Lecture 5

Notable Probability Distributions (II), Expected Value, Variance

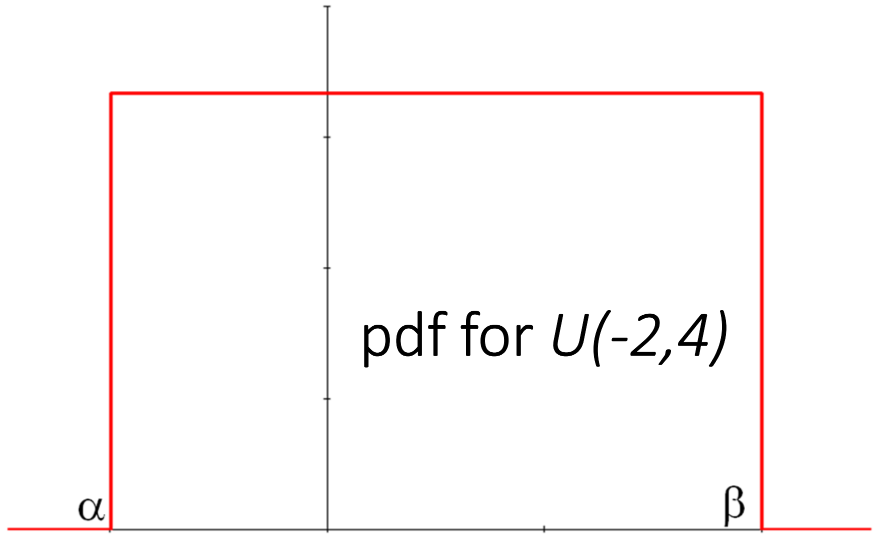
Continuous Uniform distribution

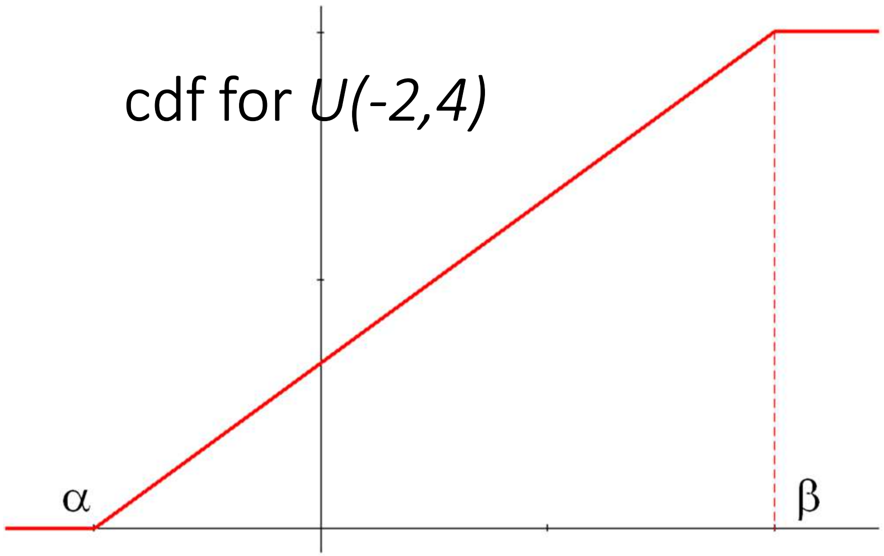
A continuous random variable X has a uniform distribution on the interval $[\alpha, \beta]$ if its probability density function f_X is given by $f_X(x) = 0$ if $x \notin [\alpha, \beta]$ and

$$f_X(x) = \frac{1}{\beta - \alpha}, \qquad x \in [\alpha, \beta].$$

We denote this distribution by $U(\alpha, \beta)$.
$U(0,1)$ is called a standard uniform distribution.

Continuous Uniform distribution





$$F_X(x) = \begin{cases} 0, & if\ x < \alpha; \\ \dfrac{x-\alpha}{\beta-\alpha}, & if\ \alpha \le x < \beta; \\ 1, & if\ x \ge \beta. \end{cases}$$

Exponential distribution

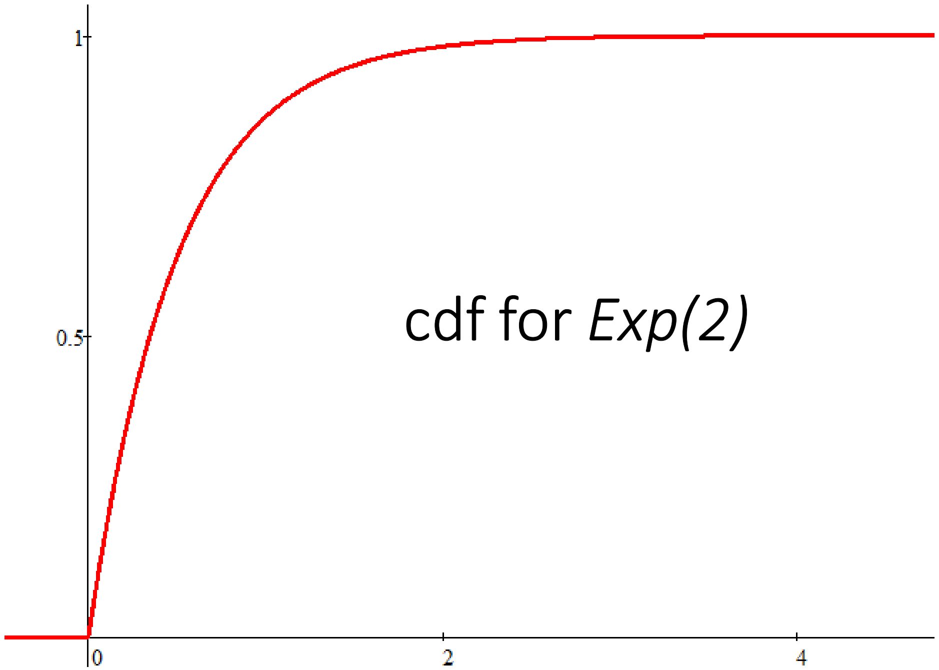
A continuous random variable X has an exponential distribution with parameter λ if its probability density function f_X is given by

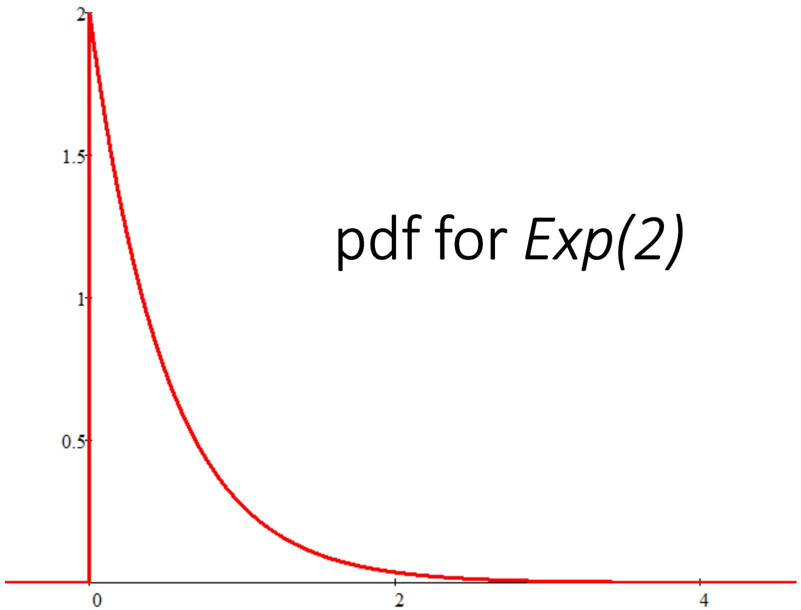
$$f_X(x) = \begin{cases} 0, & x < 0; \\ \lambda e^{-\lambda x}, & x \geq 0. \end{cases}$$

We denote this distribution by $Exp(\lambda)$.
Parameter λ is often referred to as *rate*.

