sP_6(s) = 2\lambda_B P_3(s) - 2\mu_B P_6(s) + \mu_A P_8(s) + \mathbf{0}$$

$$sP_7(s) = \lambda_A P_5(s) - (2\mu_A + \mu_B)P_7(s) + \mathbf{0}$$

$$sP_8(s) = 2\lambda_B P_5(s) - (\mu_A + 2\mu_B)P_8(s) + \mathbf{0}$$

Since the system is operable when it is in the 1st, 2nd, 3rd, or 5th state, the point availability function is

$$AV(t) = p_1(t) + p_2(t) + p_3(t) + p_5(t),$$

or, alternatively,

$$AV(t) = 1 - [p_4(t) + p_6(t) + p_7(t) + p_8(t)].$$

The system of equations for the steady-state solution:

$$0 = -(\lambda_A + \tilde{\lambda}_A + 3\lambda_B)p_1 + \mu_A p_2 + \mu_B p_3$$

$$0 = (\lambda_A + \tilde{\lambda}_A)p_1 - (\lambda_A + 3\lambda_B + \mu_A)p_2 + 2\mu_A p_4 + \mu_B p_5$$

$$0 = 3\lambda_B p_1 - (\lambda_A + \tilde{\lambda}_A + 2\lambda_B + \mu_B)p_3 + \mu_A p_5 + 2\mu_B p_6$$

$$0 = \lambda_A p_2 - 2\mu_A p_4 + \mu_B p_7$$

$$0 = 3\lambda_B p_2 + (\lambda_A + \tilde{\lambda}_A)p_3 - (\lambda_A + 2\lambda_B + \mu_A + \mu_B)p_5 + 2\mu_A p_7 + 2\mu_B p_8$$

$$0 = 2\lambda_B p_3 - 2\mu_B p_6 + \mu_A p_8$$

$$0 = \lambda_A p_5 - (2\mu_A + \mu_B)p_7$$

~~$0 = 2\lambda_B p_5 - (\mu_A + 2\mu_B)p_8$~~

$$1 = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8$$

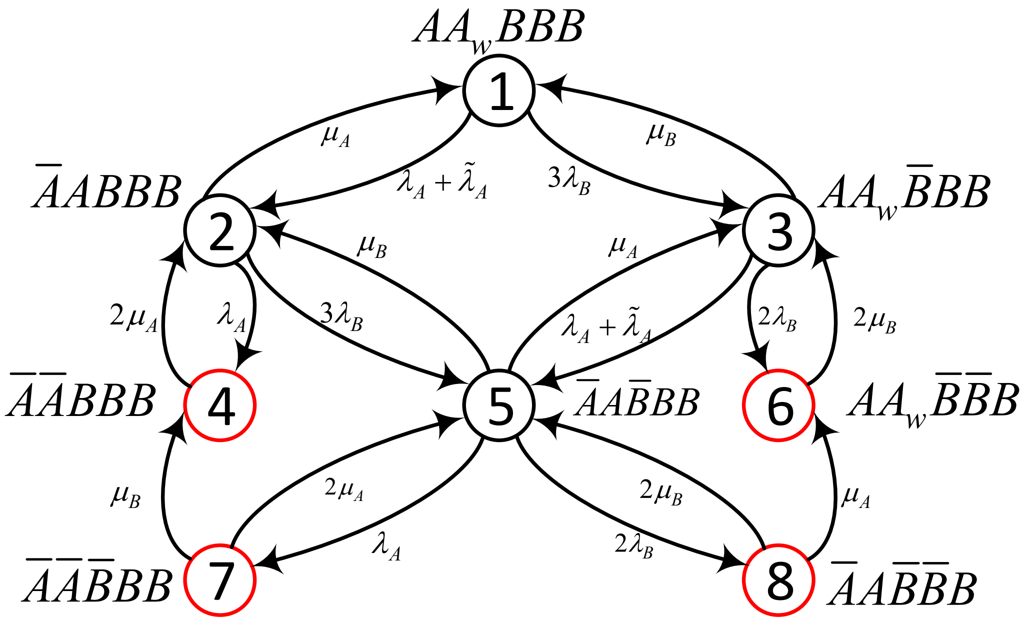
Since the system is operable when it is in the 1st, 2nd, 3rd, or 5th state, the steady-state availability is

