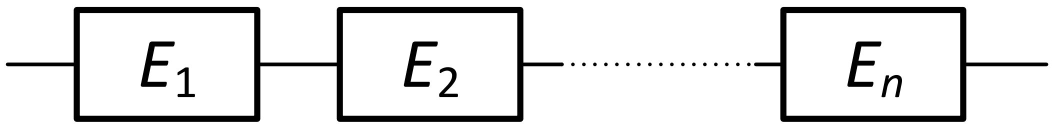
THE RELIABILITY OF SERIES SYSTEMS

The Reliability of Series Systems

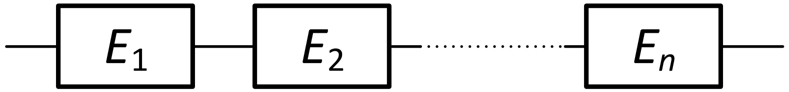
Components of a system are said to be connected in series if each one of them must be operational for the system to be operational,

i.e. *the failure of any one of its component causes the system to fail*.

The *reliability block diagram* (*RBD*) of a series system with *n* elements is as follows:



The Reliability of Series Systems



For calculation purposes it is in general tacitly assumed that for series systems, each element operates and fails independently from each other element.

Let $\{e_i\}$, $i = 1, 2,\ldots,n$ be the event

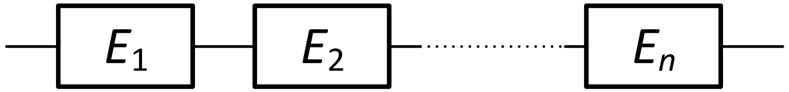
$$\{e_i\} = \{E_i \; new \; at \; t = 0 \; \cap \; E_i \; up \; in \; [0, t]\}$$

Assuming E_i is new at $t = 0$, the probability of $\{e_i\}$ is

$$Pr\{e_i\} = Pr\{E_i \; new \; at \; t = 0\} \cdot Pr\{E_i \; up \; in \; [0, t]\} = 1 \cdot R_i(t)$$

with $R_i(t)$ as reliability function of E_i.

The Reliability of Series Systems



The system does not fail in the interval [0, t] if and only if all elements E_1, E_2,…, E_n do not fail in that interval, thus

$$R_S(t) = Pr\{e_1 \cap e_2 \cap \cdots \cap e_n\}.$$

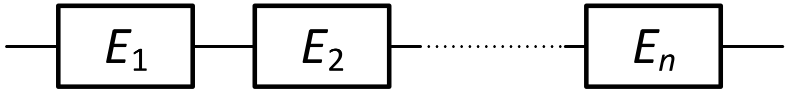
Here and in the following, S stands for system.

Due to the assumed independence among the elements E_1, E_2,…, E_n , it follows for the reliability function $R_S(t)$:

$$R_S(t) = \prod_{i=1}^{n} R_i(t)$$

Product Law of Reliabilities

The Reliability of Series Systems



Since the reliability of each component is a positive number less than one,

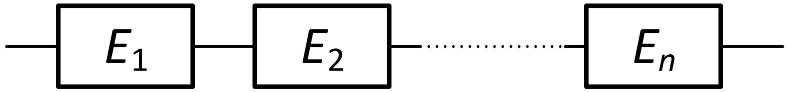
$$R_i(t) \in (0,1), t > 0$$

the product value is less then each term.

It follows that the reliability of a series system is less than the reliability of each constituent component and, hence, is less than the reliability of its least reliable component:

$$R_S(t) < \min_{i=1..n} R_i(t).$$

The Reliability of Series Systems



As we know

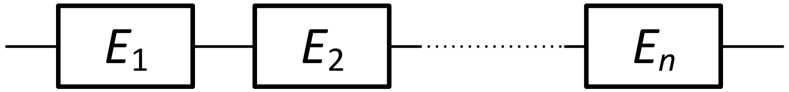
$$R(t) = e^{-\int_0^t h(\tau)d\tau}.$$

Given that, let $h_i(t)$ be the failure rate of element E_i.

Hence, for series systems

$$R_S(t) = \prod_{i=1}^{n} R_i(t) = \prod_{i=1}^{n} e^{-\int_0^t h_i(\tau)d\tau} = e^{-\int_0^t [\sum_{i=1}^{n} h_i(\tau)]d\tau}.$$

The Reliability of Series Systems



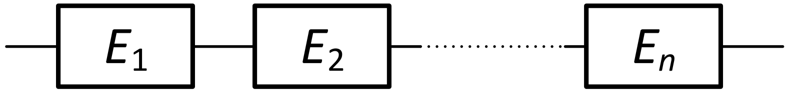
By taking

$$h_S(t) = \sum_{i=1}^{n} h_i(t)$$

we can infer:

The failure rate of a series system,
consisting of independent elements,
is the sum of the failure rates of its elements.