Notable Probability Distributions

Exponential distribution





$$F_X(x) = \begin{cases} 0, & x < 0; \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

Exponential distribution

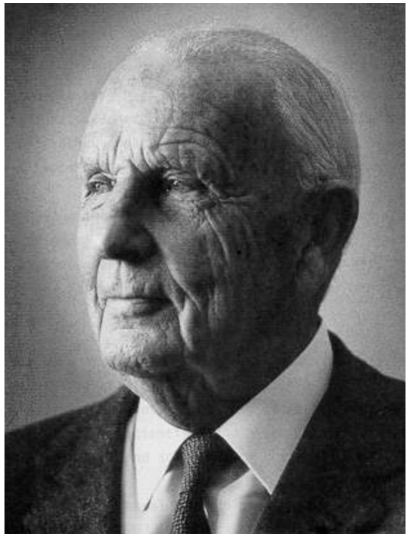
The *exponential distribution* may be viewed as a *continuous counterpart* of the *geometric distribution*, which describes the number of Bernoulli trials necessary for a discrete process to change state.

In contrast, the exponential distribution describes the time for a continuous process to change state.

Weibull distribution

A continuous random variable X has the Weibull distribution if its probability density function f_X is given by

$$f_X(x) = \begin{cases} 0, & x < 0; \\ \dfrac{\beta}{\eta}\left(\dfrac{x}{\eta}\right)^{\beta-1} e^{-\left(\frac{x}{\eta}\right)^{\beta}}, & x \geq 0. \end{cases}$$



Ernst Hjalmar Waloddi Weibull
1887-1979

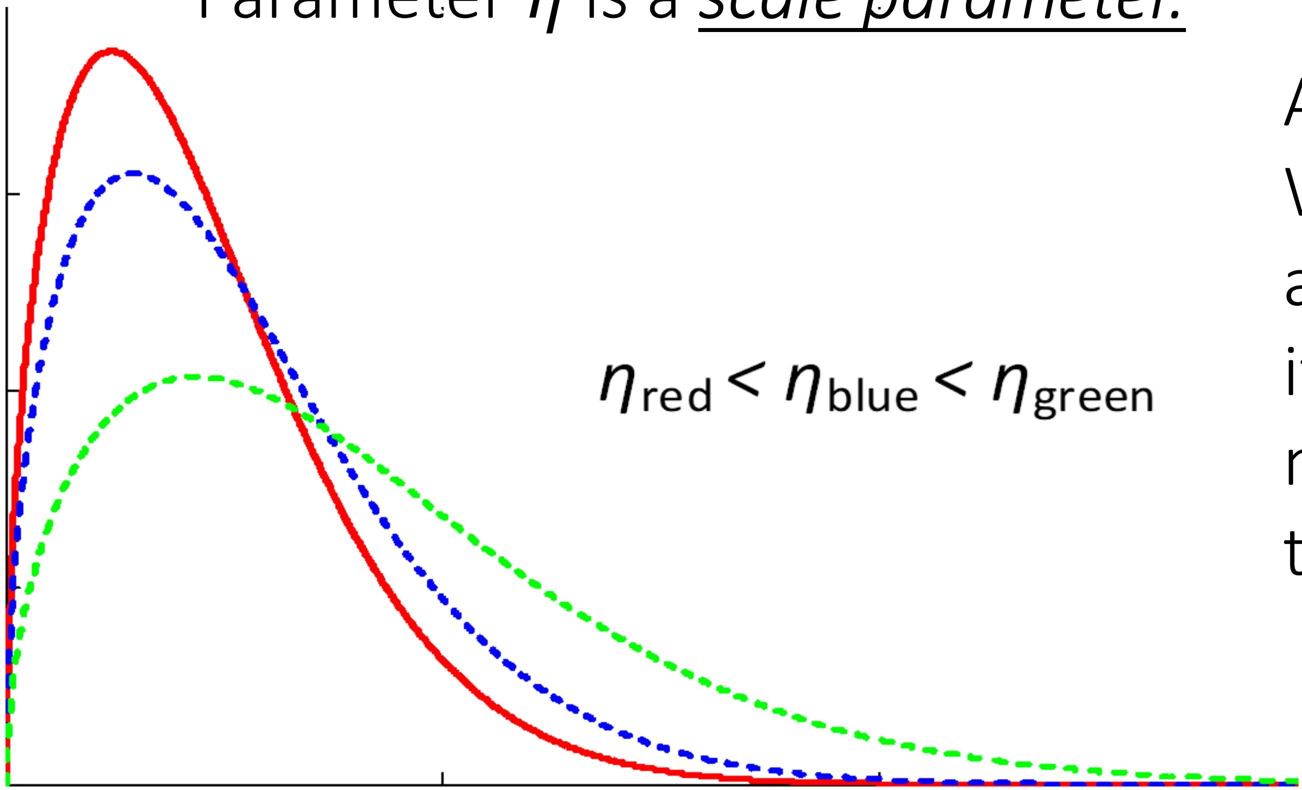
We denote this distribution by $Weib(\eta, \beta)$.

Weibull distribution

The *cdf* of the Weibull distribution is given by:

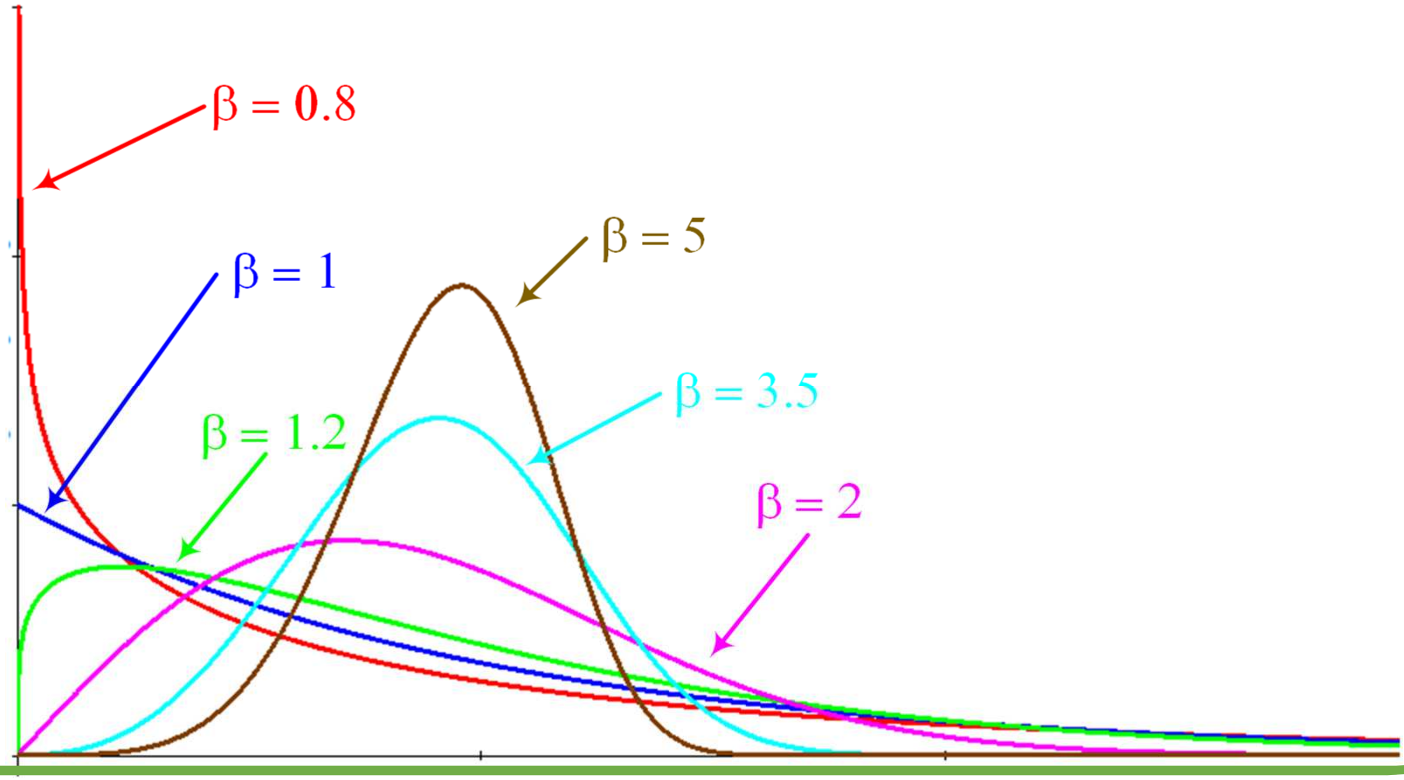
$$F_X(x) = \begin{cases} 0, & x < 0; \\ 1 - e^{-\left(\frac{x}{\eta}\right)^{\beta}}, & x \geq 0. \end{cases}$$

Parameter η is a *scale parameter.*



As it increases, the Weibull *pdf* stretches along the x-axis, while its height decreases to maintain the area under the curve constant

Parameter β is a *shape parameter*. It affects the Weibull *pdf* (and *cdf*) in a more dramatic way:



Weibull distribution

The exponential distribution can be regarded as a special case of the Weibull distribution with $\beta = 1$.