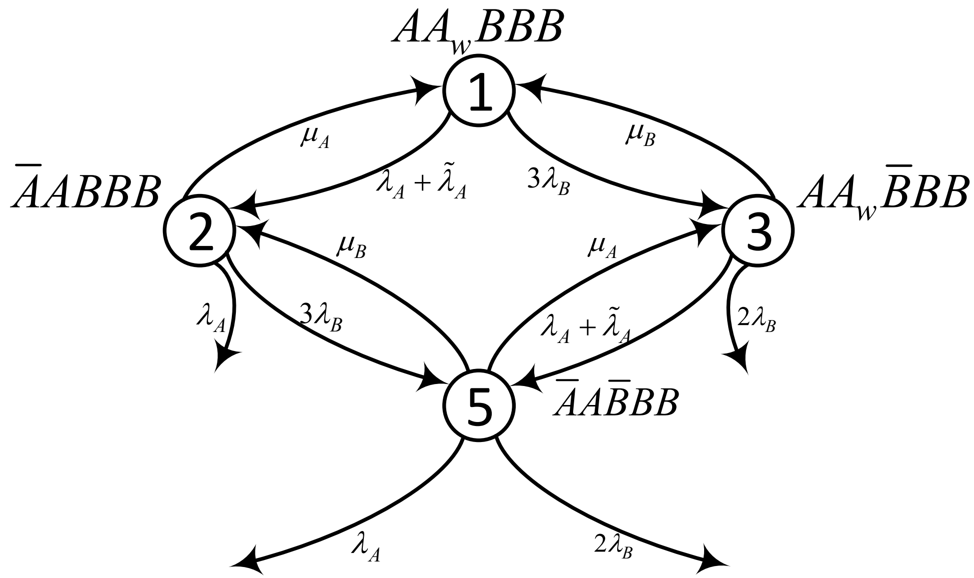
$$AV_{steady} = p_1 + p_2 + p_3 + p_5,$$

or, alternatively,

$$AV_{steady} = 1 - (p_4 + p_6 + p_7 + p_8).$$

Let’s transform the graph by removing all inoperable states and related outgoing arcs:





Now we can write down the system of equations for the calculation of the mean time to the system failure:

$$\cancel{sP_1(s)} = -\left(\lambda_A + \tilde{\lambda}_A + 3\lambda_B\right)P_1(s) + \mu_A P_2(s) + \mu_B P_3(s) \mathbf{+ 1}$$

$$\cancel{sP_2(s)} = \left(\lambda_A + \tilde{\lambda}_A\right)P_1(s) - (\lambda_A + 3\lambda_B + \mu_A)P_2(s) + \cancel{2\mu_A P_4(s)} + \mu_B P_5(s) \mathbf{+ 0}$$

$$\cancel{sP_3(s)} = 3\lambda_B P_1(s) - \left(\lambda_A + \tilde{\lambda}_A + 2\lambda_B + \mu_B\right)P_3(s) + \mu_A P_5(s) + \cancel{2\mu_B P_6(s)} \mathbf{+ 0}$$

~~$$sP_4(s) = \lambda_A P_2(s) - 2\mu_A P_4(s) + \mu_B P_7(s) \mathbf{+ 0}$$~~

$$\cancel{sP_5(s)} = 3\lambda_B P_2(s) + \left(\lambda_A + \tilde{\lambda}_A\right)P_3(s) - (\lambda_A + 2\lambda_B + \mu_A + \mu_B)P_5(s) + \cancel{2\mu_A P_7(s)} + \cancel{2\mu_B P_8(s)} \mathbf{+ 0}$$

~~$$sP_6(s) = 2\lambda_B P_3(s) - 2\mu_B P_6(s) + \mu_A P_8(s) \mathbf{+ 0}$$~~

~~$$sP_7(s) = \lambda_A P_5(s) - (2\mu_A + \mu_B)P_7(s) \mathbf{+ 0}$$~~

~~$$sP_8(s) = 2\lambda_B P_5(s) - (\mu_A + 2\mu_B)P_8(s) \mathbf{+ 0}$$~~

Replacing $P_i(s)$ with T_i, we get

$$0 = -(\lambda_A + \tilde{\lambda}_A + 3\lambda_B)T_1 + \mu_A T_2 + \mu_B T_3 + 1$$

$$0 = (\lambda_A + \tilde{\lambda}_A)T_1 - (\lambda_A + 3\lambda_B + \mu_A)T_2 + \mu_B T_5$$

$$0 = 3\lambda_B T_1 - (\lambda_A + \tilde{\lambda}_A + 2\lambda_B + \mu_B)T_3 + \mu_A T_5$$

$$0 = 3\lambda_B T_2 + (\lambda_A + \tilde{\lambda}_A)T_3 - (\lambda_A + 2\lambda_B + \mu_A + \mu_B)T_5$$

Solving the system for T_i, we obtain the mean time to the system failure as

$$MTTF = T_1 + T_2 + T_3 + T_5$$