The Reliability of Series Systems



If $h_i(t) = \lambda_i = const.$, i.e. failure times of all elements of series system are exponentially distributed r.v., we have

$$h_S(t) = \sum_{i=1}^{n} \lambda_i = \lambda_S = const.$$

Thus, <u>time to failure of the series systems</u> itself is also an exponentially distributed r.v.

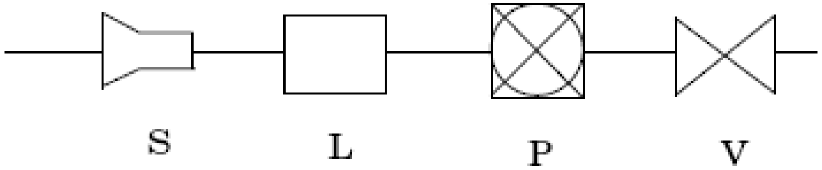
What other distributions exhibit the same property?

Rayleigh? Weibull?

The Reliability of Series Systems

Ex.: Consider a system used to maintain the fluid pressure in a tank at a constant value.

The system is composed of four components (or blocks): a pressure sensor (S), a control logic (L), a motor-pump group (P) and a valve (V).



The components are connected in series since the correct operation of each one of them is needed to guarantee the correct functioning of the system.

Often, the information that is possible to retrieve from a data bank is in the form of a (constant) failure rate, thus implicitly assuming an exponential failure time distribution.

Assume that consulting a data bank the following values have been obtained (expressed in failure per hour = f/h):

S = Pressure sensor	$\lambda_S = 2 \cdot 10^{-6}$	f / h
L = Control logic	$\lambda_L = 5 \cdot 10^{-6}$	f / h
P = Motor-pump group	$\lambda_P = 2 \cdot 10^{-5}$	f / h
V = Valve	$\lambda_V = 1 \cdot 10^{-5}$	f / h

The Reliability of Series Systems

The reliability of each component after 1 year (t=8760 h) of continuous operation is given by

$$R_S\,(t = 8760\text{ h}) = e^{-\lambda_S t} = 0.983$$
$$R_L\,(t = 8760\text{ h}) = e^{-\lambda_L t} = 0.957$$
$$R_P\,(t = 8760\text{ h}) = e^{-\lambda_P t} = 0.839$$
$$R_V\,(t = 8760\text{ h}) = e^{-\lambda_V t} = 0.916$$

and for the series system

$$R_S(t = 8760\text{ h}) = 0.983 \cdot 0.957 \cdot 0.839 \cdot 0.916 = 0.723$$

Alternatively, this result can be obtained by summing the failure rates of the constituent components:

$$\lambda_S = \lambda_R + \lambda_L + \lambda_P + \lambda_V = 3.7 \cdot 10^{-5}\, f/h$$

$$R_S(\mathrm{t} = 8760\ \mathrm{h}) = e^{-\lambda_S t} = 0.723$$

ACTIVE REDUNDANCY

Active Redundancy

When the reliability of a series system does not reach the design goal, it becomes necessary to act at the structure level and to resort to redundant configurations.

A system configuration is said to be redundant, when the occurrence of a component failure does not necessarily causes a system failure.

Various redundant architecture have been studied and applied in practice, and they will be illustrated in the following sections.

From the operating point of view, we can distinguish between:

- *Active Redundancy (parallel, hot)*: Redundant elements are subjected from the beginning to the same load as the operating elements;
- *Warm Redundancy (lightly loaded)*: Redundant elements are subjected to a lower load until they become operating;
- *Standby Redundancy (cold, unloaded)*: Redundant elements are subjected to no load until they become operating, and the failure rate in reserve (standby) state is assumed to be zero.

Active Redundancy

By default, we assume the components of the system with redundancy are statistically independent, i.e. the failure of one of them doesn't affect the reliability of the rest.

However, if this is not the case, i.e. the component(s) that are still operating assume the failed unit's portion of the load, such type of redundancy is called *load sharing*.