Indeed,

$$\boldsymbol{F_{Weib}(x)} = 1 - e^{-\left(\frac{x}{\eta}\right)^{\beta}} \overset{\beta=1}{\Longrightarrow} 1 - e^{-\frac{1}{\eta}\cdot x} \overset{\lambda=\frac{1}{\eta}}{\Longrightarrow} 1 - e^{-\lambda x} = \boldsymbol{F_{Exp}(x)}$$

$$Weib(\eta, 1) \equiv Exp\left(\frac{1}{\eta}\right)$$

Normal distribution

In probability theory, the *normal* (or *Gaussian*) distribution is a very common continuous probability distribution.

Normal distribution is important in statistics and is often used in the natural and social sciences to represent real-valued random variables whose distributions are not known.

Johann Carl Friedrich Gauß
1777-1855

Normal distribution

A continuous random variable X is normal or normally distributed with parameters μ and σ^2, (abbreviated $N(\mu, \sigma^2)$), if its *pdf* is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad for \ -\infty \leq x \leq \infty,$$

where $\mu \in \mathbb{R}$ is a *location parameter*, and $\sigma > 0$ – *shape parameter*.

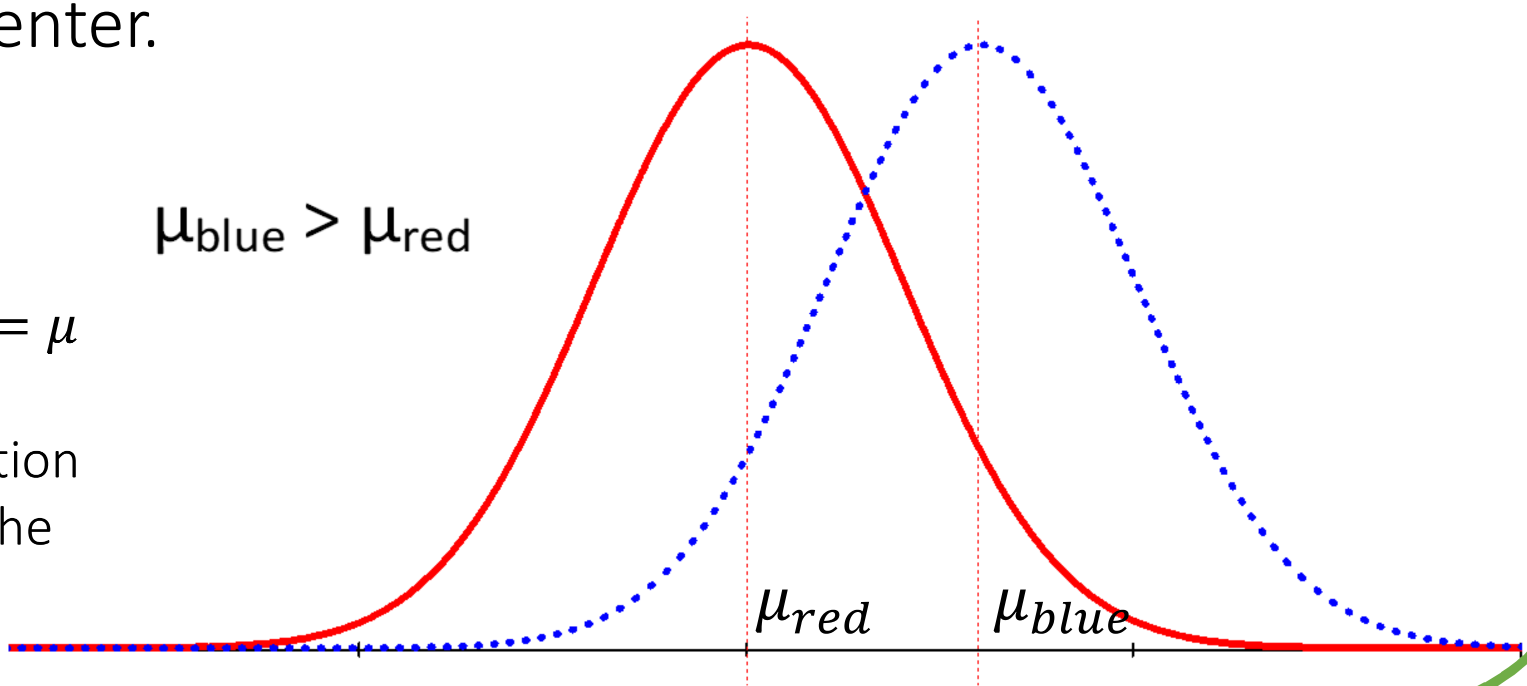
Normal distribution

Location parameter μ defines the position (locus) of the distribution's center.

$\mu_{blue} > \mu_{red}$

The distribution is symmetric about $x = \mu$

The normal distribution is informally called the bell curve



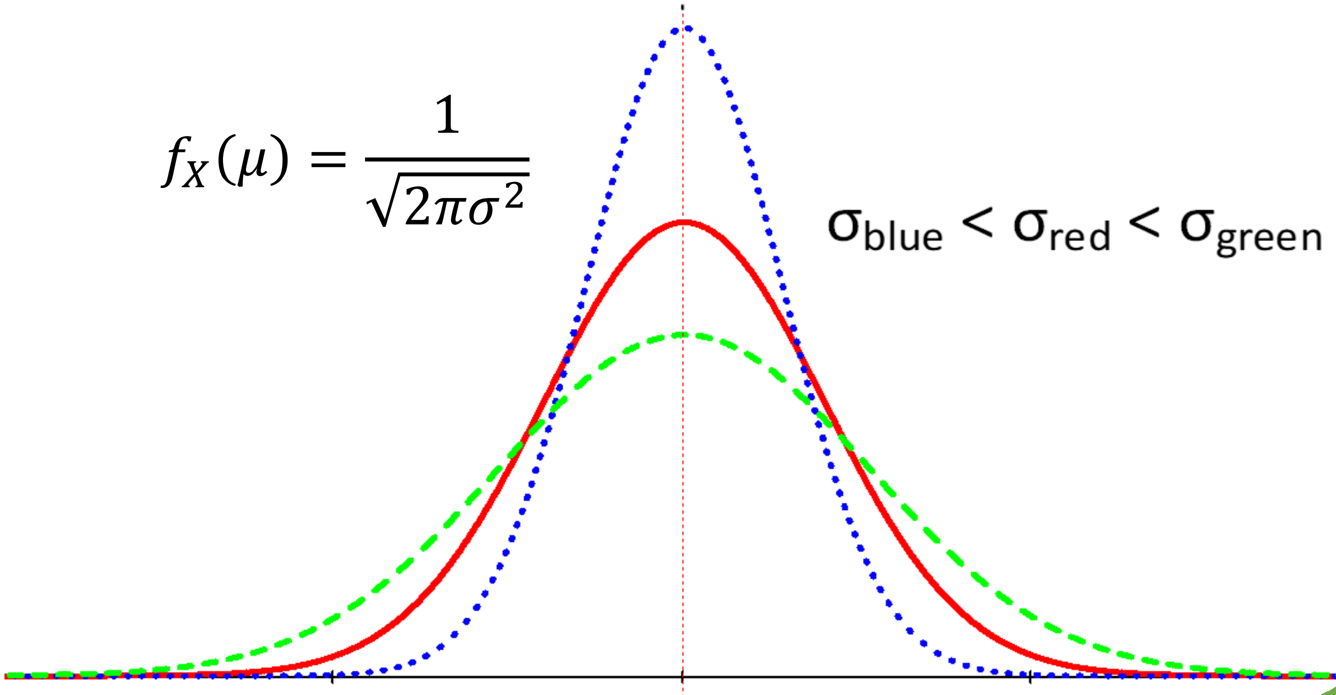
Normal distribution

Shape parameter σ affects the spread of the curve, and its height.

$$f_X(\mu) = \frac{1}{\sqrt{2\pi\sigma^2}}$$

The distribution is symmetric about $x = \mu$

The normal distribution is informally called the bell curve

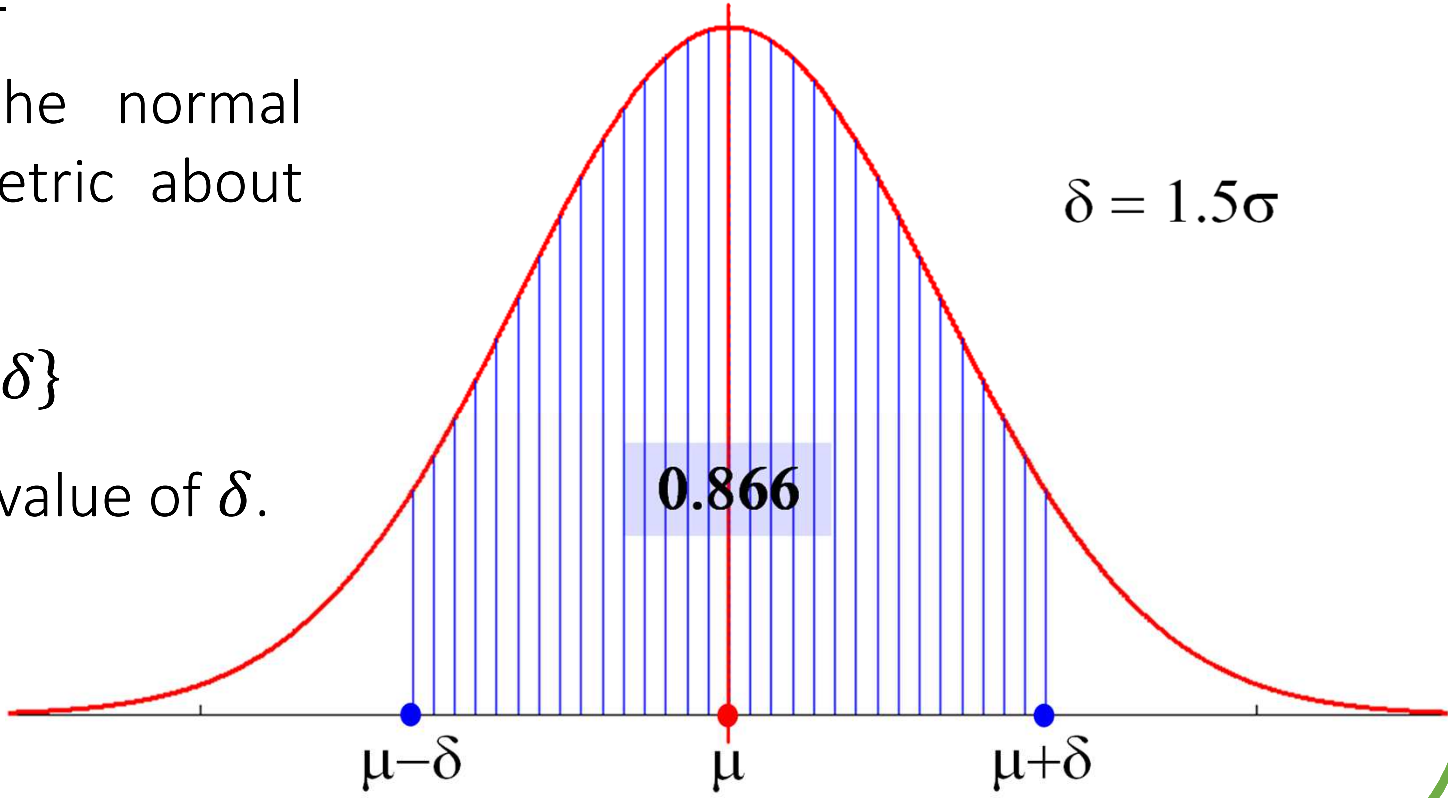


Normal distribution

Since the *pdf* of the normal distribution is symmetric about $x = \mu$, we can find

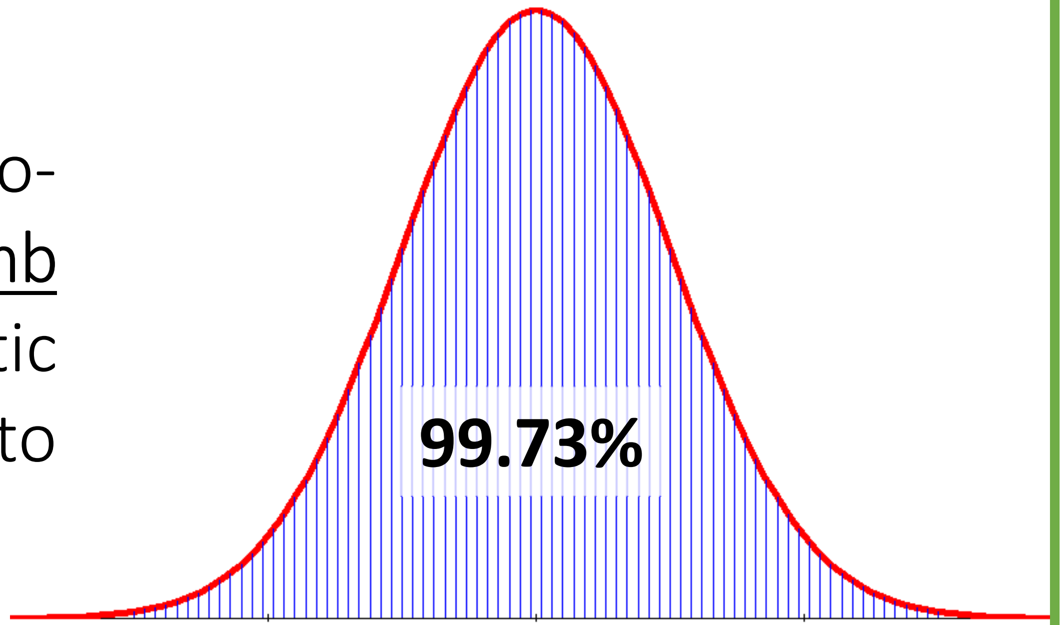
$$\Pr\{\mu - \delta \leq X \leq \mu + \delta\}$$

by specifying just the value of δ.



Normal distribution

In the empirical sciences the so-called three-sigma rule of thumb expresses a conventional heuristic that nearly all values are taken to lie within $[\mu - 3\sigma, \mu + 3\sigma]$.



It is empirically useful to treat 99.73% probability as near certainty.

Normal distribution

In the social sciences, a result may be considered “significant” if its confidence level is of the order of a two-sigma effect (95%), while in particle physics, there is a convention of a five-sigma effect (99.99994%) being required to qualify as a discovery.