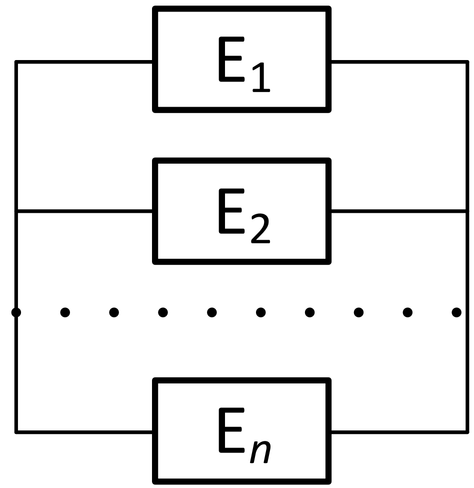
The reliability of load sharing configurations is much harder to compute.

Active Redundancy

A *parallel model* consists of n (*often statistically identical*) elements in active redundancy, of which k ($1 \leq k < n$) are necessary to perform the required function and the remaining $(n - k)$ are in reserve.

Such a structure is designated as a *k-out-of-n* (or *k-out-of-n:G*) redundancy.

1-out-of-n redundancy is also called hot redundancy.

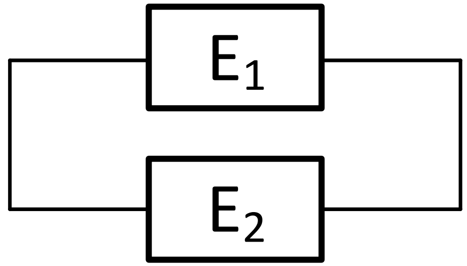


Active Redundancy

Let's consider at first the case of an active (hot) *1-out-of-2* redundancy.

The required function is fulfilled if at least one of the elements E_1 or E_2 works without failure in the interval (0, t]. In other words, the system fails if both elements failed in the interval (0, t].

Let $\{\overline{e_i}\}, i = 1,2$ be the event of *i*th element failing in (0, t], then $F_i(t) = Pr\{\overline{e_i}\}$ is the failure probability of E_i.



Then, if $F_S(t)$ denotes the failure probability of the entire system, it follows:

$$F_S(t) = Pr\{\overline{e_1} \cap \overline{e_2}\} = Pr\{\overline{e_1}\} \cdot Pr\{\overline{e_2}\} =$$
$$= F_1(t) \cdot F_2(t)$$

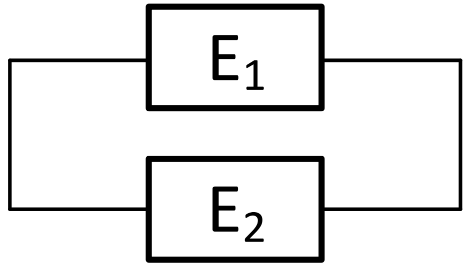
Generalizing for 1-out-of-*n* system, we obtain

$$F_S(t) = \prod_{i=1}^{n} F_i(t)$$

Product Law of Unreliabilities

Active Redundancy

Substituting $F_*(t)$ with $1 - R_*(t)$, we obtain



$$R_S(t) = R_1(t) + R_2(t) - R_1(t)R_2(t)$$

For *1-out-of-n* system, we get

$$R_S(t) = 1 - \prod_{i=1}^{n}\left(1 - R_i(t)\right)$$

Active Redundancy

As it is often that all elements in parallel system are statistically identical, i.e. $R_1(t) = R_2(t) = \cdots = R_n(t) = R(t)$, we can obtain for this particular case

$$R_S(t) = 2R(t) - R(t)^2$$ for *1-out-of-2* system

and

$$R_S(t) = 1 - \left(1 - R(t)\right)^n$$ for *1-out-of-n* system.

Active Redundancy

Thus, for parallel systems of independent components, we have a product law of "unreliabilities" analogous to the product law of reliabilities for series systems.

It follows that the reliability of a parallel system is greater than the reliability of each constituent component …

… and, hence, a parallel system is more reliable than the most reliable of its components.

Active Redundancy

Let's assume the parallel system consists of 2 identical components, and

$$R_1(t) = R_2(t) = R(t) = e^{-\lambda t},$$

i.e. the failure time of each component is an exponentially distributed r.v.

Moreover, the failure rate of each component is constant.

The reliability of the system then is

$$R_S(t) = 2R(t) - R(t)^2 = 2e^{-\lambda t} - e^{-2\lambda t}$$

Obviously, we can't find such $\lambda_S > 0$ so that

$$e^{-\lambda_S t} = 2e^{-\lambda t} - e^{-2\lambda t}$$

This means that the failure rate of the entire system is not constant, and its failure time is ***not an exponentially distributed r.v.***

To demonstrate this, let's find the expression for the system's failure rate.

Active Redundancy

We know that

$$h(t) = -\frac{R'(t)}{R(t)}$$

Given that

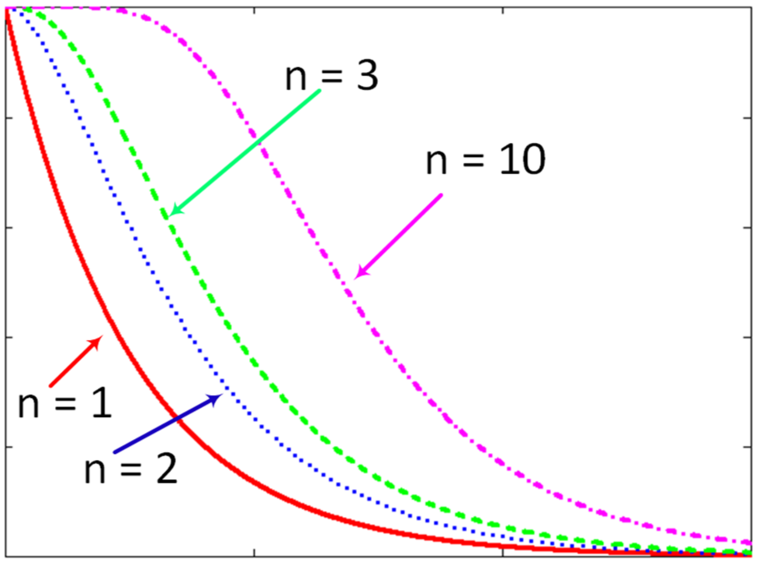
$$R_S(t) = 2e^{-\lambda t} - e^{-2\lambda t}$$

we obtain

$$h_S(t) = \frac{2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t}}{2e^{-\lambda t} - e^{-2\lambda t}} = \frac{2\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right)}{e^{-\lambda t}(2 - e^{-\lambda t})} =$$

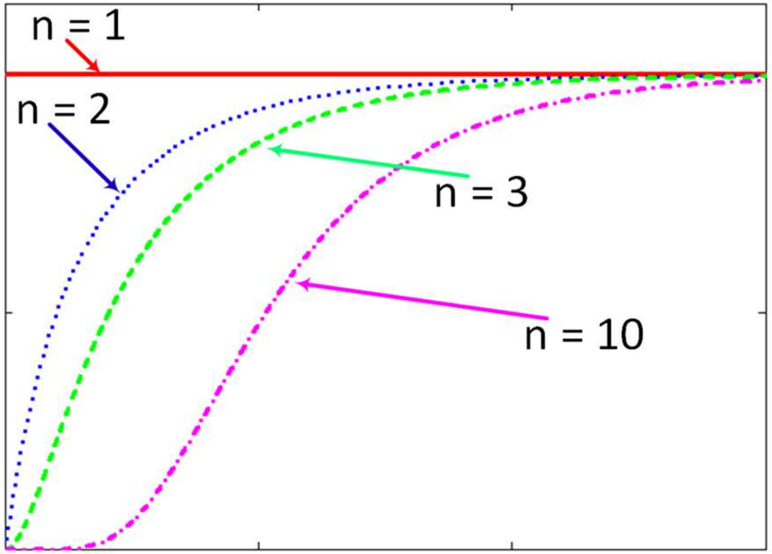
$$= \frac{2\lambda\left(1 - e^{-\lambda t}\right)}{2 - e^{-\lambda t}}.$$

Active Redundancy



Reliability

Failure Rate



As previously mentioned, a hot redundancy system is the special case of *k-out-of-n* redundancy systems (with *k=1*).

Mind that for *k-out-of-n* systems correct operation of *k* components is sufficient for the system to be operable, hence the system can tolerate *(n-k)* failures of its components.

By default we assume that all items in *k-out-of-n* systems are statistically identical.

Consider a *k-out-of-n* system ($2 \leq k < n$) with reliability of each item equals *R*.

The system is operable if it has *n, n-1, n-2, ..., k+1, k* operable components.

The probability that exactly *n-1* components are operable is given by $n \cdot R^{n-1} \cdot (1-R)^1$. Here the multiplier *n* is stands for the number of all possible combinations of operating items, i.e.

$$\binom{n}{n-1} = \frac{n!}{(n-1)!\,(n-n+1)!} = n.$$

The probability that exactly *n-2* components are operable is given by $\binom{n}{n-2} \cdot R^{n-2} \cdot (1-R)^2$.

…

Finally, the probability that exactly *k* components are operable is given by $\binom{n}{k} \cdot R^k \cdot (1-R)^{n-k}$.