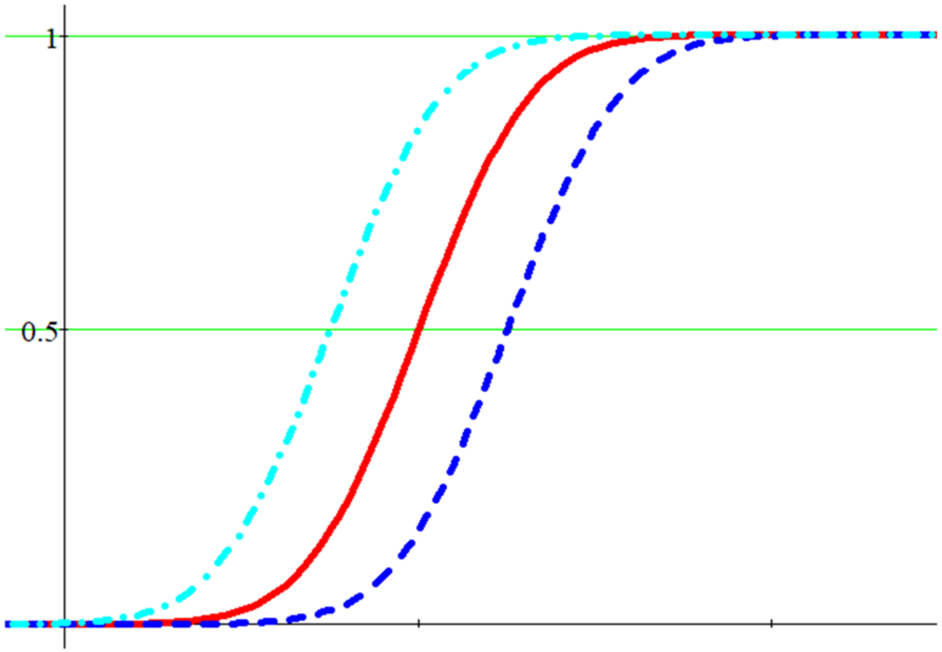
Normal distribution

If X has an $N(\mu, \sigma^2)$ distribution, then its *cdf* is given by

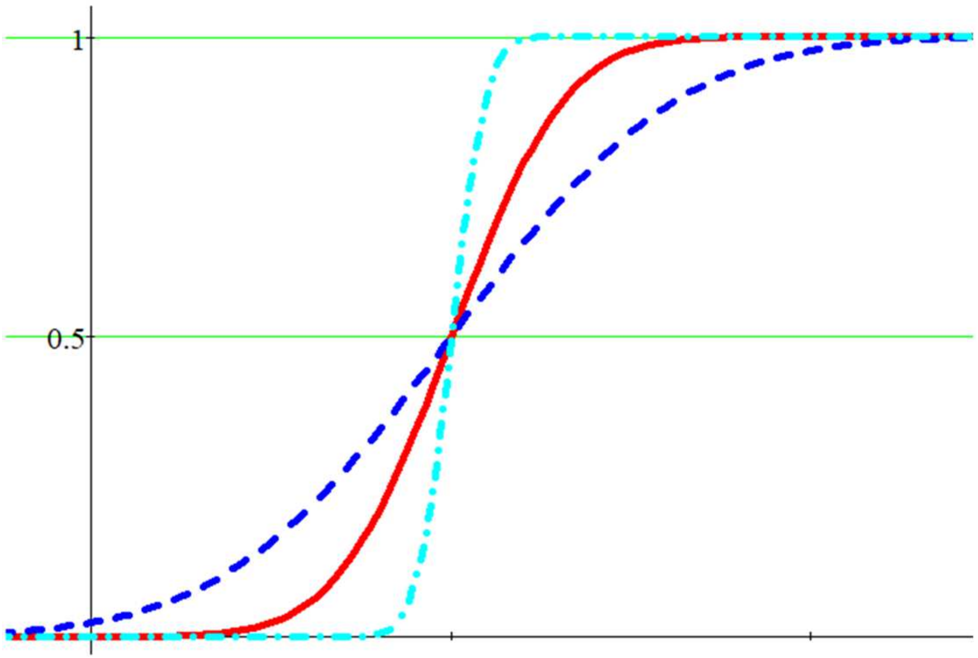
$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(\tau-\mu)^2}{2\sigma^2}} d\tau \quad \text{for } -\infty \leq x \leq \infty.$$

Unfortunately there is no explicit expression for F_X.

Normal distribution



$\mu_{cyan} < \mu_{red} < \mu_{blue}$



$\sigma_{cyan} < \sigma_{red} < \sigma_{blue}$

Normal distribution

The distribution $N(0,1)$ is called the *standard normal distribution*, and its *pdf* and *cdf* are denoted by φ and Φ, respectively, that is,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\tau^2}{2}} d\tau$$

Normal distribution

As mentioned before, the *pdf* of a normal r.v. cannot be integrated in terms of the common elementary functions, therefore, the probabilities of X falling in various intervals are obtained from tables or by computer.

It is impossible to construct tables for all μ and σ values required in applications.

Normal distribution

Fortunately, there is a way to express the normal distribution with arbitrary parameters μ and σ in terms of a standard normal distribution:

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

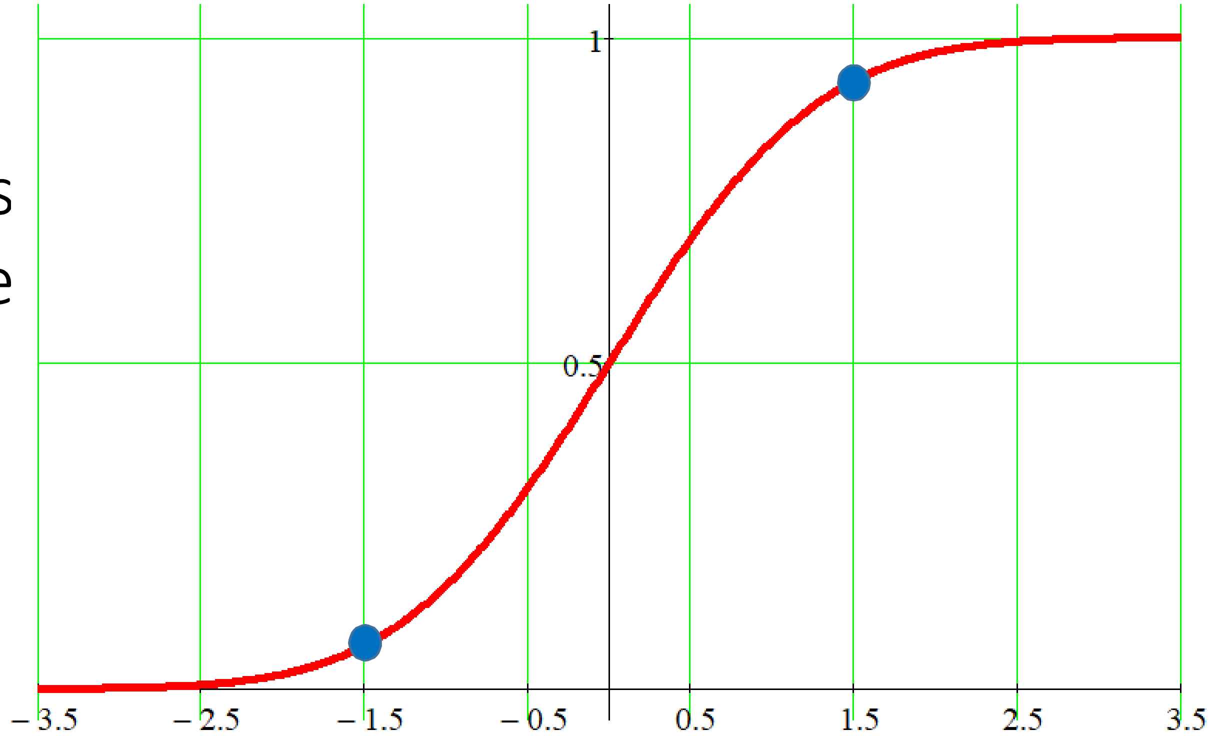
See proof in *Géza Schay. Introduction to Probability…, p. 240-241*

Normal distribution

The *cdf* of $N(0,1)$ is symmetric relative to the point $(0, 0.5)$.