Since we have listed all possible conditions for the system to be operable, the reliability of the system is given by

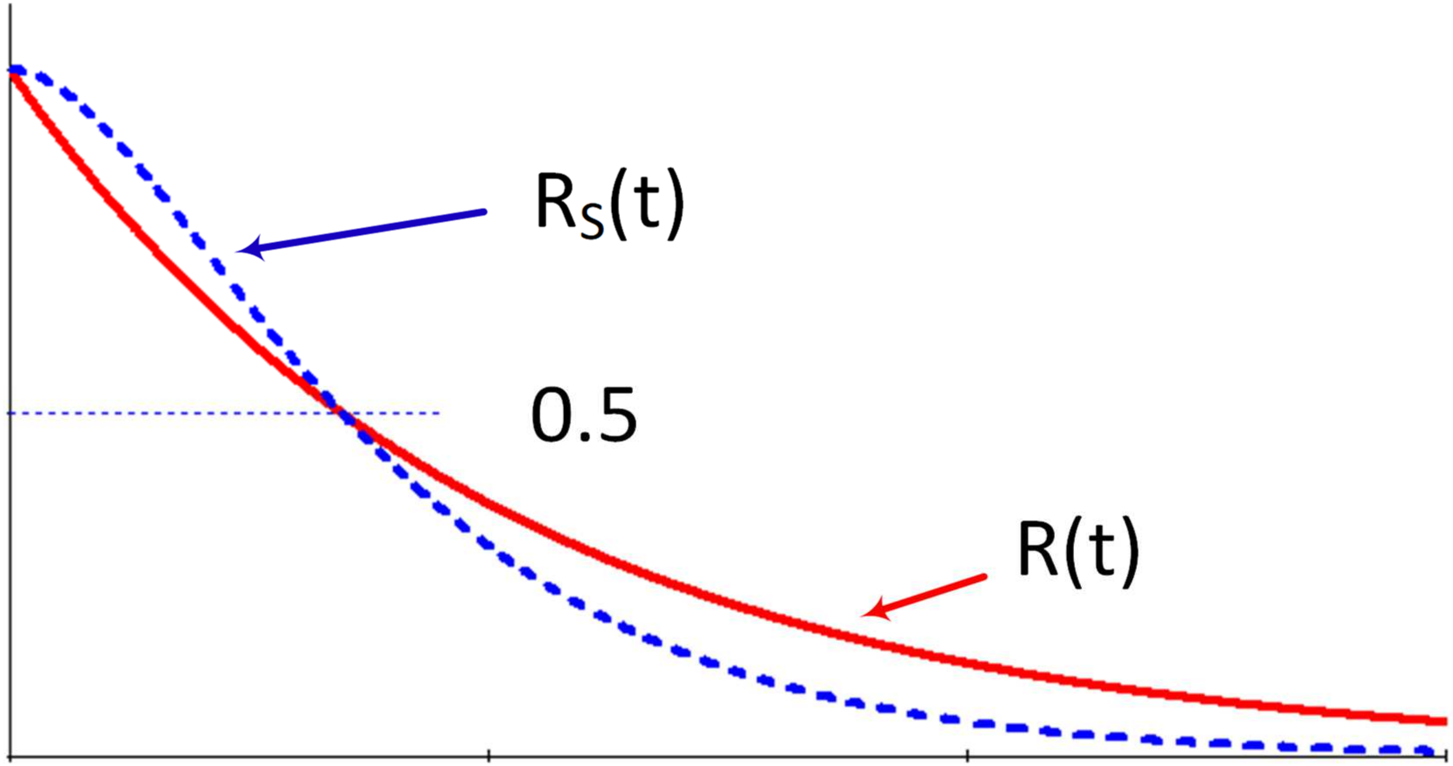
$$R_S = \sum_{i=k}^{n} \binom{n}{i} \cdot R^i \cdot (1-R)^{n-i}$$

Ex.: Consider a *2-out-of-3* system with reliability of each component given as $R(t) = e^{-\lambda t}$. Find the reliability function, MTTF and failure rate function for the entire system.

First, the reliability of the system is obtained as

$$R_S(t) = \sum_{i=2}^{3} \binom{3}{i} \cdot R(t)^i \cdot \left(1 - R(t)\right)^{3-i} =$$
$$= 3R(t)^2\left(1 - R(t)\right) + R(t)^3 =$$
$$= 3R(t)^2 - 2R(t)^3 =$$
$$= 3e^{-2\lambda t} - 2e^{-3\lambda t}$$

Active Redundancy



The system is more reliable if the reliability of each item is greater than 0,5.

Second, the MTTF is given by

$$MTTF = \int_0^{\infty} R_S(t)dt = \int_0^{\infty} \left(3e^{-2\lambda t} - 2e^{-3\lambda t}\right)dt =$$

$$= \frac{3}{2\lambda} - \frac{2}{3\lambda} = \frac{5}{6\lambda}.$$

Note that MTTF of the *2-out-of-3* system is less than the MTTF of a single item!

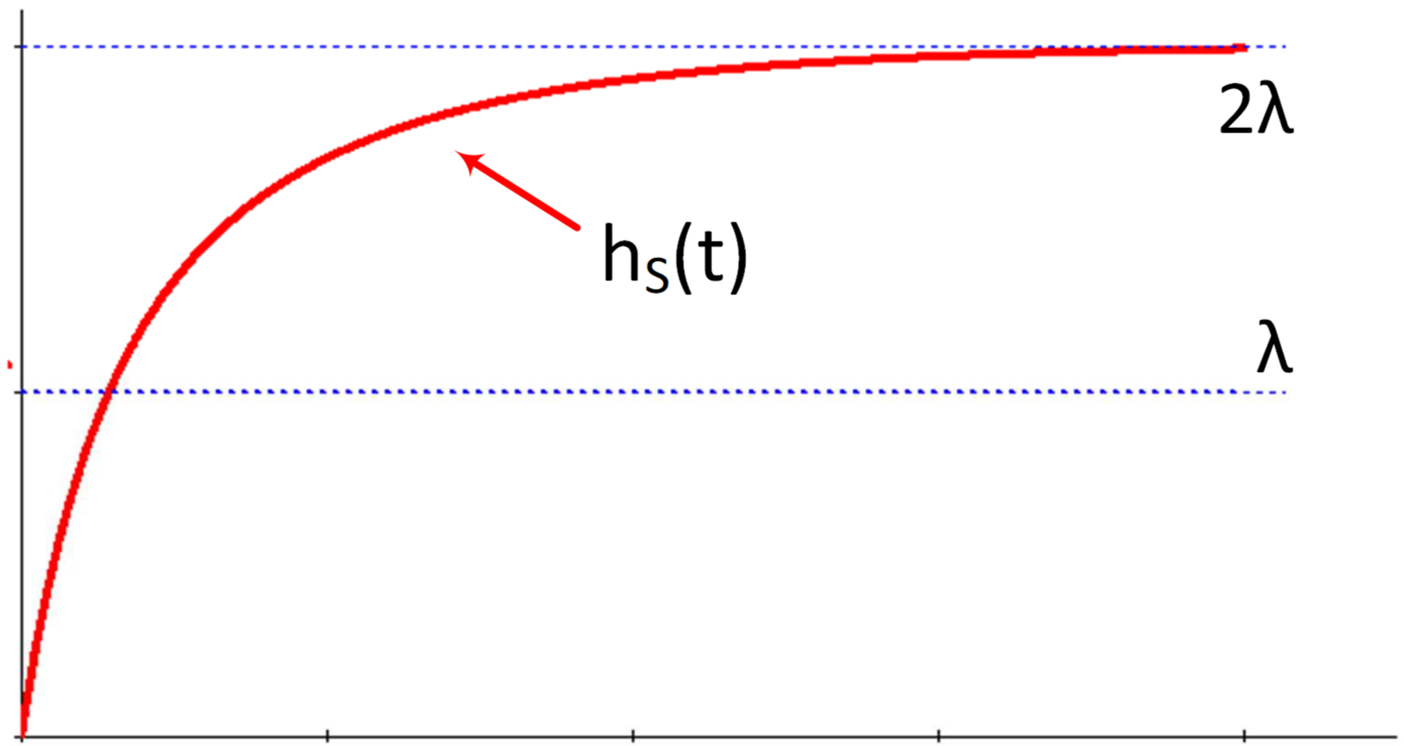
Finally, the failure rate is given by

$$h_S(t) = -\frac{R_S{}'(t)}{R_S(t)} = \frac{6\lambda e^{-2\lambda t} - 6\lambda e^{-3\lambda t}}{3e^{-2\lambda t} - 2e^{-3\lambda t}} =$$

$$= \frac{6\lambda e^{-2\lambda t}\left(1 - e^{-\lambda t}\right)}{e^{-2\lambda t}\left(3 - 2e^{-\lambda t}\right)} =$$