Therefore,

$$\Phi(-x) = 1 - \Phi(x)$$



Values of the Standard Normal cdf $\boldsymbol{\Phi(x)}$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0	0,5	0,50399	0,50798	0,51197	0,51595	0,51994	0,52392	0,5279	0,53188	0,53586
0.1	0,53983	0,5438	0,54776	0,55172	0,55567	0,55962	0,56356	0,56749	0,57142	0,57535
0.2	0,57926	0,58317	0,58706	0,59095	0,59483	0,59871	0,60257	0,60642	0,61026	0,61409
0.3	0,61791	0,62172	0,62552	0,6293	0,63307	0,63683	0,64058	0,64431	0,64803	0,65173
0.4	0,65542	0,6591	0,66276	0,6664	0,67003	0,67364	0,67724	0,68082	0,68439	0,68793
0.5	0,69146	0,69497	0,69847	0,70194	0,7054	0,70884	0,71226	0,71566	0,71904	0,7224
0.6	0,72575	0,72907	0,73237	0,73565	0,73891	0,74215	0,74537	0,74857	0,75175	0,7549
0.7	0,75804	0,76115	0,76424	0,7673	0,77035	0,77337	0,77637	0,77935	0,7823	0,78524
0.8	0,78814	0,79103	0,79389	0,79673	0,79955	0,80234	0,80511	0,80785	0,81057	0,81327
0.9	0,81594	0,81859	0,82121	0,82381	0,82639	0,82894	0,83147	0,83398	0,83646	0,83891
1	0,84134	0,84375	0,84614	0,84849	0,85083	0,85314	0,85543	0,85769	0,85993	0,86214
1.1	0,86433	0,8665	0,86864	0,87076	0,87286	0,87493	0,87698	0,879	0,881	0,88298
1.2	0,88493	0,88686	0,88877	0,89065	0,89251	0,89435	0,89617	0,89796	0,89973	0,90147
1.3	0,9032	0,9049	0,90658	0,90824	0,90988	0,91149	0,91309	0,91466	0,91621	0,91774
1.4	0,91924	0,92073	0,9222	0,92364	0,92507	0,92647	0,92785	0,92922	0,93056	0,93189
1.5	0,93319	0,93448	0,93574	0,93699	0,93822	0,93943	0,94062	0,94179	0,94295	0,94408
1.6	0,9452	0,9463	0,94738	0,94845	0,9495	0,95053	0,95154	0,95254	0,95352	0,95449
1.7	0,95543	0,95637	0,95728	0,95818	0,95907	0,95994	0,9608	0,96164	0,96246	0,96327
1.8	0,96407	0,96485	0,96562	0,96638	0,96712	0,96784	0,96856	0,96926	0,96995	0,97062
1.9	0,97128	0,97193	0,97257	0,9732	0,97381	0,97441	0,975	0,97558	0,97615	0,9767

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
2	0,97725	0,97778	0,97831	0,97882	0,97932	0,97982	0,9803	0,98077	0,98124	0,98169
2.1	0,98214	0,98257	0,983	0,98341	0,98382	0,98422	0,98461	0,985	0,98537	0,98574
2.2	0,9861	0,98645	0,98679	0,98713	0,98745	0,98778	0,98809	0,9884	0,9887	0,98899
2.3	0,98928	0,98956	0,98983	0,9901	0,99036	0,99061	0,99086	0,99111	0,99134	0,99158
2.4	0,9918	0,99202	0,99224	0,99245	0,99266	0,99286	0,99305	0,99324	0,99343	0,99361
2.5	0,99379	0,99396	0,99413	0,9943	0,99446	0,99461	0,99477	0,99492	0,99506	0,9952
2.6	0,99534	0,99547	0,9956	0,99573	0,99585	0,99598	0,99609	0,99621	0,99632	0,99643
2.7	0,99653	0,99664	0,99674	0,99683	0,99693	0,99702	0,99711	0,9972	0,99728	0,99736
2.8	0,99744	0,99752	0,9976	0,99767	0,99774	0,99781	0,99788	0,99795	0,99801	0,99807
2.9	0,99813	0,99819	0,99825	0,99831	0,99836	0,99841	0,99846	0,99851	0,99856	0,99861
3	0,99865	0,99869	0,99874	0,99878	0,99882	0,99886	0,99889	0,99893	0,99896	0,999
3.1	0,99903	0,99906	0,9991	0,99913	0,99916	0,99918	0,99921	0,99924	0,99926	0,99929
3.2	0,99931	0,99934	0,99936	0,99938	0,9994	0,99942	0,99944	0,99946	0,99948	0,9995
3.3	0,99952	0,99953	0,99955	0,99957	0,99958	0,9996	0,99961	0,99962	0,99964	0,99965
3.4	0,99966	0,99968	0,99969	0,9997	0,99971	0,99972	0,99973	0,99974	0,99975	0,99976
3.5	0,99977	0,99978	0,99978	0,99979	0,9998	0,99981	0,99981	0,99982	0,99983	0,99983
3.6	0,99984	0,99985	0,99985	0,99986	0,99986	0,99987	0,99987	0,99988	0,99988	0,99989
3.7	0,99989	0,9999	0,9999	0,9999	0,99991	0,99991	0,99992	0,99992	0,99992	0,99992
3.8	0,99993	0,99993	0,99993	0,99994	0,99994	0,99994	0,99994	0,99995	0,99995	0,99995
3.9	0,99995	0,99995	0,99996	0,99996	0,99996	0,99996	0,99996	0,99996	0,99997	0,99997

Notable Probability Distributions

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0	0,5	0,50399	0,50798	0,51197	0,51595	0,51994	0,52392	0,5279	0,53188	0,53586
0.1	0,53983	0,5438	0,54776	0,55172	0,55567	0,55962	0,56356	0,56749	0,57142	0,57535
0.2	0,57926	0,58317	0,58706	0,59095	0,59483	0,59871	0,60257	0,60642	0,61026	0,61409
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0.4	0,65542	0,6591	0,66276	0,6664	0,67003	0,67364	0,67724	0,68082	0,68439	0,68793
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0.6	0,72575	0,72907	0,73237	0,73565	0,73891	0,74215	0,74537	0,74857	0,75175	0,7549
0.7	0,75804	0,76115	0,76424	0,7673	0,77035	0,77337	0,77637	0,77935	0,7823	0,78524
0.8	0,78814	0,7910[illegible]	0,79389	0,79673	0,79955	0,80234	0,80511	0,80785	0,81057	0,81327
0.9	0,81594	0,81859	0,82121	0,82381	0,82639	0,82894	0,83147	0,83398	0,83646	0,83891
1	0,84134	0,84375	0,84614	0,84849	0,85083	0,85314	0,85543	0,85769	0,85993	0,86214
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1.2	0,88493	0,88686	0,88877	0,89065	0,89251	0,89435	0,89617	0,89796	0,89973	0,90147
1.3	0,9032	0,9049	0,90658	0,90824	0,90988	0,91149	0,913[illegible]9	0,91466	0,91621	0,91774
1.4	0,91924	0,92073	0,9222	0,92364	0,92507	0,92647	0,92785	0,92922	0,93056	0,93189
1.5	0,93319	0,93448	0,93574	0,93699	0,93822	0,93943	0,94062	0,94179	0,94295	0,94408
1.6	0,9452	0,9463	0,94738	0,94845	0,9495	0,95053	0,95154	0,95254	0,95352	0,95449
1.7	0,95543	0,95637	0,95728	0,95818	0,95907	0,95994	0,9608	0,96164	0,96246	0,96327
1.8	0,96407	0,96485	0,96562	0,96638	0,96712	0,96784	0,96856	0,96926	0,96995	0,97062
1.9	0,97128	0,97193	0,97257	0,9732	0,97381	0,97441	0,975	0,97558	0,97615	0,9767

$\Phi(0.82) = ?$

$\Phi(0.82) = 0.79389$

$\Phi(1.376) = ?$

$$\Phi(1.37) + \Delta = \Phi(1.37) + \frac{6}{10}\big(\Phi(1.38) - \Phi(1.37)\big) = 0.91466 + \frac{6 \cdot 0.00156}{10} =$$

0.9156

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0	0,5	0,50399	0,50798	0,51197	0,51595	0,51994	0,52392	0,5279	0,53188	0,53586
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1.4	0,91924	0,92073	0,9222	0,92364	0,92507	0,92647	0,92785	0,92922	0,93056	0,93189
1.5	0,93319	0,93448	0,93574	0,93699	0,93822	0,93943	0,94062	0,94179	0,94295	0,94408
1.6	0,9452	0,9463	0,94738	0,94845	0,9495	0,95053	0,95154	0,95254	0,95352	0,95449
1.7	0,95543	0,95637	0,95728	0,95818	0,95907	0,95994	0,9608	0,96164	0,96246	0,96327
1.8	0,96407	0,96485	0,96562	0,96638	0,96712	0,96784	0,96856	0,96926	0,96995	0,97062
1.9	0,97128	0,97193	0,97257	0,9732	0,97381	0,97441	0,975	0,97558	0,97615	0,9767

$$\Phi(-0.82) = ?$$

$$\boldsymbol{\Phi(-x) = 1 - \Phi(x)}$$

$$\Phi(0.82) = 0.79389$$

$$\Phi(-0.82) = 1 - \Phi(0.82)$$
$$= 1 - 0.79389 = 0.20611$$

Height Distribution of Men

Assume* that the height X, in inches, of a randomly selected man in a certain population is normally distributed with $\mu = 69$ and $\sigma = 2.6$.