$$= 6\lambda \cdot \frac{1 - e^{-\lambda t}}{3 - 2e^{-\lambda t}}.$$

Active Redundancy



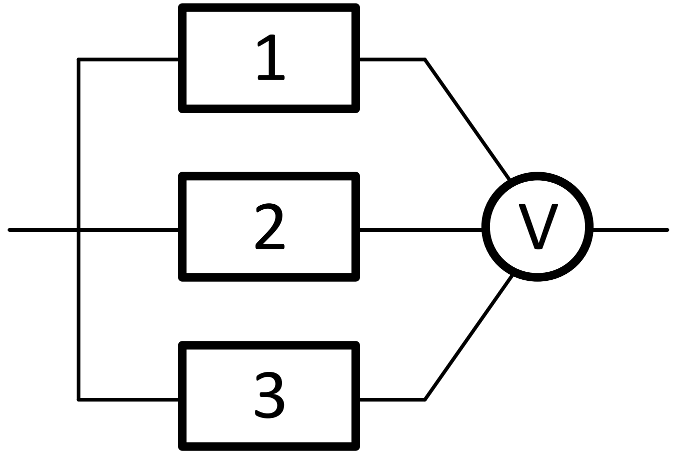
The *2-out-of-3* system is the most common case of *k-out-of-n* redundancy, since all other variants are more costly.

Often, *2-out-of-3* system is called TMR system, where TMR stands for triple modular redundancy.

Also, there is a subclass of *k-out-of-n* systems called majority voting systems. Here *n* is always odd number, and $k = \frac{n+1}{2}$.

Active Redundancy

The RBD for 2-out-of-3 system is as follows:



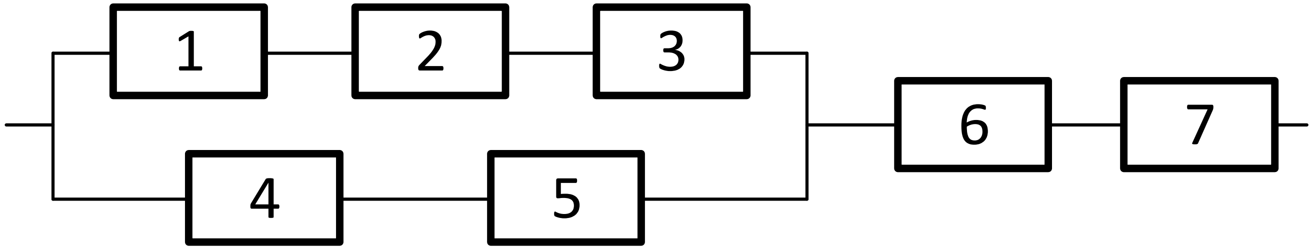
Often, the element *V* (*voter*) is assumed to be perfectly reliable, i.e. $R_V(t)=1$.

The formulae for the reliability computation of series and parallel systems can be used in combination to compute the reliability of a system having both series and parallel parts (*series-parallel systems*).

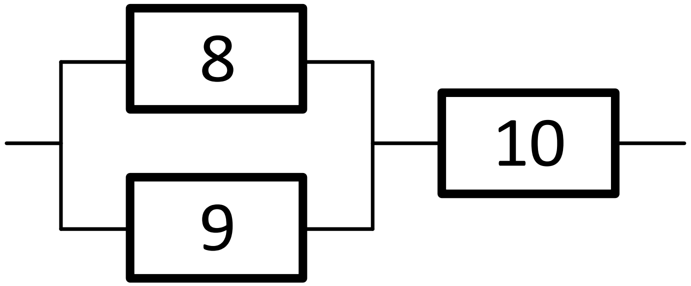
The computational procedure consists of a progressive reduction of the system complexity by substituting blocks of components in series/parallel with a single equivalent block.

Active Redundancy

Ex.: Consider a system with the following RBD:



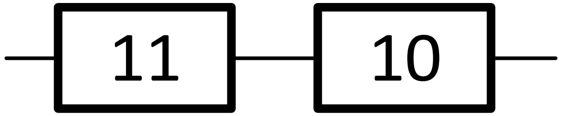
1st step: replace all series elements with equivalent ones:



$$R_8 = R_1 R_2 R_3$$
$$R_9 = R_4 R_5$$
$$R_{10} = R_6 R_7$$

Active Redundancy

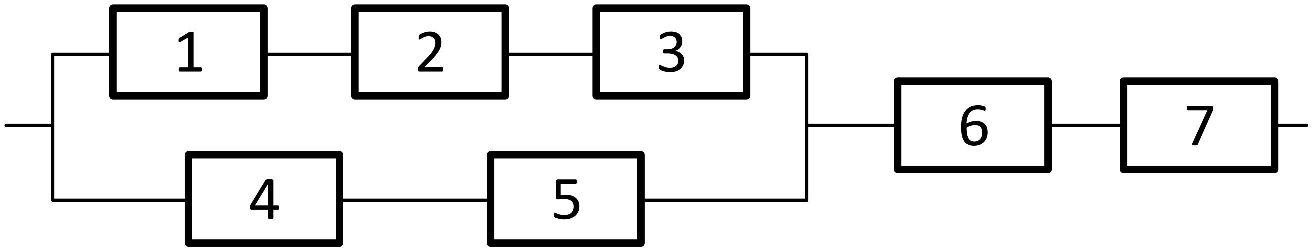
2^{nd} step: replace parallel elements 8 and 9 with equivalent one:



$R_{11} = R_8 + R_9 - R_8 R_9$

Finally, reliability of the entire system is given by

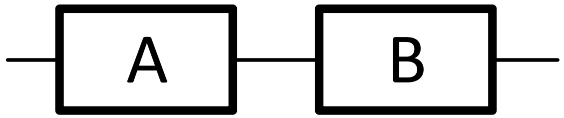
$$R_S = R_{10}R_{11} = (R_1R_2R_3 + R_4R_5 - R_1R_2R_3R_4R_5) \cdot R_6R_7$$



Active Redundancy

When discussing series-parallel systems, we should address another topic, namely, ***system redundancy vs. component redundancy***.

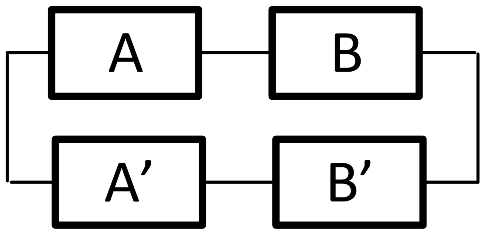
Consider a system composed of two series components A and B:



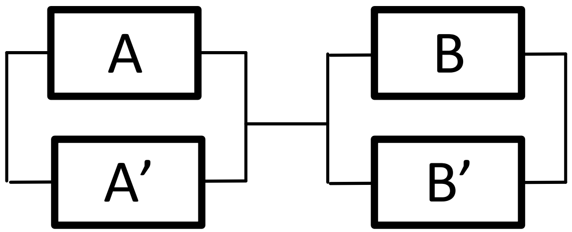
Let’s denote its reliability as $R_1 = R_A R_B$.

Active Redundancy

If we decide to improve the reliability of the system by applying redundancy using one single replica for each component, two solutions are possible. Either we replicate the complete line (system redundancy):

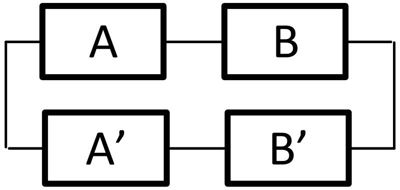


or we replicate each component individually (component redundancy):



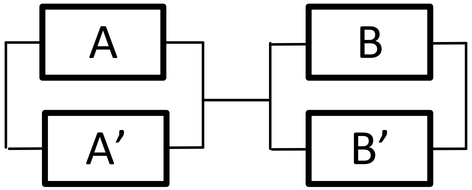
Active Redundancy

Reliability of the system with system redundancy is



$$R_2 = 2R_AR_B-(R_AR_B)^2 = R_AR_B(2 - R_AR_B)$$

Reliability of the system with component redundancy:



$$R_3 = (2R_A - R_A{}^2)(2R_B - R_B{}^2) =$$

$$= R_AR_B\,[4 – 2(R_A + R_B) + R_AR_B]$$

Active Redundancy

$R_1 = R_A R_B$

$R_2 = R_A R_B (2 - R_A R_B)$

$R_3 = R_A R_B [4 – 2(R_A + R_B) + R_A R_B]$

It is easy to see that $R_2 > R_1$ and $R_3 > R_1$; both redundant configurations are more reliable than the original system.

Next, we compare between the two redundant configurations

$$\frac{R_3}{R_2} = \frac{4 - 2(R_A + R_B) + R_A R_B}{2 - R_A R_B} = 1 + \frac{2(1 - R_A)(1 - R_B)}{2 - R_A R_B} > 1$$

It should, however, be noted that configuration 3 is more complex than configuration 2, since each type A component need to be possibly connected with any type B component.

This higher complexity requires an additional control logic (not considered in the formulae) that may reduce the benefits calculated from the equation.

Active Redundancy

The reason why configuration 3 is more reliable than configuration 2 can be also explained on a qualitative basis, noticing that there are failure combinations of basic blocks that cause failure of configuration 2, but not of configuration 3.