Find:

a) $\Pr(X \leq 72)$;
b) $\Pr(X \geq 75)$;
c) $\Pr(|X - \mu| \leq 1)$.

*Any such assumption is always just an approximation that is usually valid only within $\mu \pm 3\sigma \div 4\sigma$. But that is the range where almost all of the probability of the normal distribution falls.

As a practical matter we can ignore the fact that normal distribution gives nonzero probabilities to impossible events such as people having negative heights or heights over ten feet.

In each case, we transform the inequalities so that X will be standardized and use the table of the values of the Standard normal distribution (Φ-table) to find the required probabilities.

$$X = N(\mu, \sigma^2) \Longrightarrow \frac{X - \mu}{\sigma} = Z = N(0,1)$$

$\mu = 69; \sigma = 2.6;$

a) $\Pr(X \leq 72) = \Pr\left(Z \leq \frac{72-69}{2.6}\right) \approx \Pr(Z \leq 1.154) = \Phi(1.154)$

	.04	.05	.06	.07
1	0,85083	0,85314	0,85543	0,85769
1.1	0,87286	0,87493	0,87698	0,879
1.2	0,89251	0,89435	0,89617	0,89796

$$\Phi(1.154) \approx 0.87493 + \frac{4}{10}(0.87698 - 0.87493)$$

$$\Pr(X \leq 72) = \Phi(1.154) \approx 0.87575$$

$\mu = 69; \sigma = 2.6;$

b) $\Pr(X \geq 75) = \Pr\left(Z \geq \frac{75-69}{2.6}\right) \approx \Pr(Z \geq 2.308) =$

$= 1 - \Pr(Z \leq 2.308) = 1 - \Phi(2.308).$

	.00	.01
2.2	0,9861	0,98645
2.3	0,98928	0,98956
2.4	0,9918	0,99202

$$\Phi(2.308) \approx 0.98928 + \frac{8}{10}(0.98956 - 0.98928)$$

$$\Phi(2.308) \approx 0.9895$$

$$\Pr(X \geq 75) = 1 - \Phi(2.308) \approx 0.0105$$

$\underline{\mu = 69; \sigma = 2.6;}$

c) $\Pr(|X - \mu| \leq 1) = \Pr(68 \leq X \leq 70) =$

$= \Pr\left(\frac{68-69}{2.6} \leq Z \leq \frac{70-69}{2.6}\right) = \Pr\left(-\frac{1}{2.6} \leq Z \leq \frac{1}{2.6}\right) \approx$

$\approx \Pr(-0.385 \leq Z \leq 0.385) = \Phi(0.385) - \left(1 - \Phi(0.385)\right) =$

$= 2 \cdot \Phi(0.385) - 1.$

$\mu = 69; \sigma = 2.6;$

c) $\Pr(|X - \mu| \leq 1) = 2 \cdot \Phi(0.385) - 1.$

	.07	.08	.09
0.2	0,60642	0,61026	0,61409
0.3	0,64431	0,64803	0,65173
0.4	0,68082	0,68439	0,68793

$$\Phi(0.385) \approx 0.64803 + \frac{5}{10}(0.65173 - 0.64803)$$

$$\Phi(0.385) \approx 0.64988$$

$$\Pr(|X - \mu| \leq 1) = 2 \cdot \Phi(0.385) - 1 \approx 0.29976$$

Normal distribution

The solution for the last problem gives you another useful formula:

$$\Pr(\mu - \Delta \leq X \leq \mu + \Delta) = 2\Phi\left(\frac{\Delta}{\sigma}\right) - 1$$

Expected Value

Random variables are complicated objects, containing a lot of information on the experiments that are modeled by them. If we want to summarize a random variable by a single number, then this number should undoubtedly be its *expected value*.

The expected value, also called the *expectation* or *mean*, gives the center—in the sense of average value—of the distribution of the random variable.

Expected Value

The expected value of a discrete r.v. X with *pmf* $p_X(x)$ defined on a domain $D_X = \{x_i \in \mathbb{R} : p_X(x_i) > 0\}$ is the number

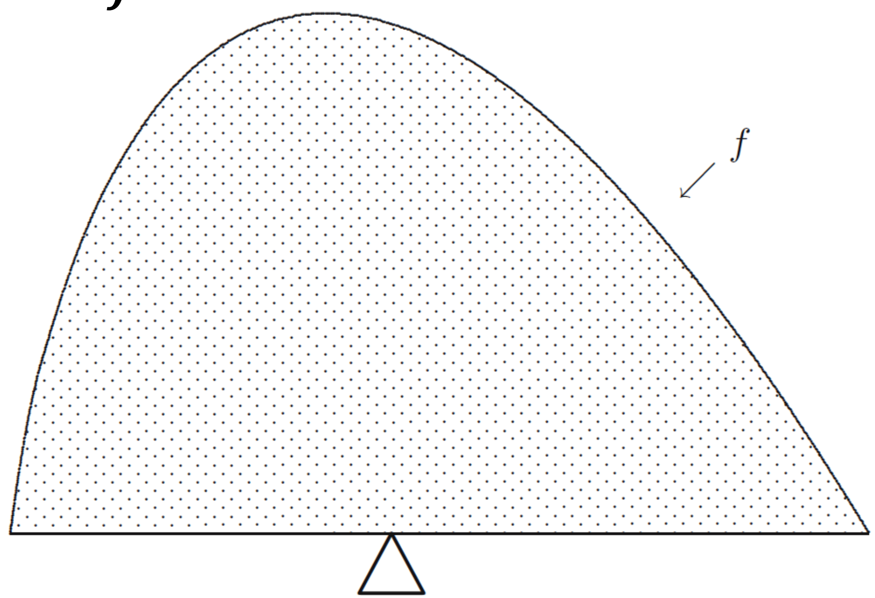
$$E[X] = \sum_{D_X} x_i \cdot p_X(x_i)\,.$$

The expected value of a continuous r.v. X with *pdf* $f_X(x)$ defined on a domain $D_X = \{x \in \mathbb{R} : f_X(x) > 0\}$ is the number

$$E[X] = \int_{D_X} x \cdot f_X(x) dx\,.$$

The expected value has the straightforward physical interpretation: $E[X]$ is the center of gravity of the mass distribution described by the function f.

$$E[X] = \frac{\int_{-\infty}^{\infty} x \cdot f_X(x)dx}{\int_{-\infty}^{\infty} f_X(x)dx}$$



NB! The expected value may not exist!

Expected Value

Properties of $E[X]$

If $C = const.$, $E[C] = C$.

If $a = const.$, $E[a \cdot X] = a \cdot E[X]$.

If $a = const.$, $E[X \pm a] = E[X] \pm a$.

$E\big[X - E[X]\big] = E[X] - E[X] = 0$.

$E[\sum_{k=1}^{n} X_k] = \sum_{k=1}^{n} E[X_k]$ for $n < \infty$.

If X_i and X_j are independent r.v., $E[\prod_{k=1}^{n} X_k] = \prod_{k=1}^{n} E[X_k]$.

Alternative notations for $E[X]$:
$M[X]$, μ_X

Properties of $E[X]$

If the distribution of a random variable is symmetric about a point α, that is, its *pdf* satisfies $f_X(\alpha - x) = f_X(\alpha + x)$ for all x, and $E[X]$ exists, then $E[X] = \alpha$.

See proof in *Géza Schay. Introduction to Probability…, p. 176*

Markov's Inequality

If X is a nonnegative random variable with expected value μ_X and a is any positive number, then,

$$\Pr(X \geq a) \leq \frac{\mu_X}{a}.$$

Andrey Andreyevich Markov Sr.
1856-1922

See proof in *Géza Schay. Introduction to Probability…, p. 177*

Roulette

In roulette, a wheel with 38 numbered pockets is spun around, and a ball is rolled around its rim in the opposite direction until it falls at random into one of the pockets.
Eighteen of the numbers are black and 18 are red, while two are green.
One of the possible betting combinations is that of betting on red with a $1 payout for every $1 bet. Let us compute the expected gain from such a bet.

Roulette

If we denote the amount won or lost in a single play of \$1 by X, then $\Pr(X = 1) = \frac{18}{38}$ and $\Pr(X = -1) = \frac{20}{38}$.

Thus,

$$E[X] = \frac{18}{38} \cdot 1 + \frac{20}{38} \cdot (-1) \approx -0.526 = -5.26\, cents.$$

This result means that in the long run, the players will lose about 5.26 cents on every dollar bet.

Expected Value

Substituting *pdf* (*pmf*) of a certain distribution into general formula for the expected value we can obtain the expression for computing the expectation for the said distribution.

$$E[X] = \sum_{D_X} x_i \cdot p_X(x_i).$$

$$E[X] = \int_{D_X} x \cdot f_X(x) dx.$$

Expected Value

Distribution	Parameters	Expected Value	Remarks
Bernoulli	$0 \leq p \leq 1$	p	
Binomial	$n \geq 1, 0 \leq p \leq 1$	np	
Geometric	$0 \leq p \leq 1$	$\frac{1}{p}$	X – number of trials until success
Uniform	$a, b \in \mathbb{Z}$, or $a, b \in \mathbb{R}$	$\frac{a+b}{2}$	
Exponential	$\lambda > 0$	$\frac{1}{\lambda}$	
Weibull	$\eta, \beta > 0$	$\eta \cdot \Gamma\left(1 + \frac{1}{\beta}\right)$	$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$
Normal	$\mu, \sigma \in \mathbb{R}$	μ	

Suppose you are offered an opportunity for an investment whose expected return is \$500. If you are given the extra information that this expected value is the result of a 50% chance of a \$450 return and a 50% chance of a \$550 return, then you would not hesitate to spend \$450 on this investment.

However, if the expected return were the result of a 50% chance of a \$0 return and a 50% chance of a \$1000 return, then most people would be reluctant to spend such an amount.

This demonstrates that the spread (around the mean) of a r.v. is of great importance. Usually this is measured by the expected squared deviation from the mean.

Variance

Let X be any random variable with expected value $\mathrm{E}[X] = \mu_X$.

We define its *variance*, $Var(X)$, and *standard deviation*, σ_X, as

$$Var(X) = E[(X - \mu_X)^2]$$

and

$$\sigma_X = \sqrt{Var(X)},$$

provided $E[(X - \mu_X)^2]$ exists as a finite quantity.

Applying the formulae for the expected value of the discrete and continuous r.v. we get

$$Var(X) = \sum_{D_X} (x_i - \mu_X)^2 \cdot p_X(x_i) \quad \text{for discrete r.v.;}$$

$$Var(X) = \int_{D_X} (x - \mu_X)^2 \cdot f_X(x) dx \quad \text{for continuous r.v.}$$

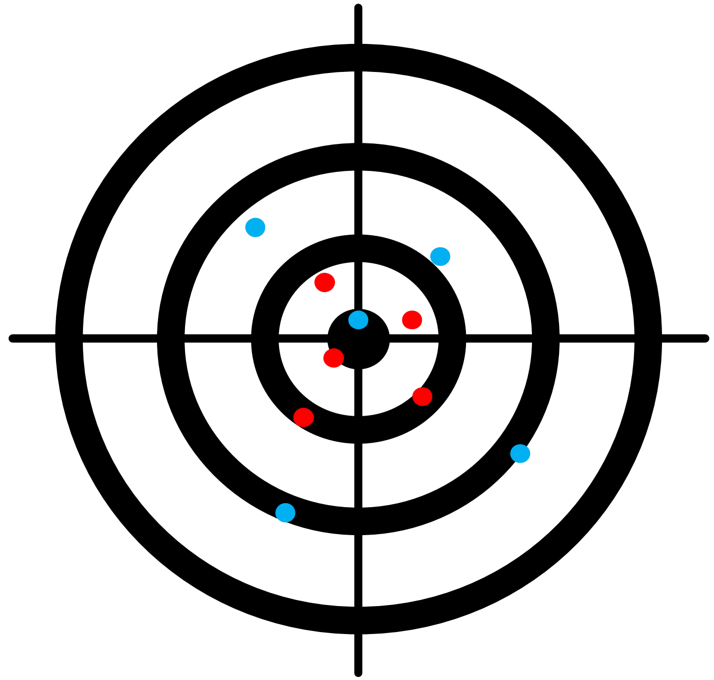
Note that $Var(X) \geq 0$. Obviously, $\sigma_X \geq 0$.

It is often preferable to work with the standard deviation rather than with the variance of a random variable, because it is easier to interpret.

Indeed, the standard deviation has the same dimension as X, whereas the unit of the variance are the squared units of X.

Variance

The variance is often interpreted geometrically as a measure of spread (or deviation) of a random variable about its expected value.



Red hit marks are grouped more compactly around the center of the target than the blue ones.

We say the shots of the red shooter have lesser variance (less dispersed) then the shots of the blue shooter.

Alternative notation for $Var(X)$: $D[X]$

Properties of $Var(X)$

If $C = const.$, $Var(C) = 0$.

If $a = const.$, $Var(X \pm a) = Var(X)$.

If $a = const.$, $Var(a \cdot X) = a^2 \cdot Var(X)$.

If X_i and X_j are independent r.v., $D[\sum_{k=1}^{n} X_k] = \sum_{k=1}^{n} D[X_k]$.

If X and Y are independent r.v., $D[X - Y] = D[X] + D[Y]$.

Properties of $Var(X)$

If X and Y are independent r.v.,

$$D[XY] = D[X] \cdot D[Y] + \mu_X^2 \cdot D[Y] + \mu_Y^2 \cdot D[X]$$

Equivalently,

$$E[X] \equiv \mu_X$$

$$D[XY] = E[X^2] \cdot E[Y^2] - \mu_X^2 \cdot \mu_Y^2$$

$$D[X] = E[X^2] - \mu_X^2$$

Variance

Substituting *pdf* (*pmf*) of a certain distribution into general formula for the variance we can obtain the expression for computing the variance for the said distribution.

$$D[X] = \sum_{D_X} (x_i - \mu_X)^2 \cdot p_X(x_i)$$

$$D[X] = \int_{D_X} (x - \mu_X)^2 \cdot f_X(x) dx$$

Variance

Distribution	Parameters	Variance	Remarks
Bernoulli	$0 \le p \le 1$	$p(1-p)$	
Binomial	$n \ge 1, 0 \le p \le 1$	$np(1-p)$	
Geometric	$0 \le p \le 1$	$\frac{1-p}{p^2}$	X – number of trials until success
Disc. Uniform	$a, b \in \mathbb{Z}$	$\frac{(b-a+1)^2-1}{12}$	
Cont. Uniform	$a, b \in \mathbb{R}$	$\frac{(b-a)^2}{12}$	
Exponential	$\lambda > 0$	$\frac{1}{\lambda^2}$	
Weibull	$\eta, \beta > 0$	$\eta^2 \left[\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma\left(1+\frac{1}{\beta}\right)^2\right]$	
Normal	$\mu, \sigma \in \mathbb{R}$	σ^2	

Textbook Assignment

Géza Schay. *Introduction to Probability…*

❖Chapter 5. 173-197 pp.

F.M. Dekking et al. *A Modern Introduction to…*

❖Chapter 7. 89-102 pp.